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Generalized Poincaré Condition and Convergence of Formal Solutions of Some Nonlinear Totally Characteristic Equations

Dedicated to Professor Ken-ichi SHINODA

Hidetoshi TAHARA

Sophia University

Abstract. This paper discusses a holomorphic nonlinear singular partial differential equation $(t\partial_t)^m u = F(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{j+\alpha\le m, j<m})$ that is of nonlinear totally characteristic type. The Newton Polygon at $x = 0$ of the equation is defined, and by means of this polygon we define a generalized Poincaré condition (GP) and a condition (R) that the equation has a regular singularity at $x = 0$. Under these conditions, (GP) and (R), it is proved that every formal power series solution is convergent in a neighborhood of the origin.

1. Introduction

We set $\mathbf{N} = \{0, 1, 2, \ldots\}$ and $\mathbf{N}^* = \{1, 2, \ldots\}$. Let $m \in \mathbf{N}^*$, and set $I_m = \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}\,;\ j + \alpha \le m, j < m\}$. Let $(t, x) \in \mathbf{C}_t \times \mathbf{C}_x$, $z = \{z_{j,\alpha}\}_{(j,\alpha)\in I_m} \in \mathbf{C}^N$ (with $N = \#I_m = m(m+3)/2$), and $F(t, x, z)$ be a function defined in a polydisk Δ centered at the origin of $\mathbf{C}_t \times \mathbf{C}_x \times \mathbf{C}_z^N$. In this paper, we consider the nonlinear partial differential equation

$$(1.1) \qquad (t\partial_t)^m u = F\big(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m}\big)$$

under the following assumptions:

A$_1$) $F(t, x, z)$ is holomorphic in Δ, and
A$_2$) $F(0, x, 0) \equiv 0$ in $\Delta_0 = \Delta \cap \{t = 0, z = 0\}$.

We set $I_m(+) = \{(j, \alpha) \in I_m\,;\ \alpha > 0\}$. Then the situation is divided into the following three cases:

Case 1: $(\partial F/\partial z_{j,\alpha})(0, x, 0) \equiv 0$ on Δ_0 for any $(j, \alpha) \in I_m(+)$,

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Case 2: $(\partial F/\partial z_{j,\alpha})(0,0,0) \neq 0$ for some $(j,\alpha) \in I_m(+)$,
Case 3: the other case.

In Case 1, the equation (1.1) is called a nonlinear Fuchsian type partial differential equation and it was studied quite well by Baouendi-Goulaouic [1], Gérard-Tahara [6, 7], Madi-Yoshino [9], Tahara-Yamazawa [15] and Tahara-Yamane [14]. In Case 2, a kind of Goursat problem appears. Gérard-Tahara [8] discussed a particular class in this case and proved the existence of holomorphic solutions as well as singular solutions of (1.1).

In Case 3, the equation (1.1) is called a nonlinear totally characteristic type partial differential equation. The main theme of this paper is to discuss this case under the following condition:

A$_3$) $(\partial F/\partial z_{j,\alpha})(0,x,0) = O(x^\alpha)$ (as $x \longrightarrow 0$) for any $(j,\alpha) \in I_m(+)$.

Under A$_3$) and a Poincaré condition, the unique solvability in the space of holomorphic functions was proved by Chen-Tahara [5] and Tahara [13]. Similar equations were studied by Chen-Luo [2, 3], Chen-Luo-Tahara [4], Miyake-Shirai [10, 11] and Shirai [12].

Let us recall the main result in [5]. By the condition A$_3$), for any $(j,\alpha) \in I_m(+)$, we can express $(\partial F/\partial z_{j,\alpha})(0,x,0) = x^\alpha c_{j,\alpha}(x)$ for some function $c_{j,\alpha}(x)$ holomorphic in a neighborhood of $x = 0 \in \mathbf{C}$. We set

$$L(\lambda,\rho) = \lambda^m - \sum_{(j,\alpha)\in I_m} c_{j,\alpha}(0)\lambda^j \rho(\rho-1)\cdots(\rho-\alpha+1)\,,$$

$$L_m(X) = X^m - \sum_{j+\alpha=m,\, j<m} c_{j,\alpha}(0) X^j\,.$$

Denote by $\lambda_1,\ldots,\lambda_m$ the roots of the equation $L_m(X) = 0$. Consider the following conditions:

(N)(Non-resonance condition). $L(k,l) \neq 0$ for any $(k,l) \in \mathbf{N}^* \times \mathbf{N}$.
(P)(Poincaré condition). $\lambda_i \in \mathbf{C} \setminus [0,\infty)$ for $i = 1,2,\ldots,m$.

THEOREM 1 (Chen-Tahara [5]). *Suppose* A$_1$), A$_2$) *and* A$_3$) *hold. If the non-resonance condition* (N) *is satisfied, the equation* (1.1) *has a unique formal solution* $u(t,x) \in \mathbf{C}[[t,x]]$ *satisfying* $u(0,x) \equiv 0$. *In addition, if the Poincaré condition* (P) *is satisfied, this unique formal solution is convergent in a neighborhood of* $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$.

In this paper, we will define a generalized Poincaré condition (GP) (see Section 2) by means of the Newton polygon at $x = 0$ of the equation (1.1), and then we will show the convergence of the formal solution under (GP) and the condition that (1.1) has a regular singularity at $x = 0$.

Throughout this paper, $\mathbf{C}[[x]]$ denotes the ring of formal power series in x, and $\mathbf{C}[[t,x]]$ denotes the ring of formal power series in (t,x).

2. Main Theorem

Suppose the conditions A$_1$), A$_2$), and A$_3$) hold. Let $L(\lambda, \rho)$ be as in Section 1. Set $c_{j,\alpha}(x) = c_{j,\alpha}(0) + xb_{j,\alpha}(x)$. Then the equation (1.1) can be rewritten in the form

$$L(t\partial_t, x\partial_x)u = \sum_{(j,\alpha)\in I_m} xb_{j,\alpha}(x)(t\partial_t)^j(x\partial_x)(x\partial_x - 1)\cdots(x\partial_x - \alpha + 1)u \\ + a(x)t + R_2\big(t, x, \{(t\partial_t)^j\partial_x^\alpha u\}_{(j,\alpha)\in I_m}\big), \tag{2.1}$$

where $R_2(t, x, z)$ is a holomorphic function on Δ whose Taylor expansion in (t, z) has the form

$$R_2(t, x, z) = \sum_{i+|\nu|\geq 2} a_{i.\nu}(x) \prod_{(j,\alpha)\in I_m} {z_{j,\alpha}}^{\nu_{j,\alpha}},$$

where $\nu = \{\nu_{j,\alpha}\}_{(j,\alpha)\in I_m} \in \mathbf{N}^N$ and $|\nu| = \sum_{(j,\alpha)\in I_m} \nu_{j,\alpha}$.

Set $c_{m,0}(x) = -1$ and

$$\Lambda_0 = \{(m, 0)\} \cup \{(j, \alpha) \in I_m\,;\ c_{j,\alpha}(0) \neq 0\}. \tag{2.2}$$

Let us define the *Newton polygon* $\mathcal{N}_0$ *at* $x = 0$ of the equation (2.1). For $(a, b) \in \mathbf{R}^2$, we write $C(a, b) = \{(x, y) \in \mathbf{R}^2\,;\ x \leq a,\ y \leq b\}$. The Newton polygon $\mathcal{N}_0$ at $x = 0$ is defined as the convex hull of the union of the sets $C(j, \alpha)$ ($(j, \alpha) \in \Lambda_0$) in $\mathbf{R}^2$, that is,

$$\mathcal{N}_0 = \text{the convex hull of } \bigcup_{(j,\alpha)\in\Lambda_0} C(j, \alpha).$$

A picture of the Newton polygon $\mathcal{N}_0$ is in Figure 1.

DEFINITION 1. We say that equation (2.1) has a *regular singularity* at $x = 0$ if the following condition is satisfied:

(R)(Regularity condition). If $c_{j,\alpha}(0) = 0$ and $c_{j,\alpha}(x) \not\equiv 0$ we have $(j, \alpha) \in \mathcal{N}_0$.

If otherwise, that is, if (R) is not satisfied then we say that equation (2.1) has an *irregular singularity* at $x = 0$.

As seen in Figure 1, the vertices of $\mathcal{N}_0$ are the p points

$$(m_1, n_1) = (m, 0), \quad (m_2, n_2), \cdots, (m_{p-1}, m_{p-1}), \quad (m_p, n_p),$$

and the boundary of $\mathcal{N}_0$ consists of the vertical half-line Γ_0, the $(p - 1)$-segments $\Gamma_1, \Gamma_2, \ldots, \Gamma_{p-1}$, and the horizontal half-line Γ_p. Obviously, Γ_i ($0 \leq i \leq p$) are closed subsets of $\mathbf{R}^2$. We denote the slope of Γ_i by $-s_i$ ($i = 0, 1, 2, \ldots, p$). Then we have

$$s_0 = \infty > s_1 > s_2 > \cdots > s_{p-1} > s_p = 0.$$

If $p = 1$ and the regularity condition (R) is satisfied, we have $c_{j,\alpha}(x) \equiv 0$ for any $(j, \alpha) \in I_m$ with $\alpha > 0$. This means that the equation is nothing but a nonlinear Fuchsian

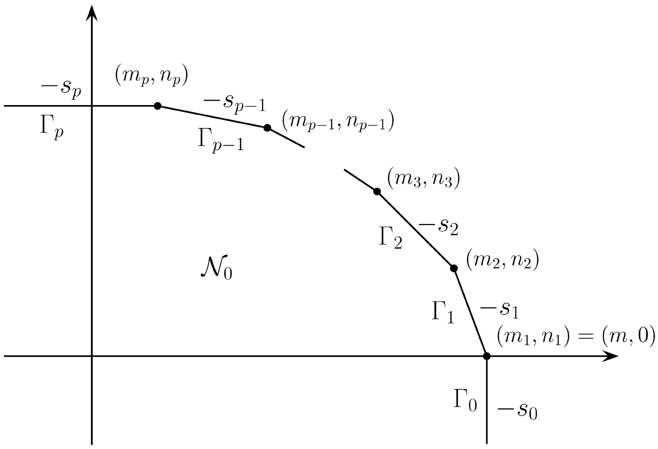

FIGURE 1. Newton polygon $\mathcal{N}_0$ at $x = 0$

type equation and so every formal power series solution is convergent (by Gerard-Tahara [6]). Thus, from now we may discuss only the case $p \geq 2$.

Suppose $p \geq 2$. For $1 \leq i \leq p-1$ we set

$$\begin{aligned} P_i(X) &= \sum_{(j,\alpha)\in\Lambda_0\cap\Gamma_i} c_{j,\alpha}(0)X^{j-m_{i+1}} \\ &= c_{m_i,n_i}(0)X^{m_i-m_{i+1}} + \cdots + c_{m_{i+1},n_{i+1}}(0) \end{aligned}$$

and call this $P_i(X)$ the characteristic polynomial on Γ_i. We denote by $\lambda_{i,q}$ ($1 \leq q \leq m_i - m_{i+1}$) the roots of $P_i(X) = 0$ which are called the characteristic roots on Γ_i. In the case $i = p$, the characteristic polynomial on Γ_p is defined by $P_p(X) = 1$ if $m_p = 0$, and by

$$P_p(X) = \sum_{(j,\alpha)\in\Lambda_0\cap\Gamma_p} c_{j,\alpha}(0)X^j = c_{m_p,n_p}(0)X^{m_p} + \cdots, \quad \text{if } m_p \geq 1\,.$$

In the case $m_p \geq 1$, the roots $\lambda_{p,q}$ ($1 \leq q \leq m_p$) of $P_p(X) = 0$ are called the characteristic roots on Γ_p. We define a *generalized Poincaré condition* as follows:

(GP)(Generalized Poincaré condition)
(i) $\lambda_{i,q} \in \mathbf{C} \setminus [0,\infty)$ for all $1 \leq i \leq p-1$ and $1 \leq q \leq m_i - m_{i+1}$,
(ii) $\lambda_{p,q} \in \mathbf{C} \setminus \mathbf{N}^*$ for $1 \leq q \leq m_p$.

Then we have the following result:

THEOREM 2 (Main Theorem). *Suppose* A$_1$), A$_2$), *and* A$_3$) *hold. If the additional conditions* (R) *and* (GP) (*in the case* $p \geq 2$) *are satisfied*, *every formal power series solution*

$u(t,x) \in \mathbf{C}[[t,x]]$ of equation (2.1) satisfying $u(0,x) \equiv 0$ is convergent in a neighborhood of $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$.

COROLLARY 1. *In addition to the assumption in Theorem 2, if the non-resonance condition (N) is satisfied, the equation (2.1) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$ satisfying $u(0,x) \equiv 0$.*

EXAMPLE 1. Let us consider

$$(2.3) \qquad (t\partial_t)^4 u + (x\partial_x)^2 u = a(x)t + c(x)(t\partial_t)^2(x\partial_x)^k u + R_2\big(t,x,\{(t\partial_t)^j \partial_x^\alpha u\}_{j+\alpha\le 4, j<4}\big), \quad k=0,1,2$$

where $a(x)$ and $c(x)$ are holomorphic functions in a neighborhood of $x=0$, and $R_2(t,x,z)$ is the same as in (2.1). This equation does not satisfy the usual Poincaré condition (P); but we have the following results (the proof is given in Section 5).

(1) If $k=0$ we have $p=2$, $\mathcal{N}_0=\{(x,y)\,;\, x\le 4, y\le 2, x/4+y/2\le 1\}$, (R), (GP) and (N). Hence, equation (2.3) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0)\in \mathbf{C}_t\times\mathbf{C}_x$ satisfying $u(0,x)=0$.

(2) If $k=1$ we have $p=2$, $\mathcal{N}_0=\{(x,y)\,;\, x\le 4, y\le 2, x/4+y/2\le 1\}$ and (R). In this case, (GP) is equivalent to the condition $c(0)\notin[2,\infty)$: moreover, if $c(0)\notin[2,\infty)$ holds, we have (N). Hence, if $k=1$ and $c(0)\notin[2,\infty)$, equation (2.3) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0)\in\mathbf{C}_t\times\mathbf{C}_x$ satisfying $u(0,x)=0$.

(3) If $k=2$ and $c(0)\ne 0$ we have $p=2$, $\mathcal{N}_0=\{(x,y)\,;\, x\le 4, y\le 2, x/4+y/4\le 1\}$ and (R). In this case, (GP) is equivalent to the condition $c(0)\notin[0,\infty)$: moreover, if $c(0)\notin[0,\infty)$ holds, we have (N). Hence, if $k=2$ and $c(0)\notin[0,\infty)$, equation (2.3) has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0)\in\mathbf{C}_t\times\mathbf{C}_x$ satisfying $u(0,x)=0$.

(4) If $k=2$ and $c(0)=0$ (with $c(x)\not\equiv 0$) we have $p=2$, $\mathcal{N}_0=\{(x,y)\,;\, x\le 4, y\le 2, x/4+y/2\le 1\}$, and the equation (2.3) satisfies (GP) and (N), but not (R). In this case, (2.3) has an irregular singularity at $x=0$, and the formal solution is not convergent in general (see Lemma 10 in Section 6).

The remaining part of this paper is organized as follows. In Section 3, we give an interpretation of the condition (GP), and then in Sections 4 and 5, we give a proof of Theorem 2, by using the result in Section 3. In the last section, we will discuss the results in Example 1; in particular, we will show the divergence of the formal solution of (2.3) by means of a very special example.

3. Interpretation of (GP)

We denote by Γ the boundary of $\mathcal{N}_0$, and by V_0 the set of the vertices of $\mathcal{N}_0$, that is,

$$\Gamma = \Gamma_0\cup\Gamma_1\cup\cdots\cup\Gamma_p\,,$$

$$V_0=\{(m_1,n_1),\ldots,(m_p,n_p)\}\,.$$

Recall that we have set $(m_1, n_1) = (m, 0)$ and $c_{m,0}(x) = -1$. Let $L(\lambda, \rho)$ be as in Section 1, Λ_0 be as in (2.2), and set

$$L_\Gamma(\lambda, \rho) = -\sum_{(j,\alpha)\in\Lambda_0\cap\Gamma} c_{j,\alpha}(0)\lambda^j\rho^\alpha = \lambda^m + \cdots ,$$

$$\phi(\lambda, \rho) = \sum_{(j,\alpha)\in V_0} \lambda^j\rho^\alpha = \lambda^m + \lambda^{m_2}\rho^{n_2} + \cdots + \lambda^{m_p}\rho^{n_p} .$$

PROPOSITION 1. *The following three conditions* (P1), (P2) *and* (P3) *are equivalent.*

(P1) *The generalized Poincaré condition* (GP) *is satisfied.*

(P2) *There are $c > 0$ and a finite subset S of $\mathbf{N}^* \times \mathbf{N}$ such that*

$$\left|L_\Gamma(k, l)\right| \geq c\phi(k, l) \quad \text{for any } (k, l) \in \mathbf{N}^* \times \mathbf{N} \setminus S . \tag{3.1}$$

(P3) *There are $c > 0$ and a finite subset S of $\mathbf{N}^* \times \mathbf{N}$ such that*

$$\left|L(k, l)\right| \geq c\phi(k, l) \quad \text{for any } (k, l) \in \mathbf{N}^* \times \mathbf{N} \setminus S . \tag{3.2}$$

3.1. Some lemmas. Let us present some basic lemmas which are needed in the proof of Proposition 1.

LEMMA 1. (1) *Let $s \geq 0$, $0 \leq j < a$ and $0 \leq b \leq \alpha$ be such that $(\alpha - b)/(a - j) \leq s$. Then for any $\varepsilon > 0$ and $\delta > 0$ we can find an $N \in \mathbf{N}^*$ such that*

$$k^j l^\alpha \leq \varepsilon k^a l^b \quad \text{on } W_1 = \{(k, l) \in \mathbf{N}^* \times \mathbf{N} ;\ k \geq l^{s+\delta}, l \geq N\}. \tag{3.3}$$

(2) *Let $s > 0$, $0 \leq a < j$ and $0 \leq \alpha < b$ be such that $(b - \alpha)/(j - a) \geq s$. Then for any $\varepsilon > 0$ and $0 < \delta < s$ we can find an $N \in \mathbf{N}^*$ such that*

$$k^j l^\alpha \leq \varepsilon k^a l^b \quad \text{on } W_2 = \{(k, l) \in \mathbf{N}^* \times \mathbf{N} ;\ k \leq l^{s-\delta}, k \geq N\}. \tag{3.4}$$

(3) *Let $0 \leq j < a$ and $0 \leq \alpha < b$ be such that $j/a + \alpha/b \leq 1 - \delta$ for some $\delta > 0$. Then for any $\varepsilon > 0$ we can find an $N \in \mathbf{N}^*$ such that*

$$k^j l^\alpha \leq \varepsilon(k^a + l^b) \quad \text{on } W_3 = W_{3,1} \cup W_{3,2} \tag{3.5}$$

with $W_{3,1} = \{(k, l) \in \mathbf{N}^* \times \mathbf{N} ;\ k^a \geq l^b, k \geq N\}$ and $W_{3,2} = \{(k, l) \in \mathbf{N}^* \times \mathbf{N} ;\ k^a \leq l^b, l \geq N\}$.

PROOF. Let us show (1). For $\varepsilon > 0$ and $\delta > 0$ we take $N \in \mathbf{N}^*$ such that $N \geq 2$ and

$$\log N \geq \frac{-\log\varepsilon}{\delta(a - j)} .$$

Then the condition (3.3) is verified in the following way.

Take any $(k, l) \in W$ and fix it. Let us show that $k^j l^\alpha \leq \varepsilon k^a l^b$. By setting $d = \log k/\log l$, we have $k = l^d$ and $d \geq s + \delta$. Since $\delta \leq d - s \leq d - (\alpha - b)/(a - j)$, by using the definition of N we obtain

$$\frac{-\log\varepsilon}{(a-j)} \leq \delta \times \log N \leq \left(d - \frac{\alpha - b}{a - j}\right)\log N \leq \left(d - \frac{\alpha - b}{a - j}\right)\log l\,,$$

that is,

$$\frac{-\log\varepsilon}{(a-j)} \leq \left(d - \frac{\alpha - b}{a - j}\right)\log l$$

which is equivalent to $l^{jd+\alpha} \leq \varepsilon l^{ad+b}$. Since $k = l^d$, it follows that $k^j l^\alpha \leq \varepsilon k^a l^b$. Thus, the condition (3.3) is proved.

The proof of (2) is as follows. Take any $\varepsilon > 0$, $0 < \delta < s$, and $\delta_1 > 0$ such that $1/s + \delta_1 = 1/(s-\delta)$. By interchanging k and l in the proof of (1), we have the result that if $N \in \mathbf{N}^*$ satisfies $N \geq 2$ and $\log N \geq (-\log\varepsilon)/(\delta_1(b-\alpha))$ then

$$k^j l^\alpha \leq \varepsilon k^a l^b \quad \text{on } W = \{(k,l) \in \mathbf{N}^* \times \mathbf{N}^*\,;\ l \geq k^{1/s+\delta_1}, k \geq N\}.$$

This proves (3.4) since $1/s + \delta_1 = 1/(s-\delta)$.

Let us show (3). If $(k,l) \in W_{3,1}$ we have $l \leq k^{a/b}$ and so

$$k^j l^\alpha \leq k^{j+\alpha(a/b)} = (k^{j/a+\alpha/b})^a = (k^{1-\delta})^a \leq \varepsilon k^a$$

provided that $\log N \geq (-\log\varepsilon)/\delta a$. Similarly, if $(k,l) \in W_{3,2}$ and $\log N \geq (-\log\varepsilon)/\delta b$ then we have $k^i l^\alpha \leq \varepsilon l^b$. This proves (3.5). □

The second lemma is as follows. Let $a, b \in \mathbf{N}^*$. We denote by Γ^0 the segment in $\mathbf{R}^2$ connecting the two points $(a,0)$ and $(0,b)$, and by $-s$ the slope of the line containing Γ^0. Then we have $s = b/a$. Let $J = \{(j,\alpha) \in \mathbf{N}^2\,;\ (j,\alpha) \in \Gamma^0\}$, $c_{j,\alpha} \in \mathbf{C}$ $((j,\alpha) \in J)$, and suppose $c_{a,0} \neq 0$. We also set

$$L^0(\lambda,\rho) = \sum_{(j,\alpha)\in J} c_{j,\alpha}\lambda^j\rho^\alpha = c_{a,0}\lambda^a + \cdots\,,$$

$$P^0(X) = \sum_{(j,\alpha)\in J} c_{j,\alpha}X^j = c_{a,0}X^a + \cdots\,.$$

Since $s = b/a$, it is easy to see that $L^0(\lambda,\rho)/\rho^b = P^0(\lambda/\rho^s)$. Denote by $\lambda_1,\ldots,\lambda_a$ the roots of $P^0(X) = 0$. Then we have the following:

LEMMA 2. (1) *Let* $0 < d_1 < b/a < d_2$, *and let* $N \in \mathbf{N}^*$ *be sufficiently large such that* $l^{d_2} > l^{d_1}$ *holds for any* $l \geq N$. *Set* $W = \{(k,l) \in \mathbf{N}^2\,;\ l^{d_2} \geq k \geq l^{d_1}, l \geq N\}$. *If the estimate*

$$|L^0(k,l)| \geq c(k^a + l^b) \quad \text{on } W \tag{3.6}$$

holds for some $c > 0$, *we have* $\lambda_q \in \mathbf{C} \setminus [0,\infty)$ *for* $q = 1, 2, \ldots, a$.

(2) *Conversely, if* $\lambda_q \in \mathbf{C} \setminus [0, \infty)$ *for* $q = 1, 2, \dots, a$, *we have the estimate*

$$|L^0(k,l)| \geq c(k^a + l^b) \quad \text{on } \mathbf{N}^* \times \mathbf{N} \tag{3.7}$$

for some $c > 0$.

PROOF. Let us show (1). Suppose the condition (3.6) holds. Then by setting $s = b/a$ and $X = k/l^s$ for $(k,l) \in W$ we obtain

$$|P^0(X)| = \left|\frac{L^0(k,l)}{l^b}\right| \geq c((k/l^s)^a + 1) = c(X^a + 1)\,. \tag{3.8}$$

To show (1), it is enough to prove the following:

$$\lambda_q \neq 0 \text{ for } q = 1, 2, \dots, a\,; \tag{3.9}$$

$$\lambda_q \notin (0, \infty) \text{ for } q = 1, 2, \dots, a\,. \tag{3.10}$$

Let us show (3.9). Suppose $\lambda_q = 0$ for some q. Then we have $P^0(X) = X^d(X - \lambda_{j_1}) \cdots (X - \lambda_{j_{a-d}})$ with $d \geq 1$ and $\lambda_{j_1} \neq 0, \dots, \lambda_{j_{a-d}} \neq 0$. Combining this with (3.8) gives

$$X^d \geq \frac{c(X^a + 1)}{(X + |\lambda_{j_1}|) \cdots (X + |\lambda_{j_{a-d}}|)} \quad \text{with } X = k/l^s\,, \tag{3.11}$$

for any $(k,l) \in W$. Here for any sufficiently large l ($\geq N$) we set $k_l = [l^{d_1}] + 1$ (where $[x]$ denotes the integer part of x). Then we have $l^{d_1} < [l^{d_1}] + 1 = k_l \leq l^{d_1} + 1 < l^{d_2}$ and so $(k_l, l) \in W$. Moreover, if we set $X_l = k_l/l^s$ we have

$$X_l = \frac{k_l}{l^s} \leq \frac{l^{d_1} + 1}{l^s} \longrightarrow 0 \quad (\text{as } l \longrightarrow \infty)\,.$$

Thus, by substituting $X = X_l$ into (3.11) and letting $l \longrightarrow \infty$ we obtain

$$0 \geq \frac{c}{|\lambda_{j_1}| \cdots |\lambda_{j_{a-d}}|}\,,$$

which is a contradiction. Thus, we have proven (3.9).

Let us show (3.10). Suppose $\lambda_1 > 0$. By (3.8) we have

$$|X - \lambda_1| \geq \frac{c(X^a + 1)}{(X + |\lambda_2|) \cdots (X + |\lambda_a|)} \quad \text{with } X = k/l^s\,, \tag{3.12}$$

for any $(k,l) \in W$. We note here that $l^{d_1} < \lambda_1 l^s < l^{d_2}$ if l ($\geq N$) is sufficiently large, and so we can take $k_l \in \mathbf{N}^*$ such that $l^{d_1} < k_l < l^{d_2}$ and $|k_l - \lambda_1 l^s| \leq 1$. This means that $(k_l, l) \in W$ and $|k_l/l^s - \lambda_1| \leq 1/l^s$. Moreover, if we set $X_l = k_l/l^s$ we have $|X_l - \lambda_1| \longrightarrow 0$ (as $l \longrightarrow \infty$). Thus, by substituting $X = X_l$ into (3.12) and letting $l \longrightarrow \infty$ we obtain

$$0 \geq \frac{c((\lambda_1)^a + 1)}{(\lambda_1 + |\lambda_2|) \cdots (\lambda_1 + |\lambda_a|)}\,,$$

which is a contradiction. Thus, we conclude that $\lambda_1 \notin (0, \infty)$.

By the same argument, we can show also that $\lambda_q \notin (0, \infty)$ for $q = 2, \ldots, a$, which is condition (3.10). This completes the proof of (1).

Next, let us show (2). Suppose that $\lambda_q \in \mathbf{C} \setminus [0, \infty)$ for $q = 1, 2, \ldots, a$. Let $l(1, -\lambda_q)$ be the segment that connects the two points 1 and $-\lambda_q$ in the complex plane $\mathbf{C}$. Since $-\lambda_q \notin (-\infty, 0]$ is assumed, we have $0 \notin l(1, -\lambda_q)$. This means that the distance d_q from 0 to $l(1, -\lambda_q)$ is positive.

Take any $(k, l) \in \mathbf{N}^* \times \mathbf{N}$. Then we have

$$\frac{k - \lambda_q l^s}{k + l^s} \in l(1, -\lambda_q)$$

and consequently

$$|k - \lambda_q l^s| \geq d_q (k + l^s) .$$

Therefore, we have

$$|k - \lambda_1 l^s| \cdots |k - \lambda_a l^s| \geq (d_1 \cdots d_a)(k + l^s)^a \quad \text{on } \mathbf{N}^* \times \mathbf{N} . \tag{3.13}$$

Since $s = b/a$, we have $(k + l^s)^a \geq k^a + l^b$ and

$$\begin{aligned} |k - \lambda_1 l^s| \cdots |k - \lambda_a l^s| &= \left| \frac{k}{l^s} - \lambda_1 \right| \cdots \left| \frac{k}{l^s} - \lambda_a \right| \times l^b = |P^0(k/l^s)| \times l^b \\ &= \frac{|L^0(k, l)|}{l^b} \times l^b = |L^0(k, l)| . \end{aligned}$$

Thus, by (3.13) we arrive at the desired inequality (3.7). □

Let us give a generalization of Lemma 2. Let $(a_1, b_1), (a_2, b_2) \in \mathbf{N} \times \mathbf{N}^*$ with $a_1 > a_2 \geq 0$ and $b_2 > b_1 \geq 0$. Denote by Γ the segment in $\mathbf{R}^2$ connecting the two points (a_1, b_1) and (a_2, b_2), and by $-s$ the slope of the line containing Γ. Then we have $s = (b_2 - b_1)/(a_1 - a_2)$. Set $J = \{(j, \alpha) \in \mathbf{N}^2 ;\ (j, \alpha) \in \Gamma\}$. Let $c_{j,\alpha} \in \mathbf{C}$ ($(j, \alpha) \in J$) and suppose $c_{a_1,b_1} \neq 0$. We also set

$$L(\lambda, \rho) = \sum_{(j,\alpha) \in J} c_{j,\alpha} \lambda^j \rho^\alpha = c_{a_1,b_1} \lambda^{a_1} \rho^{b_1} + \cdots ,$$

$$P(X) = \sum_{(j,\alpha) \in J} c_{j,\alpha} X^{j - a_2} = c_{a_1,b_1} X^{a_1 - a_2} + \cdots .$$

We denote by $\lambda_1, \ldots, \lambda_{a_1 - a_2}$ the roots of $P(X) = 0$. Then, we have the following:

LEMMA 3. (1) *Let $N \in \mathbf{N}^*$, $0 < d_1 < s < d_2$, and set $W = \{(k, l) \in \mathbf{N}^2 ;\ l^{d_2} \geq k \geq l^{d_1}, l \geq N\}$. If the estimate*

$$|L(k, l)| \geq c(k^{a_1} l^{b_1} + k^{a_2} l^{b_2}) \quad \text{on } W \tag{3.14}$$

holds for some $c > 0$, we have $\lambda_q \in \mathbf{C} \setminus [0, \infty)$ for $q = 1, 2, \ldots, a_1 - a_2$.

(2) *Conversely, if* $\lambda_q \in \mathbf{C} \setminus [0, \infty)$ *for* $q = 1, 2, \dots, a_1 - a_2$, *we have the estimate*

$$|L(k,l)| \geq c(k^{a_1} l^{b_1} + k^{a_2} l^{b_2}) \quad \text{on } \mathbf{N}^* \times \mathbf{N}\,. \tag{3.15}$$

PROOF. Set

$$L^0(\lambda, \rho) = \sum_{(j,\alpha)\in J} c_{j,\alpha} \lambda^{j-a_2} \rho^{\alpha - b_1} = c_{a_1,b_1} \lambda^{a_1-a_2} + \cdots\,.$$

Then, the condition (3.14) is equivalent to

$$|L^0(k,l)| \geq c(k^{a_1-a_2} + l^{b_2-b_1}) \quad \text{on } W. \tag{3.16}$$

By applying Lemma 2 to (3.16) we obtain the desired result. □

3.2. Proof of Proposition 1. In this section, we give a proof of Proposition 1. In our Newton polygon $\mathcal{N}_0$ at $x = 0$ we have

$$\Lambda_0 \cap \Gamma_i = \{(m_i, n_i), \dots, (m_{i+1}, n_{i+1})\}\,, \quad 1 \leq i \leq p-1,$$

$$\Lambda_0 \cap \Gamma_p = \{(m_p, n_p), \dots\}$$

and $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_p$. Recall also that $-s_i$ denotes the slope of the line containing Γ_i. For $1 \leq i \leq p$, we let $N_i \in \mathbf{N}^*$ and define the sets Ω_i as follows:

$$\Omega_1 = \{(k,l) \in \mathbf{N}^* \times \mathbf{N}\,;\ k \geq l^{s_2+\delta} \ \text{ and } \ l \geq N_1\}\,,$$

$$\Omega_i = \{(k,l) \in \mathbf{N}^* \times \mathbf{N}\,;\ l^{s_{i+1}+\delta} \leq k \leq l^{s_{i-1}-\delta} \ \text{ and } \ l \geq N_i\}\,, \quad 2 \leq i \leq p-1\,,$$

$$\Omega_p = \{(k,l) \in \mathbf{N}^* \times \mathbf{N}\,;\ k \leq l^{s_{p-1}-\delta} \ \text{ and } \ k \geq N_p\}\,.$$

By applying (1) and (2) of Lemma 1 to the above situation we obtain the following:

LEMMA 4. (1) *For any* $\varepsilon > 0$ *and* $\delta > 0$ *we can find an* $N_1 \in \mathbf{N}^*$ *such that for any* $(j, \alpha) \in \Lambda_0 \cap (\Gamma \setminus \Gamma_1)$ *we have*

$$k^j l^\alpha \leq \varepsilon k^{m_2} l^{n_2} \quad \text{on } \Omega_1\,.$$

(2) *Let* $2 \leq i \leq p-1$. *For any* $\varepsilon > 0$ *and* $0 < \delta < s_{i-1}$ *we can find an* $N_i \in \mathbf{N}^*$ *such that for any* $(j, \alpha) \in \Lambda_0 \cap (\Gamma \setminus \Gamma_i)$ *we have*

$$k^j l^\alpha \leq \varepsilon (k^{m_i} l^{n_i} + k^{m_{i+1}} l^{n_{i+1}}) \quad \text{on } \Omega_i\,.$$

(3) *For any* $\varepsilon > 0$ *and* $0 < \delta < s_{p-1}$ *we can find an* $N_p \in \mathbf{N}^*$ *such that for any* $(j, \alpha) \in \Lambda_0 \cap (\Gamma \setminus \Gamma_p)$ *we have*

$$k^j l^\alpha \leq \varepsilon k^{m_p} l^{n_p} \quad \text{on } \Omega_p\,.$$

For $1 \leq i \leq p$ we set

$$L_{\Gamma_i}(\lambda, \rho) = \sum_{(j,\alpha)\in \Lambda_0 \cap \Gamma_i} c_{j,\alpha}(0) \lambda^j \rho^\alpha = c_{m_i,n_i}(0) \lambda^{m_i} \rho^{n_i} + \cdots\,.$$

Then we have the following consequence on Lemma 4.

LEMMA 5. (1) *For any $\varepsilon > 0$ and $\delta > 0$ we can find an $N_1 \in \mathbf{N}^*$ such that the following estimates hold on Ω_1*:

$$|L_\Gamma(k,l) - L_{\Gamma_1}(k,l)| \le \varepsilon k^{m_2} l^{n_2}\,,$$

$$|\phi(k,l) - k^m - k^{m_2} l^{n_2}| \le \varepsilon k^{m_2} l^{n_2}\,.$$

(2) *Let $2 \le i \le p-1$. For any $\varepsilon > 0$ and $0 < \delta < s_{i-1}$ we can find an $N_i \in \mathbf{N}^*$ such that the following estimates hold on Ω_i*:

$$|L_\Gamma(k,l) - L_{\Gamma_i}(k,l)| \le \varepsilon(k^{m_i} l^{n_i} + k^{m_{i+1}} l^{n_{i+1}})\,,$$

$$|\phi(k,l) - k^{m_i} l^{n_i} - k^{m_{i+1}} l^{n_{i+1}}| \le \varepsilon(k^{m_i} l^{n_i} + k^{m_{i+1}} l^{n_{i+1}})\,.$$

(3) *For any $\varepsilon > 0$ and $0 < \delta < s_{i-1}$ we can find an $N_p \in \mathbf{N}^*$ such that the following estimates hold on Ω_p*:

$$|L_\Gamma(k,l) - L_{\Gamma_p}(k,l)| \le \varepsilon k^{m_p} l^{n_p}\,,$$

$$|\phi(k,l) - k^{m_p} l^{n_p}| \le \varepsilon k^{m_p} l^{n_p}\,.$$

Since $0 = s_p < s_{p-1} < \cdots < s_2 < s_1 < s_0 = \infty$, we can take $\delta > 0$ sufficiently small so that

$$\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_p \supset \{(k,l) \in \mathbf{N}^2\,;\ k \ge N \ \text{ and } \ l \ge N\} \tag{3.17}$$

for some $N \in \mathbf{N}^*$ sufficiently large. By using Lemma 5 and (3.17), let us show the equivalence of (P1), (P2) and (P3).

PROOF OF "(P1) $\Longrightarrow$ (P2)". Suppose the condition (P1) holds. Then we can find a $c_1 > 0$ such that the following conditions are satisfied on $\mathbf{N}^* \times \mathbf{N}$:

$$|L_{\Gamma_i}(k,l)| \ge c_1(k^{m_i} l^{n_i} + k^{m_{i+1}} l^{n_{i+1}})\,, \quad 1 \le i \le p-1, \tag{3.18}$$

$$|L_{\Gamma_p}(k,l)| \ge c_1 k^{m_p} l^{n_p}\,. \tag{3.19}$$

Note that (3.18) is a consequence of (2) of Lemma 3, and (3.19) follows from the fact that (3.19) is equivalent to the condition $|P_p(k)| \ge c_1 k^{m_p}$ (for any $k \in \mathbf{N}^*$) which is easily verified by the assumption (ii) of (GP).

We take $\varepsilon > 0$ and $\delta > 0$ sufficiently small. Then by (1) of Lemma 5 and (3.18) with $i = 1$ we have

$$\begin{aligned}
|L_\Gamma(k,l)| &\ge |L_{\Gamma_1}(k,l)| - |L_\Gamma(k,l) - L_{\Gamma_1}(k,l)| \\
&\ge c_1(k^m + k^{m_2} l^{n_2}) - \varepsilon k^{m_2} l^{n_2} \ge (c_1 - \varepsilon)(k^m + k^{m_2} l^{n_2}) \\
&\ge (c_1 - \varepsilon)\big(\phi(k,l) - |\phi(k,l) - k^m - k^{m_2} l^{n_2})|\big) \\
&\ge (c_1 - \varepsilon)\big(\phi(k,l) - \varepsilon(k^m + k^{m_2} l^{n_2})\big) \\
&\ge (c_1 - \varepsilon)(1 - \varepsilon)\phi(k,l) \quad \text{on } \Omega_1.
\end{aligned}$$

By the same argument, we see that there is a $c_2 > 0$ such that

$$|L_\Gamma(k,l)| \geq c_2\phi(k,l) \quad \text{on } \Omega_i, \quad 1 \leq i \leq p,$$

and so by (3.17) there is an $N \in \mathbf{N}$ such that

$$|L_\Gamma(k,l)| \geq c_2\phi(k,l) \quad \text{on } \{(k,l) \in \mathbf{N}^* \times \mathbf{N}\,;\ k \geq N \ \text{ and } \ l \geq N\}.$$

Thus, to complete the proof of "(P1) $\Longrightarrow$ (P2)", it is enough to prove the following assertion: there are $c_3 > 0$ and $N^0 \in \mathbf{N}^*$ such that

$$|L_\Gamma(k,l)| \geq c_3\phi(k,l) \quad \text{for } k \geq N^0 \text{ and } 0 \leq l < N, \tag{3.20}$$

$$|L_\Gamma(k,l)| \geq c_3\phi(k,l) \quad \text{for } 1 \leq k < N \text{ and } l \geq N^0. \tag{3.21}$$

Let us show this assertion.

We note that if $l = 0$ we have $|L_\Gamma(k,0)| = k^m = \phi(k,0)$. If l $(1 \leq l < N)$ is fixed, $L_\Gamma(k,l)$ and $\phi(k,l)$ are monic polynomials of degree m with respect to k and so it is clear that $|L_\Gamma(k,l)| \geq (1/2)\phi(k,l)$ for sufficiently large k. This proves (3.20).

Similarly, if $1 \leq k < N$ is fixed,

$$L_\Gamma(k,l) = P_p(k)l^{n_p} + \cdots, \quad \text{and}$$
$$\phi(k,l) = k^{m_p}l^{n_p} + \cdots$$

are polynomials of degree n_p with respect to l, in which $P_p(k) \neq 0$ by the assumption (ii) of (GP). Therefore, if we take $c_3 > 0$ such that $|P_p(k)| > c_3k^{m_p}$ for $1 \leq k < N$, we have $|L_\Gamma(k,l)| \geq c_3\phi(k,l)$ for sufficiently large l. This proves (3.21).

Thus, we have proven that (P1) implies (P2) for some $c > 0$ and a finite subset S of $\mathbf{N}^* \times \mathbf{N}$. For example, in the case $N^0 \geq N$, we may take $S = \{(k,l) \in \mathbf{N}^* \times \mathbf{N}\,;\ k \leq N^0 \ \text{and} \ l \leq N^0\}$. □

PROOF OF "(P2) $\Longrightarrow$ (P1)". Suppose that (3.1) holds for some $c > 0$ and a finite subset S of $\mathbf{N}^* \times \mathbf{N}$. Then, by (1) of Lemma 5 we have

$$\begin{aligned}
|L_{\Gamma_1}(k,l)| &\geq |L_\Gamma(k,l)| - |L_\Gamma(k,l) - L_{\Gamma_1}(k,l)| \\
&\geq c\phi(k,l) - \varepsilon k^{m_2}l^{n_2} \geq (c-\varepsilon)\phi(k,l) \\
&\geq (c-\varepsilon)\big(k^m + k^{m_2}l^{n_2} - |\phi(k,l) - k^m - k^{m_2}l^{n_2})|\big) \\
&\geq (c-\varepsilon)\big((k^m + k^{m_2}l^{n_2}) - \varepsilon(k^m + k^{m_2}l^{n_2})\big) \\
&= (c-\varepsilon)(1-\varepsilon)(k^m + k^{m_2}l^{n_2}) \quad \text{on } \Omega_1 \setminus S.
\end{aligned}$$

By the same argument we can see that there is a $c_1 > 0$ satisfying

$$|L_{\Gamma_i}(k,l)| \geq c_1(k^{m_i}l^{n_i} + k^{m_{i+1}}l^{n_{i+1}}) \quad \text{on } \Omega_i \setminus S, \quad 1 \leq i \leq p-1, \tag{3.22}$$

$$|L_{\Gamma_p}(k,l)| \geq c_1k^{m_p}l^{n_p} \quad \text{on } \Omega_p \setminus S. \tag{3.23}$$

Then, by applying Lemma 3 to (3.22) we obtain the condition $\lambda_{i,q} \in \mathbf{C} \setminus [0,\infty)$ $(1 \le q \le m_i - m_{i+1})$ for $1 \le i \le p-1$. This proves (i) of (GP).

Let us show (ii) of (GP) when $m_p \ge 1$. In this case, by (3.23) we have $|P_p(k)| \ge c_1 k^{m_p}$ for any $k \ge N_p$ (where N_p is an integer satisfying (3) of Lemma 5). This shows that $\lambda_{p,q} \notin \{k \in \mathbf{N}\,;\, k \ge N_p\}$ $(1 \le q \le m_p)$.

For $1 \le k < N_p$, by directly using condition (3.1) we have $|L_\Gamma(k,l)|/l^{n_p} \ge c\phi(k,l)/l^{n_p}$ for any sufficiently large l. Therefore, by letting $l \longrightarrow \infty$ we obtain $|P_p(k)| \ge ck^{m_p}$ for $1 \le k < N_p$. This shows $\lambda_{p,q} \notin \{k \in \mathbf{N}\,;\, 1 \le k < N_p\}$ $(1 \le q \le m_p)$. Thus, we have proven (ii) of (GP). □

We denote by $int(\mathcal{N}_0)$ the interior of the set $\mathcal{N}_0$. Then, to show the equivalence of (P2) and (P3) it is sufficient to prove the following lemma.

LEMMA 6. *Let $(j,\alpha) \in I_m$ be such that $(j,\alpha) \in int(\mathcal{N}_0)$. Then for any $\varepsilon > 0$ we can find a finite subset S of $\mathbf{N}^* \times \mathbf{N}$ such that*

$$k^j l^\alpha \le \varepsilon\phi(k,l) \quad on \ \mathbf{N}^* \times \mathbf{N} \setminus S\,. \tag{3.24}$$

PROOF. Let $(j,\alpha) \in int(\mathcal{N}_0)$. If $l = 0$ and $\alpha = 0$, we have $k^j l^\alpha = k^j \le \varepsilon k^m = \varepsilon\phi(k,0)$ for any $k \in \mathbf{N}^*$ satisfying $\log k \ge -\log\varepsilon/(m-j)$. If $l = 0$ and $\alpha > 0$, we have $k^j l^\alpha = 0 \le \varepsilon k^m = \varepsilon\phi(k,0)$ for any $k \in \mathbf{N}^*$. Therefore, to show (3.24) it is enough to prove the following: for any $\varepsilon > 0$ we can find a finite subset S of $\mathbf{N}^* \times \mathbf{N}^*$ such that

$$k^j l^\alpha \le \varepsilon\phi(k,l) \quad \text{on} \ \mathbf{N}^* \times \mathbf{N}^* \setminus S\,. \tag{3.25}$$

Let us show this assertion. We note that the situation is divided into the following three cases:

Case 1) $m_{i+1} \le j < m_i$, $n_i \le \alpha < n_{i+1}$ and $\alpha < -s_i(j - m_i) + n_i$ for some $1 \le i \le p-1$.
Case 2) $m_{i+1} \le j < m_i$, $0 \le \alpha < n_i$ for some $1 \le i \le p-1$.
Case 3) $0 \le j < m_p$ and $0 \le \alpha < n_p$ (if $m_p \ge 1$).

Take any $\varepsilon > 0$. Let us first consider Case 1). Note that $-s_i$ is the slope of Γ_i and $y = -s_i(x - m_i) + n_i$ defines the line containing Γ_i. Since $s_i = (n_{i+1} - n_i)/(m_i - m_{i+1})$ holds, we see that $\alpha < -s_i(j - m_i) + n_i$ is equivalent to the condition

$$\frac{j - m_{i+1}}{m_i - m_{i+1}} + \frac{\alpha - n_i}{n_{i+1} - n_i} = 1 - \delta \tag{3.26}$$

for some $\delta > 0$. Thus in this case we have $0 \le j - m_{i+1} < m_i - m_{i+1}$, $0 \le \alpha - n_i < n_{i+1} - n_i$ and (3.26). By (3) of Lemma 1 we can find a finite subset S of $\mathbf{N}^* \times \mathbf{N}$ such that, on $\mathbf{N}^* \times \mathbf{N} \setminus S$,

$$k^{j-m_{i+1}} l^{\alpha - n_i} \le \varepsilon(k^{m_i - m_{i+1}} + l^{n_{i+1} - n_i})$$

which yields

$$k^j l^\alpha \le \varepsilon(k^{m_i} l^{n_i} + k^{m_{i+1}} l^{n_{i+1}})\,.$$

We then have (3.25) since $k^{m_i}l^{n_i} + k^{m_{i+1}}l^{n_{i+1}} \le \phi(k,l)$.

Now, in Case 2), we have $k^j l^\alpha \le k^j l^{n_i}$ and so by applying the result in Case 1) (with α replaced by n_i) we again obtain (3.25). Finally, in Case 3), we have

$$k^j l^\alpha \le \varepsilon k^{m_p} l^{n_p} \quad \text{if } k \ge (-\log\varepsilon)/(m_p - j) \text{ or } l \ge (-\log\varepsilon)/(n_p - \alpha)$$

and this also yields (3.25). □

4. Proof of Theorem 2 under (N)

In this section, we prove Theorem 2 under the additional condition (N). The general case will be proved in the next section.

Suppose A_1), A_2) and A_3) hold. Let $L(\lambda,\rho)$ and $\phi(\lambda,\rho)$ be as in Sections 1 and 3, and suppose the conditions (N), (R), $p \ge 2$ and (GP) are satisfied. By Proposition 1 and the condition (N) we have

$$|L(k,l)| \ge c\phi(k,l) \quad \text{on } \mathbf{N}^* \times \mathbf{N} \tag{4.1}$$

for some $c > 0$. We set $\mathcal{J} = \{(j,\alpha) \in I_m\,;\, b_{j,\alpha}(x) \not\equiv 0\}$. If $c_{j,\alpha}(0) \ne 0$, by the definition of $\mathcal{N}_0$ we have $(j,\alpha) \in \mathcal{N}_0$. If $c_{j,\alpha}(0) = 0$ and $b_{j,\alpha}(x) \not\equiv 0$, we have $c_{j,\alpha}(x) \not\equiv 0$ and so by the assumption (R) we have $(j,\alpha) \in \mathcal{N}_0$. Thus, we have $\mathcal{J} \subset \mathcal{N}_0$. We also set

$$\mathscr{L}(x,\lambda,\rho) = L(\lambda,\rho) - \sum_{(j,\alpha)\in\mathcal{J}} x b_{j,\alpha}(x)\lambda^j \rho(\rho-1)\cdots(\rho-\alpha+1)\,.$$

Then our equation (2.1) can be written in the form

$$\mathscr{L}(x, t\partial_t, x\partial_x)u = a(x)t + R_2\big(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{(j,\alpha)\in I_m}\big)\,. \tag{4.2}$$

By the condition (N), as we already know in Theorem 1, the equation (4.2) has a unique formal power series solution $u(t,x) \in \mathbf{C}[[t,x]]$ satisfying $u(0,x) \equiv 0$. Our purpose is to show the convergence of this formal solution.

4.1. Some lemmas.

First, let us present some lemmas which are needed in the proof of Theorem 2.

LEMMA 7. (1) *Let $a, b \in \mathbf{N}^*$, $0 \le j \le a$, $0 \le \alpha \le b$, and suppose $j/a + \alpha/b \le 1$. Then we have*

$$k^j l^\alpha \le k^a + l^b \quad \textit{on } \mathbf{N}^* \times \mathbf{N}\,. \tag{4.3}$$

(2) *Let $(j,\alpha) \in I_m$. If $(j,\alpha) \in \mathcal{N}_0$ then we have*

$$k^j l^\alpha \le \phi(k,l) \quad \textit{on } \mathbf{N}^* \times \mathbf{N}\,. \tag{4.4}$$

PROOF. Let us show (1). If $k^a \ge l^b$ then we have $k^j l^\alpha \le k^{j+\alpha(a/b)} = (k^{j/a+\alpha/b})^a \le k^a \le k^a + l^b$. If $k^a \le l^b$, we also have $k^j l^\alpha \le l^{j(b/a)+\alpha} = (l^{j/a+\alpha/b})^b \le l^b \le k^a + l^b$. This proves (4.3).

By using the result (1) and by the similar argument as in the proof of Lemma 6, we can easily verify the result (2). □

For a formal power series $f(x) = \sum_{l \geq 0} f_l x^l \in \mathbf{C}[[x]]$ we define a (formal) norm $\|f\|_\rho$ of $f(x)$ by

$$\|f\|_\rho = \sum_{l \geq 0} |f_l| \rho^l .$$

We have the following Nagumo-type lemma:

LEMMA 8. *Let $R > 0$. If $f(x) \in \mathbf{C}[[x]]$ satisfies*

$$\|f\|_\rho \leq \frac{C}{(R-\rho)^a} \quad \textit{for any } \ 0 < \rho < R$$

for some $C > 0$ and $a \geq 0$, we have

$$\left\|\partial_x f\right\|_\rho \leq \frac{(a+1)eC}{(R-\rho)^{a+1}} \quad \textit{for any } \ 0 < \rho < R.$$

Since $l\rho^{l-1} \leq (\rho+h)^l/h$ holds for any $l \geq 1$, $\rho > 0$ and $h > 0$, we have $\|\partial_x f\|_\rho \leq \|f\|_{\rho+h}/h$. Therefore, by a standard argument we can show this lemma. For details, see the proof of [Lemma 5 in [13]].

4.2. On the equation $\mathscr{L}(x, k, x\partial_x)w = g$. Let $k \in \mathbf{N}^*$ and consider

$$\mathscr{L}(x, k, x\partial_x)w = g(x) \in \mathbf{C}[[x]] . \tag{4.5}$$

PROPOSITION 2. *There are $K > 0$ and $R > 0$ which satisfy the following: for any $k \in \mathbf{N}^*$ and $g(x) \in \mathbf{C}[[x]]$, the equation* (4.5) *has a unique solution $w(x) \in \mathbf{C}[[x]]$ that satisfies*

$$\|w\|_\rho \leq \frac{K}{k^m}\|g\|_\rho \quad \textit{for any } \ 0 < \rho \leq R \tag{4.6}$$

provided that $\|g\|_R < \infty$. In particular, if $g(x)$ is a holomorphic function in a neighborhood of $\{x \in \mathbf{C}\,;\ |x| \leq R\}$, (4.5) *has a unique holomorphic solution in a neighborhood of $\{x \in \mathbf{C}\,;\ |x| \leq R\}$.*

PROOF. We set

$$w(x) = \sum_{l \geq 0} w_l x^l , \quad g(x) = \sum_{l \geq 0} g_l x^l , \quad b_{j,\alpha}(x) = \sum_{h \geq 0} b_{j,\alpha,h} x^h .$$

Then, by substituting these series into (4.5) and comparing the coefficients of x^l in both sides we obtain the following recursive formulas:

$$L(k,0)w_0 = g_0$$

and for $l \geq 1$,

$$L(k,l)w_l = g_l + \sum_{(j,\alpha)\in\mathcal{J}} \sum_{h+i=l-1} b_{j,\alpha,h} k^j i(i-1)\cdots(i-\alpha+1) w_i \,.$$

Therefore, by the condition (N) we see that w_l ($l = 0, 1, 2, \ldots$) are uniquely determined, inductively on l. Moreover, by the condition (4.1) we have the estimates

$$|w_0| \leq \frac{1}{c\phi(k,0)}|g_0| = \frac{1}{ck^m}|g_0|$$

and for $l \geq 1$,

$$\begin{aligned} |w_l| &\leq \frac{1}{c\phi(k,l)}|g_l| \\ &\quad + \frac{1}{c}\sum_{(j,\alpha)\in\mathcal{J}} \sum_{h+i=l-1} |b_{j,\alpha,h}| \times \frac{k^j i(i-1)\cdots(i-\alpha+1)}{\phi(k,l)}|w_i| \\ &\leq \frac{1}{ck^m}|g_l| + \frac{M}{c}\sum_{(j,\alpha)\in\mathcal{J}} \sum_{h+i=l-1} |b_{j,\alpha,h}||w_i| \end{aligned}$$

for some $M > 0$ which is independent of (j, α), k and l. In the above computations, we used the condition $\mathcal{J} \subset \mathcal{N}_0$ and (2) of Lemma 7. Hence, for any $\rho > 0$ we have

$$\|w\|_\rho \leq \frac{1}{ck^m}\|g\|_\rho + \frac{M}{c}\rho \sum_{(j,\alpha)\in\mathcal{J}} \|b_{j,\alpha}\|_\rho \|w\|_\rho \,.$$

Since $b_{j,\alpha}(x)$ ($(j,\alpha) \in \mathcal{J}$) are holomorphic functions in a neighborhood of $x = 0$, by taking $R > 0$ sufficiently small we have

$$\frac{M}{c}R \sum_{(j,\alpha)\in\mathcal{J}} \|b_{j,\alpha}\|_R \leq \frac{1}{2}$$

and so

$$\|w\|_\rho \leq \frac{1}{ck^m}\|g\|_\rho + \frac{1}{2}\|w\|_\rho \quad \text{for any } \ 0 < \rho \leq R\,.$$

Thus, by setting $K = 2/c$ we arrive at (4.6). □

4.3. Formal solution of (4.2). Note that our equation (4.2) can be written in the form

$$\mathscr{L}(x, t\partial_t, x\partial_x)u = a(x)t + \sum_{i+|\nu|\geq 2} a_{i,\nu}(x)t^i \prod_{(j,\alpha)\in I_m} \left[(t\partial_t)^j \partial_x^\alpha u\right]^{\nu_{j,\alpha}} , \tag{4.7}$$

where $a(x)$ and $a_{i,\nu}(x)$ $(i + |\nu| \geq 2)$ are all holomorphic functions in a neighborhood of $\{x \in \mathbf{C}\,;\ |x| \leq R\}$ for some $R > 0$ and the series

$$\sum_{i+|\nu|\geq 2} \|a_{i,\nu}\|_R t^i X^{|\nu|}$$

is convergent in a neighborhood of $(t, X) = (0, 0)$. Take a small $R > 0$ and fix it. Without loss of generality we may assume that $0 < R \leq 1$ and that Proposition 2 is valid also for this R.

Let

$$u(t, x) = \sum_{k\geq 1} u_k(x) t^k\,, \quad u_k(x) \in \mathbf{C}[[x]]\ \ (k \geq 1)\,, \tag{4.8}$$

be the unique formal solution of (4.7). By substituting this into (4.7) and comparing the coefficients of t^k in both sides we obtain the following recursive formulas:

$$\mathscr{L}(x, 1, x\partial_x)u_1 = a(x) \tag{4.9}$$

and for $k \geq 2$,

$$\mathscr{L}(x, k, x\partial_x)u_k = \sum_{2\leq i+|\nu|\leq k} a_{i,\nu}(x) \sum_{i+|k(\nu)|=k} \prod_{(j,\alpha)\in I_m} \prod_{h=1}^{\nu_{j,\alpha}} \big[(k_{j,\alpha}(h))^j \partial_x^\alpha u_{k_{j,\alpha}(h)}\big]\,, \tag{4.10}$$

where

$$|k(\nu)| = \sum_{(j,\alpha)\in I_m} (k_{j,\alpha}(1) + \cdots + k_{j,\alpha}(\nu_{j,\alpha}))\,.$$

By applying Proposition 2 to (4.9) and (4.10), we can see that $u_k(x)$ $(k \geq 1)$ are uniquely determined, inductively on k. Moreover, $u_k(x)$ $(k \geq 1)$ are all holomorphic in a neighborhood of $\{x \in \mathbf{C}\,;\ |x| \leq R\}$.

4.4. Convergence of the formal solution. Let us show the convergence of the formal solution (4.8). The argument below is almost the same as in [Chapter 5 of [7]]; but for self-containedness of the paper, we discuss it again here.

Set $A_{i,\nu} = \|a_{i,\nu}\|_R$ $(i + |\nu| \geq 2)$ and take $A > 0$ such that

$$\|\partial_x^\alpha u_1\|_R \leq A \quad \text{for any } \alpha \leq m\,, \tag{4.11}$$

and let us consider the following functional equation with respect to (t, Y):

$$Y = At + \frac{K}{(R-\rho)^m} \sum_{i+|\nu|\geq 2} \frac{A_{i,\nu}}{(R-\rho)^{m(i+|\nu|-2)}} t^i (BY)^{|\nu|}\,, \tag{4.12}$$

where $B = (em)^m$ and ρ is a parameter with $0 < \rho < R$. Since this is an analytic functional equation, the implicit function theorem implies that (4.12) has a unique holomorphic solution

$Y = Y(t)$ in a neighborhood of $t = 0$ satisfying $Y(0) = 0$. If we expand this into Taylor series $Y(t) = \sum_{k\geq 1} Y_k t^k$, it is easy to see that Y_k ($k \geq 1$) are determined by the following recursive formulas:

$$Y_1 = A \tag{4.13}$$

and for $k \geq 2$,

$$Y_k = \frac{K}{(R-\rho)^m} \sum_{2\leq i+|\nu|\leq k} \frac{A_{i,\nu}}{(R-\rho)^{m(i+|\nu|-2)}} \sum_{i+|k(\nu)|=k} \prod_{(j,\alpha)\in I_m} \prod_{h=1}^{\nu_{j,\alpha}} (BY_{k_{j,\alpha}(h)})\,. \tag{4.14}$$

Moreover, by induction on k we can see that Y_k has the form

$$Y_k = \frac{C_k}{(R-\rho)^{m(k-1)}}\,, \quad k \geq 1\,,$$

where $C_1 = A$ and $C_k \geq 0$ ($k \geq 2$) are constants which are independent of the parameter ρ. Let us write $Y_k = Y_k(\rho)$ to emphasize that Y_k depends on ρ. The following lemma guarantees that $Y(t)$ is a majorant series of the formal solution (4.8).

LEMMA 9. *For any $k \geq 1$ we have*

$$\|k^j \partial_x^\alpha u_k\|_\rho \leq BY_k(\rho) \quad \textit{for any } \ 0 < \rho < R \ \textit{ and } \ (j,\alpha) \in I_m\,. \tag{4.15}$$

PROOF. In the case $k = 1$, (4.15) is clear from (4.11), (4.13) and the fact $B > 1$. Let us show (4.15) in the general case by induction on k.

Let $k \geq 2$, and suppose that (4.15) (with k replaced by n) is true for all $n < k$. Then, by applying Proposition 2 to (4.10) and by using the induction hypothesis we obtain

$$\|u_k\|_\rho \leq \frac{K}{k^m} \sum_{2\leq i+|\nu|\leq k} A_{i,\nu} \sum_{i+|k(\nu)|=k} \prod_{(j,\alpha)\in I_m} \prod_{h=1}^{\nu_{j,\alpha}} BY_{k_{j,\alpha}(h)}(\rho) \tag{4.16}$$

for any $0 < \rho < R$. Since $0 < R \leq 1$, by comparing (4.16) with (4.14) we have

$$\|u_k\|_\rho \leq \frac{(R-\rho)^m}{k^m} Y_k = \frac{1}{k^m} \frac{C_k}{(R-\rho)^{m(k-2)}} \quad \text{for any } \ 0 < \rho < R\,.$$

Hence, applying Lemma 8 to this estimate yields

$$\begin{aligned}\|k^j \partial_x^\alpha u_k\|_\rho &\leq \frac{k^j}{k^m} \frac{(m(k-2)+1)\cdots(m(k-2)+\alpha)e^\alpha C_k}{(R-\rho)^{m(k-2)+\alpha}} \\ &\leq \frac{k^{j+\alpha}}{k^m} \frac{(me)^\alpha C_k}{(R-\rho)^{m(k-2)+m}} \leq \frac{(me)^m C_k}{(R-\rho)^{m(k-1)}} = BY_k(\rho)\end{aligned}$$

for any $0 < \rho < R$ and $(j,\alpha) \in I_m$. This completes the proof of the lemma. □

COMPLETION OF THE PROOF OF THEOREM 2 UNDER (N). By Lemma 9 we have

$$\sum_{k\geq 1}\|u_k\|_\rho |t|^k \leq B\sum_{k\geq 1} Y_k |t|^k = BY(|t|)\,.$$

If $0 < \rho < R$ is fixed, $Y(|t|)$ is convergent in a neighborhood of $|t| = 0$. This proves the convergence of our formal solution (4.8). □

5. Proof of Theorem 2 in the general case

In this section, we prove Theorem 2 in the general case without the condition (N). In this case, we have

$$|L(k,l)| \geq c\phi(k,l) \quad \text{on } \{(k,l) \in \mathbf{N}^* \times \mathbf{N}\,;\ k \geq N \ \text{ or } \ l \geq N\} \tag{5.1}$$

for some $c > 0$ and a sufficiently large $N \in \mathbf{N}^*$ (by Proposition 1). We set

$$\mathcal{M} = \{(k,l) \in \mathbf{N}^* \times \mathbf{N}\,;\ k \geq N \ \text{ or } \ l \geq N\}\,.$$

PROPOSITION 3. *Suppose* A_1), A_2), A_3), (R), $p \geq 2$, (GP), *and* (5.1) *hold. If* $u(t,x) \in \mathbf{C}[[t,x]]$ *is a formal solution of* (4.2) *of the form*

$$u(t,x) = \sum_{(k,l)\in\mathcal{M}} u_{k,l} t^k x^l\,, \tag{5.2}$$

then it is convergent in a neighborhood of $(0,0) \in \mathbf{C}_t \times \mathbf{C}_x$.

PROOF. Since $u(t,x)$ is a formal solution of (4.2), we have $a(x) = O(x^N)$ (as $x \longrightarrow 0$) and $a_{i,0}(x) = O(x^N)$ (as $x \longrightarrow 0$) for $2 \leq i < N$. Therefore, we can apply the same argument as in Section 4. □

Generally, if

$$u(t,x) = \sum_{k\geq 1, l\geq 0} u_{k,l} t^k x^l$$

is a formal solution of (4.2) and we set

$$w(t,x) = u(t,x) - \varphi(t,x) \quad \text{with } \ \varphi(t,x) = \sum_{1\leq k<N, 0\leq l<N} u_{k,l} t^k x^l\,,$$

then $w(t,x)$ is of the form (5.2) and becomes a formal solution of the equation

$$\begin{aligned}\mathscr{L}(x, t\partial_t, x\partial_x)w = a(x)t + R_2\big(t, x, \{(t\partial_t)^j \partial_x^\alpha (w+\varphi)\}_{(j,\alpha)\in I_m}\big)\\ - \mathscr{L}(x, t\partial_t, x\partial_x)\varphi\,.\end{aligned} \tag{5.3}$$

Thus, by applying Proposition 3 to (5.3), we have the convergence of $w(t,x)$.

6. Proof of Example 1

Consider the equation

$$(6.1)\qquad (t\partial_t)^4 u + (x\partial_x)^2 u = a(x)t + c(x)(t\partial_t)^2(x\partial_x)^k u + R_2\big(t, x, \{(t\partial_t)^j \partial_x^\alpha u\}_{j+\alpha\le 4,\, j<4}\big), \quad k = 0, 1, 2\,.$$

Let us show the results in Example 1.

The case $k = 0$ can be easily verified.

In the case $k = 1$, we have $P_1(X) = X^4 - c(0)X^2 + 1$. Thus, to show the equivalence of (GP) and $c(0) \notin [2, \infty)$, it is enough to prove that $P_1(X) \neq 0$ on $[0, \infty)$ if and only if $c(0) \notin [2, \infty)$, that is, $P_1(\lambda) = 0$ for some $\lambda \in [0, \infty)$ if and only if $c(0) \in [2, \infty)$. This can be verified as follows. If $P_1(\lambda) = 0$ for some $\lambda \in [0, \infty)$, we have $\lambda \neq 0$ and $c(0) = (\lambda - 1/\lambda)^2 + 2 \in [2, \infty)$. Conversely, if $c(0) \in [2, \infty)$, we can find a $\lambda \in (0, \infty)$ which satisfies $c(0) = (\lambda - 1/\lambda)^2 + 2$ and so $P_1(\lambda) = 0$. Since $L(k, l) = k^4 - c(0)k^2 l + l^2$, we have $L(k, 0) \neq 0$ (for $k \geq 1$) and $L(k, l)/l^2 = (k^2/l)^2 - c(0)(k^2/l) + 1$ (for $l \geq 1$). Therefore, we can see the condition (N) under $c(0) \notin [2, \infty)$ in the same way.

In the case $k = 2$ and $c(0) \neq 0$, we have $P_1(X) = X^2 - c(0)$ and $P_2(X) = c(0)X^2 - 1$. It is easy to see that $P_1(X) \neq 0$ on $[0, \infty)$ if and only if $c(0) \notin [0, \infty)$ and that $P_2(X) \neq 0$ on $\mathbf{N}^*$ if and only if $c(0) \notin \{1, 1/2^2, 1/3^2, \ldots\}$. This shows the equivalence of (GP) and $c(0) \notin [0, \infty)$. Since $L(k, l) = k^4 - c(0)k^2 l^2 + l^2$, we have $L(k, 0) \neq 0$ (for $k \geq 1$) and $L(k, l)/l^2 = (k^2/l)^2 - (c(0)l)(k^2/l) + 1$ (for $l \geq 1$). Therefore, we can see the condition (N) under $c(0) \notin [0, \infty)$ in the same way.

Finally, let us show that the formal solution of (6.1) is not convergent in general when $k = 2$ and $c(0) = 0$ (with $c(x) \not\equiv 0$). Consider the following particular case with $a(x) = x$, $c(x) = x$ and $R_2 = xt^2/(1 - t)$. We have the following result:

LEMMA 10. *The equation*

$$(6.2)\qquad (t\partial_t)^4 u + (x\partial_x)^2 u = xt/(1 - t) + x(t\partial_t)^2(x\partial_x)^2 u\,, \quad u(0, x) = 0$$

has a unique formal power series solution

$$u(t, x) = \sum_{k\geq 1, l\geq 1} \frac{k^{2(l-1)}(l - 1)!^2}{(k^4 + 1^2)(k^4 + 2^2)\cdots(k^4 + l^2)}\, t^k x^l$$

which is divergent.

PROOF. We show that the above formal solution is divergent. Take any $0 < \rho < 1$. Then if $l = k^2$ we have $\rho^k \geq \rho^l$ and so

$$u(\rho, \rho) \geq \sum_{l=k^2, k\geq 1} \frac{k^{2(l-1)}(l - 1)!^2}{(k^4 + 1^2)(k^4 + 2^2)\cdots(k^4 + l^2)}\, \rho^k \rho^l$$

$$\geq \sum_{l=k^2, k\geq 1} \frac{k^{2(l-1)}(l-1)!^2}{(k^4+l^2)^l} \rho^l \rho^l = \sum_{l=k^2, k\geq 1} \frac{l^{l-1}(l-1)!^2}{(2l^2)^l} (\rho^2)^l \,.$$

We can see by Stirling's formula that the last series is divergent for any $0 < \rho < 1$. □

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Present Address:
DEPARTMENT OF INFORMATION AND COMMUNICATION SCIENCES,
SOPHIA UNIVERSITY,
KIOICHO, CHIYODA-KU, TOKYO 102–8554, JAPAN.
e-mail: h-tahara@sophia.ac.jp