

Toeplitz Operators and the Roe-Higson Type Index Theorem in Riemannian Surfaces

Tatsuki SETO

Nagoya University

(Communicated by K. Ahara)

Abstract. Let M be a non-compact complete Riemannian manifold of dimension two and N a circle in M . We assume that M is partitioned by N . We define a unital C^* -algebra $C_b^*(M)$, which is slightly larger than the Roe algebra of M . We also construct $[u_\phi]$ in $K_1(C_b^*(M))$, which is a counter part of Roe's odd index class. We prove that Connes' pairing of Roe's cyclic one-cocycle with $[u_\phi]$ is equal to the Fredholm index of a Toeplitz operator on N . It is a part of an extension of the Roe-Higson index theorem to even-dimensional partitioned manifolds.

1. Introduction

Let M be a complete Riemannian manifold. We assume that M is a partitioned manifold, that is, there exists a closed hypersurface N in M such that M is decomposed by N into two submanifolds M^+ and M^- and we have $N = M^+ \cap M^- = \partial M^+ = \partial M^-$; see Definition 2.1. Let $S \rightarrow M$ be a Clifford bundle in the sense of [10, Definition 3.4] and D the Dirac operator on S defined by its Clifford structure. Denote by S_N the restriction of S to N and by ν the unit normal vector field on N pointing from M^- into M^+ . Then S_N can be equipped with a $\mathbf{Z}_2 (= \mathbf{Z}/2\mathbf{Z})$ -graded Clifford bundle structure, where a \mathbf{Z}_2 -graded structure of S_N is induced by the Clifford action of ν . Denote by D_N the graded Dirac operator on S_N .

Let $C^*(M)$ be the Roe algebra of M . $C^*(M)$ is a non-unital C^* -algebra, which is introduced by Roe [9]. In [9], He also defined the odd index class $\text{odd-ind}(D) \in K_1(C^*(M))$ out of D . It is given by $[u_D] - [1]$ in $K_1(C^*(M))$, where u_D is the Cayley transform of D . Note that $\text{odd-ind}(D)$ vanishes for a closed manifold M since we obtain $K_1(C^*(M)) = K_1(\mathcal{K}(L^2(S))) = 0$. He also defined the cyclic one-cocycle ζ on a dense subalgebra of $C^*(M)$, which is called the Roe cocycle. Recall that there is a pairing of cyclic cohomology with K -theory due to Connes [3]. In [9], Roe proved that Connes' pairing $\langle \text{odd-ind}(D), \zeta \rangle$ is equal to the Fredholm index of D_N^+ up to a certain constant multiple. In [6], Higson gave an alternative proof of Roe's theorem, thus we call it the Roe-Higson index theorem in this

Received April 6, 2015

2000 *Mathematics Subject Classification*: 19K56 (Primary), 46L87 (Secondary)

Key words and phrases: Index theory, Noncommutative geometry, Coarse geometry, Hilbert transformation, Roe algebra

paper. Higson calculated $\text{index}(1 - \varphi + \varphi u_D)$ as follows, where φ is a smooth function on M which is equal to the characteristic function of M^+ outside of the compact set. First, he proved $\text{index}(1 - \varphi + \varphi u_D) = \text{index}(D_N^+)$ the case when $M = \mathbf{R} \times N$ by proving the dimension of the kernel of $D \pm i(2\varphi - 1)$, a Callias-type operator, is equal to that of D_N^\pm , respectively. Second, he reduce the proof for a general partitioned manifold M to the case when $M = \mathbf{R} \times N$.

On the other hand, $\text{index}(D_N^+)$ is equal to zero when N is odd-dimensional (see, for instance, [10, Proposition 11.14]). This implies that the Roe-Higson index $\langle \text{odd-ind}(D), \zeta \rangle$ is trivial when M is of even dimension. However, Connes' pairing of the Roe cocycle ζ with an element in $K_1(C^*(M))$ is non trivial in general. In this paper, we shall develop an index theorem on even-dimensional partitioned manifold analogous to the Roe-Higson index theorem. For this purpose, we replace two parts, Roe's odd index class $\text{odd-ind}(D) \in K_1(C^*(M))$ and the Dirac operator D_N^+ by an index class $[u_\phi] \in K_1(C_b^*(M))$ and a Toeplitz operator on N , respectively. Here, $C_b^*(M)$ is a C^* -algebra which is slightly larger than $C^*(M)$; see Definition 2.5. Then it turns out that Connes' pairing $\langle [u_\phi], \zeta \rangle$ is equal to the Fredholm index of a Toeplitz operator on N up to a certain constant multiple. The precise statement is as follows.

THEOREM 1.1 (see Theorem 2.13). *Let M be an oriented complete Riemannian manifold. We assume that M is of dimension two and is partitioned manifold as previously. Let S be a \mathbf{Z}_2 -graded spin bundle over M with the grading ε and denote by D the graded Dirac operator on S . Denote by $\phi \in C^1(M; GL_1(\mathbf{C}))$ a $GL_1(\mathbf{C})$ -valued map of C^1 -class defined on M . Suppose that ϕ is bounded with bounded gradient and ϕ^{-1} is also bounded. Set*

$$u_\phi := (D + \varepsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D + \varepsilon).$$

Then the following formula holds:

$$\langle [u_\phi], \zeta \rangle = -\frac{1}{8\pi i} \text{index}(T_{\phi|_N}).$$

The outline of the proof is as follows. First, when M is a cylinder $\mathbf{R} \times S^1$, we carry out an explicit computation on $\text{index}(T_\phi)$ by using the Hilbert transformation and a standard basis $\{e^{ikx}\}_k$ of $L^2(S^1)$ in order to prove the equality. Then the proof for a general case can be reduced to that of a cylinder $\mathbf{R} \times S^1$ by applying a similar argument in Higson [6].

Certainly, it is interesting to extend our theorem to higher dimensional cases. In order to do this, we need a more general method. It is discussed in a forthcoming paper [11], where we also discuss a KK -theoretic construction of our index class $[u_\phi] \in K_1(C_b^*(M))$.

2. Main Theorem

In this section, we state our main theorem.

DEFINITION 2.1. Let M be an oriented complete Riemannian manifold. We assume that the triple (M^+, M^-, N) satisfies the following conditions:

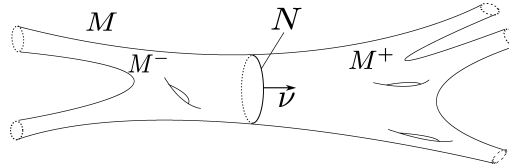


FIGURE 1. Partitioned manifold

- M^+ and M^- are submanifolds of M of the same dimension as M , $\partial M^+ \neq \emptyset$ and $\partial M^- \neq \emptyset$,
- $M = M^+ \cup M^-$,
- N is a closed submanifold of M of codimension one,
- $N = M^+ \cap M^- = -\partial M^+ = \partial M^-$.

Then we call (M^+, M^-, N) a partition of M . M is also called a partitioned manifold.

In this paper, we assume that (M, g) is an oriented two-dimensional complete Riemannian manifold (i.e. a complete Riemannian surface) that admits a partition (M^+, M^-, N) . Let S be a spin bundle of M ¹ with the \mathbf{Z}_2 -grading ε . Denote by c the Clifford action on S and denote by D the graded Dirac operator on S . For the simplicity, we assume that M is connected and that N is isometric to the unit circle S^1 . Then we also assume $S|_N$, that is a vector bundle on N of rank two, is isomorphic to a product bundle. Denote by ν the unit normal vector field on N pointing from M^- to M^+ . We introduce coordinate $x \in N$ with $\{\nu, \partial/\partial x\}$ is an orthonormal vector field on $N \subset M$. In these notation, we also assume

$$c(\nu)c(d/dx) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \text{ on } S|_N.$$

REMARK 2.2. We are interested in the case that M is non compact, then S is isomorphic to a product bundle: $S \cong M \times \mathbf{C}^2$. Especially, if $M = \mathbf{R}^2 \cong \mathbf{C}$ with standard metric, then one has

$$D = 2 \begin{bmatrix} 0 & -\partial/\partial \bar{z} \\ \partial/\partial z & 0 \end{bmatrix}.$$

DEFINITION 2.3 [9, p.191]. Let $\mathcal{L}(L^2(S))$ be the set of all bounded operators on the L^2 -sections of S . We denote by \mathcal{X} the $*$ -subalgebra of $\mathcal{L}(L^2(S))$ with the element has a smooth integral-kernel and finite propagation. We denote by $C^*(M)$ the completion of \mathcal{X} . We call $C^*(M)$ the Roe algebra.

The definition of $C^*(M)$ in Definition 2.3 is Roe’s first definition. In fact, we get same algebra of the definition in [7, Definition 6.3.8]. In the following, we collect some properties of the Roe algebra which we shall need.

¹Every orientable surface admits a spin structure [8, p.88]. We fix one.

PROPOSITION 2.4 [7, 9]. *We assume that M, S and D are as above. The followings hold.*

- (i) *Let $f \in C_0(\mathbf{R})$ be a continuous function on \mathbf{R} vanishing at infinity and $\lambda \in \mathbf{R}$. Set $D' := D + \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$. Then one has $f(D') \in C^*(M)$.*
- (ii) *Let $D^*(M)$ be the unital C^* -algebra generated by all pseudolocal operators on $L^2(S)^2$ with finite propagation. Then $C^*(M)$ is a closed bisided $*$ -ideal of $D^*(M)$.*
- (iii) *For all $u \in C^*(M)$ and $f \in C_0(M)$, one has $fu \sim 0$ and $uf \sim 0$. Here, define $T \sim S$ for $T, S \in \mathcal{L}(L^2(S))$ if $T - S$ is a compact operator.*
- (iv) *Let ϖ be the characteristic function of M^+ . Then one has $[\varpi, u] \sim 0$ for all $u \in C^*(M)$. Here we consider ϖ as a multiplication operator.*

By using Proposition 2.4, we define elements in a K_1 group. First, we define the unital C^* -algebra $C_b^*(M)$. $C_b^*(M)$ contains $C^*(M)$ as a closed bisided $*$ -ideal and our elements are in $K_1(C_b^*(M))$.

DEFINITION 2.5. Let $C_b(M)$ be the set of all bounded continuous functions on M . We consider any bounded functions on M as a multiplication operator on $L^2(S)$. Set

$$C_b^*(M) := \left\{ u + \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} ; u \in C^*(M), f, g \in C_b(M) \right\}.$$

Set $\|f\| := \sup_{x \in M} |f(x)|$ for $f \in C(M)$ and $\|X\| := \sup_{x \in M} \sqrt{g_x(X, X)}$ for $X \in \mathcal{X}(M)$.

PROPOSITION 2.6. *Let $\phi \in C^1(M; GL_l(\mathbf{C}))$ be a continuously differentiable map from M to the general linear group $GL_l(\mathbf{C})$. We assume that $\|\phi\| < \infty$, $\|\text{grad}(\phi)\| < \infty$ and $\|\phi^{-1}\| < \infty$. Set*

$$u_\phi := (D + \varepsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D + \varepsilon). \tag{1}$$

One has $u_\phi \in GL_l(C_b^(M))$. Thus one has $[u_\phi] \in K_1(C_b^*(M))$.*

PROOF. It suffices to show the case when $l = 1$. First, we obtain $(D + \varepsilon)^{-1} \in C^*(M)$ and $\|(D + \varepsilon)^{-1}\| \leq 2$ since $(D + \varepsilon)^{-1} = (D^2 + 1)^{-1}(D + \varepsilon) \in C^*(M)$. On the other hand, it follows

$$\begin{aligned} u_\phi \sigma &= (D + \varepsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & D^- \\ D^+ & -1 \end{bmatrix} \sigma \\ &= (D + \varepsilon)^{-1} \begin{bmatrix} \phi & D^- \phi - D^- \phi + \phi D^- \\ D^+ & -1 \end{bmatrix} \sigma \end{aligned}$$

² $T \in \mathcal{L}(L^2(S))$ is pseudolocal if $[f, T] \sim 0$ for all $f \in C_0(M)$, that is, $[f, T]$ is compact.

$$\begin{aligned} &= (D + \varepsilon)^{-1} \left((D + \varepsilon) \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \begin{bmatrix} \phi - 1 & [\phi, D^-] \\ 0 & \phi - 1 \end{bmatrix} \right) \sigma \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \sigma + (D + \varepsilon)^{-1} \begin{bmatrix} \phi - 1 & -c(\text{grad}(\phi))^- \\ 0 & \phi - 1 \end{bmatrix} \sigma \end{aligned}$$

for any $\sigma \in C_c^\infty(S)$, where $c(\text{grad}(\phi))^-$ is the restriction of $c(\text{grad}(\phi))$ to S^- . This implies

$$\|u_\phi \sigma\|_{L^2} \leq 3(\|\phi\| + \|\text{grad}(\phi)\| + 1)\|\sigma\|_{L^2}.$$

Thus u_ϕ is uniquely extended to a bounded operator on $L^2(S)$ since $C_c^\infty(S)$ is dense in $L^2(S)$. This proves $u_\phi \in C_b^*(M)$.

Second, since $\text{grad}(\phi^{-1}) = -\phi^{-2}\text{grad}(\phi)$, so we obtain $\|\text{grad}(\phi^{-1})\| < \infty$. This implies $u_{\phi^{-1}} \in C_b^*(M)$ and $u_{\phi^{-1}} = (u_\phi)^{-1}$. So we get $u_\phi \in GL_1(C_b^*(M))$ and $[u_\phi] \in K_1(C_b^*(M))$. \square

REMARK 2.7. Due to Proposition 2.6, one has

$$[u_\phi] - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \in K_1(C^*(M)).$$

But we do not use this point of view for the simplicity of Connes' pairing.

Next, we see Connes' pairing of the Roe's cyclic one-cocycle with $[u_\phi] \in K_1(C_b^*(M))$. Let ϖ be the characteristic function of M^+ , and set $\chi := 2\varpi - 1$. We note that $[\chi, u]$ is a compact operator for all $u \in C_b^*(M)$ since $[\chi, f] = 0$ for all $f \in C_b(M)$. We define a Banach algebra such as

$$\mathcal{A}_b := \{A \in C_b^*(M) ; [\chi, A] \text{ is of trace class}\}$$

with norm $\|A\|_{\mathcal{A}_b} := \|A\| + \|[\chi, A]\|_1$, where $\|\cdot\|$ is the operator norm on $L^2(S)$ and $\|\cdot\|_1$ is the trace norm. We define a cyclic one-cocycle on \mathcal{A}_b and take the pairing of it with a element in $K_1(C_b^*(M))$.

DEFINITION 2.8. For any $A, B \in \mathcal{A}_b$, set

$$\zeta(A, B) := \frac{1}{4} \text{Tr}(\chi[\chi, A][\chi, B]). \tag{2}$$

We call ζ the Roe cocycle.

PROPOSITION 2.9 [9, Proposition 1.6]. ζ is a cyclic one-cocycle on \mathcal{A}_b .

In order to take Connes' pairing of the Roe cocycle ζ with a element in $K_1(C_b^*(M))$, we need the following:

PROPOSITION 2.10. \mathcal{A}_b is dense and closed under holomorphic functional calculus in $C_b^*(M)$. So the inclusion $i : \mathcal{A}_b \rightarrow C_b^*(M)$ induces the isomorphism $i_* : K_1(\mathcal{A}_b) \cong K_1(C_b^*(M))$.

PROOF. Set

$$\mathcal{X}_b := \left\{ u + \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}; u \in \mathcal{X}, f, g \in C_b(M) \right\}.$$

\mathcal{X}_b is a dense subalgebra in $C_b^*(M)$ and we have $\mathcal{X}_b \subset \mathcal{A}_b$ [9, Proposition 1.6]. So \mathcal{A}_b is dense in $C_b^*(M)$.

The rest of proof is in [3, p.92]. □

Using Proposition 2.10, we can take the pairing of the Roe cocycle with an element in $K_1(C_b^*(M))$ through the isomorphism $i_* : K_1(\mathcal{A}_b) \cong K_1(C_b^*(M))$ as follows:

DEFINITION 2.11 [3, p.109]. Define the map

$$\langle \cdot, \zeta \rangle : K_1(C_b^*(M)) \rightarrow \mathbf{C}$$

by $\langle [u], \zeta \rangle := \frac{1}{8\pi i} \sum_{i,j} \zeta((u^{-1})_{ji}, u_{ij})$, where we assume $[u]$ is represented by an element of $GL_l(\mathcal{A}_b)$ and u_{ij} is the (i, j) -component of u . We note that this is Connes' pairing of cyclic cohomology with K -theory, and $\frac{1}{8\pi i}$ is a constant multiple appears in Connes' pairing.

The goal of this paper is to prove that the result of this pairing with $[u_\phi]$ is the Fredholm index of a Toeplitz operator. We review Toeplitz operators on S^1 to fix notations.

PROPOSITION 2.12 [4]. *Let $\phi \in C(S^1; GL_l(\mathbf{C}))$ be a continuous map from S^1 to $GL_l(\mathbf{C})$. Set $\mathcal{H} := \text{Span}_{\mathbf{C}}\{e^{ikx}; k = 0, 1, 2, \dots\} \subset L^2(S^1)$ ³ and let $P : L^2(S^1)^l \rightarrow \mathcal{H}^l$ be the projection. Then for any $f \in \mathcal{H}^l$, we define Toeplitz operator $T_\phi : \mathcal{H}^l \rightarrow \mathcal{H}^l$ by $T_\phi f := P(\phi f)$. Then T_ϕ is a Fredholm operator and the Fredholm index satisfies $\text{index}(T_\phi) = -\text{deg}(\det(\phi))$. Here $\text{deg}(\det(\phi))$ is the degree of the map $\det(\phi) : S^1 \rightarrow \mathbf{C}^\times$.*

We note that the Hardy space \mathcal{H} is generated by non-negative eigenfunctions of $-i\partial/\partial x$, which is a Dirac operator on S^1 . See also [1, p.160].

Using the above notation, we state our main theorem as follows. For the definition of u_ϕ , see (1) in Proposition 2.6 and for the definition of ζ , see (2) in Definition 2.8.

THEOREM 2.13. *We denote the restriction of $\phi \in C^1(M; GL_l(\mathbf{C}))$ to N by the same letter ϕ . Then the following formula holds:*

$$\langle [u_\phi], \zeta \rangle = -\frac{1}{8\pi i} \text{index}(T_\phi).$$

Here $\text{index}(T_\phi)$ of the right hand side is the Fredholm index of the Toeplitz operator of ϕ .

By the index theorem of Toeplitz operators (Proposition 2.12), the right hand side of this theorem is calculated by the mapping degree. Thus the above theorem can be considered as an index theorem for the pairing $\langle [u_\phi], \zeta \rangle$. Moreover, due to Section 3, we obtain the following:

³ \mathcal{H} is called the Hardy space.

COROLLARY 2.14. *Using the above notation, one has*

$$\text{index}(\varpi u_\phi \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l) = -\text{deg}(\det(\phi)).$$

The proof for Theorem 2.13 will be provided in Sections 3, 4 and 5.

3. The pairing and the Fredholm index

In order to prove Theorem 2.13, we firstly describe $\zeta(u^{-1}, u) = \sum_{i,j} \zeta((u^{-1})_{ji}, u_{ij})$ in terms of the Fredholm index of a certain operator.

PROPOSITION 3.1. *For any $u \in GL_l(\mathcal{A}_b)$, one has*

$$\zeta(u^{-1}, u) = -\text{index}(\varpi u \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l).$$

PROOF. Since $u \in GL_l(\mathcal{A}_b)$ and

$$\varpi - \varpi u^{-1} \varpi u \varpi = -\varpi[\varpi, u^{-1}][\varpi, u]\varpi,$$

so $\varpi - \varpi u^{-1} \varpi u \varpi$ and $\varpi - \varpi u \varpi u^{-1} \varpi$ are of trace class on $\varpi(L^2(S))^l$. Therefore we obtain

$\text{index}(\varpi u \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l) = \text{Tr}(\varpi - \varpi u^{-1} \varpi u \varpi) - \text{Tr}(\varpi - \varpi u \varpi u^{-1} \varpi)$
by [3, p.88]. So we get

$$\begin{aligned} \text{index}(\varpi u \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l) &= \frac{1}{4} \text{Tr}(\chi[\chi, u][\chi, u^{-1}]) \\ &= \frac{1}{4} \sum_{i,j} \text{Tr}(\chi[\chi, u_{ij}][\chi, (u^{-1})_{ji}]) = -\zeta(u^{-1}, u). \end{aligned}$$

□

Due to Proposition 3.1 and homotopy invariance of the Fredholm index, we obtain the following:

$$\langle [u_\phi], \zeta \rangle = -\frac{1}{8\pi i} \text{index}(\varpi u_\phi \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l). \tag{3}$$

So we shall calculate this Fredholm index.

4. The $\mathbf{R} \times S^1$ case

We firstly prove Theorem 2.13 in the case for $M = \mathbf{R} \times S^1$. In this section, we assume that the product $M = \mathbf{R} \times S^1$ is partitioned by $(\mathbf{R}_+ \times S^1, \mathbf{R}_- \times S^1, \{0\} \times S^1)$, where $\mathbf{R}_+ :=$

$\{t \in \mathbf{R}; t \geq 0\}$ and $\mathbf{R}_- := \{t \in \mathbf{R}; t \leq 0\}$ are half lines. Then the Dirac operator D on $S = \mathbf{R} \times S^1 \times \mathbf{C}^2$ is given by the following formula:

$$D = \begin{bmatrix} 0 & -\partial/\partial t - i\partial/\partial x \\ \partial/\partial t - i\partial/\partial x & 0 \end{bmatrix},$$

where we use the coordinate $(t, x) \in \mathbf{R} \times S^1$. Given a continuously differentiable map $\phi \in C^1(S^1; GL_l(\mathbf{C}))$, we define the map $\tilde{\phi} : \mathbf{R} \times S^1 \rightarrow GL_l(\mathbf{C})$ by $\tilde{\phi}(t, x) := \phi(x)$. We often denote $\tilde{\phi}$ by ϕ in the sequel.

In order to calculate $\text{index}(\varpi u_\phi \varpi : \varpi(L^2(S))^l \rightarrow \varpi(L^2(S))^l)$, we firstly perturb this operator by a homotopy.

PROPOSITION 4.1. *For any $s \in [0, 1]$, set*

$$D_s := \begin{bmatrix} 0 & -\partial/\partial t + s/2 - i\partial/\partial x \\ \partial/\partial t + s/2 - i\partial/\partial x & 0 \end{bmatrix} = D + \begin{bmatrix} 0 & s/2 \\ s/2 & 0 \end{bmatrix}$$

$$\text{and } u_{\phi,s} := (D_s + (1-s)\varepsilon)^{-1} \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} (D_s + (1-s)\varepsilon).$$

Then $[0, 1] \ni s \mapsto u_{\phi,s} \in GL_l(C_b^*(M))$ is continuous.

PROOF. It suffices to show the case when $l = 1$. We note that $\|D_s f\|_{L^2} \geq s\|f\|_{L^2}/2$ for any $f \in \text{domain}(D_s) = \text{domain}(D)$ and $s \in (0, 1]$. Moreover D_s is self-adjoint. Therefore the spectrum of D_s and $(-s/2, s/2)$ are disjoint, especially $D_s^{-1} \in \mathcal{L}(L^2(S))$.

Since $(D_s + (1-s)\varepsilon)^2 = D_s^2 + (1-s)^2$, so we obtain $(D_s + (1-s)\varepsilon)^{-1} \in C^*(M)$. Therefore $u_{\phi,s}$ is well defined as a closed operator densely defined on $\text{domain}(u_{\phi,s}) = \text{domain}(D)$. By a similar proof of Proposition 2.6, we obtain

$$u_{\phi,s} = \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + (D_s + (1-s)\varepsilon)^{-1} \begin{bmatrix} (1-s)(\phi - 1) & i\partial\phi/\partial x \\ 0 & (1-s)(\phi - 1) \end{bmatrix}$$

and $u_{\phi,s} \in GL_1(C_b^*(M))$.

Next we show $\|u_{\phi,s} - u_{\phi,s'}\| \rightarrow 0$ as $s \rightarrow s'$ for all $s' \in [0, 1]$. First, we show $\{\|(D_s + (1-s)\varepsilon)^{-1}\|\}_{s \in [0,1]}$ is a bounded set. Set $f_s(x) := \frac{x}{x^2+(1-s)^2}$ and $g_s(x) := \frac{1}{x^2+(1-s)^2}$ for $x \in \mathbf{R} \setminus (-s/2, s/2)$. By simple computation, we can show

$$\sup_{|x| \geq s/2} |f_s(x)| \leq \frac{5}{2} \quad \text{and} \quad \sup_{|x| \geq s/2} |g_s(x)| \leq \frac{5}{4}.$$

Therefore we obtain

$$\begin{aligned} \|(D_s + (1-s)\varepsilon)^{-1}\| &\leq \|(D_s^2 + (1-s)^2)^{-1} D_s\| + \|(1-s)(D_s^2 + (1-s)^2)^{-1}\| \\ &\leq \sup_{|x| \geq s/2} |f_s(x)| + \sup_{|x| \geq s/2} |g_s(x)| \leq 15/4 \end{aligned} \quad (*)$$

for all $s \in [0, 1]$. On the other hand, we have

$$\begin{aligned} & u_{\phi,s} - u_{\phi,s'} \\ &= \{(D_s + (1-s)\varepsilon)^{-1} - (D_{s'} + (1-s')\varepsilon)^{-1}\} \begin{bmatrix} (1-s)(\phi-1) & i\phi' \\ 0 & (1-s)(\phi-1) \end{bmatrix} \\ & \quad + (D_{s'} + (1-s')\varepsilon)^{-1} \begin{bmatrix} (s'-s)(\phi-1) & 0 \\ 0 & (s'-s)(\phi-1) \end{bmatrix} \end{aligned}$$

and the second term converges to 0 with the operator norm as $s \rightarrow s'$, thus it suffices to show $\|(D_s + (1-s)\varepsilon)^{-1} - (D_{s'} + (1-s')\varepsilon)^{-1}\| \rightarrow 0$ as $s \rightarrow s'$. But this is proved by (*) as follows:

$$\begin{aligned} & \|(D_s + (1-s)\varepsilon)^{-1} - (D_{s'} + (1-s')\varepsilon)^{-1}\| \\ &= \|(D_s + (1-s)\varepsilon)^{-1}((s-s')\varepsilon + D_{s'} - D_s)(D_{s'} + (1-s')\varepsilon)^{-1}\| \\ & \leq \frac{3}{2}|s-s'| \|(D_{s'} + (1-s')\varepsilon)^{-1}\| \|(D_s + (1-s)\varepsilon)^{-1}\| \\ & \leq 32|s-s'| \rightarrow 0. \end{aligned}$$

□

Due to Proposition 4.1, we obtain

$$\text{index}(\varpi u_\phi \varpi) = \text{index}(\varpi u_{\phi,0} \varpi) = \text{index}(\varpi u_{\phi,1} \varpi).$$

Set

$$\mathcal{T}_\phi := \varpi(-\partial/\partial t + 1/2 - i\partial/\partial x)^{-1} \phi(-\partial/\partial t + 1/2 - i\partial/\partial x) \varpi.$$

Since we have $\varpi u_{\phi,1} \varpi = \begin{bmatrix} \varpi & 0 \\ 0 & \mathcal{T}_\phi \end{bmatrix}$, so $\text{index}(\varpi u_\phi \varpi)$ equals to

$$\text{index}(\mathcal{T}_\phi : \varpi(L^2(\mathbf{R})) \otimes L^2(S^1)^l \rightarrow \varpi(L^2(\mathbf{R})) \otimes L^2(S^1)^l).$$

Next, we treat the Fredholm index of \mathcal{T}_ϕ . Let $\mathcal{F} : L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$ be the Fourier transformation:

$$\mathcal{F}[f](\xi) := \int_{\mathbf{R}} e^{-ix\xi} f(x) dx.$$

Let $H : L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$ be the Hilbert transformation⁴:

$$Hf(t) := \frac{i}{\pi} \text{p.v.} \int_{\mathbf{R}} \frac{f(y)}{t-y} dy,$$

⁴In literature, the coefficient of the Hilbert transformation in the right hand side is usually $1/\pi$. We need a coefficient i/π in order to get $H^2 = \text{id}$.

where p.v. is Cauchy's principal value. H can be verified $H^2 = \text{id}$. Then we denote by $\hat{P} : L^2(\mathbf{R}) \rightarrow \mathcal{H}_-$ the projection to the (-1) -eigenspace \mathcal{H}_- of H , that is, $\hat{P} := \frac{1}{2}(\text{id} - H)$. Since \mathcal{F} induces a invertible operator from $\varpi(L^2(\mathbf{R}))$ to \mathcal{H}_- , so we obtain

$$\begin{aligned} & \text{index}\left(\mathcal{T}_\phi : \varpi(L^2(\mathbf{R})) \otimes L^2(S^1) \rightarrow \varpi(L^2(\mathbf{R})) \otimes L^2(S^1)\right) \\ &= \text{index}\left(\mathcal{F}\mathcal{T}_\phi\mathcal{F}^{-1} : \mathcal{H}_- \otimes L^2(S^1) \rightarrow \mathcal{H}_- \otimes L^2(S^1)\right). \end{aligned}$$

Set

$$\hat{\mathcal{T}}_\phi := \mathcal{F}\mathcal{T}_\phi\mathcal{F}^{-1} = \hat{P}(-it + 1/2 - i\partial/\partial x)^{-1}\phi(-it + 1/2 - i\partial/\partial x)\hat{P}^*.$$

In order to calculate the Fredholm index of $\hat{\mathcal{T}}_\phi$, we use a basis of $L^2(\mathbf{R})$ consisting of eigenvectors of the Hilbert transformation.

PROPOSITION 4.2 [12, Theorem 1]. Define $\rho_n \in L^2(\mathbf{R})$ by

$$\rho_n(t) := \frac{(t - i)^n}{(t + i)^{n+1}}.$$

Then $\{\rho_n/\sqrt{\pi}\}$ is an orthonormal basis of $L^2(\mathbf{R})$ and one has

$$H\rho_n = \begin{cases} \rho_n & \text{if } n < 0 \\ -\rho_n & \text{if } n \geq 0 \end{cases}.$$

Due to Proposition 4.2, we obtain $\mathcal{H}_- = \text{Span}_{\mathbf{C}}\{\rho_n; n \geq 0\}$ and we can calculate following Fredholm indices.

LEMMA 4.3. For any $\alpha, \beta \neq 0$, $\hat{P}\frac{t+i\beta}{t+i\alpha}\hat{P}^* \in \mathcal{L}(\mathcal{H}_-)$ is a Fredholm operator and one has

$$\text{index}\left(\hat{P}\frac{t+i\beta}{t+i\alpha}\hat{P}^*\right) = \begin{cases} 0 & \text{if } \alpha\beta > 0 \\ -1 & \text{if } \alpha > 0, \beta < 0 \\ 1 & \text{if } \alpha < 0, \beta > 0 \end{cases}.$$

PROOF. Basically, our proof is adopted from [2, p.99]. Let $c : \mathbf{R} \rightarrow S^1(\subset \mathbf{C})$ be the Cayley transformation: $c(t) := (t - i)(t + i)^{-1}$. Set $\Phi(g)(t) := (t + i)^{-1}g(c(t))$ for any $g \in L^2(S^1)$. Then $\Phi : L^2(S^1) \rightarrow L^2(\mathbf{R})$ is an invertible bounded linear operator with $\|\Phi\| = 1/\sqrt{2}$ and $\Phi^{-1}(f)(z) = (c^{-1}(z) + i)f(c^{-1}(z))$ for all $f \in L^2(\mathbf{R})$ and $z \in S^1 \setminus \{1\}$. Since $\Phi(e^{inx}) = \rho_n$, so $\hat{P}\hat{\mathcal{T}}_\phi\hat{P}^*$ is a Fredholm operator on \mathcal{H}_- for any $\phi \in C^\infty(\mathbf{R}; \mathbf{C}^\times)$ with $\lim_{t \rightarrow \infty} \phi(t) = \lim_{t \rightarrow -\infty} \phi(t) \in \mathbf{C}^\times$. Now, we can calculate

$$\left|\frac{t+i\beta}{t+i\alpha}\right|^2 = \frac{t^2 + \beta^2}{t^2 + \alpha^2} > 0$$

and $\lim_{t \rightarrow \pm\infty} \frac{t+i\beta}{t+i\alpha} = 1$. Therefore $\hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^*$ is a Fredholm operator.

We calculate $\text{index}(\hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^*)$. Set $\text{sgn}(\alpha) := \begin{cases} 1 & \text{if } \alpha \geq 0 \\ -1 & \text{if } \alpha < 0 \end{cases}$. Then we define a

homotopy of Fredholm operators from $\hat{P} \frac{t+i\beta}{t+i\alpha} \hat{P}^*$ to $\hat{P} \frac{t+i\text{sgn}(\beta)}{t+i\text{sgn}(\alpha)} \hat{P}^*$ by

$$\hat{P} \frac{t + i(s\beta + (1-s)\text{sgn}(\beta))}{t + i(s\alpha + (1-s)\text{sgn}(\alpha))} \hat{P}^*$$

for $s \in [0, 1]$. Therefore we obtain

$$\text{index} \left(\hat{P} \frac{t + i\beta}{t + i\alpha} \hat{P}^* \right) = \text{index} \left(\hat{P} \frac{t + i\text{sgn}(\beta)}{t + i\text{sgn}(\alpha)} \hat{P}^* \right) = \begin{cases} 0 & \text{if } \alpha\beta > 0 \\ -1 & \text{if } \alpha > 0, \beta < 0 \\ 1 & \text{if } \alpha < 0, \beta > 0 \end{cases}$$

by $\mathcal{H}_- = \text{Span}_{\mathbf{C}}\{\rho_n; n \geq 0\}$. □

Set $\phi_k(x) = e^{ikx}$ on S^1 for $k \in \mathbf{Z}$. By using Lemma 4.3, we calculate

$$\text{index}(\hat{\mathcal{T}}_{\phi_k} : \mathcal{H}_- \otimes L^2(S^1) \rightarrow \mathcal{H}_- \otimes L^2(S^1)),$$

which turns out to be $\text{index}(T_{\phi_k})$ for the classical Toeplitz operator.

PROPOSITION 4.4. *One has $\text{index}(\varpi u_{\phi_k} \varpi) = \text{index}(\hat{\mathcal{T}}_{\phi_k}) = -k = \text{index}(T_{\phi_k})$.*

PROOF. The first equality is proved above in this section and the last equality is well known. So it suffices to show the second equality. Let $E_\lambda := \mathbf{C}\{e^{i\lambda x}\}$ be the λ -eigenspace of $-i\partial/\partial x$. On $\mathcal{H}_- \otimes E_\lambda$, $\hat{\mathcal{T}}_{\phi_k}$ acts as

$$\hat{P}(-it + 1/2 + \lambda + k)^{-1}(-it + 1/2 + \lambda)\hat{P}^* \otimes \phi_k$$

and $\hat{\mathcal{T}}_{\phi_k}(\mathcal{H}_- \otimes E_\lambda)$ is contained in $\mathcal{H}_- \otimes E_{\lambda+k}$. Therefore we obtain

$$\begin{aligned} & \text{index}(\hat{\mathcal{T}}_{\phi_k}) \\ &= \sum_{\lambda=-\infty}^{\infty} \text{index} \left(\hat{P} \frac{t + i(\lambda + 1/2)}{t + i(\lambda + k + 1/2)} \hat{P}^* \otimes \phi_k : \mathcal{H}_- \otimes E_\lambda \rightarrow \mathcal{H}_- \otimes E_{\lambda+k} \right) \\ &= -k \end{aligned}$$

by Lemma 4.3. □

Next, let $\phi \in C^1(S^1; GL_l(\mathbf{C}))$ be any mapping of C^1 -class. We prove $\text{index}(\hat{\mathcal{T}}_\phi) = \text{index}(T_\phi)$. For this purpose, we reduce to Proposition 4.4.

As well known (see, for instance, [5, Example 4.55]), the inclusion $i : C(S^1; S^1) \rightarrow C(S^1; GL_l(\mathbf{C}))$ defined by $i(f) := \begin{bmatrix} f & 0 \\ 0 & 1_{l-1} \end{bmatrix}$ induces the isomorphism on fundamental

groups $i_* : \pi_1(S^1) \rightarrow \pi_1(GL_l(\mathbf{C}))$. In fact, the homotopy class of ϕ_k is corresponding to $k \in \mathbf{Z} \cong \pi_1(S^1)$. Therefore ϕ is homotopic to $\begin{bmatrix} \phi_k & 0 \\ 0 & 1_{l-1} \end{bmatrix}$ in $C(S^1; GL_l(\mathbf{C}))$ for some $k \in \mathbf{Z}$. We denote this homotopy by ψ_s . Moreover, since $C^1(S^1)$ is dense and closed under holomorphic functional calculus in $C(S^1)$, so we can take this homotopy ψ_s in $C^1(S^1; GL_l(\mathbf{C}))$.

PROPOSITION 4.5. u_{ψ_s} is a continuous path in $\mathcal{L}(L^2(S)^l)$.

PROOF. By the proof of Proposition 2.6, we obtain

$$u_{\psi_s} = \begin{bmatrix} 1 & 0 \\ 0 & \psi_s \end{bmatrix} + (D + \varepsilon)^{-1} \begin{bmatrix} \psi_s - 1 & i\psi'_s \\ 0 & \psi_s - 1 \end{bmatrix}.$$

This implies

$$\|u_{\psi_s} - u_{\psi_{s'}}\| \leq 3\|\psi_s - \psi_{s'}\|_{C^1} \rightarrow 0$$

as $s \rightarrow s'$ since ψ_s is a continuous path in $C^1(S^1; GL_l(\mathbf{C}))$. □

PROOF OF THEOREM 2.13 FOR $M = \mathbf{R} \times S^1$. Due to Proposition 4.5, we have

$$\text{index}(\varpi u_\phi \varpi) = \text{index} \left(\varpi \begin{bmatrix} u_{\phi_k} & 0 \\ 0 & 1_{l-1} \end{bmatrix} \varpi \right) = \text{index}(\varpi u_{\phi_k} \varpi)$$

for some $k \in \mathbf{Z}$. On the other hand, since ϕ is homotopic to $\begin{bmatrix} \phi_k & 0 \\ 0 & 1_{l-1} \end{bmatrix}$, so we obtain $\text{index}(T_{\phi_k}) = \text{index}(T_\phi)$. Due to Proposition 4.4, we obtain $\text{index}(\varpi u_\phi \varpi) = \text{index}(T_\phi)$, which completes the proof with (3) in Section 3. □

5. The general two-manifold case

In this section we reduce the proof for the general two-dimensional manifold to the proof for $\mathbf{R} \times S^1$. Our argument is similar to Higson's in [6]. Firstly, we shall show cobordism invariance of the pairing. See also [6, Lemma 1.4].

LEMMA 5.1. Let (M^+, M^-, N) and $(M^{+'}, M^{-'}, N')$ be two partitions of M . We assume these two partitions are cobordant, that is, symmetric differences $M^\pm \Delta M^{\mp'}$ are compact. Let ϖ and ϖ' be the characteristic function of M^+ and $M^{+'}$, respectively. We assume $\phi \in C^1(M; GL_l(\mathbf{C}))$ satisfies $\|\phi\| < \infty$, $\|\text{grad}(\phi)\| < \infty$ and $\|\phi^{-1}\| < \infty$. Then one has $\text{index}(\varpi u_\phi \varpi) = \text{index}(\varpi' u_\phi \varpi')$.

PROOF. It suffices to show the case when $l = 1$. Because of $[\phi, \varpi] = 0$ and $[u_\phi, \varpi] \sim 0$, we obtain

$$\text{index}(\varpi u_\phi \varpi : \varpi(L^2(S)) \rightarrow \varpi(L^2(S)))$$

$$\begin{aligned} &= \text{index} \left((1 - \varpi) \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \varpi u_\phi \varpi : L^2(S) \rightarrow L^2(S) \right) \\ &= \text{index} \left((1 - \varpi) \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \varpi u_\phi : L^2(S) \rightarrow L^2(S) \right) \\ &= \text{index} \left(\begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} + \varpi v_\phi : L^2(S) \rightarrow L^2(S) \right), \end{aligned}$$

where set $v_\phi := u_\phi - \begin{bmatrix} 1 & 0 \\ 0 & \phi \end{bmatrix} \in C^*(M)$. Therefore it suffices to show $\varpi v_\phi \sim \varpi' v_\phi$. Now, since $M^\pm \triangle M^{\mp'}$ are compact, there exists $f \in C_0(M)$ such that $\varpi - \varpi' = (\varpi - \varpi')f$. So we obtain $\varpi v_\phi - \varpi' v_\phi = (\varpi - \varpi')f v_\phi \sim 0$. □

Secondly, we shall prove an analogue of Higson’s lemma [6, Lemma 3.1].

LEMMA 5.2. *Let M_1 and M_2 be two oriented complete Riemannian manifolds with a partition (M_1^+, M_1^-, N_1) and (M_2^+, M_2^-, N_2) , respectively, and S_j a Hermitian vector bundle over M_j . Let ϖ_j be the characteristic function of M_j^+ . We assume that there exists an isometry $\gamma : M_2^+ \rightarrow M_1^+$ which lifts an isomorphism $\gamma^* : S_1|_{M_1^+} \rightarrow S_2|_{M_2^+}$. We denote the Hilbert space isometry defined by γ^* by the same letter $\gamma^* : \varpi_1(L^2(S_1)) \rightarrow \varpi_2(L^2(S_2))$. We assume $u_j \in GL_1(C_b^*(M_j))$ satisfies $\gamma^* u_1 \varpi_1 \sim \varpi_2 u_2 \gamma^*$. Then one has $\text{index}(\varpi_1 u_1 \varpi_1) = \text{index}(\varpi_2 u_2 \varpi_2)$.*

Similarly, if there exists an isometry $\gamma : M_2^- \rightarrow M_1^-$ which lifts an isomorphism $\gamma^* : S_1|_{M_1^-} \rightarrow S_2|_{M_2^-}$ and $\gamma^* u_1 \varpi_1 \sim \varpi_2 u_2 \gamma^*$, then one has $\text{index}(\varpi_1 u_1 \varpi_1) = \text{index}(\varpi_2 u_2 \varpi_2)$.

PROOF. It suffices to show the case when $l = 1$. Let $v : (1 - \varpi_1)(L^2(S_1)) \rightarrow (1 - \varpi_2)(L^2(S_2))$ be any invertible operator. Then $V := \gamma^* \varpi_1 + v(1 - \varpi_1) : L^2(S_1) \rightarrow L^2(S_2)$ is also invertible. Hence we obtain

$$\begin{aligned} &V((1 - \varpi_1) + \varpi_1 u_1 \varpi_1) - ((1 - \varpi_2) + \varpi_2 u_2 \varpi_2)V \\ &= -\gamma^* \varpi_1 + \varpi_2 \gamma^* + \gamma^* \varpi_1 u_1 \varpi_1 - \varpi_2 u_2 \varpi_2 \gamma^* \\ &\sim \gamma^* u_1 \varpi_1 - \varpi_2 u_2 \gamma^* \sim 0. \end{aligned}$$

Therefore we obtain $\text{index}(\varpi_1 u_1 \varpi_1) = \text{index}(\varpi_2 u_2 \varpi_2)$ since V is an invertible operator and we have $\text{index}(\varpi_j u_j \varpi_j) = \text{index}((1 - \varpi_j) + \varpi_j u_j \varpi_j)$ for $j = 1, 2$. □

Applying Lemma 5.2, we prove the following:

COROLLARY 5.3. *Let M_j be an oriented complete Riemannian surface with a partition (M_j^+, M_j^-, N_j) and S_j a graded spin bundle over M_j with the grading ε_j . We assume that there exists an isometry $\gamma : M_2^+ \rightarrow M_1^+$ which defines the Hilbert space isometry $\gamma^* : \varpi_1(L^2(S_1)) \rightarrow \varpi_2(L^2(S_2))$ as in Lemma 5.2 and γ^* satisfies $D_2 \gamma^* \sim \gamma^* D_1$*

and $\varepsilon_2\gamma^* \sim \gamma^*\varepsilon_1$ on $\varpi_1(L^2(S_1))$. Let $\phi_j \in C^1(M_j; GL_1(\mathbf{C}))$ satisfies $\|\phi_j\| < \infty$, $\|\text{grad}(\phi_j)\| < \infty$ and $\|\phi_j^{-1}\| < \infty$ as in Proposition 2.6. Assume that ϕ_1 and ϕ_2 satisfy $\phi_1(\gamma(x)) = \phi_2(x)$ for all $x \in M_2^+$. Then we obtain $\text{index}(\varpi_1 u_{\phi_1} \varpi_1) = \text{index}(\varpi_2 u_{\phi_2} \varpi_2)$.

PROOF. It suffices to show $\gamma^* u_{\phi_1} \varpi_1 \sim \varpi_2 u_{\phi_2} \gamma^*$. Let φ_1 be a smooth function on M_1 such that $\text{Supp}(\varphi_1) \subset M_1^+$ and there exists a compact set $K_1 \subset M_1$ such that $\varphi_1 = \varpi_1$ on $M_1 \setminus K_1$. Set $\varphi_2(x) := \varphi_1(\gamma(x))$ for all $x \in M_2^+$ and $\varphi_2 = 0$ on M_2^- . Then φ_2 is a smooth function on M_2 such that $\text{Supp}(\varphi_2) \subset M_2^+$ and there exists a compact set $K_2 \subset M_2$ such that $\varphi_2 = \varpi_2$ on $M_2 \setminus K_2$. Set $v_{\phi_j} := u_{\phi_j} - \begin{bmatrix} 1 & 0 \\ 0 & \phi_j \end{bmatrix}$. Then we obtain $\gamma^* v_{\phi_1} \varpi_1 \sim \gamma^* v_{\phi_1} \varphi_1$ and $\varpi_2 v_{\phi_2} \gamma^* \sim \varphi_2 v_{\phi_2} \gamma^*$. So if $\gamma^* v_{\phi_1} \varphi_1 \sim \varphi_2 v_{\phi_2} \gamma^*$, then we obtain

$$\gamma^* u_{\phi_1} \varpi_1 \sim \gamma^* v_{\phi_1} \varphi_1 + \gamma^* \begin{bmatrix} 1 & 0 \\ 0 & \phi_1 \end{bmatrix} \varpi_1 \sim \varphi_2 v_{\phi_2} \gamma^* + \varpi_2 \begin{bmatrix} 1 & 0 \\ 0 & \phi_2 \end{bmatrix} \gamma^* \sim \varpi_2 u_{\phi_2} \gamma^*.$$

So it suffices to show $\gamma^* v_{\phi_1} \varphi_1 \sim \varphi_2 v_{\phi_2} \gamma^*$. In fact, we obtain

$$\begin{aligned} & \gamma^* v_{\phi_1} \varphi_1 - \varphi_2 v_{\phi_2} \gamma^* \\ &= \gamma^*(D_1 + \varepsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \varphi_1 \\ & \quad - \varphi_2 (D_2 + \varepsilon_2)^{-1} \begin{bmatrix} \phi_2 - 1 & -c(\text{grad}(\phi_2))^- \\ 0 & \phi_2 - 1 \end{bmatrix} \gamma^* \\ &= \{\gamma^*(D_1 + \varepsilon_1)^{-1} \varphi_1 - \varphi_2 (D_2 + \varepsilon_2)^{-1} \gamma^*\} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\ & \sim \{\gamma^* \varphi_1 (D_1 + \varepsilon_1)^{-1} - (D_2 + \varepsilon_2)^{-1} \gamma^* \varphi_1\} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\ &= (D_2 + \varepsilon_2)^{-1} \{(D_2 + \varepsilon_2) \gamma^* \varphi_1 - \gamma^* \varphi_1 (D_1 + \varepsilon_1)\} (D_1 + \varepsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\ & \sim (D_2 + \varepsilon_2)^{-1} \gamma^* [D_1, \varphi_1] (D_1 + \varepsilon_1)^{-1} \begin{bmatrix} \phi_1 - 1 & -c(\text{grad}(\phi_1))^- \\ 0 & \phi_1 - 1 \end{bmatrix} \\ & \sim 0 \end{aligned}$$

since $\text{grad}(\varphi_1)$ has a compact support and we have $[D_1, \varphi_1] = c(\text{grad}(\varphi_1))$. Thus we obtain $\gamma^* u_{\phi_1} \varpi_1 \sim \varpi_2 u_{\phi_2} \gamma^*$. Therefore we get $\text{index}(\varpi_1 u_{\phi_1} \varpi_1) = \text{index}(\varpi_2 u_{\phi_2} \varpi_2)$ by Lemma 5.2. □

PROOF OF THEOREM 2.13, THE GENERAL CASE. Firstly, let $a \in C^\infty([-1, 1]; [-1, 1])$

satisfies

$$a(t) = \begin{cases} -1 & \text{if } -1 \leq t \leq -3/4 \\ 0 & \text{if } -2/4 \leq t \leq 2/4 \\ 1 & \text{if } 3/4 \leq t \leq 1 \end{cases} .$$

Let $(-4\delta, 4\delta) \times N$ be diffeomorphic to a tubular neighborhood of N in M satisfies

$$\sup_{(t,x),(s,y) \in [-3\delta, 3\delta] \times N} |\phi(t, x) - \phi(s, y)| < \|\phi^{-1}\|^{-1} .$$

Set $\psi(t, x) := \phi(4\delta a(t), x)$ on $(-4\delta, 4\delta) \times N$ and $\psi = \phi$ on $M \setminus (-4\delta, 4\delta) \times N$. Then we obtain $\psi \in C^1(M; GL_l(\mathbf{C}))$ and $\|\psi - \phi\| < \|\phi^{-1}\|^{-1}$. Thus a map $[0, 1] \ni t \mapsto \psi_t := t\psi + (1 - t)\phi$ satisfies $\psi_t \in C^1(M; GL_l(\mathbf{C}))$, $\|\psi_t\| < \infty$, $\|grad(\psi_t)\| < \infty$, $\|\psi_t^{-1}\| < \infty$, $\|\psi_t - \psi_{t'}\| \rightarrow 0$ as $t \rightarrow t' \in [0, 1]$ and $\|grad(\psi_t) - grad(\psi_{t'})\| \rightarrow 0$ as $t \rightarrow t' \in [0, 1]$. Therefore it suffices to show the case of which ϕ satisfies $\phi(t, x) = \phi(0, x)$ on $(-2\delta, 2\delta) \times N$. Due to Lemma 5.1, we may change a partition of M to $(M^+ \cup ([-\delta, 0] \times N), M^- \setminus ((-\delta, 0] \times N), \{-\delta\} \times N)$ without changing $index(\varpi u_\phi \varpi)$. Then due to Corollary 5.3, we may change $M^+ \cup ([-\delta, 0] \times N)$ to $[-\delta, \infty) \times N$ without changing $index(\varpi u_\phi \varpi)$. Here ϕ is equals to $\phi(0, x)$ on $[-\delta, \infty) \times N$ and the metric on $[\delta, \infty) \times N$ is product. We denote this manifold by $M' := ([-\delta, \infty) \times N) \cup (M^- \setminus ((-\delta, 0] \times N))$. M' is partitioned by $([-\delta, \infty) \times N, M^- \setminus ((-\delta, 0] \times N), \{-\delta\} \times N)$. We apply a similar argument to M' , we may change M' to a product manifold $\mathbf{R} \times N$ without changing $index(\varpi u_\phi \varpi)$. Now we have changed M to $\mathbf{R} \times N = \mathbf{R} \times S^1$. □

6. Example

In this section, we exhibit an example of a partitioned manifold that is not diffeomorphic to $\mathbf{R} \times S^1$ with non-trivial pairing $\langle [u_\phi], \zeta \rangle$.

Let Σ_2 be a closed Riemannian surface of genus two and C and C' two disjoint submanifolds of Σ_2 , respectively, whose homology class give generators of $H_1(\Sigma_2; \mathbf{Z})$ (see Figure 2). Then we cut Σ_2 along C and C' and embed it to \mathbf{R}^3 as in Figure 3.

We assume this embedded surface S is an oriented Riemannian submanifold in \mathbf{R}^3 . Then we paste many copies of S like the Cayley graph of F_2 , where we denote by F_2 a free group with two generators. Denote this surface by M . Then M is a covering space on Σ_2 and F_2 acts freely on M as a deck transformation. Denote by $\pi : M \rightarrow \Sigma_2$ this covering map.

Let $N \subset \pi^{-1}(C)$ be a connected component of $\pi^{-1}(C)$. Then M is decomposed by N into two components. So we can define M^+ and M^- which satisfy $N = \partial M^-$. Therefore M is a partitioned manifold.

On the other hand, there exists a C^1 -map $\varphi : \Sigma_2 \rightarrow GL_l(\mathbf{C})$ such that $deg(\det(\varphi|_C)) \neq 0$ since $[C] \neq 0$. For example, we choose $\psi : S^1 \rightarrow GL_l(\mathbf{C})$ such that $deg(\det(\psi)) \neq 0$, and

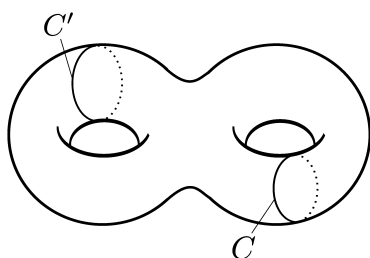


FIGURE 2

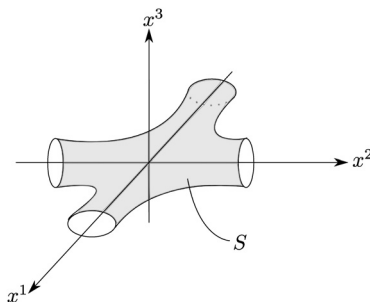


FIGURE 3

we extend on $T^2 = S^1 \times S^1$ trivially. Then we can define such φ on Σ_2 through $\Sigma_2 = T^2 \# T^2$. Set $\phi := \varphi \circ \pi$, then ϕ satisfies assumptions in Theorem 2.13.

In above setting, we obtain $\deg(\det(\phi|_N)) = \deg(\det(\varphi|_C))$. Therefore we get $\langle [u_\phi], \zeta \rangle \neq 0$.

ACKNOWLEDGMENTS. The author would like to thank his advisor Professor Hitoshi Moriyoshi for helpful comments.

References

- [1] P. BAUM and R. G. DOUGLAS, Toeplitz operators and Poincaré duality, In *Toeplitz centennial* (Tel Aviv, 1981), Vol. 4 of Operator Theory: Adv. Appl., 137–166. Birkhäuser, Basel, 1982.
- [2] B. BOOSS-BAVNBK and D. D. BLEECKER, *Topology and analysis*, Universitext. Springer-Verlag, New York, 1985. The Atiyah-Singer index formula and gauge-theoretic physics, Translated from the German by Bleecker and A. Mader.
- [3] A. CONNES, Noncommutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360.
- [4] I. C. GOHBERG and M. G. KREIN, Systems of integral equations on a half-line with kernels depending on the difference of arguments, *Uspehi Mat. Nauk (N.S.)* **13** (1958), No. 2 (80), 3–72.
- [5] A. HATCHER, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
- [6] N. HIGSON, A note on the cobordism invariance of the index, *Topology* **30** (1991), No. 3, 439–443.
- [7] N. HIGSON and J. ROE, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000, Oxford Science Publications.
- [8] H. B. LAWSON, JR. and M.-L. MICHELSON, *Spin geometry*, Vol. 38 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1989.
- [9] J. ROE, Partitioning noncompact manifolds and the dual Toeplitz problem, In *Operator algebras and applications, Vol. 1*, Vol. 135 of London Math. Soc. Lecture Note Ser., pp. 187–228, Cambridge Univ. Press, Cambridge, 1988.
- [10] J. ROE, *Elliptic operators, topology and asymptotic methods*, Vol. 395 of Pitman Research Notes in Mathematics Series, Longman, Harlow, second edition, 1998.
- [11] T. SETO, Toeplitz operators and the Roe-Higson type index theorem, 2014, arXiv:1405.4852.

- [12] J. A. C. WEIDEMAN, Computing the Hilbert transform on the real line, *Math. Comp.* **64** (1995), No. 210, 745–762.

Present Address:

GRADUATE SCHOOL OF MATHEMATICS,
NAGOYA UNIVERSITY,
FUROCHO, CHIKUSA-KU, NAGOYA, JAPAN.
e-mail: m11034y@math.nagoya-u.ac.jp