TOKYO J. MATH.
VOL. 39, NO. 3, 2017

# On the Construction of Continued Fraction Normal Series in Positive Characteristic

Dedicated to Ken-ichi Shinoda

Dong Han KIM, Hitoshi NAKADA and Rie NATSUI

*Dongguk University, Keio University and Japan Women's University*

**Abstract.** Motivated by the famous Champernowne construction of a normal number, R. Adler, M. Keane, and M. Smorodinsky constructed a normal number with respect to the simple continued fraction transformation. In this paper, we follow their idea and construct a normal series for the Artin continued fraction expansion in positive characteristic. A normal series for Lüroth expansion is also discussed.

## 1. Introduction

After D. G. Champernowne [4], a number of works have been done for constructions of normal numbers for various types of expansions of numbers, mostly by finitely many digits. In this paper, we are interested in constructions of normal formal power series with respect to expansions with countably many digits (polynomials).

We consider a sequence of rational numbers $\{r_n\}$ given as follows

$$r_1 = \frac{1}{2}\,,\ r_2 = \frac{1}{3}\,,\ r_3 = \frac{2}{3}\,,\ r_4 = \frac{1}{4}\,,\ r_5 = \frac{2}{4}\,,\ r_6 = \frac{3}{4}\,,\ r_7 = \frac{1}{5}\,,\ r_8 = \frac{2}{5}\,,\ \ldots\,.$$

For each rational number $r_n$, $n \geq 1$, we expand it as a simple continued expansion as follows:

$$r_n = \frac{1\,|}{|\,a_{n,1}} + \frac{1\,|}{|\,a_{n,2}} + \cdots + \frac{1\,|}{|\,a_{n,k_n}}\,, \quad a_{n,k_n} \neq 1\,.$$

For examples, $k_1 = 1$, $k_2 = 1$, $k_3 = 2$ and $a_{1,1} = 2$, $a_{2,1} = 3$, $a_{3,1} = 1$, $a_{3,2} = 2$. Then, we define a real number $\hat{x}$ by

Received April 14, 2015; revised July 25, 2015
*Mathematics Subject Classification*: 11K16, 11J70
*Key words and phrases*: normal numbers, continued fractions

$$\hat{x} = \frac{1|}{|2} + \frac{1|}{|3} + \frac{1|}{|1} + \frac{1|}{|2} + \cdots + \frac{1|}{|a_{n,1}} + \cdots + \frac{1|}{|a_{n,k_n}} + \frac{1|}{|a_{n+1,1}} + \cdots .$$

In 1981, R. Adler, M. Keane, and M. Smorodinsky ([1]) showed that $\hat{x}$ is normal in the sense that for any sequence of positive integers $(b_1, \dots, b_\ell)$ the sequence of partial quotient $a_n$ of $\hat{x}$ satisfies that

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \le n \le N : (a_n, \dots, a_{n+\ell-1}) = (b_1, \dots, b_\ell)\} = \mu_G(\langle b_1, \dots, b_\ell\rangle).$$

Here, $\mu_G$ is the absolutely continuous invariant measure, which we call the Gauss measure, for the continued fraction transformation and

$$\langle b_1, \dots, b_\ell\rangle = \{x \in (0,1) : a_1(x) = b_1, \dots, a_\ell(x) = b_\ell\},$$

where $a_n(x)$ denotes the $n$th partial coefficient of the simple continued fraction expansion of $x$. We can regard the construction of $\hat{x}$ as a continued fraction version of the Champernowne normal number construction [4].

Now, let $\mathbb{F}$ be a finite field of $q$ elements, $\mathbb{F}[X]$ be the set of polynomials of $\mathbb{F}$-coefficients, $\mathbb{F}(X)$ be the set of rational functions induced by $\mathbb{F}[X]$, and $\mathbb{F}((X^{-1}))$ is the set of formal power series of $\mathbb{F}$-coefficients. For $f \in \mathbb{F}((X^{-1}))$, we define

$$\deg f = \begin{cases} k, & \text{if} \quad f = a_k X^k + a_{k-1}X^{k-1} + a_{k-2}X^{k-2} + \cdots \text{ with } a_k \ne 0, \\ -\infty, & \text{if} \quad f = 0 \end{cases}$$

and

$$|f| = q^{\deg f}, \quad \mathbb{L} = \{f \in \mathbb{F}((X^{-1})) : \deg f < 0\}.$$

For $f \in \mathbb{L}$, there exists a sequence of polynomials $A_1(f), A_2(f), \dots,$ in $\mathbb{F}[X]$ such that $\deg A_n(f) \ge 1$ and

$$f = \frac{1|}{|A_1} + \frac{1|}{|A_2} + \cdots,$$

which means

$$\lim_{N\to\infty} \left| f - \frac{1|}{|A_1} + \frac{1|}{|A_2} + \cdots \frac{1|}{|A_N} \right| = 0.$$

We call this continued fraction expansion of $f$ the Artin continued fraction expansion of $f$ (see [2]). Indeed, we can uniquely obtain $\{A_n(f), n \ge 1\}$ in the following way: For $f = a_k X^k + a_{k-1}X^{k-1} + \cdots \in \mathbb{F}((X^{-1}))$, we put

$$[f] = \begin{cases} a_k X^k + \cdots a_1 X + a_0, & \text{when } \deg f = k \ge 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\{f\} = f - [f].$$

We call $[f]$ and $\{f\}$ the polynomial part and the fractional part of $f$ respectively. For $f \in \mathbb{L}$ we define the continued fraction transformation $T$ of $\mathbb{L}$ (the Artin map) by

$$T(f) = \begin{cases} \left\{\frac{1}{f}\right\} = \frac{1}{f} - \left[\frac{1}{f}\right], & \text{if } f \neq 0, \\ 0, & \text{if } f = 0. \end{cases}$$

Then, we have $A_n(f) = \left[\frac{1}{T^{n-1}(f)}\right]$ for $T^{n-1}(f) \neq 0$, $n \geq 1$. If $f \in \mathbb{L} \cap \mathbb{F}(X)$ then we have

$$f = \frac{1}{|A_1} + \frac{1}{|A_2} + \cdots \frac{1}{|A_n}$$

when $T^k(f) \neq 0$, $0 \leq k \leq n-1$ and $T^n(f) = 0$. Let $\mu$ denote the normalized Haar measure of $\mathbb{L}$ with respect to the addition. It is well-known that $\mu$ is an invariant measure for $T$ (e.g. see [3]). For each $B \in \mathbb{F}[X]$ with $\deg B = k \geq 1$,

$$\mu(\{f \in \mathbb{L} : A_1(f) = B\}) = \frac{1}{q^{2k}}.$$

Moreover, for each $B_m \in \mathbb{F}[X]$, $1 \leq m \leq \ell$,

$$\mu(\langle B_1, \ldots B_m \rangle) = \mu(\{f : A_1(f) = B_1, \ldots, A_m(f) = B_m\}) = \frac{1}{q^{2\sum_{j=1}^{m} \deg B_j}}.$$

We say that $f \in \mathbb{L}$ is continued fraction normal if the partial quotient $A_n$ of $f$ satisfies that

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N : A_n = B_1, \ldots, A_{n+m-1} = B_m\} = \mu(\langle B_1, \ldots, B_m \rangle)$$

for any choice of $B_1, \ldots, B_m \in \mathbb{F}[X]$ with $\deg B_j \geq 1$, $1 \leq j \leq m$.

Now suppose that a linear order $\prec$ on $\mathbb{F}$ is given. We extend the linear order to $\mathbb{F}[X]$. If $\deg(P) < \deg(Q)$, then we set $P \prec Q$; For $P = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$, $Q = b_n X^n + b_{n-1}X^{n-1} + \cdots + b_1 X + b_0$, polynomials of the same degree, we say $P \prec Q$ when $a_k = b_k$, $k > k_0$ and $a_{k_0} \prec b_{k_0}$ for some $0 \leq k_0 \leq n$.

We list up all polynomials $P \in \mathbb{F}[X] \setminus \{0\}$ of $\deg P < n$ by

$$P^{n,1} \prec P^{n,2} \prec \cdots \prec P^{n,q^n-1}$$

and also all monic polynomials $Q \in \mathbb{F}[X] \setminus \{0\}$ of $\deg Q = n$ by

$$Q_{n,1} \prec Q_{n,2} \prec \cdots \prec Q_{n,q^n}.$$

Then we have a sequence of fractions

$$\frac{P^{1,1}}{Q_{1,1}}, \dots, \frac{P^{1,q-1}}{Q_{1,1}}, \dots, \frac{P^{1,1}}{Q_{1,q}}, \dots, \frac{P^{1,q-1}}{Q_{1,q}}, \frac{P^{2,1}}{Q_{2,1}}, \dots, \frac{P^{2,q^2-1}}{Q_{2,1}}, \dots .$$

Note that the denominators are monic polynomials arranged in the increasing order starting from degree 1 polynomials. For each denominator the numerator runs over polynomials whose degree is less than the degree of the denominator. Also note that the number of $P/Q$'s such that $Q$ is monic of $\deg Q = n$ and $\deg P < n$, $P \neq 0$ is $q^n(q^n - 1) = q^{2n} - q^n$. For an example, if $\mathbb{F} = \{0, 1\}$ $(q = 2)$ with $0 \prec 1$, then

$$\frac{1}{X}, \frac{1}{X+1}, \frac{1}{X^2}, \frac{X}{X^2}, \frac{X+1}{X^2}, \frac{1}{X^2+1}, \frac{X}{X^2+1}, \frac{X+1}{X^2+1}, \frac{1}{X^2+X}, \dots .$$

We expand each $\frac{P_{n,k}}{Q_{n,m}}$, $n \geq 1$, $1 \leq m \leq q^n$, $1 \leq k \leq q^n - 1$ as

$$\frac{P_{n,k}}{Q_{n,m}} = \frac{1}{\left| A_1^{(n,m,k)} \right.} + \frac{1}{\left| A_2^{(n,m,k)} \right.} + \cdots + \frac{1}{\left| A_{k_{n,m,k}}^{(n,m,k)} \right.}$$

and define $h \in \mathbb{L}$ as

$$\begin{aligned} h = & \frac{1}{\left| A_1^{(1,1,1)} \right.} + \cdots + \frac{1}{\left| A_{k_{1,1,q-1}}^{(1,1,q-1)} \right.} + \frac{1}{\left| A_1^{(1,2,1)} \right.} + \cdots + \frac{1}{\left| A_{k_{1,2,q-1}}^{(1,2,q-1)} \right.} + \cdots \\ & + \frac{1}{\left| A_{k_{n-1,q^{n-1},q^{n-1}-1}}^{(n-1,q^{n-1},q^{n-1}-1)} \right.} + \frac{1}{\left| A_1^{(n,1,1)} \right.} + \cdots + \frac{1}{\left| A_{k_{n,q^n,q^n-1}}^{(n,q^n,q^n-1)} \right.} + \cdots \\ =: & \frac{1}{\left| A_1^\sharp \right.} + \frac{1}{\left| A_2^\sharp \right.} + \frac{1}{\left| A_3^\sharp \right.} + \cdots . \end{aligned}$$

THEOREM 1. *For any order $\prec$ on $\mathbb{F}$, $h \in \mathbb{L}$ constructed in the above is continued fraction normal.*

Main point of the proof of this theorem is the following. In the case of real numbers, the cardinality of (not necessarily irreducible) fractions in (0, 1) with denominators less than or equals to $n$ is $1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2} = O(n^2)$ and that of fractions in (0, 1) with denominator $n$ is $n - 1$. On the other hand, in the formal Laurent series, the cardinality of rational functions $\frac{R}{S}$ of $0 \leq \deg R < \deg S < n$ is $O(q^{2n})$ which is the same order as that of polynomials of $0 \leq \deg R < \deg S = n$ ($S$ monic). Moreover, each irreducible rational $\frac{R}{S}$ of $0 \leq \deg R < \deg S < n$ equals to $q^{n-\deg S}$ fractional functions of $\frac{R'}{S'}$ with $\deg S' = n$, $S'$ monic and $\deg R' < n$. This might destroy "normality" if we have chosen a "bad" order $\prec$ since there could be a long sequences of $A_1^{(n,m,k)}, \dots, A_{k_n}^{(n,m,k)}$ of "bad normality". We will show that this never happens because there are sufficiently many "good" rationals $\frac{R}{S}$ if $n$ is sufficiently large.

It is also possible to construct the normal series $h^\flat$ by listing up only irreducible $\frac{R}{S}$. The proof of the normality of this case is easier than that of $h$. In the sequel, we start with $h^\flat$ showing it being continued fraction normal, in §2. Then we show, in §3, that $h$ is also continued fraction normal. Finally, in §4, we give a brief comment concerning Lüroth series in the set of formal power series $\mathbb{L}$. Originally, Lüroth series is a sort of a linear version of the simple continued fractions. Later on, A. Knopfmacher and J. Knopfmacher [7] consider its formal power series version. Then its metric property was discussed in [6], [8], and S. Kristensen [9]. It is not difficult to see that the method discussed in §2 also works here. We discuss this point in §4.

## 2. Irreducible construction

In this section, we start with the explicit definition of $h^\flat$. We put

$$\mathcal{P}_n = \{(U, V) : V \text{ is monic}, \ 0 \le \deg U < \deg V = n\}$$

and

$$\mathcal{P}_n^* = \{(U, V) \in \mathcal{P}_n : U \text{ and } V \text{ are coprime}\}.$$

We list up all rational functions $\frac{U}{V}$, $(U, V) \in \mathcal{P}_n^*$ :

$$\frac{U_{n,1}}{V_{n,1}}, \frac{U_{n,2}}{V_{n,2}}, \ldots, \frac{U_{n,q^{2n}-q^{2n-1}}}{V_{n,q^{2n}-q^{2n-1}}}.$$

Here we note that the cardinality of the set $\mathcal{P}_n^*$ is $q^{2n} - q^{2n-1}$, see [5] for example. We can choose any order for $\left\{\frac{U_{n,\ell}}{V_{n,\ell}}\right\}$. For each $\frac{U_{n,\ell}}{V_{n,\ell}}$, we consider its Artin continued fraction expansion

$$\frac{U_{n,\ell}}{V_{n,\ell}} = \frac{1}{|A_{n,\ell,1}} + \frac{1}{|A_{n,\ell,2}} + \cdots + \frac{1}{|A_{n,\ell,\gamma(n,\ell)}}.$$

We denote by $\gamma(n, \ell)$ the length of the Artin continued fraction expansion of $\frac{U_{n,\ell}}{V_{n,\ell}}$. We define $h^\flat \in \mathbb{L}$ by

$$\begin{aligned} h^\flat = & \frac{1}{|A_{1,1,1}} + \cdots + \frac{1}{|A_{n,\ell,1}} + \frac{1}{|A_{n,\ell,2}} + \cdots + \frac{1}{|A_{n,\ell,\gamma(n,\ell)}} + \cdots \\ & + \frac{1}{|A_{n,q^{2n}-q^{2n-1},1}} + \cdots + \frac{1}{|A_{n,q^{2n}-q^{2n-1},\gamma(n,q^{2n}-q^{2n-1})}} + \frac{1}{|A_{n+1,1,1}} + \cdots \\ =: & \frac{1}{|A_1^\flat} + \frac{1}{|A_2^\flat} + \cdots . \end{aligned}$$

THEOREM 2. *The powers series $h^\flat \in \mathbb{L}$ is continued fraction normal.*

We denote by $\frac{P_n(f)}{Q_n(f)}$ the $n$th convergent of the Artin continued fraction expansion of $f \in \mathbb{L}$, that is $P_n(f)$ and $Q_n(f)$ are given by

$$\begin{pmatrix} P_{n-1}(f) & P_n(f) \\ Q_{n-1}(f) & Q_n(f) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & A_1(f) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & A_n(f) \end{pmatrix} .$$

LEMMA 1. *For every $(U, V) \in \mathcal{P}_n^*$, we have*

$$\mu\left(\left\{ f \in \mathbb{L} : \frac{P_k(f)}{Q_k(f)} = \frac{U}{V} \quad \text{for some } k \geq 1 \right\}\right) = \frac{1}{q^{2n}} .$$

PROOF. Since $\left|f - \frac{U}{V}\right| < \frac{1}{|V|^2}$ implies $\frac{U}{V} = \frac{P_k(f)}{Q_k(f)}$ for some $k \geq 1$ (see [11]) and

$$\left| f - \frac{P_k(f)}{Q_k(f)} \right| < \frac{1}{|Q_k(f)|^2} = \frac{1}{q^{2n}} ,$$

the first $2n$ coefficients $a_1, \ldots, a_{2n}$ of $f = a_1 X^{-1} + a_2 X^{-2} + \ldots$ are determined by $\frac{U}{V}$. On the other hand, for every $f = a_1 X^{-1} + a_2 X^{-2} + \cdots \in \mathbb{L}$ such that $a_1, \ldots, a_{2n}$ are the same as those of $\frac{U}{V}$ has the same $k$th convergent $\frac{P_k(f)}{Q_k(f)} = \frac{U}{V}$. □

To prove Theorem 2, we show that for any finite sequence of polynomials $\mathbf{B} = (B_1, \ldots, B_s)$ with $\deg B_j \geq 1$, $1 \leq j \leq s$

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N : (A_n^\flat, \ldots, A_{n+s-1}^\flat) = \mathbf{B}\} = \frac{1}{q^{2\sum_{j=1}^{s} \deg B_j}} .$$

LEMMA 2. *The number of irreducible $\frac{U}{V}$ such that $(U, V) \in \mathcal{P}_n^*$ such that their Artin continued fractions have length $k$, $1 \leq k \leq n$, is*

$$\binom{n-1}{k-1} (q-1)^k q^n .$$

PROOF. By the assumption of this lemma, all $\frac{U}{V}$ under consideration are of the form

$$\frac{1|}{|A_1} + \cdots + \frac{1|}{|A_k} .$$

Thus, the leading coefficients of $A_1, \ldots, A_k$ have $(q-1)^k$ choices. Since $\deg A_j \geq 1$ and $\sum_{j=1}^{k} \deg A_j = n$, we have the assertion of this lemma. □

Note that

$$\sum_{k=1}^{n} \binom{n-1}{k-1} (q-1)^k q^n = \sum_{k=0}^{n-1} \binom{n-1}{k} (q-1)^{k+1} q^n = q^{2n-1}(q-1) = \#\mathcal{P}_n^* . \qquad (1)$$

LEMMA 3. *The sum of lengths of Artin continued fraction expansions of irreducible* $\frac{U}{V}$ *with* $(U,V) \in \mathcal{P}_n^*$ *is*

$$\sum_{\ell=1}^{q^{2n}-q^{2n-1}} \gamma(n,\ell) = q^{2n-2}\,(q-1)(n(q-1)+1)$$

PROOF. From Lemma 2, we have the left hand side of the assertion is equal to

$$\begin{aligned}\sum_{k=1}^{n} k\binom{n-1}{k-1}(q-1)^k q^n &= \sum_{k=1}^{n}\binom{n-1}{k-1}(q-1)^k q^n + \sum_{k=1}^{n}(k-1)\binom{n-1}{k-1}(q-1)^k q^n \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}(q-1)^{k+1} q^n + \sum_{k=0}^{n-1} k\binom{n-1}{k}(q-1)^{k+1} q^n \\ &= q^{2n-1}(q-1) + (n-1)q^{2n-2}(q-1)^2 \\ &= q^{2n-2}\,(q-1)(n(q-1)+1)\,. \end{aligned}$$

□

We fix $\mathbf{B} = (B_1, \ldots, B_s)$, $B_j \in \mathbb{F}[X]$, $1 \le j \le s$ in the subsequent discussion. For $\varepsilon > 0$,

$$\frac{U}{V} = \frac{1 \,|}{|\, A_1} + \cdots + \frac{1 \,|}{|\, A_k}$$

is said to be $\varepsilon$-good if

$$\left| \frac{1}{k-s+1}\,\#\{0 \le i \le k-s \,:\, A_{i+1} = B_1, \ldots, A_{i+s} = B_s\} - \mu(\mathbf{B}) \right| < \varepsilon\,.$$

In this case, we say also that $(A_1, \ldots, A_k)$ is $\varepsilon$-good.

LEMMA 4. *For any* $\varepsilon > 0$ *and* $\eta > 0$, *there exist a measurable subset* $E_\varepsilon$ *of* $\mathbb{L}$ *and a positive integer* $k_0$ *such that* $\mu(E_\varepsilon) > 1 - \eta$ *and for each* $f \in E_\varepsilon$ *the kth convergents* $\frac{P_k(f)}{Q_k(f)}$ *are* $\varepsilon$*-good for all* $k \ge k_0$.

PROOF. By the Birkhoff ergodic theorem (see [10] for the ergodicity of $T$),

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} 1_{\mathbf{B}}(T^i f) = \mu(\mathbf{B}) \quad \text{for almost every } f \in \mathbb{L}\,.$$

This means for all $\varepsilon > 0$ there exists $k_0 = k_0(f)$ such that

$$\left| \frac{1}{k}\sum_{i=0}^{k-1} 1_{\mathbf{B}}(T^i f) - \mu(\mathbf{B}) \right| < \varepsilon \tag{2}$$

for any $k \geq k_0$. We put

$$E_{\varepsilon,K} = \{f \in \mathbb{L} \,:\, (2) \text{ holds for all } k \geq K - s + 1\}.$$

Then $E_{\varepsilon,K}$ is a measurable set and $f \in E_{\varepsilon,K}$ for some $K > 0$ for almost every $f \in \mathbb{L}$. Thus we see $\mu(\cup_{K=1}^{\infty} E_{\varepsilon,K}) = 1$, which shows the assertion of the theorem. □

LEMMA 5. *For each $\eta > 0$ and $k_0$ there exists a positive integer $n_0$ such that*

$$\mu(\{f \in \mathbb{L} : \deg Q_{k_0}(f) \leq n_0\}) > 1 - \eta.$$

PROOF. Put

$$D_{k_0,m} = \{f \in \mathbb{L} \,:\, \deg Q_{k_0}(f) = m\}.$$

Then $\mathbb{L} = \cup_{m=k_0}^{\infty} D_{k_0,m}$. Thus there exists $n_0$ such that

$$\mu\left(\bigcup_{m=k_0}^{n_0} D_{k_0,m}\right) > 1 - \eta. \qquad \square$$

LEMMA 6. *For each $\eta > 0$, there exists a positive integer $n_0$ such that*

$$\#\left\{(U, V) \in \mathcal{P}_n^* : \frac{U}{V} \textit{ is not } \varepsilon\textit{-good}\right\} < \eta q^{2n}$$

*holds for any $n \geq n_0$.*

PROOF. By Lemma 4, we have $k_0$ and $E_{\varepsilon,k_0}$ with $\mu(E_{\varepsilon,k_0}) > 1 - \frac{\eta}{2}$ and by Lemma 5 we have $n_0$ such that

$$\mu(\{f \in \mathbb{L} : \deg Q_{k_0}(f) > n_0\}) < \frac{\eta}{2}.$$

For $n \geq n_0$, let $\frac{U}{V}$, $(U, V) \in \mathcal{P}_n^*$, be not $\varepsilon$-good. Then each $f \in \mathbb{L}$ with $P_k(f)/Q_k(f) = U/V$ satisfies $f \notin E_{\varepsilon,k_0}$ or $\deg Q_{k_0}(f) > n \geq n_0$. Therefore, we have

$$\bigcup_{\substack{(U,V)\in\mathcal{P}_n^* \\ U/V \text{ not } \varepsilon\text{-good}}} \left\{f \in \mathbb{L} : \frac{P_k(f)}{Q_k(f)} = \frac{U}{V} \text{ for some } k\right\} \subset \{f \in \mathbb{L} : \deg Q_{k_0}(f) > n_0\} \cup E_{\varepsilon,k_0}^c.$$

By Lemma 1, we have

$$\frac{\#\{(U, V) \in \mathcal{P}_n^* : \frac{U}{V} \text{ is not } \varepsilon\text{-good}\}}{q^{2n}} < \mu(E_{\varepsilon,k_0}^c) + \mu(\{f \in \mathbb{L} : \deg Q_{k_0}(f) > n_0\}) < \eta.$$

□

PROPOSITION 3. *For any $\varepsilon > 0$, there exists $n_1$ such that*

$$\left(A_{n,1,1}, \ldots, A_{n,q^{2n}-q^{2n-1},\gamma(n,q^{2n}-q^{2n-1})}\right)$$

*is $\varepsilon$-good for any $n \geq n_1$.*

PROOF. We may assume that $\varepsilon < 1$. We apply Lemma 6 with $\frac{\varepsilon}{2}$ and $\eta < \frac{\varepsilon}{4q^2}$ for a given $\varepsilon$. Then there exists $n_0$ such that for $n \geq n_0$ the number of all non $\varepsilon$-good $(A_1, \dots, A_k)$ with $\sum_{i=1}^{k} \deg A_i = n$ is less than $\eta q^{2n}$. Thus, the sum of the length of all non $\varepsilon$-good sequences $(A_1, \dots, A_k)$ of length $k$, $1 \leq k \leq n$ is smaller than $\eta n q^{2n}$.

Let $W_n$ be the number of occurrence of $\mathbf{B} = (B_1, \dots, B_s)$ in the sequence $\big(A_{n,1,1}, \dots, A_{n,q^{2n}-q^{2n-1},\gamma(n,q^{2n}-q^{2n-1})}\big)$. Then, from (1) ($\#\mathcal{P}_n^* = q^{2n} - q^{2n-1}$), Lemma 3, and Lemma 6, we see

$$W_n \geq \Big(q^{2n-2}(q-1)(n(q-1)+1) - \eta n q^{2n} - (s-1)(q^{2n} - q^{2n-1})\Big)\Big(\mu(\mathbf{B}) - \frac{\varepsilon}{2}\Big)$$
$$\geq q^{2n-2}(q-1)(n(q-1)+1)\Big(\mu(\mathbf{B}) - \frac{\varepsilon}{2}\Big) - \Big(\eta n q^{2n} + s(q^{2n} - q^{2n-1})\Big).$$

Since $\eta < \frac{\varepsilon}{4q^2}$, for $n \geq \frac{4qs}{\varepsilon}$ we have

$$W_n \geq \Big(q^{2n-2}(q-1)(n(q-1)+1) - s + 1\Big)(\mu(\mathbf{B}) - \varepsilon).$$

On the other hand, by the similar way we also see that

$$W_n \leq \Big(q^{2n-2}(q-1)(n(q-1)+1) - s + 1\Big)(\mu(\mathbf{B}) + \varepsilon)$$

holds for any sufficiently large $n$. Hence, there exists $n_1$ such that

$$\left| \frac{W_n}{q^{2n-2}(q-1)(n(q-1)+1) - s + 1} - \mu(\mathbf{B}) \right| \leq \varepsilon$$

holds for any $n \geq n_1$. □

PROPOSITION 4. *For any $\varepsilon > 0$, there exists $n_2$ such that*

$$\big(A_{1,1,1}, \dots, A_{n,q^{2n}-q^{2n-1},\gamma(n,q^{2n}-q^{2n-1})}\big)$$

*is $\varepsilon$-good for any $n \geq n_2$.*

PROOF. We may assume $0 < \varepsilon < 1$. Then, from Proposition 3, we find $n_1$ such that

$$\big(A_{n,1,1}, \dots, A_{n,q^{2n}-q^{2n-1},\gamma(n,q^{2n}-q^{2n-1})}\big), \quad n \geq n_1$$

are all $\frac{\varepsilon}{3}$-good. From Lemma 3,

$$\sum_{k=1}^{n_1} \sum_{\ell=1}^{q^{2k}-q^{2k-1}} \gamma(k,\ell) = O\big(n_1 q^{2n_1}\big).$$

Thus we can find $n_2$ so that $n_2 \geq n_1$ and for any $n \geq n_2$

$$\frac{\sum_{k=1}^{n_1} \sum_{\ell=1}^{q^{2k}-q^{2k-1}} \gamma(k,\ell)}{\sum_{k=1}^{n_2} \sum_{\ell=1}^{q^{2k}-q^{2k-1}} \gamma(k,\ell)} < \frac{\varepsilon}{3}, \qquad \frac{n(s-1)}{\sum_{k=1}^{n_2} \sum_{\ell=1}^{q^{2k}-q^{2k-1}} \gamma(k,\ell)} < \frac{\varepsilon}{3}.$$

This shows the assertion of this proposition. □

PROOF OF THEOREM 2. For $\frac{\varepsilon}{2} > 0$, we apply Proposition 4. Then there exists $n_2(\frac{\varepsilon}{2})$ such that for $N \geq n_2(\frac{\varepsilon}{2})$

$$(A_1^\flat, A_2^\flat, \ldots, A_L^\flat) \quad \text{with} \quad L = \sum_{n=1}^{N} \sum_{\ell=1}^{q^{2n}-q^{2n-1}} \gamma(n,\ell)$$

is $\frac{\varepsilon}{2}$-good. For $N \geq n_2(\frac{\varepsilon}{2})$ consider $L$ with

$$\sum_{n=1}^{N} \sum_{\ell=1}^{q^{2n}-q^{2n-1}} \gamma(n,\ell) + \sum_{\ell=1}^{M-1} \gamma(N+1,\ell) < L \leq \sum_{n=1}^{N} \sum_{\ell=1}^{q^{2n}-q^{2n-1}} \gamma(n,\ell) + \sum_{\ell=1}^{M} \gamma(N+1,\ell), \tag{3}$$

where $M \leq q^{2(N+1)} - q^{2N+1}$. Here $\frac{U_{N+1,1}}{V_{N+1,1}}, \frac{U_{N+1,2}}{V_{N+1,2}}, \ldots, \frac{U_{N+1,M}}{V_{N+1,M}}$ contains at most $K < \frac{\varepsilon}{2} q^{2(N+1)}$ non $\frac{\varepsilon}{2}$-good rational functions.

Then we have

$$\begin{aligned} \#\{1 \leq j \leq L-s+1 : (A_j^\flat, A_{j+1}^\flat, \ldots, A_{j+s-1}^\flat) = \mathbf{B}\} \\ \geq (L - K(N+1) - M(s-1)) \left(\mu(\mathbf{B}) - \frac{\varepsilon}{2}\right). \end{aligned} \tag{4}$$

From Lemma 3, $L \geq Nq^{2N}(1-\frac{1}{q})^2$, and $M \leq q^{2N+2} - q^{2N+1}$, the right hand side of (4) is less than $L(1-\frac{\varepsilon}{2})(\mu(\mathbf{B}) - \frac{\varepsilon}{2})$ if we choose $n_3 \geq n_2(\frac{\varepsilon}{2})$ sufficiently large and $N \geq n_3$.

We can show the estimate from above by the same way. This shows that $(A_1^\flat, A_2^\flat, \ldots, A_L^\flat)$ is $\varepsilon$-good for $L$ of (3) with $N \geq n_3$. □

## 3. Proof of Theorem 1

As in the previous section, we fix $\mathbf{B} = (B_1, \ldots, B_s)$, where $B_j \in \mathbb{F}[X]$ with $\deg B_j \geq 1$, $1 \leq j \leq s$. For any positive number $\varepsilon < 1$ we consider $n_0$ in Lemma 6.

LEMMA 7. *For any $\eta > 0$, there exists a positive integer $n_1 \geq n_0$ such that*

$$\frac{\#\{(RU, RV) \in \mathcal{P}_N \ : \ R \text{ monic } (U,V) \in \mathcal{P}_k^* \text{ for some } 1 \leq k \leq n_0\}}{\#\mathcal{P}_N} < \eta \tag{5}$$

*holds for any* $N \geq n_1$.

PROOF. Since there are $q^{2k} - q^{2k-1}$ pairs $(U, V) \in \mathcal{P}_k^*$ and $q^{N-k}$ monic polynomials of degree $N - k$, the numerator of (5) is

$$\sum_{k=1}^{n_0} (q^{2k} - q^{2k-1}) q^{N-k} = q^N (q^{n_0} - 1) . \tag{6}$$

Since $\#\mathcal{P}_N = q^{2N} - q^N$, we complete the proof. □

LEMMA 8. *The sum of the lengths of Artin continued fraction expansions of* $\frac{U}{V}$, $(U, V) \in \mathcal{P}_n$, *is equal to* $n(q-1)q^{2n-1}$.

PROOF. For any $(RU, RV) \in \mathcal{P}_n$ with $(U, V) \in \mathcal{P}_k^*$ and a monic polynomial $R$, $\deg R = n - k$, the Artin continued fraction expansion of $\frac{RU}{RV}$ is the same as that of $\frac{U}{V}$. Thus, from Lemma 3, the sum of the lengths is calculated as

$$\begin{aligned} \sum_{k=1}^{n} \sum_{\ell=1}^{q^{2k}-q^{2k-1}} \gamma(k, \ell) q^{n-k} &= \sum_{k=1}^{n} q^{2k-2}(q-1)(k(q-1)+1) q^{n-k} \\ &= (q-1) q^{n-1} \sum_{k=1}^{n} \left( k q^k - (k-1) q^{k-1} \right) \\ &= n(q-1) q^{2n-1} . \end{aligned}$$
□

LEMMA 9. *The sum of the lengths of Artin continued fraction expansions of* $\frac{U}{V}$, $(U, V) \in \mathcal{P}_k$, $1 \leq k \leq n$, *is*

$$\frac{q\left(nq^{2n+2} - (n+1)q^{2n} + 1\right)}{(q+1)(q^2-1)} \; (=: \hat{W}_n) .$$

PROOF. This follows directly from simple calculation by Lemma 8 :

$$\sum_{k=1}^{n} k(q-1) q^{2k-1} = \frac{nq^{2n+1}}{q+1} - \frac{q\left(q^{2n}-1\right)}{(q+1)(q^2-1)} .$$
□

PROOF OF THEOREM 1. First we consider fractions by polynomial pairs in $\mathcal{P}_N$. The total length of their Artin continued fraction expansions is $(q-1)Nq^{2N-1}$ as shown in Lemma 8. Lemma 7 shows that there exists a positive integer $n_1 \geq n_0$ such that the total length of reducible polynomials $\frac{RU}{RV}$, with $(RU, RV) \in \mathcal{P}_N$ and $\deg V \leq n_0$, is less than $n_0(q^{2N} - q^N)\eta$ for any $N \geq n_1$. By Lemma 6, among other rational functions $\frac{U}{V}$, $(U, V) \in \mathcal{P}_N$, at most

$$\eta \cdot \sum_{j=0}^{N-n_0} q^j q^{2(N-j)} \tag{7}$$

rational functions are not $\varepsilon$-good. This shows that the sum of lengths of Artin continued fraction expansions of all those rational functions is less than $\frac{\eta N q^{2N+1}}{q-1}$. Thus the number of occurrence of $\mathbf{B}$ in the sequence of polynomials by the concatenations of Artin continued fraction expansions of all elements in $\mathcal{P}_N$ is estimated from below by

$$\left(\hat{W}_N - \frac{\eta N q^{2N+1}}{q-1} - (q^{2N} - q^N)(s-1) - n_0(q^{2N} - q^N)\eta\right) \cdot (\mu(\mathbf{B}) - \varepsilon)\,.$$

Then dividing by $\hat{W}_N$, we see the frequency of $\mathbf{B}$ is larger than

$$\left(1 - C \cdot \eta - O\left(\frac{1}{N}\right)\right)(\mu(\mathbf{B}) - \varepsilon) \quad \text{as} \quad N \to \infty\,,$$

where $C$ is a positive constant. Similar to the proof of Proposition 3, we choose an appropriate $\eta$ and have a positive integer $n_2 \geq n_1$ so that the frequency of $\mathbf{B}$ in the above is larger than $\mu(\mathbf{B}) - 2\varepsilon$ for any $N \geq n_2$. We can estimate from above by the same way and see that the sequence of polynomials arising from the concatenations of Artin continued fractions of all elements in $\mathcal{P}_N$ is $2\varepsilon$-good for $N \geq n_2$. From Lemma 9, we can find $n_3 \geq n_2$ such that

$$\frac{\sum_{j=1}^{n_0} \hat{W}_j}{\sum_{j=1}^{n} \hat{W}_j} < \varepsilon \quad \text{for} \quad n \geq n_3\,.$$

Now we consider positive integer $L$ such that

$$\sum_{j=1}^{n} \hat{W}_j \leq L < \sum_{j=1}^{n+1} \hat{W}_j$$

for some $n \geq n_3$. We put

$$Z(L) = \#\left\{1 \leq j \leq L - s + 1 \,:\, \left(A_j^{\sharp}, A_{j+1}^{\sharp}, \ldots, A_{j+s-1}^{\sharp}\right) = \mathbf{B}\right\}\,.$$

Note that there are at most $q^{2n+3}\frac{1}{q-1}\eta$ non $\varepsilon$-good rational functions $\frac{U}{V}$ such that

$$(U, V) \in \{(U, V) \in \mathcal{P}_{n+1} \,:\, \deg V \geq n_0\} \quad \text{(see (7))}$$

and there are at most $q^{n+1}(q^{n_0} - 1)$ rational functions $\frac{U}{V}$ such that

$$(U, V) \in \{(U, V) \in \mathcal{P}_{n+1} \,:\, \deg V < n_0\} \quad \text{(see (6))}\,.$$

Then we have

$$\begin{aligned} Z(L) \geq &(L - s + 1)(\mu(\mathbf{B}) - 2\varepsilon) - n_0 \sum_{j=1}^{n_0} \hat{W}_j - \frac{\eta(n+1)q^{2(n+1)}}{q-1} \\ &- n_0 q^{n+1}(q^{n_0} - 1) - (s-1)(q^{2(n+1)} - q^{n+1})\,. \end{aligned}$$

Dividing by $L$, we see that

$$\frac{Z(L)}{L-s+1} \geq \mu(\mathbf{B}) - 3\varepsilon ,$$

where we had chosen $\eta$ appropriately. The estimate from above also follows in the same way. Consequently, we have the assertion of the theorem. □

## 4. Lüroth series

In this section we apply our method adopted in §2 to Lüroth series in positive characteristic, which was introduced in A. Knopfmacher and J. Knopfmacher [7]. Let $S$ be a map of $\mathbb{L}$ onto itself by

$$S(f) = \begin{cases} \left(\left[\frac{1}{f}\right]-1\right)\left(\left[\frac{1}{f}\right]f-1\right), & \text{if} \quad f \neq 0, \\ 0, & \text{if} \quad f = 0, \end{cases}$$

for $f \in \mathbb{L}$. We put $A_n(f) = \left[\frac{1}{S^{n-1}(f)}\right]$ and have the expansion of $f$ by the following

$$f = \frac{1}{A_1(f)} + \sum_{n=2}^{\infty} \frac{1}{A_1(f)(A_1(f)-1)A_2(f)(A_2(f)-1)\cdots A_{n-1}(f)(A_{n-1}(f)-1)A_n(f)} ,$$

which we call Lüroth expansion of $f$. The $n$th convergent of Lüroth expansion is

$$\frac{1}{A_1(f)} + \sum_{k=2}^{n} \frac{1}{A_1(f)(A_1(f)-1)A_2(f)(A_2(f)-1)\cdots A_{k-1}(f)(A_{k-1}(f)-1)A_k(f)}$$

and the degree of its denominator polynomial is $\deg A_n(f) + \sum_{k=1}^{n-1} \deg A_k(f)$.

It is easy to see that the Lüroth expansion of a rational function may not be finite. Indeed, for example, a rational function $\frac{A-1}{(A-1)A-1}$ is a fixed point of $S$ and have the expansion

$$\frac{1}{A} + \frac{1}{(A-1)AA} + \frac{1}{(A-1)A(A-1)AA} + \frac{1}{(A-1)A(A-1)A(A-1)AA}$$

for any $A \in \mathbb{F}[X]$ with $\deg A \geq 1$. However, we have the following proposition.

PROPOSITION 5. *For any rational function $\frac{U}{V} \in \mathbb{L}$, there exists positive integers $n$ and $m$ ($n \neq m$) such that $S^n(\frac{U}{V}) = S^m(\frac{U}{V})$.*

PROOF. Due to the definition of $S$, $S(\frac{U}{V})$ is also a rational function and the denominator of its degree is less than $\deg V$. There are only finitely many polynomials of degree less that $\deg V$, which shows the assertion of this proposition. □

For this reason, we do not use all rational functions to construct normal series associated with the Lüroth expansion. The simple idea is that making use of cylinder sets. We arrange a sequence of polynomials by concatenating sequences of cylinder sets to construct the normal series with respect to Lüroth series.

It has been shown that $S$ is $\mu$-preserving and

$$\mu\left(\{f \in \mathbb{L} \,:\, A_1(f) = B_1,\ \ldots,\ A_n(f) = B_n\}\right) = \mu(\langle B_1, \ldots, B_m\rangle) = \frac{1}{q^{2\sum_{j=1}^{n} \deg B_j}} \tag{8}$$

for any finite sequence of positive degree polynomials $B_1, \ldots B_n \in \mathbb{F}[X]$. In this sense, we can define the normality of Lüroth series in positive characteristic : $f \in \mathbb{L}$ is said to be Lüroth normal if

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N \,:\, A_n = B_1, \ldots, A_{n+m-1} = B_m\} = \mu(\langle B_1, \ldots, B_m\rangle)$$

for any choice of $B_1, \ldots, B_m \in \mathbb{F}[X]$ with $\deg B_j \ge 1$, $1 \le j \le m$, where $A_j$ denotes the $j$-th coefficient of the Lüroth expansion of $f$ for $j \ge 1$. We infer form these that the sequence of polynomials constructed in §2 also gives the Lüroth normal series. Indeed the following theorem is a direct consequence of (8) :

THEOREM 6. *For any sequence of polynomials* $\{A_n\}$ *in* $\mathbb{F}[X]$ *with* $\deg A_n \ge 1$ *for* $n \ge 1$,

$$\frac{1 \mid}{\mid A_1} + \frac{1 \mid}{\mid A_2} + \cdots$$

*is continued fraction normal if and only if*

$$\frac{1}{A_1} + \sum_{n=2}^{\infty} \frac{1}{A_1(A_1-1)A_2(A_2-1)\cdots A_{n-1}(A_{n-1}-1)A_n},$$

*is Lüroth normal.*

Now let's define the set of cylinder sets as

$$\Xi_n = \left\{\langle B_1,\ \ldots,\ B_s\rangle \,:\, \text{cylinder sets such that } \sum_{j=1}^{s} \deg B_j = n\right\}.$$

Then $\#\Xi_n = \#\mathcal{P}_n^* = q^{2n} - q^{2n-1}$. We arrange all elements of $\Xi_n$ in any order and list up their components (polynomials) just like we did in §2. Furthermore we concatenate these finite sequences of polynomials, $n \ge 1$. Then we get an infinite sequence of polynomials $C_1, C_2, \ldots$.

COROLLARY 7. *Let $C_i \in \mathbb{F}[X]$ be given as above. Then*

$$h^* = \frac{1}{C_1} + \sum_{n=2}^{\infty} \frac{1}{C_1(C_1-1)C_2(C_2-1)\cdots C_{n-1}(C_{n-1}-1)C_n}$$

*is Lüroth normal.*

The proof of this proposition is exactly the same as that of §2.

ACKNOWLEDGMENTS. The first author was supported by the National Research Foundation of Korea (NRF) (2012R1A1A2004473). The second and the third authors were supported in part by the Grant-in-Aid for Scientific research No. 24340020 and No. 15K17559, respectively, the Japan Society for the Promotion of Science.

## References

[1] R. ADLER, M. KEANE and M. SMORODINSKY, A construction of a normal number for the continued fraction transformation, J. Number Theory **13** (1981), 95–105.

[2] E. ARTIN, Ein mechanisches system mit quasiergodischen bahnen, Abh. Math. Sem. Univ. Hamburg **3** (1924), 170–175.

[3] V. BERTHÉ and H. NAKADA, On continued fraction expansions in positive characteristic: equivalence relations and some metric properties, Expo. Math. **18** (2000), 257–284.

[4] D. G. CHAMPERNOWNE, The construction of decimal normal in the scale of ten, J. London Math. Soc. **8** (1933), 254–260.

[5] K. INOUE and H. NAKADA, On metric Diophantine approximation in positive characteristic, Acta Arith. **110** (2003), 205–218.

[6] J. KNOPFMACHER, Ergodic properties of some inverse polynomial series expansions of Laurent series, Acta Math. Hungar. **60** (1992), 241–246.

[7] A. KNOPFMACHER and J. KNOPFMACHER, Inverse polynomial expansions of Laurent series, Constr. Approx. **4** (1988), 379–389.

[8] A. KNOPFMACHER and J. KNOPFMACHER, Metric properties of algorithms inducing Lüroth series expansions of Laurent series, Astérisque **15** (1992), 237–246.

[9] S. KRISTENSEN, Some metric properties of Lüroth expansions over the field of Laurent series, Bull. Austral. Math. Soc. **64** (2001), 345–351.

[10] R. PAYSANT-LEROUX and E. DUBOIS, Ètude mètrique de l'algorithme de Jacobi-Perron dans un corps de sèries formelles, C. R. Acad. Sci. Paris Ser. A-B **275** (1972), A683–A686.

[11] W. M. SCHMIDT, On continued fractions and diophantine approximation in power series fields, Acta Arith. **95** (2000), 139–166.

*Present Addresses*:
DONG HAN KIM
DEPARTMENT OF MATHEMATICS EDUCATION,
DONGGUK UNIVERSITY-SEOUL,
SEOUL 04620, KOREA.
*e-mail*: kim2010@dongguk.edu

HITOSHI NAKADA
DEPARTMENT OF MATHEMATICS,
KEIO UNIVERSITY,
YOKOHAMA 223–8522, JAPAN.
*e-mail*: nakada@math.keio.ac.jp

RIE NATSUI
DEPARTMENT OF MATHEMATICS,
JAPAN WOMEN'S UNIVERSITY,
TOKYO 112–8681, JAPAN.
*e-mail*: natsui@fc.jwu.ac.jp