B_w^u-function Spaces and Their Interpolation

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Abstract. We introduce B_w^u-function spaces which unify Lebesgue, Morrey-Campanato, Lipschitz, B^p, CMO, local Morrey-type spaces, etc., and investigate the interpolation property of B_w^u-function spaces. We also apply it to the boundedness of linear and sublinear operators, for example, the Hardy-Littlewood maximal and fractional maximal operators, singular and fractional integral operators with rough kernel, the Littlewood-Paley operator, Marcinkiewicz operator, and so on.

1. Introduction

The purpose of this paper is to introduce B_w^u-function spaces which unify many function spaces, Lebesgue, Morrey-Campanato, Lipschitz, B^p, CMO, local Morrey-type spaces, etc. We investigate the interpolation property of B_w^u-function spaces and apply it to the boundedness of linear and sublinear operators, for example, the Hardy-Littlewood maximal operator, singular and fractional integral operators, and so on, which contains previous results and extends them to B_w^u-function spaces.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote by Q_r the open cube centered at the origin and sidelength $2r$, or the open ball centered at the origin and of radius r, that is,

$$Q_r = \left\{ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \max_{1 \le i \le n} |y_i| < r \right\} \quad \text{or}$$

$$Q_r = \{ y \in \mathbb{R}^n : |y| < r \} \,.$$

For each $r \in (0, \infty)$, let $E(Q_r)$ be a function space on Q_r with quasi-norm $\| \cdot \|_{E(Q_r)}$. Let $E_Q(\mathbb{R}^n)$ be the set of all measurable functions f on $\mathbb{R}^n$ such that $f|_{Q_r} \in E(Q_r)$ for all $r > 0$. We assume the following *restriction property*:

$$f|_{Q_r} \in E(Q_r) \quad \text{and} \quad 0 < t < r < \infty$$
$$\Rightarrow f|_{Q_t} \in E(Q_t) \quad \text{and} \quad \|f\|_{E(Q_t)} \le C_E \|f\|_{E(Q_r)} \,, \tag{1.1}$$

where C_E is a positive constant independent of r, t and f. For example, $E = L^p$, Lip_α, BMO, etc. Then, for a weight function $w : (0, \infty) \to (0, \infty)$ and an exponent $u \in (0, \infty]$, we define function spaces $B_w^u(E) = B_w^u(E)(\mathbb{R}^n)$ and $\dot{B}_w^u(E) = \dot{B}_w^u(E)(\mathbb{R}^n)$ as the sets of all functions $f \in E_Q(\mathbb{R}^n)$ such that $\|f\|_{B_w^u(E)} < \infty$ and $\|f\|_{\dot{B}_w^u(E)} < \infty$, respectively, where

$$\|f\|_{B_w^u(E)} = \big\| w(r)\|f\|_{E(Q_r)} \big\|_{L^u([1,\infty), dr/r)},$$

$$\|f\|_{\dot{B}_w^u(E)} = \big\| w(r)\|f\|_{E(Q_r)} \big\|_{L^u((0,\infty), dr/r)}.$$

In the above we abbreviated $\|f|_{Q_r}\|_{E(Q_r)}$ to $\|f\|_{E(Q_r)}$.

In this paper we always assume that w has some decreasingness condition. Note that, if $w(r) \to \infty$ as $r \to \infty$, then $B_w^u(E) = \dot{B}_w^u(E) = \{0\}$. In particular, if $w(r) = r^{-\sigma}$, $\sigma \geq 0$ and $u = \infty$, we denote $B_w^u(E)(\mathbb{R}^n)$ and $\dot{B}_w^u(E)(\mathbb{R}^n)$ by $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$, respectively, which were introduced recently by Komori-Furuya, Matsuoka, Nakai and Sawano [25]. These B_σ-function spaces unify several function spaces, see the following Examples 1–4. Moreover, if $E = L^p$, then $\dot{B}_w^u(L^p)(\mathbb{R}^n)$ is the local Morrey-type space introduced by Burenkov and Guliyev [7], see Example 5.

EXAMPLE 1.   Beurling [3] introduced the space $B^p(\mathbb{R}^n)$ together with its predual $A^p(\mathbb{R}^n)$ so-called the Beurling algebra. Later, to extend Wiener's ideas [46, 47] which describe the behavior of functions at infinity, Feichtinger [16] gave an equivalent norm on $B^p(\mathbb{R}^n)$, which is a special case of norms to describe non-homogeneous Herz spaces $K_{p,r}^\alpha(\mathbb{R}^n)$ introduced in [22]. The function space $B^p(\mathbb{R}^n)$ and its homogeneous version $\dot{B}^p(\mathbb{R}^n)$ are characterized by the following norms, respectively:

$$\|f\|_{B^p} = \sup_{r \geq 1} \left(\frac{1}{|Q_r|} \int_{Q_r} |f(x)|^p \, dx \right)^{1/p} \quad \text{and}$$

$$\|f\|_{\dot{B}^p} = \sup_{r > 0} \left(\frac{1}{|Q_r|} \int_{Q_r} |f(x)|^p \, dx \right)^{1/p},$$

where $|Q_r|$ is the Lebesgue measure of Q_r. In this case $B^p(\mathbb{R}^n) = B_\sigma(L^p)(\mathbb{R}^n)$ and $\dot{B}^p(\mathbb{R}^n) = \dot{B}_\sigma(L^p)(\mathbb{R}^n)$ with $\sigma = n/p$.

EXAMPLE 2.   Chen and Lau [13] and García-Cuerva [18] introduced the central mean oscillation space $\text{CMO}^p(\mathbb{R}^n)$ with the norm

$$\|f\|_{\text{CMO}^p} = \sup_{r \geq 1} \left(\frac{1}{|Q_r|} \int_{Q_r} |f(x) - f_{Q_r}|^p \, dx \right)^{1/p},$$

and Lu and Yang [28, 29] introduced the central bounded mean oscillation space $\text{CBMO}^p(\mathbb{R}^n)$ with the norm

$$\|f\|_{\text{CBMO}^p} = \sup_{r > 0} \left(\frac{1}{|Q_r|} \int_{Q_r} |f(x) - f_{Q_r}|^p \, dx \right)^{1/p},$$

where f_{Q_r} is the mean value of f on Q_r. Then $\mathrm{CMO}^p(\mathbb{R}^n)$ and $\mathrm{CBMO}^p(\mathbb{R}^n)$ are expressed by $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$, respectively, with $E = L^p$ (modulo constants), $\|f\|_{E(Q_r)} = \|f - f_{Q_r}\|_{L^p(Q_r)}$ and $\sigma = n/p$.

EXAMPLE 3. García-Cuerva and Herrero [19] and Alvarez, Guzmán-Partida and Lakey [2] introduced the non-homogeneous central Morrey space $B^{p,\lambda}(\mathbb{R}^n)$, the central Morrey space $\dot{B}^{p,\lambda}(\mathbb{R}^n)$, the λ-central mean oscillation space $\mathrm{CMO}^{p,\lambda}(\mathbb{R}^n)$ and the λ-central bounded mean oscillation space $\mathrm{CBMO}^{p,\lambda}(\mathbb{R}^n)$ as an extension of $B^p(\mathbb{R}^n)$, $\dot{B}^p(\mathbb{R}^n)$, $\mathrm{CMO}^p(\mathbb{R}^n)$ and $\mathrm{CBMO}^p(\mathbb{R}^n)$, respectively, with the following norms:

$$\|f\|_{B^{p,\lambda}} = \sup_{r \geq 1} \frac{1}{r^\lambda} \left(\frac{1}{|Q_r|} \int_{Q_r} |f(x)|^p \, dx \right)^{1/p},$$

$$\|f\|_{\dot{B}^{p,\lambda}} = \sup_{r > 0} \frac{1}{r^\lambda} \left(\frac{1}{|Q_r|} \int_{Q_r} |f(x)|^p \, dx \right)^{1/p},$$

$$\|f\|_{\mathrm{CMO}^{p,\lambda}} = \sup_{r \geq 1} \frac{1}{r^\lambda} \left(\frac{1}{|Q_r|} \int_{Q_r} |f(x) - f_{Q_r}|^p \, dx \right)^{1/p} \quad \text{and}$$

$$\|f\|_{\mathrm{CBMO}^{p,\lambda}} = \sup_{r > 0} \frac{1}{r^\lambda} \left(\frac{1}{|Q_r|} \int_{Q_r} |f(x) - f_{Q_r}|^p \, dx \right)^{1/p}.$$

Then these spaces are expressed by $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$ with $E = L^p$ (or $E = L^p$ (modulo constants)) and $\sigma = n/p + \lambda$.

EXAMPLE 4. If $E = L_{p,\lambda}$ (Morrey space) or $\mathcal{L}_{p,\lambda}$ (Campanato space), then the function spaces $B_\sigma(L_{p,\lambda})(\mathbb{R}^n)$, $\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)$, $B_\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ and $\dot{B}_\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ unify the function spaces in above examples and the usual Morrey-Campanato and Lipschitz spaces. Actually, if $\lambda = -n/p$, then $L_{p,\lambda} = L^p$. If $\sigma = 0$, then $B_0(L_{p,\lambda})(\mathbb{R}^n) = \dot{B}_0(L_{p,\lambda})(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n)$ and $B_0(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \dot{B}_0(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \mathcal{L}_{p,\lambda}(\mathbb{R}^n)$. If $\lambda = 0$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ for all $p \in [1, \infty)$ (John and Nirenberg [23]). If $\lambda = \alpha \in (0, 1]$, then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$ for all $p \in [1, \infty)$ (Campanato [12], Meyers [31], Spanne [45]). B_σ-Morrey-Campanato spaces were investigated in [24, 25, 26, 30]. For the definitions of $L_{p,\lambda}$ and $\mathcal{L}_{p,\lambda}$, see Subsection 3.2.

EXAMPLE 5. Burenkov and Guliyev [7] introduced local Morrey-type space $LM_{p\theta,w}(\mathbb{R}^n)$ with the (quasi-)norm

$$\|f\|_{LM_{p\theta,w}} = \big\| w(r) \|f\|_{L^p(Q_r)} \big\|_{L^\theta(0,\infty)},$$

and investigated the boundedness of the Hardy-Littlewood maximal operator. $LM_{p\theta,\tilde{w}}(\mathbb{R}^n)$ is expressed by $\dot{B}^u_w(E)(\mathbb{R}^n)$ with $E = L^p$ and $\tilde{w}(r) = w(r)/r$. For recent progress of local Morrey-type spaces, see [4, 5]. See also [6, 10] for interpolation spaces for local Morrey-type spaces.

In this paper we investigate the interpolation property of B_w^u-function spaces

$$\left(\dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n), \ \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n)\right)_{\theta,u} = \dot{B}_w^u(E)(\mathbb{R}^n) \, .$$

Moreover, we give the interpolation property with $w = w_0 \, \Theta(w_1/w_0)$ for some pseudocon-cave function Θ (Theorem 1). To do this we assume that, for any $f \in E_Q(\mathbb{R}^n)$ and for any $r > 0$, there exists a decomposition $f = f_0^r + f_1^r$ such that

$$\|f_0^r\|_{E(Q_t)} \leq \begin{cases} C_E\|f\|_{E(Q_t)} & (0 < t < r) \, , \\ C_E\|f\|_{E(Q_{ar})} & (r \leq t < \infty) \, , \end{cases} \tag{1.2}$$

and

$$\|f_1^r\|_{E(Q_t)} \leq \begin{cases} 0 & (0 < t < cr) \, , \\ C_E\|f\|_{E(Q_{bt})} & (cr \leq t < \infty) \, , \end{cases} \tag{1.3}$$

where C_E, a, b, c are positive constants independent of r, t and f. We call such property the *decomposition property*. For example, Lebesgue, Orlicz, Lorentz and Morrey spaces have the decomposition property. Actually, $f = f\chi_r + f(1 - \chi_r)$ is the desired decomposition, where χ_r is the characteristic function of Q_r. Moreover, we prove that Campanato and Lipschitz spaces also have the decomposition property (Proposition 1).

As applications of the interpolation property, we also give the boundedness of linear and sublinear operators. It is known that the Hardy-Littlewood maximal operator, fractional maximal operators, singular and fractional integral operators are bounded on B_σ-Morrey-Campanato spaces, see [24, 25, 26, 30]. Using these boundedness, we get the boundedness of these operators on $B_w^u(L_{p,\lambda}), \dot{B}_w^u(L_{p,\lambda}), B_w^u(\mathcal{L}_{p,\lambda})$ and $\dot{B}_w^u(\mathcal{L}_{p,\lambda})$, which are also generaliza-tion of the results on the local Morrey-type spaces $LM_{pu,w}(\mathbb{R}^n)$.

We give notation and definitions in Section 2 to state main results in Section 3. We prove them in Section 4 and give applications for the boundedness of linear and sublinear operators in Section 5.

2. Notation and definitions

In this section we give several notation and definitions to state main result.

A function $w : (0, \infty) \to (0, \infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$w(r) \leq Cw(s) \quad (w(r) \geq Cw(s)) \quad \text{for} \quad r \leq s \, . \tag{2.1}$$

A function $w : (0, \infty) \to (0, \infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{w(r)}{w(s)} \leq C \quad \text{for} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2 \, . \tag{2.2}$$

For functions $w_1, w_2 : (0, \infty) \to (0, \infty)$, we write $w_1 \sim w_2$ if there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{w_1(r)}{w_2(r)} \le C \quad \text{for} \quad r > 0 . \tag{2.3}$$

Note that, if $w_1 \sim w_2$, then $B_{w_1}^u(E) = B_{w_2}^u(E)$ and $\dot{B}_{w_1}^u(E) = \dot{B}_{w_2}^u(E)$ with equivalent norms. Note also that, if w satisfies the doubling condition, then, for any $\eta > 0$, $\|w(r)\|f\|_{E(Q_r)}\|_{L^u([\eta,\infty),dr/r)}$ and $\|w(r)\|f\|_{E(Q_r)}\|_{L^u([1,\infty),dr/r)}$ are equivalent each other, by the restriction property of $\{E(Q_r)\}$.

We denote by $\mathcal{W}^u$, $u \in (0, \infty]$, the set of all almost decreasing functions $w : (0, \infty) \to (0, \infty)$ such that w satisfies the doubling condition and $w \in L^u([1, \infty), dr/r)$. Note that, if $w \notin L^u([1, \infty), dr/r)$, then $B_w^u(E) = \dot{B}_w^u(E) = \{0\}$. We also denote by $\mathcal{W}^*$ the set of all almost decreasing functions $w : (0, \infty) \to (0, \infty)$ such that w satisfies the doubling condition and

$$\int_r^\infty w(t) \frac{dt}{t} \le C w(r) , \quad r \in (0, \infty) , \tag{2.4}$$

where C is a positive constant independent of r. If w satisfies the doubling condition, then

$$w(r) \le C \int_r^\infty w(t) \frac{dt}{t} , \quad r \in (0, \infty) ,$$

for some positive constant C independent of r, that is, the condition (2.4) implies that $w(r) \sim \int_r^\infty w(t) \, dt/t$. Then the condition (2.4) is equivalent that there exists a positive constant ε such that $w(r)r^\varepsilon$ is almost decreasing, see [38, Lemma 7.1]. Therefore, we have the relation

$$\mathcal{W}^* \subset \mathcal{W}^{u_1} \subset \mathcal{W}^{u_2} \subset \mathcal{W}^\infty , \quad 0 < u_1 < u_2 < \infty .$$

Moreover, if w satisfies the doubling condition, then there exists a positive constant ν such that $w(r)r^\nu$ is almost increasing. Actually, take ν such that $C \le 2^\nu$, here C is the doubling constant in (2.2). Then, for $r \le s$, choosing an integer k such that $2^{k-1}r \le s < 2^k r$, we have

$$w(r)r^\nu \le C^k w(s)r^\nu \le 2^{\nu k} w(s)(s/2^{k-1})^\nu = 2^\nu w(s)s^\nu .$$

We say that a function $\Theta : (0, \infty) \to (0, \infty)$ is pseudoconcave if there exists a concave function $\tilde{\Theta} : (0, \infty) \to (0, \infty)$ such that $\Theta \sim \tilde{\Theta}$. All pseudoconcave functions satisfy the doubling condition. Let $\boldsymbol{\Theta}_*$ be the set of all functions $\Theta : (0, \infty) \to (0, \infty)$ such that, for some constants $C \in (0, \infty)$ and $\varepsilon, \varepsilon' \in (0, 1)$,

$$\frac{\Theta(tr)}{\Theta(r)} \le C \max(t^\varepsilon, t^{\varepsilon'}) \quad \text{for all} \quad r, t \in (0, \infty) .$$

Then all functions $\Theta \in \boldsymbol{\Theta}_*$ are pseudoconcave, see [41]. Note that $\Theta \in \boldsymbol{\Theta}_*$ if and only if there exist constants $\varepsilon, \varepsilon' \in (0, 1)$ such that $\Theta(r)r^{-\varepsilon}$ is almost increasing and that $\Theta(r)r^{-\varepsilon'}$ is almost decreasing. In this case $\varepsilon \le \varepsilon'$.

We consider a couple $(A_0, A_1) = (\dot{B}^{u_0}_{w_0}(E), \dot{B}^{u_1}_{w_1}(E))$ or $(B^{u_0}_{w_0}(E), B^{u_1}_{w_1}(E))$. For $f \in A_0 + A_1$, let

$$K(r, f; A_0, A_1) = \inf_{f=f_0+f_1} \left(\|f_0\|_{A_0} + r\|f_0\|_{A_1} \right) \quad (0 < r < \infty),$$

where the infimum is taken over all decompositions $f = f_0 + f_1$ in $A_0 + A_1$. For a pseudo-concave function Θ and $u \in (0, \infty]$, let

$$(A_0, A_1, \Theta)_u = \left\{ f \in E_Q(\mathbb{R}^n) : \left\| \Theta(r^{-1}) K(r, f; A_0, A_1) \right\|_{L^u((0,\infty), dr/r)} < \infty \right\}.$$

We also consider the following:

$$(A_0, A_1, \Theta)_{u, [1,\infty)} = \left\{ f \in E_Q(\mathbb{R}^n) : \left\| \Theta(r^{-1}) K(r, f; A_0, A_1) \right\|_{L^u([1,\infty), dr/r)} < \infty \right\}.$$

In particular, for $\Theta(r) = r^\theta$, $\theta \in (0, 1)$, we denote $(A_0, A_1, \Theta)_u$ and $(A_0, A_1, \Theta)_{u, [1,\infty)}$ by $(A_0, A_1)_{\theta, u}$ and $(A_0, A_1)_{\theta, u, [1,\infty)}$, respectively.

3. Main results

In this section we investigate the interpolation properties of $\dot{B}^u_w(E) = \dot{B}^u_w(E)(\mathbb{R}^n)$ and $B^u_w(E) = B^u_w(E)(\mathbb{R}^n)$, using the restriction and decomposition properties (1.1), (1.2) and (1.3) of $\left\{ (E(Q_r), \| \cdot \|_{E(Q_r)}) \right\}_{0<r<\infty}$.

3.1. Interpolation.   The main theorem is the following:

THEOREM 1. *Assume that a family* $\left\{ (E(Q_r), \| \cdot \|_{E(Q_r)}) \right\}_{0<r<\infty}$ *has the restriction and decomposition properties. Let* $u_0, u_1, u \in (0, \infty]$, $w_0, w_1 \in \mathcal{W}^\infty$, $\Theta \in \boldsymbol{\Theta}_*$ *and*

$$w = w_0 \, \Theta(w_1/w_0).$$

For each $i = 0, 1$, *if* $\min(u_i, u) < \infty$, *then we assume that* $w_i \in \mathcal{W}^*$. *Assume also that, for some positive constant* ε, $(w_0(r)/w_1(r))r^{-\varepsilon}$ *is almost increasing, or,* $(w_1(r)/w_0(r))r^{-\varepsilon}$ *is almost increasing. Then*

$$\left(\dot{B}^{u_0}_{w_0}(E)(\mathbb{R}^n), \; \dot{B}^{u_1}_{w_1}(E)(\mathbb{R}^n), \; \Theta \right)_u = \dot{B}^u_w(E)(\mathbb{R}^n),$$

and

$$\left(B^{u_0}_{w_0}(E)(\mathbb{R}^n), \; B^{u_1}_{w_1}(E)(\mathbb{R}^n), \; \Theta \right)_{u, [1,\infty)} = B^u_w(E)(\mathbb{R}^n).$$

REMARK 1. The function $w = w_0 \, \Theta(w_1/w_0)$ in Theorem 1 is in $\mathcal{W}^\infty$, since the function $R(r, s) = r \, \Theta(s/r)$ is almost increasing with respect to both r and s. For properties of pseudoconcave functions, see [21]. If $(w_0(r)/w_1(r))r^{-\varepsilon}$ is almost increasing, then $w_1(r)r^\varepsilon$ is almost decreasing, that is, $w_1 \in \mathcal{W}^*$. Similarly, if $(w_1(r)/w_0(r))r^{-\varepsilon}$ is almost increasing, then $w_0 \in \mathcal{W}^*$.

Take $u_0 = u_1 = \infty$, $w_0(r) = r^{-\sigma_0}$, $w_1(r) = r^{-\sigma_1}$ in Theorem 1. Then we have the following:

COROLLARY 1.   *Assume that a family $\left\{(E(Q_r), \|\cdot\|_{E(Q_r)})\right\}_{0<r<\infty}$ has the restriction and decomposition properties. Let $u \in (0, \infty]$, $\sigma_0, \sigma_1 \in [0, \infty)$ with $\sigma_0 \neq \sigma_1$, $\Theta \in \boldsymbol{\Theta}_*$ and*

$$w(r) = r^{-\sigma_0} \Theta(r^{\sigma_0-\sigma_1}). \tag{3.1}$$

If $u < \infty$, we assume that $\sigma_0, \sigma_1 \in (0, \infty)$. Then

$$\left(\dot{B}_{\sigma_0}(E)(\mathbb{R}^n), \ \dot{B}_{\sigma_1}(E)(\mathbb{R}^n), \ \Theta\right)_u = \dot{B}_w^u(E)(\mathbb{R}^n),$$

and

$$\left(B_{\sigma_0}(E)(\mathbb{R}^n), \ B_{\sigma_1}(E)(\mathbb{R}^n), \ \Theta\right)_{u, [1,\infty)} = B_w^u(E)(\mathbb{R}^n).$$

REMARK 2.   For any $w \in \mathcal{W}^*$, there exist $\sigma_0, \sigma_1 \in [0, \infty)$ and $\Theta \in \boldsymbol{\Theta}_*$ such that (3.1) holds. Actually, since $w(r)r^\nu$ is almost increasing and $w(r)r^\eta$ is almost decreasing for some positive constants ν and η with $\nu > \eta$, choosing $\sigma_0, \sigma_1 \in [0, \infty)$ and $\varepsilon, \varepsilon' \in (0, 1)$ such that

$$\sigma_0 > \sigma_1, \quad \varepsilon < \varepsilon', \quad \sigma_0 - (\sigma_0 - \sigma_1)\varepsilon = \nu, \quad \sigma_0 - (\sigma_0 - \sigma_1)\varepsilon' = \eta, \tag{3.2}$$

and setting Θ as

$$\Theta(r^{\sigma_0-\sigma_1}) = w(r)r^{\sigma_0},$$

we have

$$\Theta(r^{\sigma_0-\sigma_1})r^{(\sigma_0-\sigma_1)(-\varepsilon)} = w(r)r^\nu, \quad \Theta(r^{\sigma_0-\sigma_1})r^{(\sigma_0-\sigma_1)(-\varepsilon')} = w(r)r^\eta. \tag{3.3}$$

These show that $\Theta(r)r^{-\varepsilon}$ is almost increasing and $\Theta(r)r^{-\varepsilon'}$ is almost decreasing, that is $\Theta \in \boldsymbol{\Theta}_*$.

Conversely, for any $\Theta \in \boldsymbol{\Theta}_*$ and $\sigma_0, \sigma_1 \in [0, \infty)$ with $\sigma_0 > \sigma_1$, the function w defined by (3.1) is in $\mathcal{W}^*$ by the relations (3.2) and (3.3).

EXAMPLE 6.   Let $\sigma_0, \sigma_1 \in [0, \infty)$, $\sigma_0 > \sigma_1$, $w_0(r) = r^{\sigma_0}$, $w_1(r) = r^{\sigma_1}$, $\alpha, \beta \in (0, 1)$, and let

$$w = w_0 \Theta(w_1/w_0), \quad \Theta(r) = \max(r^\alpha, r^\beta).$$

Then

$$w(r) = \max(r^{-(\sigma_0+\alpha(\sigma_1-\sigma_0))}, r^{-(\sigma_0+\beta(\sigma_1-\sigma_0))}),$$

and $\Theta \in \boldsymbol{\Theta}_*$, since

$$\frac{\Theta(tr)}{\Theta(r)} \leq \max(t^\alpha, t^\beta) \quad \text{for all } r, t \in (0, \infty).$$

EXAMPLE 7. Let $\mathcal{L}$ be the set of all continuous functions $\ell : (0, \infty) \to (0, \infty)$ for which there exists a constant $c \geq 1$ such that

$$c^{-1} \leq \frac{\ell(s)}{\ell(r)} \leq c \quad \text{whenever} \quad \frac{1}{2} \leq \frac{\log s}{\log r} \leq 2 \,. \tag{3.4}$$

If $\ell \in \mathcal{L}$, then, for all $\alpha > 0$, there exists a constant $c_\alpha \geq 1$ such that

$$c_\alpha^{-1} \ell(r) \leq \ell(r^\alpha) \leq c_\alpha \ell(r) \quad \text{for} \quad 0 < r < \infty \,. \tag{3.5}$$

For other properties on functions $\ell \in \mathcal{L}$, see [33, Section 7]. For example, the following function ℓ_{β_1, β_2} is in $\mathcal{L}$:

$$\ell_{\beta_1, \beta_2}(r) = \begin{cases} \left(\log \frac{1}{r} \right)^{-\beta_1} & (0 < r < e^{-1}) \,, \\ 1 & (e^{-1} \leq r \leq e) \,, \quad \beta_1, \beta_2 \in (-\infty, \infty) \,. \\ (\log r)^{\beta_2} & (e < r) \,, \end{cases}$$

Let $\sigma_0, \sigma_1 \in [0, \infty)$, $\sigma_0 > \sigma_1$, $w_0(r) = r^{-\sigma_0}$, $w_1(r) = r^{-\sigma_1}$, $\theta \in (0, 1)$, and let

$$w = w_0 \, \Theta(w_1/w_0) \,, \quad \Theta(r) = r^\theta \ell(r) \,, \quad \ell \in \mathcal{L} \,.$$

Then $\Theta \in \boldsymbol{\Theta}_*$ and

$$w(r) \sim r^{-\sigma} \ell(r) \,, \quad \sigma = (1 - \theta)\sigma_0 + \theta \sigma_1 \,.$$

We can take ℓ_{β_1, β_2} as ℓ.

Take $u = \infty$ and $\Theta(r) = r^\theta$ in Corollary 1. Then we have the following:

COROLLARY 2. *Assume that a family $\left\{ (E(Q_r), \| \cdot \|_{E(Q_r)}) \right\}_{0 < r < \infty}$ has the restriction and decomposition properties. Let $\sigma_0, \sigma_1 \in [0, \infty)$ with $\sigma_0 \neq \sigma_1$, $\theta \in (0, 1)$ and*

$$\sigma = (1 - \theta)\sigma_0 + \theta \sigma_1 \,.$$

Then

$$\left(\dot{B}_{\sigma_0}(E)(\mathbb{R}^n), \ \dot{B}_{\sigma_1}(E)(\mathbb{R}^n) \right)_{\theta, \infty} = \dot{B}_\sigma(E)(\mathbb{R}^n) \,,$$

and

$$\left(B_{\sigma_0}(E)(\mathbb{R}^n), \ B_{\sigma_1}(E)(\mathbb{R}^n) \right)_{\theta, \infty, [1, \infty)} = B_\sigma(E)(\mathbb{R}^n) \,.$$

Let $E = L^p$. Then, using Corollaries 1 and 2 we have the following:

EXAMPLE 8. Take $\sigma_0 = \sigma \in (0, \infty)$, $\sigma_1 = 0$ and $\tau = (1 - \theta)\sigma$ with $\theta \in (0, 1)$ in Corollary 2. Then, since $B_0(L^p)(\mathbb{R}^n) = L^p(\mathbb{R}^n)$,

$$\left(\dot{B}_\sigma(L^p)(\mathbb{R}^n), \ L^p(\mathbb{R}^n) \right)_{\theta, \infty} = \dot{B}_\tau(L^p)(\mathbb{R}^n) \,,$$

and

$$\left(B_{\sigma_1}(L^p)(\mathbb{R}^n),\ L^p(\mathbb{R}^n)\right)_{\theta,\infty,\,[1,\infty)} = B_\tau(L^p)(\mathbb{R}^n)\,.$$

EXAMPLE 9. Take $u = \infty$, $\sigma_0 = \sigma \in (0,\infty)$, $\sigma_1 = 0$, $w(r) = r^{-\sigma}\Theta(r^\sigma)$ with $w \in \mathcal{W}^*$ and $\Theta \in \boldsymbol{\Theta}_*$, in Corollary 1. Then

$$\left(\dot{B}_\sigma(L^p)(\mathbb{R}^n),\ L^p(\mathbb{R}^n),\ \Theta\right)_\infty = \dot{B}_w^\infty(L^p)(\mathbb{R}^n)\,,$$

and

$$\left(B_{\sigma_1}(L^p)(\mathbb{R}^n),\ L^p(\mathbb{R}^n),\ \Theta\right)_{\infty,\,[1,\infty)} = B_w^\infty(L^p)(\mathbb{R}^n)\,.$$

EXAMPLE 10. Take $u \in (0,\infty)$, $\sigma_0, \sigma_1 \in (0,\infty)$, $w(r) = r^{-\sigma_0}\Theta(r^{\sigma_0-\sigma_1})$ with $w \in \mathcal{W}^*$ and $\Theta \in \boldsymbol{\Theta}_*$, in Corollary 1. Then

$$\left(\dot{B}_{\sigma_0}(L^p)(\mathbb{R}^n),\ \dot{B}_{\sigma_1}(L^p)(\mathbb{R}^n),\ \Theta\right)_u = \dot{B}_w^u(L^p)(\mathbb{R}^n)\,,$$

and

$$\left(B_{\sigma_0}(L^p)(\mathbb{R}^n),\ B_{\sigma_1}(L^p)(\mathbb{R}^n),\ \Theta\right)_{u,\,[1,\infty)} = B_w^u(L^p)(\mathbb{R}^n)\,.$$

In this case $\dot{B}_w^u(L^p)(\mathbb{R}^n)$ is the local Morrey-type space $LM_{pu,\tilde{w}}(\mathbb{R}^n)$ with $\tilde{w}(r) = w(r)/r$.

3.2. Morrey, Campanato and Lipschitz spaces. In this subsection, we consider Morrey, Campanato and Lipschitz spaces as concrete examples of the function space E which does not satisfy the lattice condition (3.11). Let

$$Q(x,r) = x + Q_r = \{x + y : y \in Q_r\}\,.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively. We also abbreviate χ_{Q_r} to χ_r.

For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a measurable set $G \subset \mathbb{R}^n$ with $|G| > 0$, let

$$f_G = \frac{1}{|G|}\int_G f(y)\,dy\,. \tag{3.6}$$

For a measurable function f on $\mathbb{R}^n$, a measurable set $G \subset \mathbb{R}^n$ with $|G| > 0$ and $t \in [0,\infty)$, let

$$m(G,f,t) = |\{y \in G : |f(y)| > t\}|\,. \tag{3.7}$$

We recall the definitions of Morrey, weak Morrey, Campanato and Lipschitz spaces below. These function spaces have the restriction properties. The first two have also the support property (3.10) and the lattice property (3.11), and then the decomposition property. The last two also have the decomposition property by Theorem 2 and Proposition 1. Therefore, we can take these function spaces as E in Theorem 1 and Corollaries 1 and 2.

DEFINITION 1. Let $U = \mathbb{R}^n$ or $U = Q_r$ with $r > 0$. For $p \in [1, \infty)$, $\lambda \in \mathbb{R}$ and $\alpha \in (0, 1]$, let $L_{p,\lambda}(U)$, $WL_{p,\lambda}(U)$, $\mathcal{L}_{p,\lambda}(U)$ and $\mathrm{Lip}_\alpha(U)$ be the sets of all functions f such that the following functionals are finite, respectively:

$$\|f\|_{L_{p,\lambda}(U)} = \sup_{Q(x,s) \subset U} \frac{1}{s^\lambda} \left(\frac{1}{|Q(x,s)|} \int_{Q(x,s)} |f(y)|^p \, dy \right)^{1/p},$$

$$\|f\|_{WL_{p,\lambda}(U)} = \sup_{Q(x,s) \subset U} \frac{1}{s^\lambda} \left(\frac{\sup_{t>0} t^p \, m(Q(x,s), f, t)}{|Q(x,s)|} \right)^{1/p},$$

$$\|f\|_{\mathcal{L}_{p,\lambda}(U)} = \sup_{Q(x,s) \subset U} \frac{1}{s^\lambda} \left(\frac{1}{|Q(x,s)|} \int_{Q(x,s)} |f(y) - f_{Q(x,s)}|^p \, dy \right)^{1/p},$$

and

$$\|f\|_{\mathrm{Lip}_\alpha(U)} = \sup_{x,y \in U, \, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Then $L_{p,\lambda}(U)$ is a Banach space and $WL_{p,\lambda}(U)$ is a complete quasi-normed space. In this paper we regard $\mathcal{L}_{p,\lambda}(U)$ and $\mathrm{Lip}_\alpha(U)$ as spaces of functions modulo constant functions. Then $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ are Banach spaces equipped with the norms $\|f\|_{\mathcal{L}_{p,\lambda}}$ and $\|f\|_{\mathrm{Lip}_\alpha}$, respectively.

By the definition, if $\lambda = -n/p$, then $L_{p,-n/p}(U) = L^p(U)$ and $WL_{p,-n/p}(U) = WL^p(U)$, the weak L^p space. If $p = 1$ and $\lambda = 0$, then $\mathcal{L}_{1,0}(U)$ is the usual BMO(U).

REMARK 3. We note that $B_\sigma(L_{p,\lambda})(\mathbb{R}^n)$ unifies $L_{p,\lambda}(\mathbb{R}^n)$ and $B^{p,\lambda}(\mathbb{R}^n)$ and that $B_\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ unifies $\mathcal{L}_{p,\lambda}(\mathbb{R}^n)$ and CMO$^{p,\lambda}(\mathbb{R}^n)$. Actually, we have the following relations:

$$B_0(L_{p,\lambda})(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n), \quad B_0(\mathcal{L}_{p,\lambda})(\mathbb{R}^n) = \mathcal{L}_{p,\lambda}(\mathbb{R}^n), \tag{3.8}$$

$$B_{\lambda+n/p}(L_{p,-n/p})(\mathbb{R}^n) = B^{p,\lambda}(\mathbb{R}^n), \quad B_{\lambda+n/p}(\mathcal{L}_{p,-n/p})(\mathbb{R}^n) = \text{CMO}^{p,\lambda}(\mathbb{R}^n). \tag{3.9}$$

In the above relations, the first three follow immediately from their definitions, and the last one follows from Theorem 3 below. We also have the same properties for the function spaces $\dot{B}_\sigma(L_{p,\lambda})(\mathbb{R}^n)$ and $\dot{B}_\sigma(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$.

Here we state two known theorems which give the relations among Morrey, Campanato and Lipschitz spaces. For the proofs of Theorems 2 and 3 below, see [12, 31, 45] and [32, 37], respectively. For other relations among function spaces in Remark 3, see [25, Proposition 1].

THEOREM 2. *If $p \in [1, \infty)$ and $\lambda = \alpha \in (0, 1]$, then, for each $r > 0$, $\mathcal{L}_{p,\lambda}(Q_r) = \mathrm{Lip}_\alpha(Q_r)$ modulo null-functions and there exists a positive constant C, dependent only on n and λ, such that*

$$C^{-1}\|f\|_{\mathcal{L}_{p,\lambda}(Q_r)} \leq \|f\|_{\mathrm{Lip}_\alpha(Q_r)} \leq C\|f\|_{\mathcal{L}_{p,\lambda}(Q_r)}.$$

The same conclusion holds on $\mathbb{R}^n$.

THEOREM 3. *If $p \in [1, \infty)$ and $\lambda \in [-n/p, 0)$, then, for each $r > 0$, $\mathcal{L}_{p,\lambda}(Q_r) \cong L_{p,\lambda}(Q_r)$. More precisely, the map $f \mapsto f - f_{Q_r}$ is bijective and bicontinuous from $\mathcal{L}_{p,\lambda}(Q_r)$ to $L_{p,\lambda}(Q_r)$, that is, there exists a positive constant C, dependent only on n and λ, such that*

$$C^{-1}\|f\|_{\mathcal{L}_{p,\lambda}(Q_r)} \le \|f - f_{Q_r}\|_{L_{p,\lambda}(Q_r)} \le C\|f\|_{\mathcal{L}_{p,\lambda}(Q_r)}.$$

The same conclusion holds on $\mathbb{R}^n$ by using $\lim_{r \to \infty} f_{Q_r}$ instead of f_{Q_r}.

Now we consider the decomposition property. Recall that $E_Q(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$ such that $f|_{Q_r} \in E(Q_r)$ for all $r > 0$. If the family $\{E(Q_r)\}$ has the restriction property and the following two conditions, then it has the decomposition property.

$$f \in E(Q_t), \ 0 < r < t < \infty \ \text{ and } \ \text{supp} f \subset Q_r \ \Rightarrow \ \|f\|_{E(Q_t)} \le C_E\|f\|_{E(Q_r)}, \quad (3.10)$$

$$g \in E(Q_r) \text{ and } |f(x)| \le |g(x)| \text{ for a.e. } x \in Q_r$$
$$\Rightarrow \ f \in E(Q_r) \ \text{ and } \ \|f\|_{E(Q_r)} \le C_E\|g\|_{E(Q_r)}. \quad (3.11)$$

Actually, for $f \in E_Q(\mathbb{R}^n)$, letting

$$f_0^r = f\chi_r, \quad f_1^r = f - f_0^r,$$

we have the desired decomposition with $a = b = c = 1$, where χ_r is the characteristic function of Q_r. Lebesgue, Orlicz and Lorentz spaces satisfy these conditions. Moreover, Morrey and weak Morrey spaces also satisfy them.

Next we prove the decomposition property of Campanato spaces. For $r > 0$, let

$$h_r(x) = h(x/r), \quad h(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2, \end{cases} \quad \|h\|_{\text{Lip}_1(\mathbb{R}^n)} \le 1. \quad (3.12)$$

PROPOSITION 1. *Let $p \in [1, \infty)$ and $\lambda \in [-n/p, 1]$. Then the family $\{\mathcal{L}_{p,\lambda}(Q_r)\}$ has the decomposition property. More precisely, for any $f \in (\mathcal{L}_{p,\lambda})_Q(\mathbb{R}^n)$ and for any $r > 0$, let*

$$f_0^r = (f - f_{Q_{2r}})h_r, \quad f_1^r = f - (f - f_{Q_{2r}})h_r.$$

Then $f = f_0^r + f_1^r$,

$$\|f_0^r\|_{\mathcal{L}_{p,\lambda}(Q_t)} \le \begin{cases} C\|f\|_{\mathcal{L}_{p,\lambda}(Q_t)} & (0 < t < r) \\ C\|f\|_{\mathcal{L}_{p,\lambda}(Q_{3r})} & (r \le t < \infty), \end{cases}$$

and

$$\|f_1^r\|_{\mathcal{L}_{p,\lambda}(Q_t)} \le \begin{cases} 0 & (0 < t < r) \\ C\|f\|_{\mathcal{L}_{p,\lambda}(Q_{3t})} & (r \le t < \infty), \end{cases}$$

where C is a positive constant independent of r, t and f.

PROOF. If $0 < t < r$, then $f_0^r = f - f_{Q_{2r}}$, $f_1^r = f_{Q_{2r}}$ and

$$\|f_0^r\|_{\mathcal{L}_{p,\lambda(Q_t)}} = \|f\|_{\mathcal{L}_{p,\lambda(Q_t)}}, \quad \|f_1^r\|_{\mathcal{L}_{p,\lambda(Q_t)}} = 0.$$

If $r \leq t < \infty$, then, by the same argument as [30, Lemma 3.5] we have

$$\|f_0^r\|_{\mathcal{L}_{p,\lambda(Q_t)}} \leq C\|f\|_{\mathcal{L}_{p,\lambda(Q_{3r})}},$$

and

$$\begin{aligned}
\|f_1^r\|_{\mathcal{L}_{p,\lambda(Q_t)}} &\leq \|f\|_{\mathcal{L}_{p,\lambda(Q_t)}} + \|f_0^r\|_{\mathcal{L}_{p,\lambda(Q_t)}} \\
&\leq \|f\|_{\mathcal{L}_{p,\lambda(Q_t)}} + C\|f\|_{\mathcal{L}_{p,\lambda(Q_{3r})}} \\
&\leq C\|f\|_{\mathcal{L}_{p,\lambda(Q_{3t})}}.
\end{aligned}$$

Then we have the conclusion. $\qquad\square$

By Theorem 2 we have the following:

COROLLARY 3. *Let $\alpha \in (0, 1]$. Then the family $\{\mathrm{Lip}_\alpha(Q_r)\}$ has the decomposition property.*

Therefore, it turned out that we can take $L_{p,\lambda}$, $WL_{p,\lambda}$, $\mathcal{L}_{p,\lambda}$, BMO and Lip_α instead of L^p in Examples 8, 9 and 10. Actually, we have the following:

EXAMPLE 11. Take $\sigma_0 = \sigma \in (0, \infty)$, $\sigma_1 = 0$ and $\tau = (1 - \theta)\sigma$ with $\theta \in (0, 1)$ in Corollary 2. Then

$$\left(\dot{B}_\sigma(E)(\mathbb{R}^n),\ E(\mathbb{R}^n)\right)_{\theta,\infty} = \dot{B}_\tau(E)(\mathbb{R}^n),$$

and

$$\left(B_{\sigma_1}(E)(\mathbb{R}^n),\ E(\mathbb{R}^n)\right)_{\theta,\infty,\,[1,\infty)} = B_\tau(E)(\mathbb{R}^n),$$

where $E = L_{p,\lambda}$, $WL_{p,\lambda}$, $\mathcal{L}_{p,\mu}$, BMO, or Lip_α, with $p \in [1, \infty)$, $\lambda \in [-n/p, 0]$, $\mu \in [-n/p, 1]$ and $\alpha \in (0, 1]$.

EXAMPLE 12. Take $u = \infty$, $\sigma_0 = \sigma \in (0, \infty)$, $\sigma_1 = 0$, $w(r) = r^{-\sigma}\Theta(r^\sigma)$ with $w \in \mathcal{W}^*$ and $\Theta \in \boldsymbol{\Theta}_*$, in Corollary 1. Then

$$\left(\dot{B}_\sigma(E)(\mathbb{R}^n),\ E(\mathbb{R}^n),\ \Theta\right)_\infty = \dot{B}_w^\infty(E)(\mathbb{R}^n),$$

and

$$\left(B_{\sigma_1}(E)(\mathbb{R}^n),\ E(\mathbb{R}^n),\ \Theta\right)_{\infty,\,[1,\infty)} = B_w^\infty(E)(\mathbb{R}^n),$$

where $E = L_{p,\lambda}$, $WL_{p,\lambda}$, $\mathcal{L}_{p,\mu}$, BMO, or Lip_α, with $p \in [1, \infty)$, $\lambda \in [-n/p, 0]$, $\mu \in [-n/p, 1]$ and $\alpha \in (0, 1]$.

EXAMPLE 13. Take $u \in (0, \infty)$, $\sigma_0, \sigma_1 \in (0, \infty)$, $w(r) = r^{-\sigma_0} \Theta(r^{\sigma_0 - \sigma_1})$ with $w \in \mathcal{W}^*$ and $\Theta \in \boldsymbol{\Theta}_*$, in Corollary 1. Then

$$\left(\dot{B}_{\sigma_0}(E)(\mathbb{R}^n), \ \dot{B}_{\sigma_1}(E)(\mathbb{R}^n), \ \Theta \right)_u = \dot{B}_w^u(E)(\mathbb{R}^n),$$

and

$$\left(B_{\sigma_0}(E)(\mathbb{R}^n), \ B_{\sigma_1}(E)(\mathbb{R}^n), \ \Theta \right)_{u, \, [1, \infty)} = B_w^u(E)(\mathbb{R}^n),$$

where $E = L_{p, \lambda}$, $WL_{p, \lambda}$, $\mathcal{L}_{p, \mu}$, BMO, or Lip_α, with $p \in [1, \infty)$, $\lambda \in [-n/p, 0]$, $\mu \in [-n/p, 1]$ and $\alpha \in (0, 1]$.

EXAMPLE 14. Let $p \in [1, \infty)$, $\lambda_0, \lambda_1 \in [-n/p, \infty)$, $\theta \in (0, 1)$ and $\lambda = (1 - \theta)\lambda_0 + \theta\lambda_1$. Then

$$\left(\dot{B}^{p, \lambda_0}(\mathbb{R}^n), \ \dot{B}^{p, \lambda_1}(\mathbb{R}^n) \right)_{\theta, \infty} = \dot{B}^{p, \lambda}(\mathbb{R}^n),$$

$$\left(\text{CBMO}^{p, \lambda_0}(\mathbb{R}^n), \ \text{CBMO}^{p, \lambda_1}(\mathbb{R}^n) \right)_{\theta, \infty} = \text{CBMO}^{p, \lambda}(\mathbb{R}^n),$$

and

$$\left(B^{p, \lambda_0}(\mathbb{R}^n), \ B^{p, \lambda_1}(\mathbb{R}^n) \right)_{\theta, \infty, \, [1, \infty)} = B^{p, \lambda}(\mathbb{R}^n),$$

$$\left(\text{CMO}^{p, \lambda_0}(\mathbb{R}^n), \ \text{CMO}^{p, \lambda_1}(\mathbb{R}^n) \right)_{\theta, \infty, \, [1, \infty)} = \text{CMO}^{p, \lambda}(\mathbb{R}^n).$$

4. Proof of the main theorem

To prove the main theorem we need several lemmas. We also use a weighted Hardy's inequality by Muckenhoupt [35].

LEMMA 1. *Let $0 < u_0 < u_1 \leq \infty$ and $w : (0, \infty) \to (0, \infty)$. If w satisfies the doubling condition, then*

$$B_w^{u_0}(E)(\mathbb{R}^n) \subset B_w^{u_1}(E)(\mathbb{R}^n) \quad and \quad \dot{B}_w^{u_0}(E)(\mathbb{R}^n) \subset \dot{B}_w^{u_1}(E)(\mathbb{R}^n)$$

with

$$\|f\|_{B_w^{u_1}(E)} \leq C \|f\|_{B_w^{u_0}(E)} \quad and \quad \|f\|_{\dot{B}_w^{u_1}(E)} \leq C \|f\|_{\dot{B}_w^{u_0}(E)},$$

respectively, where C is independent of f.

PROOF. Let $f \in \dot{B}_w^{u_1}(E)$.

$$\begin{aligned}
\|f\|_{\dot{B}_w^{u_1}(E)} &= \left\| w(r) \|f\|_{E(Q_r)} \right\|_{L^{u_1}((0, \infty), dr/r)} \\
&= \left\| \left\{ \left\| w(r) \|f\|_{E(Q_r)} \right\|_{L^{u_1}([2^{j-1}, 2^j), dr/r)} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{u_1}} \\
&\lesssim \left\| \left\{ w(2^j) \|f\|_{E(Q_{2^j})} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{u_1}} \\
&\leq \left\| \left\{ w(2^j) \|f\|_{E(Q_{2^j})} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{u_0}}
\end{aligned}$$

$$\lesssim \left\| \left\{ \left\| w(r) \| f \|_{E(Q_r)} \right\|_{L^{u_0}([2^j, 2^{j+1}), dr/r)} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^{u_0}}$$

$$= \left\| w(r) \| f \|_{E(Q_r)} \right\|_{L^{u_0}((0,\infty), dr/r)} = \| f \|_{\dot{B}_w^{u_0}(E)}.$$

For $f \in B_w^{u_1}(E)$, take $j \geq 1$ instead of $j \in \mathbb{Z}$ in the above calculation. □

LEMMA 2. *Let functions* $\phi, G : (0, \infty) \to (0, \infty)$ *satisfy the doubling condition,* $\varepsilon > 0$ *and* $u \in (0, \infty]$. *Assume that* $\phi(r) r^{-\varepsilon}$ *is almost increasing or* $\phi(r) r^{\varepsilon}$ *is almost decreasing. Then*

$$C^{-1} \| G \|_{L^u((0,\infty), dr/r)} \leq \| G \circ \phi \|_{L^u((0,\infty), dr/r)} \leq C \| G \|_{L^u((0,\infty), dr/r)},$$

and

$$C^{-1} \| G \|_{L^u([1,\infty), dr/r)} \leq \| G \circ \phi \|_{L^u([1,\infty), dr/r)} \leq C \| G \|_{L^u([1,\infty), dr/r)},$$

where C *is a positive constant depending only on* ε, u *and the doubling constants of* ϕ *and* G.

PROOF. If ϕ satisfies the doubling condition and $\phi(r) r^{-\varepsilon}$ is almost increasing, then $\phi(r) \sim \int_0^r \phi(t) \, dt/t$. Let $\phi_1(r) = \int_0^r \phi(t) \, dt/t$. Then ϕ_1 is continuous and $\phi \sim \phi_1$, that is, ϕ_1 satisfies the doubling condition and $\phi_1(r) r^{-\varepsilon}$ is almost increasing. Let $\phi_2(r) = \int_0^r \phi_1(t) \, dt/t$. Then ϕ_2 is differentiable, strictly increasing and $\phi \sim \phi_2$. In this case $\phi_2(r) r^{-\varepsilon}$ is almost increasing, and then $\lim_{r \to 0} \phi_2(r) = 0$ and $\lim_{r \to \infty} \phi_2(r) = \infty$. Therefore, ϕ_2 is bijective from $(0, \infty)$ to itself. Moreover,

$$\frac{\phi_2'(r)}{\phi_2(r)} = \frac{\phi_1(r)/r}{\phi_2(r)} \sim \frac{1}{r}.$$

Using the doubling condition of G, we have

$$\| G \circ \phi \|_{L^u((0,\infty), dr/r)} \sim \| G \circ \phi_2 \|_{L^u((0,\infty), dr/r)}$$

$$\sim \| G \circ \phi_2 \|_{L^u((0,\infty), (\phi_2'(r)/\phi_2(r)) dr)}$$

$$= \| G \|_{L^u((0,\infty), dr/r)}.$$

Further, let $\phi_3(r) = \phi_2(r)/\phi_2(1)$. Then $\phi_3(1) = 1$ and ϕ_3 has the same properties as ϕ_2. Hence, using ϕ_3, we have

$$\| G \circ \phi \|_{L^u([1,\infty), dr/r)} \sim \| G \|_{L^u([1,\infty), dr/r)}.$$

If $\phi(r) r^{\varepsilon}$ is almost decreasing, letting $\phi_1(r) = \int_r^\infty \phi(t) \, dt/t$ and $\phi_2(r) = \int_r^\infty \phi_1(t) \, dt/t$, we see that ϕ_2 is differentiable and bijective from $(0, \infty)$ to itself, and

$$\lim_{r \to 0} \phi_2(r) = \infty, \quad \lim_{r \to \infty} \phi_2(r) = 0, \quad -\frac{\phi_2'(r)}{\phi_2(r)} = \frac{\phi_1(r)/r}{\phi_2(r)} \sim \frac{1}{r}.$$

In this case, we also have the same conclusion. □

THEOREM 4 (Muckenhoupt [35]). *Let $p \in [1, \infty]$ and $1/p + 1/p' = 1$. Let $F^*(r) = \int_0^r f(t)\,dt$ and $F_*(r) = \int_r^\infty f(t)\,dt$. Then*

$$\|U F^*\|_{L^p(0,\infty)} \le C \|V f\|_{L^p(0,\infty)}$$

if and only if

$$\sup_{r>0} \left(\int_r^\infty |U(t)|^p \, dt \right)^{1/p} \left(\int_0^r |V(t)|^{-p'} \, dt \right)^{1/p'} < \infty. \tag{4.1}$$

Also,

$$\|U F_*\|_{L^p(0,\infty)} \le C \|V f\|_{L^p(0,\infty)}$$

if and only if

$$\sup_{r>0} \left(\int_0^r |U(t)|^p \, dt \right)^{1/p} \left(\int_r^\infty |V(t)|^{-p'} \, dt \right)^{1/p'} < \infty. \tag{4.2}$$

In the above, we use the obvious modification when $p = \infty$ or $p' = \infty$. That is, we regard $\left(\int_a^b |W(t)|^{\pm\infty} \, dt \right)^{1/\infty}$ as $\sup_{t \in (a,b)} |W(t)|^{\pm 1}$ for $0 \le a < b \le \infty$.

LEMMA 3. *Let $u_0, u_1, u \in (0, \infty]$, $\max(u_0, u_1) \le u$, $w_0, w_1 \in \mathcal{W}^\infty$, $\Theta \in \boldsymbol{\Theta}_*$, and let*

$$w = w_0 \, \Theta(w_1/w_0), \quad w_* = w_0/w_1.$$

(i) *Let $\max(u_0, u_1) < \infty$ and $w_0, w_1 \in \mathcal{W}^*$. Assume that $w_*(r) r^{-\varepsilon}$ is almost increasing for some positive constant ε. For $f \in \dot{B}_w^u(E)$, let*

$$F_0(t) = w_0(t)^{u_0} \|f\|_{E(Q_t)}^{u_0} t^{-1}, \quad U_0(r) = \left(\Theta(w_*(r)^{-1}) \right)^{u_0} r^{-u_0/u},$$

and

$$F_1(t) = w_1(t)^{u_1} \|f\|_{E(Q_t)}^{u_1} t^{-1}, \quad U_1(r) = \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{u_1} r^{-u_1/u}.$$

Then

$$\left\| U_0(r) \int_0^r F_0(t)\,dt \right\|_{L^{u/u_0}(0,\infty)}^{1/u_0} + \left\| U_1(r) \int_r^\infty F_1(t)\,dt \right\|_{L^{u/u_1}(0,\infty)}^{1/u_1} \le C \|f\|_{\dot{B}_w^u(E)},$$

where C is independent of f.

(ii) *Let $u_0 = u = \infty$. Assume that $w_*(r)$ is almost increasing. For $f \in \dot{B}_w^u(E)$, let*

$$F_0(t) = w_0(t) \|f\|_{E(Q_t)}, \quad U_0(r) = \Theta(w_*(r)^{-1}).$$

Then

$$\left\| U_0(r) \left(\sup_{t \in (0,r)} F_0(t) \right) \right\|_{L^\infty(0,\infty)} \leq C \|f\|_{\dot{B}_w^u(E)},$$

where C is independent of f.

(iii) *Let $u_1 = u = \infty$. Assume that $w_*(r)$ is almost increasing. For $f \in \dot{B}_w^u(E)$, let*

$$F_1(t) = w_1(t) \|f\|_{E(Q_t)}, \quad U_1(r) = w_*(r) \Theta(w_*(r)^{-1}).$$

Then

$$\left\| U_1(r) \left(\sup_{t \in (r,\infty)} F_1(t) \right) \right\|_{L^\infty(0,\infty)} \leq C \|f\|_{\dot{B}_w^u(E)},$$

where C is independent of f.

REMARK 4. In the definition of F_0 and F_1 of Lemma 3, using $\|f\|_{E(Q_r)} \chi_{[1,\infty)}(r)$ instead of $\|f\|_{E(Q_r)}$, we have the result for $f \in B_w^u(E)$.

PROOF OF LEMMA 3. (i) We may assume that $w_*(r) r^{-\varepsilon}$ and $\Theta(r) r^{-\varepsilon}$ are almost increasing and $\Theta(r) r^{\varepsilon-1}$ is almost decreasing for the same small ε. First note that, using these properties and the doubling condition of Θ, we have that, for $a > 0$,

$$\int_0^r \left(\Theta(w_*(t)^{-1}) \right)^{-a} \frac{dt}{t} = \int_0^r \left(\Theta(w_*(t)^{-1}) w_*(t)^\varepsilon \right)^{-a} \left(w_*(t)^{-1} t^\varepsilon \right)^{-\varepsilon a} t^{\varepsilon^2 a} \frac{dt}{t}$$

$$\lesssim \left(\Theta(w_*(r)^{-1}) w_*(r)^\varepsilon \right)^{-a} \left(w_*(r)^{-1} r^\varepsilon \right)^{-\varepsilon a} \int_0^r t^{\varepsilon^2 a} \frac{dt}{t}$$

$$\sim \left(\Theta(w_*(r)^{-1}) \right)^{-a},$$

and

$$\int_0^r \left(w_*(t) \Theta(w_*(t)^{-1}) \right)^a \frac{dt}{t} = \int_0^r \left(w_*(t)^{1-\varepsilon} \Theta(w_*(t)^{-1}) \right)^a \left(w_*(t) t^{-\varepsilon} \right)^{\varepsilon a} t^{\varepsilon^2 a} \frac{dt}{t}$$

$$\lesssim \left(w_*(r)^{1-\varepsilon} \Theta(w_*(r)^{-1}) \right)^a \left(w_*(r) r^{-\varepsilon} \right)^{\varepsilon a} \int_0^r t^{\varepsilon^2 a} \frac{dt}{t}$$

$$\sim \left(w_*(r) \Theta(w_*(r)^{-1}) \right)^a.$$

Similarly, we can get

$$\int_r^\infty \left(\Theta(w_*(t)^{-1}) \right)^a \frac{dt}{t} \lesssim \left(\Theta(w_*(r)^{-1}) \right)^a,$$

and

$$\int_r^\infty \left(w_*(t) \Theta(w_*(t)^{-1}) \right)^{-a} \frac{dt}{t} \lesssim \left(w_*(r) \Theta(w_*(r)^{-1}) \right)^{-a}.$$

Let

$$V_0(r) = \left(\Theta(w_*(r)^{-1})\right)^{u_0} r^{1-u_0/u}, \quad V_1(r) = \left(w_*(r)\Theta(w_*(r)^{-1})\right)^{u_1} r^{1-u_1/u}.$$

<u>Part 1</u>. Proof of

$$\left\| U_0(r) \int_0^r F_0(t)\,dt \right\|_{L^{u/u_0}(0,\infty)}^{1/u_0} \leq C \|f\|_{B_w^u(E)}.$$

Case 1: $u_0 < u < \infty$.

$$\left(\int_r^\infty U_0(t)^{u/u_0}\,dt \right)^{u_0/u} \left(\int_0^r V_0(t)^{-u/(u-u_0)}\,dt \right)^{(u-u_0)/u}$$

$$= \left(\int_r^\infty \left(\Theta(w_*(t)^{-1})\right)^u \frac{dt}{t} \right)^{u_0/u} \left(\int_0^r \left(\Theta(w_*(t)^{-1})\right)^{-u_0 u/(u-u_0)} \frac{dt}{t} \right)^{(u-u_0)/u}$$

$$\lesssim \left(\Theta(w_*(r)^{-1})\right)^{u_0} \left(\Theta(w_*(r)^{-1})\right)^{-u_0} = 1.$$

Case 2: $u_0 = u < \infty$.

$$\left(\int_r^\infty U_0(t)\,dt \right) \left(\sup_{t\in(0,r)} V_0(t)^{-1} \right)$$

$$= \left(\int_r^\infty \left(\Theta(w_*(t)^{-1})\right)^{u_0} \frac{dt}{t} \right) \left(\sup_{t\in(0,r)} \left(\Theta(w_*(t)^{-1})\right)^{-u_0} \right)$$

$$\lesssim \left(\Theta(w_*(r)^{-1})\right)^{u_0} \left(\Theta(w_*(r)^{-1})\right)^{-u_0} = 1.$$

Case 3: $u_0 < u = \infty$. In this case

$$U_0(r) = \left(\Theta(w_*(r)^{-1})\right)^{u_0}, \quad V_0(r) = \left(\Theta(w_*(r)^{-1})\right)^{u_0} r.$$

Then

$$\left(\sup_{t\in(r,\infty)} U_0(t) \right) \left(\int_0^r V_0(t)^{-1}\,dt \right)$$

$$= \left(\sup_{t\in(r,\infty)} \left(\Theta(w_*(t)^{-1})\right)^{u_0} \right) \left(\int_0^r \left(\Theta(w_*(t)^{-1})\right)^{-u_0} \frac{dt}{t} \right)$$

$$\sim \left(\Theta(w_*(r)^{-1})\right)^{u_0} \left(\Theta(w_*(r)^{-1})\right)^{-u_0} = 1.$$

In any cases, U_0 and V_0 satisfy the condition (4.1) in Theorem 4. Since

$$V_0(r)F_0(r) = \left(\Theta(w_*(r)^{-1})\right)^{u_0} r^{1-u_0/u} w_0(r)^{u_0} \|f\|_{E(Q_r)}^{u_0} r^{-1}$$

$$= w(r)^{u_0} \|f\|_{E(Q_r)}^{u_0} r^{-u_0/u},$$

using Theorem 4, we have

$$\left\| U_0(r) \int_0^r F_0(t)\, dt \right\|_{L^{u/u_0}(0,\infty)}^{1/u_0} \lesssim \left\| V_0(r) F_0(r) \right\|_{L^{u/u_0}(0,\infty)}^{1/u_0}$$

$$= \left\| w(r)^{u_0} \|f\|_{E(Q_r)}^{u_0} r^{-u_0/u} \right\|_{L^{u/u_0}(0,\infty)}^{1/u_0}$$

$$= \left\| w(r)\|f\|_{E(Q_r)} \right\|_{L^u((0,\infty),dr/r)} = \|f\|_{\dot{B}_w^u(E)} .$$

<u>Part 2</u>. Proof of

$$\left\| U_1(r) \int_r^\infty F_1(t)\, dt \right\|_{L^{u/u_1}(0,\infty)}^{1/u_1} \le C \|f\|_{B_w^u(E)} .$$

Case 1: $u_1 < u < \infty$.

$$\left(\int_0^r U_1(t)^{u/u_1}\, dt \right)^{u_1/u} \left(\int_r^\infty V_1(t)^{-u/(u-u_1)}\, dt \right)^{(u-u_1)/u}$$

$$= \left(\int_0^r \left(w_*(t)\Theta(w_*(t)^{-1}) \right)^u \frac{dt}{t} \right)^{u_1/u}$$

$$\times \left(\int_r^\infty \left(w_*(t)\Theta(w_*(t)^{-1}) \right)^{-u_1 u/(u-u_1)} \frac{dt}{t} \right)^{(u-u_1)/u}$$

$$\lesssim \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{u_1} \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{-u_1} = 1.$$

Case 2: $u_1 = u < \infty$.

$$\left(\int_0^r U_1(t)\, dt \right) \left(\sup_{t\in(r,\infty)} V_1(t)^{-1} \right)$$

$$= \left(\int_0^r \left(w_*(t)\Theta(w_*(t)^{-1}) \right)^{u_1} \frac{dt}{t} \right) \left(\sup_{t\in(r,\infty)} \left(w_*(t)\Theta(w_*(t)^{-1}) \right)^{-u_1} \right)$$

$$\lesssim \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{u_1} \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{-u_1} = 1 .$$

Case 3: $u_1 < u = \infty$. In this case

$$U_1(r) = \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{u_1}, \quad V_1(r) = \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{u_1} r.$$

Then

$$\left(\sup_{t\in(0,r)} U_1(t) \right) \left(\int_r^\infty V_1(t)^{-1}\, dt \right)$$

$$= \left(\sup_{t\in(0,r)} \left(w_*(t)\Theta(w_*(t)^{-1}) \right)^{u_1} \right) \left(\int_r^\infty \left(w_*(t)\Theta(w_*(t)^{-1}) \right)^{-u_1} \frac{dt}{t} \right)$$

$$\lesssim \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{u_1} \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{-u_1} = 1 \,.$$

In any cases, U_1 and V_1 satisfy the condition (4.2) in Theorem 4. Since

$$V_1(r)F_1(r) = \left(w_*(r)\Theta(w_*(r)^{-1}) \right)^{u_1} r^{1-u_1/u} w_1(r)^{u_1} \|f\|_{E(Q_r)}^{u_1} r^{-1}$$

$$= w(r)^{u_1} \|f\|_{E(Q_r)}^{u_1} r^{-u_1/u} \,,$$

using Theorem 4, we have

$$\left\| U_1(r) \int_r^\infty F_1(t)\, dt \right\|_{L^{u/u_1}(0,\infty)}^{1/u_1} \lesssim \|V_1(r)F_1(r)\|_{L^{u/u_1}(0,\infty)}^{1/u_1}$$

$$= \left\| w(r)^{u_1} \|f\|_{E(Q_t)}^{u_1} r^{-u_1/u} \right\|_{L^{u/u_1}(0,\infty)}^{1/u_1}$$

$$= \left\| w(r)\|f\|_{E(Q_r)} \right\|_{L^u((0,\infty),dr/r)} = \|f\|_{\dot{B}_w^u(E)}.$$

(ii) Since $U_0(r) = \Theta(w_*(r)^{-1})$ is almost decreasing,

$$\left\| U_0(r)\left(\sup_{t\in(0,r)} F_0(t) \right) \right\|_{L^\infty(0,\infty)} \lesssim \left\| \left(\sup_{t\in(0,r)} U_0(t)F_0(t) \right) \right\|_{L^\infty(0,\infty)}$$

$$= \left\| U_0(t)F_0(t) \right\|_{L^\infty(0,\infty)} = \left\| w(t)\|f\|_{E(Q_t)} \right\|_{L^\infty(0,\infty)} = \|f\|_{\dot{B}_w^\infty(E)} \,.$$

(iii) Since $U_1(r) = w_*(r)\Theta(w_*(r)^{-1})$ is almost increasing,

$$\left\| U_1(r)\left(\sup_{t\in(r,\infty)} F_1(t) \right) \right\|_{L^\infty(0,\infty)} \lesssim \left\| \left(\sup_{t\in(r,\infty)} U_1(t)F_1(t) \right) \right\|_{L^\infty(0,\infty)}$$

$$= \left\| U_1(t)F_1(t) \right\|_{L^\infty(0,\infty)} = \left\| w(t)\|f\|_{E(Q_t)} \right\|_{L^\infty(0,\infty)} = \|f\|_{\dot{B}_w^\infty(E)}.$$

Therefore, we have the conclusion. $\qquad\square$

PROOF OF THEOREM 1. We may assume that $(w_0(r)/w_1(r))r^{-\varepsilon}$ is almost increasing, by changing w_0 and w_1 if need.
<u>Part 1</u>. Proof of

$$\left(\dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n),\ \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n),\ \Theta \right)_u \subset \dot{B}_w^u(E)(\mathbb{R}^n)\,. \tag{4.3}$$

Let $f \in \left(\dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n),\ \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n),\ \Theta \right)_u$ and $f = f_0 + f_1$ with $f_i \in \dot{B}_{w_i}^{u_i}(E)(\mathbb{R}^n), i = 1, 2$. Then

$$w(r)\|f\|_{E(Q_r)} \le Cw(r)\left(\|f_0\|_{E(Q_r)} + \|f_1\|_{E(Q_r)} \right)$$

$$\le C\frac{w(r)}{w_0(r)} \left(w_0(r)\|f_0\|_{E(Q_r)} + \frac{w_0(r)}{w_1(r)}w_1(r)\|f_1\|_{E(Q_r)} \right)$$

$$\leq C \frac{w(r)}{w_0(r)} \left(\|f_0\|_{\dot{B}_{w_0}^\infty(E)} + \frac{w_0(r)}{w_1(r)} \|f_1\|_{\dot{B}_{w_1}^\infty(E)} \right)$$

$$\leq C \, \Theta\left(\frac{w_1(r)}{w_0(r)} \right) \left(\|f_0\|_{\dot{B}_{w_0}^{u_0}(E)} + \frac{w_0(r)}{w_1(r)} \|f_1\|_{\dot{B}_{w_1}^{u_1}(E)} \right).$$

Then, letting $w_* = w_0/w_1$, we have

$$w(r)\|f\|_{E(Q_r)} \leq C\Theta(w_*(r)^{-1}) K\left(w_*(r), f; \dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n), \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n) \right).$$

By Lemma 2 we have

$$\|f\|_{\dot{B}_w^u(E)} = \left\| w(r)\|f\|_{E(Q_r)} \right\|_{L^u((0,\infty),dr/r)}$$

$$\lesssim \left\| \Theta(w_*(r)^{-1}) K\left(w_*(r), f; \dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n), \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n) \right) \right\|_{L^u((0,\infty),dr/r)}$$

$$\sim \left\| \Theta(r^{-1}) K\left(r, f; \dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n), \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n) \right) \right\|_{L^u((0,\infty),dr/r)}$$

$$= \|f\|_{\left(\dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n), \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n), \Theta \right)_u}.$$

This shows (4.3).

<u>Part 2</u>. Proof of

$$\left(\dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n), \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n), \Theta \right)_u \supset \dot{B}_w^u(E)(\mathbb{R}^n). \tag{4.4}$$

We may assume that $0 < \max(u_0, u_1) \leq u \leq \infty$, since

$$\left(\dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n), \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n), \Theta \right)_u \supset \left(\dot{B}_{w_0}^{\min(u_0,u)}(E)(\mathbb{R}^n), \dot{B}_{w_1}^{\min(u_1,u)}(E)(\mathbb{R}^n), \Theta \right)_u.$$

Let $f \in \dot{B}_w^u(E)(\mathbb{R}^n)$ and $r > 0$. From the decomposition property of $\{E(Q_r)\}$, we can take functions f_0^r and f_1^r satisfying $f = f_0^r + f_1^r$,

$$\|f_0^r\|_{E(Q_t)} \leq \begin{cases} C_E \|f\|_{E(Q_t)} & (0 < t < r), \\ C_E \|f\|_{E(Q_{ar})} & (r \leq t < \infty), \end{cases} \tag{4.5}$$

and

$$\|f_1^r\|_{E(Q_t)} \leq \begin{cases} 0 & (0 < t < cr), \\ C_E \|f\|_{E(Q_{bt})} & (cr \leq t < \infty). \end{cases} \tag{4.6}$$

Here we may assume that $a \geq 1$ and $b \geq 1$. We will show that $f_0^r \in \dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n)$, $f_1^r \in \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n)$ and

$$\left\| \Theta(w_*(r)^{-1}) \|f_0^r\|_{\dot{B}_{w_0}^{u_0}(E)} \right\|_{L^u((0,\infty),dr/r)}$$

$$+ \left\| w_*(r)\Theta(w_*(r)^{-1}) \|f_1^r\|_{\dot{B}_{w_1}^{u_1}(E)} \right\|_{L^u((0,\infty),dr/r)} \lesssim \|f\|_{\dot{B}_w^u(E)}. \tag{4.7}$$

Then, by Lemma 2

$$\|f\|_{\left(\dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n),\, \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n),\, \Theta\right)_u}$$

$$= \left\|\Theta(r^{-1})K\left(r,\, f;\, \dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n),\, \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n)\right)\right\|_{L^u((0,\infty),dr/r)}$$

$$\sim \left\|\Theta(w_*(r)^{-1})K\left(w_*(r),\, f;\, \dot{B}_{w_0}^{u_0}(E)(\mathbb{R}^n),\, \dot{B}_{w_1}^{u_1}(E)(\mathbb{R}^n)\right)\right\|_{L^u((0,\infty),dr/r)}$$

$$\leq \left\|\Theta(w_*(r)^{-1})\|f_0^r\|_{\dot{B}_{w_0}^{u_0}(E)} + w_*(r)\Theta(w_*(r)^{-1})\|f_1^r\|_{\dot{B}_{w_1}^{u_1}(E)}\right\|_{L^u((0,\infty),dr/r)}$$

$$\lesssim \|f\|_{\dot{B}_w^u(E)}.$$

This shows (4.4).

Now we prove (4.7). From Lemma 3 we see that

$$\left\|w_0(t)\|f\|_{E(Q_t)}\right\|_{L^{u_0}((0,2ar),dt/t)} < \infty, \quad \left\|w_1(t)\|f\|_{E(Q_t)}\right\|_{L^{u_1}([r,\infty),dt/t)} < \infty,$$

and

$$\left\|\Theta(w_*(2ar)^{-1})\left\|w_0(t)\|f\|_{E(Q_t)}\right\|_{L^{u_0}((0,2ar),dt/t)}\right\|_{L^u((0,\infty),dr/r)}$$

$$+ \left\|w_*(r)\Theta(w_*(r)^{-1})\left\|w_1(t)\|f\|_{E(Q_t)}\right\|_{L^{u_1}([r,\infty),dt/t)}\right\|_{L^u((0,\infty),dr/r)}$$

$$\lesssim \|f\|_{\dot{B}_w^u(E)}.$$

Therefore, to prove (4.7) it is enough to show

$$\left\|w_0(t)\|f_0^r\|_{E(Q_t)}\right\|_{L^{u_0}((0,\infty),dt/t)} \lesssim \left\|w_0(t)\|f\|_{E(Q_t)}\right\|_{L^{u_0}((0,2ar),dt/t)}, \tag{4.8}$$

and

$$\left\|w_1(t)\|f_1^r\|_{E(Q_t)}\right\|_{L^{u_1}((0,\infty),dt/t)} \lesssim \left\|w_1(t)\|f\|_{E(Q_t)}\right\|_{L^{u_1}([r,\infty),dt/t)}. \tag{4.9}$$

Since $w_0 \in \mathcal{W}^\infty$ if $u_0 = \infty$, or $w_0 \in \mathcal{W}^*$ if $u_0 < \infty$,

$$\|w_0(t)\|_{L^{u_0}([r,\infty),dt/t)} \lesssim w_0(r) \lesssim \|w_0(t)\|_{L^{u_0}([ar,2ar),dt/t)}.$$

From (4.5) it follows that

$$\left\|w_0(t)\|f_0^r\|_{E(Q_t)}\right\|_{L^{u_0}((0,\infty),dt/t)}$$

$$\lesssim \left\|w_0(t)\|f\|_{E(Q_t)}\right\|_{L^{u_0}((0,r),dt/t)} + \|f\|_{E(Q_{ar})}\|w_0(t)\|_{L^{u_0}([r,\infty),dt/t)}$$

$$\lesssim \left\|w_0(t)\|f\|_{E(Q_t)}\right\|_{L^{u_0}((0,2ar),dt/t)}.$$

This shows (4.8). Next we show (4.9). From (4.6) it follows that

$$\left\| w_1(t) \| f_1^r \|_{E(Q_t)} \right\|_{L^{u_1}((0,\infty),dt/t)} \lesssim \left\| w_1(t) \| f \|_{E(Q_{bt})} \right\|_{L^{u_1}([cr,\infty),dt/t)}$$

$$\sim \left\| w_1(bt) \| f \|_{E(Q_{bt})} \right\|_{L^{u_1}([cr,\infty),dt/t)}$$

$$= \left\| w_1(t) \| f \|_{E(Q_t)} \right\|_{L^{u_1}([cr/b,\infty),dt/t)}.$$

If $c/b \geq 1$, then we have (4.9). If $c/b < 1$, then

$$\left\| w_1(t) \| f \|_{E(Q_t)} \right\|_{L^{u_1}([cr/b,\infty),dt/t)}$$

$$= \left\| w_1(t) \| f \|_{E(Q_t)} \right\|_{L^{u_1}([cr/b,r),dt/t)} + \left\| w_1(t) \| f \|_{E(Q_t)} \right\|_{L^{u_1}([r,\infty),dt/t)}$$

$$\lesssim \left\| w_1(t) \| f \|_{E(Q_t)} \right\|_{L^{u_1}([r,br/c),dt/t)} + \left\| w_1(t) \| f \|_{E(Q_t)} \right\|_{L^{u_1}([r,\infty),dt/t)}$$

$$\leq 2 \left\| w_1(t) \| f \|_{E(Q_t)} \right\|_{L^{u_1}([r,\infty),dt/t)}.$$

This shows (4.9).
Part 3. Proof of

$$\left(B_{w_0}^{u_0}(E)(\mathbb{R}^n),\ B_{w_1}^{u_1}(E)(\mathbb{R}^n),\ \Theta \right)_{u,\,[1,\infty)} \subset B_w^u(E)(\mathbb{R}^n). \tag{4.10}$$

Using $L^u([1,\infty), dr/r)$ instead of $L^u((0,\infty), dr/r)$ in Part 1, we have the conclusion.
Part 4. Proof of

$$\left(B_{w_0}^{u_0}(E)(\mathbb{R}^n),\ B_{w_1}^{u_1}(E)(\mathbb{R}^n),\ \Theta \right)_{u,\,[1,\infty)} \supset B_w^u(E)(\mathbb{R}^n). \tag{4.11}$$

Instead of (4.7) we need

$$\left\| \Theta(w_*(r)^{-1}) \| f_0^r \|_{B_{w_0}^{u_0}(E)} \right\|_{L^u([1,\infty),dr/r)}$$

$$+ \left\| w_*(r)\Theta(w_*(r)^{-1}) \| f_1^r \|_{B_{w_1}^{u_1}(E)} \right\|_{L^u([1,\infty),dr/r)} \lesssim \| f \|_{B_w^u(E)}. \tag{4.12}$$

By the same way as (4.8) and (4.9) we can get, for $r \geq 1$,

$$\left\| w_0(t) \| f_0^r \|_{E(Q_t)} \right\|_{L^{u_0}([1,\infty),dt/t)} \lesssim \left\| w_0(t) \| f \|_{E(Q_t)} \right\|_{L^{u_0}([1,2ar),dt/t)},$$

and

$$\left\| w_1(t) \| f_1^r \|_{E(Q_t)} \right\|_{L^{u_1}([1,\infty),dt/t)} \lesssim \left\| w_1(t) \| f \|_{E(Q_t)} \right\|_{L^{u_1}([r,\infty),dt/t)},$$

respectively. By Remark 4 we see that (4.12) follows from these inequalities. $\square$

5. Boundedness of linear and sublinear operators

In this section we consider the boundedness of linear and sublinear operators on $B_w^u(E)(\mathbb{R}^n)$ and $\dot{B}_w^u(E)(\mathbb{R}^n)$ with $E = L_{p,\lambda}$ or $\mathcal{L}_{p,\lambda}$. It is known that some classical operators are bounded on $B_\sigma(E)(\mathbb{R}^n)$ and $\dot{B}_\sigma(E)(\mathbb{R}^n)$, see [25]. Applying the interpolation property, we extend these boundedness to $B_w^u(E)(\mathbb{R}^n)$ and $\dot{B}_w^u(E)(\mathbb{R}^n)$. We consider sublinear operators T defined on $L_{\text{comp}}^1(\mathbb{R}^n)$. That is, the operator T satisfies that, for all $f, g \in L_{\text{comp}}^1(\mathbb{R}^n)$ and for a.e. $x \in \mathbb{R}^n$,

$$|T(f+g)(x)| \le |Tf(x)| + |Tg(x)|.$$

We also assume that

$$|Tf(x) - Tg(x)| \le C|T(f-g)(x)| \tag{5.1}$$

for some positive constant C. For example, if T is linear, or, sublinear and $Tf(x) \ge 0$ for all f and a.e. x, then T satisfies the condition (5.1) with $C = 1$.

In general, for quasi-normed function spaces A_i and B_i, $i = 0, 1$, let a sublinear operator $T : A_0 + A_1 \to B_0 + B_1$ be bounded from A_i to B_i, $i = 0, 1$, and satisfy (5.1) for all $f, g \in A_0 + A_1$. If T is not linear, we also assume that B_i, $i = 0, 1$, satisfy the lattice property (3.11). Then we conclude that

$$K(r, Tf; B_0, B_1) \le C_T K(r, f; A_0, A_1),$$

where C_T is a positive constant dependent on T and C in (5.1). Therefore we can use the interpolation property for the boundedness of T. Actually, if T is linear, then

$$Tf = Tf_0 + Tf_1, \ Tf_0 \in B_0, \ Tf_1 \in B_1$$

for any decomposition $f = f_0 + f_1$ in $A_0 + A_1$. Hence

$$K(r, Tf; B_0, B_1) \le \|Tf_0\|_{B_0} + r\|Tf_1\|_{B_1} \le C_T (\|f_0\|_{A_0} + r\|f_1\|_{A_1}).$$

If T is not linear, then, using (5.1) and the lattice property, we have

$$|Tf(x) - Tf_0(x)| \le C|Tf_1(x)|$$

and

$$Tf = Tf_0 + (Tf - Tf_0), \ Tf_0 \in B_0, \ Tf - Tf_0 \in B_1.$$

Hence

$$K(r, Tf; B_0, B_1) \le \|Tf_0\|_{B_0} + r\|Tf - Tf_0\|_{B_1} \le C_T (\|f_0\|_{A_0} + r\|f_1\|_{A_1}),$$

for any decomposition $f = f_0 + f_1$ in $A_0 + A_1$.

We also point out that the condition (5.1) is important to extend L^p-bounded operators to bounded operators on Morrey spaces. Actually, there exists an L^p-bounded sublinear operator

T such that T does not satisfy (5.1) and that T cannot be extended to a bounded operator on Morrey spaces, see Remark 5.

In this section, first we give the boundedness of the Hardy-Littlewood maximal and fractional maximal operators in Subsection 5.1. Next we investigate singular and fractional integral operators and more general sublinear operators with (5.1) in Subsection 5.2. In Subsections 5.3 and 5.4 we consider singular integral operators with the cancellation property and modified fractional integral operators, respectively. Finally, we show the vector-valued boundedness in Subsection 5.5.

If $\lambda = -n/p$, then $L_{p,\lambda} = L^p$ and $\dot{B}_w^u(L_{p,\lambda})(\mathbb{R}^n) = \dot{B}_w^u(L^p)(\mathbb{R}^n) = LM_{pu,\tilde{w}}(\mathbb{R}^n)$ with $\tilde{w}(r) = w(r)/r$. Let $\dot{B}_w^u(WL^p)(\mathbb{R}^n) = WLM_{pu,\tilde{w}}(\mathbb{R}^n)$ with $\tilde{w}(r) = w(r)/r$, where WL^p is the weak L^p space.

5.1. The Hardy-Littlewood maximal and fractional maximal operators. The fractional maximal operators M_α of order $\alpha \in [0, n)$ are sublinear, which is defined as

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| \, dy \,,$$

where the supremum is taken over all cubes (or balls) Q containing $x \in \mathbb{R}^n$. If $\alpha = 0$, then M_α is the Hardy-Littlewood maximal operator denoted by M.

It is known that, for $\alpha \in [0, n)$, $p, q \in [1, \infty]$ and $-n/p + \alpha = -n/q$, the operator M_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if $p \in (1, \infty]$, and from $L^1(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n)$ if $p = 1$.

It is also known that, for $\alpha \in [0, n)$, $p, q \in [1, \infty)$, $\lambda \in [-n/p, 0)$, $\mu \in [-n/q, 0)$, $\mu = \lambda + \alpha$ and $q \le (\lambda/\mu)p$, the operator M_α is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$ if $p \in (1, \infty)$, and from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\mu}(\mathbb{R}^n)$ if $p = 1$. In particular, the Hardy-Littlewood maximal operator M is bounded from $L_{p,\lambda}(\mathbb{R}^n)$ to itself if $p \in (1, \infty)$ and from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{1,\lambda}(\mathbb{R}^n)$, see [14].

Moreover, the boundedness of M_α on B_σ-Morrey spaces is known by [25, Theorem 7]. Using this boundedness and Example 13, we have the following:

THEOREM 5. *Let* $\alpha \in [0, n)$, $p, q \in [1, \infty)$, $\lambda \in [-n/p, 0)$, $\mu \in [-n/q, 0)$, $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, *and let*

$$w(r) = r^{-\sigma} \Theta(r^\tau) \,, \quad \sigma, \tau \in (0, \infty) \ \text{with} \ \sigma > \tau \,.$$

Assume that

$$\mu = \lambda + \alpha \,, \quad q \le (\lambda/\mu)p \quad \text{and} \quad \sigma + \lambda + \alpha \le 0 \,.$$

Then the operator M_α *is bounded from* $B_w^u(L_{p,\lambda})(\mathbb{R}^n)$ *to* $B_w^u(L_{q,\mu})(\mathbb{R}^n)$ *if* $p \in (1, \infty)$, *from* $B_w^u(L_{1,\lambda})(\mathbb{R}^n)$ *to* $B_w^u(WL_{q,\mu})(\mathbb{R}^n)$ *if* $p = 1$. *The same conclusion holds for* $\dot{B}_w^u(L_{p,\lambda})(\mathbb{R}^n)$.

Taking $\lambda = -n/p$ and $\mu = -n/q$ in Theorem 5, we have the following:

COROLLARY 4. *Let $\alpha \in [0, n)$, $p, q \in [1, \infty)$, $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and let*

$$\tilde{w}(r) = w(r)/r, \quad w(r) = r^{-\sigma} \Theta(r^{\tau}), \quad \sigma, \tau \in (0, \infty) \text{ with } \sigma > \tau.$$

Assume that

$$-n/q = -n/p + \alpha \quad and \quad \sigma - n/p + \alpha \leq 0.$$

Then the operator M_α is bounded from $LM_{pu,\tilde{w}}(\mathbb{R}^n)$ to $LM_{qu,\tilde{w}}(\mathbb{R}^n)$ if $p \in (1, \infty)$, from $LM_{1u,\tilde{w}}(\mathbb{R}^n)$ to $WLM_{qu,\tilde{w}}(\mathbb{R}^n)$ if $p = 1$.

For necessary and sufficient conditions for the boundedness of M on local Morrey-type spaces, see [7].

5.2. Singular and fractional integral operators. We consider sublinear operators T which satisfy (5.1) and the following condition: There exist constants $\alpha \in [0, n)$ and $C \in (0, \infty)$ such that, for all $f \in L^1_{\text{comp}}(\mathbb{R}^n)$,

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x - y)|}{|x - y|^{n-\alpha}} |f(y)| \, dy, \quad x \notin \text{supp } f, \tag{5.2}$$

where Ω is a function on $\mathbb{R}^n$ which is homogeneous of degree zero and $\Omega \in L^{\tilde{p}}(S^{n-1})$ for some $\tilde{p} \in [1, \infty]$. For example, singular and fractional integral operators satisfy (5.2) with $\Omega \equiv 1$. More precisely, the singular integral operator T is defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy, \quad x \notin \text{supp } f, \quad f \in L^1_{\text{comp}}(\mathbb{R}^n) \tag{5.3}$$

with kernel $K(x, y)$ satisfying the condition

$$|K(x, y)| \leq C|x - y|^{-n}, \quad x \neq y, \tag{5.4}$$

and some regularity conditions. (For regularity conditions, see Yabuta [48] and references therein.) Then the singular integral operator T satisfies the condition (5.2) with $\alpha = 0$ and it is bounded on $L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, and from $L^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$. Moreover, under the assumption that $p \in [1, \infty)$ and $\lambda \in [-n/p, 0)$, T can be extended to a bounded operator on $L_{p,\lambda}(\mathbb{R}^n)$ if $p \in (1, \infty)$, and from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{1,\lambda}(\mathbb{R}^n)$ if $p = 1$, see [14, 36, 40]. Fractional integral operators I_α, $\alpha \in (0, n)$, are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy.$$

Then I_α satisfies (5.2) with this α and it is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1 < p < q < \infty$, $-n/p + \alpha = -n/q$, and from $L^1(\mathbb{R})$ to $WL^{n/(n-\alpha)}(\mathbb{R})$. Moreover, under the assumption that $p, q \in [1, \infty)$, $\lambda \in [-n/p, 0)$, $\mu \in [-n/q, 0)$, $\lambda + \alpha = \mu$ and $q \leq (\lambda/\mu)p$, I_α can be extended to a bounded operator from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$ if $p \in (1, \infty)$, and from $L_{1,\lambda}(\mathbb{R}^n)$ to $WL_{q,\mu}(\mathbb{R}^n)$ if $p = 1$, see [1, 14].

For the L^p-boundedness of Calderón-Zygmund singular integral operators

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy\,,$$

and fractional integral operators with rough kernel

$$I_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy\,,$$

see [11] and [34], respectively.

REMARK 5. Let T be a sublinear operator satisfying (5.1) and (5.2) for some $\alpha \in [0, n)$. Let $p, q \in [1, \infty)$, $\lambda \in [-n/p, 0)$, $\mu \in [-n/q, 0)$ and $\mu = \lambda + \alpha$. Assume that T is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ or to $WL^q(\mathbb{R}^n)$. Then, for $f \in L_{p,\lambda}(\mathbb{R}^n)$ and $R > 0$, $T(f\chi_R)$ is well defined and $\lim_{R\to\infty} T(f\chi_R)$ exists a.e. on $\mathbb{R}^n$, or in $L_{\text{loc}}^q(\mathbb{R}^n)$, with some additional assumption on Ω in (5.2). Actually, $f\chi_R \in L^p(\mathbb{R}^n)$ and we can prove that

$$|T(f\chi_S)(x) - T(f\chi_R)(x)| \le C|T(f(\chi_S - \chi_R)(x)| \to 0$$

as $R, S \to \infty$ for a.e. $\mathbb{R}^n$, or in $L_{\text{loc}}^q(\mathbb{R}^n)$, see [25, Lemmas 3 and 4]. Then, letting $Tf = \lim_{R\to\infty} T(f\chi_R)$ for $f \in L_{p,\lambda}(\mathbb{R}^n)$, we can define T as a bounded operator from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$ or to $WL_{q,\mu}(\mathbb{R}^n)$, see [25, Remark 15] in which we point out that we need the condition (5.1). For example, the operator $Tf = e^{i\|f\|_{L^p(\mathbb{R}^n)}} Mf$, where M is the Hardy-Littlewood maximal operator, is bounded on $L^p(\mathbb{R}^n)$ but not well defined on Morrey spaces in general.

REMARK 6. If T is a singular integral operator defined by (5.3), then the equality

$$\lim_{R\to\infty} T(f\chi_R)(x) = T(f\chi_{Q(z,2r)})(x) + \int_{\mathbb{R}^n \setminus Q(z,2r)} K(x,y) f(y)\, dy$$

holds for a.e. $x \in Q(z, r)$ and for any $Q(z, r)$, see [36, 39, 43]. See also Rosenthal and Triebel [42] for the extension of singular integral (Calderón-Zygmund) operators to Morrey spaces.

Let T be a sublinear operator defined on $L_{\text{comp}}^1(\mathbb{R}^n)$ and satisfy (5.1) and (5.2). It is known that, under some conditions, if T can be extended to a bounded operator on Morrey spaces, then T can be further extended to a bounded operator on B_σ-Morrey spaces, see [25, Theorem 8]. Using this boundedness and Example 13, we have the following:

THEOREM 6. *Let* $p, q \in [1, \infty)$, $\lambda \in [-n/p, 0)$, $\mu \in [-n/q, 0)$, $u \in (0, \infty]$, $\Theta \in \Theta_*$, *and let*

$$w(r) = r^{-\sigma} \Theta(r^\tau)\,, \quad \sigma, \tau \in (0, \infty) \ \text{with} \ \sigma > \tau\,. \tag{5.5}$$

Let T *be a sublinear operator defined on* $L_{\text{comp}}^1(\mathbb{R}^n)$ *and satisfy* (5.1) *and* (5.2) *for some* $\alpha \in [0, n)$ *and* $\Omega \in L^{\tilde{p}}(S^{n-1})$ *with* $\tilde{p} \in [1, \infty]$. *Assume one of the following conditions*:

(i) $\mu = \lambda + \alpha$, $\tilde{p} \geq p'$ and $\sigma + \lambda + \alpha < 0$,

(ii) $\mu = \lambda + \alpha$, $\tilde{p} \geq q$ and $\sigma + \lambda + n/\tilde{p} + \alpha < 0$.

Assume in addition T can be extended to a bounded operator from $L_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$ or to $WL_{q,\mu}(\mathbb{R}^n)$. Then T can be further extended to a bounded operator from $B_w^u(L_{p,\lambda})(\mathbb{R}^n)$ to $B_w^u(L_{q,\mu})(\mathbb{R}^n)$ or to $B_w^u(WL_{q,\mu})(\mathbb{R}^n)$, respectively. The same conclusion holds for $\dot{B}_w^u(L_{p,\lambda})(\mathbb{R}^n)$.

REMARK 7. Let $f \in B_w^u(L_{p,\lambda})(\mathbb{R}^n)$ and $R > 0$. Then $f\chi_R \in B_\tau(L_{p,\lambda})(\mathbb{R}^n)$ and $f(1 - \chi_R) \in B_\sigma(L_{p,\lambda})(\mathbb{R}^n)$. Hence $T(f\chi_R)$, $T(f(1-\chi_R))$ and $Tf = \lim_{R\to\infty} T(f\chi_R)$ are well defined by [25, Theorem 8], see also its proof.

COROLLARY 5. *Let T be a singular integral operator with kernel $K(x, y)$ satisfying the condition (5.4). Let $p \in [1, \infty)$, $\lambda \in [-n/p, 0)$, $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and define w by (5.5). Assume that $\sigma + \lambda < 0$. If T is bounded on $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$, then T can be extended to a bounded operator on $B_w^u(L_{p,\lambda})(\mathbb{R}^n)$. If T is bounded from $L^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$, then T can be extended to a bounded operator from $B_w^u(L_{1,\lambda})(\mathbb{R}^n)$ to $B_w^u(WL_{1,\lambda})(\mathbb{R}^n)$. The same conclusion holds for $\dot{B}_w^u(L_{p,\lambda})(\mathbb{R}^n)$.*

COROLLARY 6. *Let $\alpha \in (0, n)$, $p, q \in [1, \infty)$, $\lambda \in [-n/p, 0)$, $\mu \in [-n/q, 0)$, $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and define w by (5.5). Assume that $\lambda + \alpha = \mu$, $q \leq (\lambda/\mu)p$ and $\sigma + \mu < 0$. Then fractional integral operators I_α are bounded from $B_w^u(L_{p,\lambda})(\mathbb{R}^n)$ to $B_w^u(L_{q,\mu})(\mathbb{R}^n)$ if $p \in (1, \infty)$, and from $B_w^u(L_{1,\lambda})(\mathbb{R}^n)$ to $B_w^u(WL_{q,\mu})(\mathbb{R}^n)$ if $p = 1$. The same conclusion holds for $\dot{B}_w^u(L_{p,\lambda})(\mathbb{R}^n)$.*

Further, Theorem 6 is valid for the Calderón-Zygmund singular integral operators, fractional integral operators with rough kernel, C. Fefferman's singular multipliers, the Littlewood-Paley operator, the Marcinkiewicz operator, Ricci-Stein's oscillatory singular integral, the Bochner-Riesz operator at the critical index, and so on. For these operators, see [15, 20, 27, 44].

Taking $\lambda = -n/p$ and $\mu = -n/q$ in Theorem 6, we have the following:

COROLLARY 7. *Let $p, q \in [1, \infty)$, $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and let*

$$\tilde{w}(r) = w(r)/r, \quad w(r) = r^{-\sigma}\Theta(r^\tau), \quad \sigma, \tau \in (0, \infty) \text{ with } \sigma > \tau.$$

Let T be a sublinear operator defined on $L_{\text{comp}}^1(\mathbb{R}^n)$ and satisfy (5.1) and (5.2) for some $\alpha \in [0, n)$ and $\Omega \in L^{\tilde{p}}(S^{n-1})$ with $\tilde{p} \in [1, \infty]$. Assume one of the following conditions:

(i) $-n/q = -n/p + \alpha$, $\tilde{p} \geq p'$ and $\sigma - n/p + \alpha < 0$,

(ii) $-n/q = -n/p + \alpha$, $\tilde{p} \geq q$ and $\sigma - n/p + n/\tilde{p} + \alpha < 0$.

Assume in addition T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ or to $WL^q(\mathbb{R}^n)$. Then T can be extended to a bounded operator from $LM_{pu,\tilde{w}}(\mathbb{R}^n)$ to $LM_{qu,\tilde{w}}(\mathbb{R}^n)$ or to $WLM_{qu,\tilde{w}}(\mathbb{R}^n)$, respectively.

For the boundedness of singular and fractional integral operators on local Morrey-type spaces, see [8, 9].

5.3. Singular integral operators with the cancellation property. Let $\kappa \in (0, 1]$. In this section we consider a singular integral operator T with kernel $K(x, y)$ satisfying the following properties;

$$|K(x, y)| \leq \frac{C}{|x - y|^n} \quad \text{for } x \neq y;$$

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \leq \frac{C}{|x - y|^n} \left(\frac{|x - z|}{|x - y|}\right)^\kappa$$
$$\text{for } |x - y| \geq 2|x - z|;$$

$$\int_{r \leq |x-y| < R} K(x, y) \, dy = \int_{r \leq |x-y| < R} K(y, x) \, dy = 0$$
$$\text{for } 0 < r < R < \infty \text{ and } x \in \mathbb{R}^n,$$

where C is a positive constant independent of $x, y, z \in \mathbb{R}^n$. For $\eta > 0$, let

$$T_\eta f(x) = \int_{|x-y| \geq \eta} K(x, y) f(y) \, dy.$$

Then the integral defining $T_\eta f(x)$ is convergent whenever $f \in L^p_{\text{comp}}(\mathbb{R}^n)$ with $p \in (1, \infty)$. We assume that, for all $p \in (1, \infty)$, there exists a positive constant C_p such that for all $\eta > 0$ and $f \in L^p_{\text{comp}}(\mathbb{R}^n)$,

$$\|T_\eta f\|_p \leq C_p \|f\|_p,$$

and that

$$\lim_{\eta \to 0} T_\eta f = Tf$$

exists in $L^p(\mathbb{R}^n)$. By this assumption, the operator T can be extended to a continuous linear operator on $L^p(\mathbb{R}^n)$. We shall say the operator T satisfying the above conditions is a singular integral operator of type κ. For example, Riesz transforms R_j, $j = 1, \ldots, n$, are singular integral operators of type 1.

To define T for Campanato spaces, we first define the modified version of T_η as follows:

$$\tilde{T}_\eta f(x) = \int_{|x-y| \geq \eta} \left[K(x, y) - K(0, y)(1 - \chi_1(y))\right] f(y) \, dy.$$

Then, for $f \in \mathcal{L}_{p,\lambda}(\mathbb{R}^n)$, $p \in (1, \infty)$, $\lambda \in [-n/p, 1)$, we can show that the integral in the definition above converges absolutely for all x and that $\tilde{T}_\eta f$ converges in $L^p(Q)$ as $\eta \to 0$ for each Q (see the proof of [39, Theorem 4.1]). We denote the limit by $\tilde{T}f$.

REMARK 8. If Tf is well defined, then $\tilde{T}f$ is also well defined and $Tf - \tilde{T}f$ is a constant function. Furthermore, for the constant function 1, $T1$ is undefined, while $\tilde{T}1 = 0$. See [25, Remark 10] for details.

Let T be a singular integral operator of type $\kappa \in (0, 1]$. It is known that, under some conditions, $\tilde{T}$ can be extended to a bounded operator on Campanato spaces and on B_σ-Campanato spaces, see [39, 40] and [25, Theorem 10], respectively. Using the boundedness on B_σ-Campanato spaces and Example 13, we have the following:

THEOREM 7. *Let T be a singular integral operator of type $\kappa \in (0, 1]$. Let $p \in (1, \infty)$, $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and let*

$$w(r) = r^{-\sigma} \Theta(r^\tau), \quad \sigma, \tau \in (0, \infty) \ \text{with} \ \sigma > \tau. \tag{5.6}$$

If $-n/p + \sigma < \kappa$ and if $\lambda \in [-n/p, \kappa - \sigma)$, then $\tilde{T}$ can be extended to a bounded operator on $B_w^u(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ and $\dot{B}_w^u(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$. Moreover, if $\sigma < \kappa$ and if $\lambda \in [0, \kappa - \sigma)$, then $\tilde{T}$ can be also extended to a bounded operator on $B_w^u(\mathcal{L}_{1,\lambda})(\mathbb{R}^n)$ and $\dot{B}_w^u(\mathcal{L}_{1,\lambda})(\mathbb{R}^n)$.

Let $\lambda = 0$ in Theorem 7 we have the following.

COROLLARY 8. *Let T be a singular integral operator of type $\kappa \in (0, 1]$. Let $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and define w by (5.6). If $\sigma < \kappa$, then $\tilde{T}$ can be extended to a bounded operator on $B_w^u(\mathrm{BMO})(\mathbb{R}^n)$ and $\dot{B}_w^u(\mathrm{BMO})(\mathbb{R}^n)$.*

By Theorem 2 we have the following:

COROLLARY 9. *Let T be a singular integral operator of type $\kappa \in (0, 1]$. Let $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and define w by (5.6). If $\sigma < \sigma + \alpha < \kappa$, then $\tilde{T}$ can be extended to a bounded operator on $B_w^u(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$ and $\dot{B}_w^u(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$.*

5.4. Modified fractional integral operators. To define fractional integral operators on Campanato spaces we define the modified version of I_α, $\alpha \in (0, n)$, as follows;

$$\tilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} f(y) \left(\frac{1}{|x - y|^{n-\alpha}} - \frac{1 - \chi_1(y)}{|y|^{n-\alpha}} \right) dy.$$

If $I_\alpha f$ is well defined, then $\tilde{I}_\alpha f$ is also well defined and $I_\alpha f - \tilde{I}_\alpha f$ is a constant function. For the constant function 1, $I_\alpha 1 \equiv \infty$, while $\tilde{I}_\alpha 1$ is well defined and also a constant function, see [30, Remark 2.1] for example.

It is known that, under some conditions, $\tilde{I}_\alpha$ is bounded on B_σ-Campanato spaces, see [30, Theorem 2.6]. Using this boundedness and Example 13, we have the following:

THEOREM 8. *Let $\alpha \in (0, 1)$, $p, q \in [1, \infty)$, $\lambda \in [-n/p, 1)$, $\mu \in [-n/q, 1)$ and $\lambda + \alpha = \mu$. Let $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and let*

$$w(r) = r^{-\sigma} \Theta(r^\tau), \quad \sigma, \tau \in (0, \infty) \ \text{with} \ \sigma > \tau. \tag{5.7}$$

Assume that $\sigma + \lambda + \alpha < 1$. Assume also that p and q satisfy one of the following conditions:

    (i) $p = 1$ and $1 \leq q < n/(n-\alpha)$;

    (ii) $1 < p < n/\alpha$ and $1 \leq q \leq pn/(n-p\alpha)$;

    (iii) $n/\alpha \leq p < \infty$ and $1 \leq q < \infty$ (*in this case*, $0 \leq \mu < 1$).

Then $\tilde{I}_\alpha$ is bounded from $B_w^u(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ to $B_w^u(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$ and from $\dot{B}_w^u(\mathcal{L}_{p,\lambda})(\mathbb{R}^n)$ to $\dot{B}_w^u(\mathcal{L}_{q,\mu})(\mathbb{R}^n)$.

If $\lambda = 0$, then $\mathcal{L}_{p,\lambda} = \mathrm{BMO}$. If $0 < \lambda < 1$, then $\mathcal{L}_{p,\lambda} = \mathrm{Lip}_\lambda$. Therefore, we have the following:

COROLLARY 10. *Let $\alpha, \beta, \gamma \in (0, 1)$ and $\alpha + \beta = \gamma$. Let $u \in (0, \infty]$, $\Theta \in \boldsymbol{\Theta}_*$, and define w by (5.7) with $\alpha + \beta + \sigma < 1$. Then $\tilde{I}_\alpha$ is bounded from $B_w^u(\mathrm{BMO})(\mathbb{R}^n)$ to $B_w^u(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$, from $B_w^u(\mathrm{Lip}_\beta)(\mathbb{R}^n)$ to $B_w^u(\mathrm{Lip}_\gamma)(\mathbb{R}^n)$, from $\dot{B}_w^u(\mathrm{BMO})(\mathbb{R}^n)$ to $\dot{B}_w^u(\mathrm{Lip}_\alpha)(\mathbb{R}^n)$ and from $\dot{B}_w^u(\mathrm{Lip}_\beta)(\mathbb{R}^n)$ to $\dot{B}_w^u(\mathrm{Lip}_\gamma)(\mathbb{R}^n)$.*

5.5. Vector-valued boundedness. In this section we state the vector-valued inequalities for $B_w^u(L_{p,\lambda})(\mathbb{R}^n)$ and $\dot{B}_w^u(L_{p,\lambda})(\mathbb{R}^n)$.

DEFINITION 2. Let $U = \mathbb{R}^n$ or Q_r with $r > 0$. Let $p \in [1, \infty)$, $\lambda \in \mathbb{R}$ and $v \in (0, \infty]$. For

$$E = L^p, \quad WL^p, \quad L_{p,\lambda} \text{ or } WL_{p,\lambda},$$

let $E(\ell^v)(U)$ be the sets of all sequences of functions $\{f_j\}_{j=1}^\infty$ such that the following functional is finite:

$$\left\|\{f_j\}_{j=1}^\infty\right\|_{E(\ell^v)(U)} = \left\|\left(\sum_{j=1}^\infty |f_j|^v\right)^{1/v}\right\|_{E(U)},$$

where we use the obvious modification when $v = \infty$.

Then $\left\{(E(\ell^v)(Q_r), \|\cdot\|_{E(\ell^v)(Q_r)})\right\}_{0 < r < \infty}$ has the restriction and decomposition properties for $E = L^p$, WL^p, $L_{p,\lambda}$ or $WL_{p,\lambda}$, since

$$\left(\sum_{j=1}^\infty \left|\left. f_j\right|_{Q_r}\right|^v\right)^{1/v} = \left(\sum_{j=1}^\infty |f_j|^v\right)^{1/v}\bigg|_{Q_r} \quad \text{and}$$

$$\left(\sum_{j=1}^\infty |f_j \chi_r|^v\right)^{1/v} = \left(\sum_{j=1}^\infty |f_j|^v\right)^{1/v} \chi_r.$$

DEFINITION 3. Let $p \in [1, \infty)$, $\lambda \in \mathbb{R}$, $u, v \in (0, \infty]$ and $w \in \mathcal{W}^u$. For

$$E = L^p, \quad WL^p, \quad L_{p,\lambda} \text{ or } WL_{p,\lambda},$$

let $B_w^u(E(\ell^v))(\mathbb{R}^n)$ and $\dot{B}_w^u(E(\ell^v))(\mathbb{R}^n)$ be the sets of all sequences $\{f_j\}_{j=1}^\infty$, $f_j \in E_Q(\mathbb{R}^n)$, such that $\|\{f_j\}_{j=1}^\infty\|_{B_w^u(E(\ell^v))} < \infty$ and $\|\{f_j\}_{j=1}^\infty\|_{\dot{B}_w^u(E(\ell^v))} < \infty$, respectively, where

$$\left\|\{f_j\}_{j=1}^\infty\right\|_{B_w^u(E(\ell^v))} = \left\|w(r)\left\|\{f_j\}_{j=1}^\infty\right\|_{E(\ell^v)(Q_r)}\right\|_{L^u([1,\infty),dr/r)},$$

$$\left\|\{f_j\}_{j=1}^\infty\right\|_{\dot{B}_w^u(E(\ell^v))} = \left\|w(r)\left\|\{f_j\}_{j=1}^\infty\right\|_{E(\ell^v)(Q_r)}\right\|_{L^u((0,\infty),dr/r)}.$$

We consider sublinear operators T as in Subsection 5.2 on vector-valued function spaces, that is,

$$T : \{f_j\}_{j=1}^\infty \mapsto \{Tf_j\}_{j=1}^\infty.$$

Then the vector-valued boundedness of T on B_σ-Morrey spaces is known, see [25, Theorem 14]. Using this boundedness and Example 13, we have the following:

THEOREM 9. *Let*

$$\begin{cases} p, q \in [1, \infty), \ \lambda \in [-n/p, 0), \ \mu \in [-n/q, 0), v \in (1, \infty], u \in (0, \infty], \\ w(r) = r^{-\sigma}\Theta(r^\tau), \ \Theta \in \boldsymbol{\Theta}_*, \ and \ \sigma, \tau \in (0, \infty) \ with \ \sigma > \tau. \end{cases} \quad (5.8)$$

Let T be a sublinear operator defined on $L_{\mathrm{comp}}^1(\mathbb{R}^n)$ and satisfy (5.1) and (5.2) for some $\alpha \in [0, n)$ and $\Omega \in L^{\tilde{p}}(S^{n-1})$ with $\tilde{p} \in [1, \infty]$. Assume one of the following conditions:

  (i) $\mu = \lambda + \alpha$, $\tilde{p} \geq p'$ and $\sigma + \lambda + \alpha < 0$,
  (ii) $\mu = \lambda + \alpha$, $\tilde{p} \geq q$ and $\sigma + \lambda + n/\tilde{p} + \alpha < 0$.

If T can be extended to a bounded operator from $L_{p,\lambda}(\ell^v)(\mathbb{R}^n)$ to $L_{q,\mu}(\ell^v)(\mathbb{R}^n)$ or to $WL_{q,\mu}(\ell^v)(\mathbb{R}^n)$, then T can be further extended to a bounded operator from $B_w^u(L_{p,\lambda}(\ell^v))(\mathbb{R}^n)$ to $B_w^u(L_{q,\mu}(\ell^v))(\mathbb{R}^n)$ or to $B_w^u(WL_{q,\mu}(\ell^v))(\mathbb{R}^n)$, respectively. That is,

$$\left\|\left(\sum_{j=1}^\infty |Tf_j|^v\right)^{1/v}\right\|_{B_w^u(L_{q,\mu})} \leq C\left\|\left(\sum_{j=1}^\infty |f_j|^v\right)^{1/v}\right\|_{B_w^u(L_{p,\lambda})}, \quad if \ p \in (1, \infty),$$

and

$$\left\|\left(\sum_{j=1}^\infty |Tf_j|^v\right)^{1/v}\right\|_{B_w^u(WL_{q,\mu})} \leq C\left\|\left(\sum_{j=1}^\infty |f_j|^v\right)^{1/v}\right\|_{B_w^u(L_{1,\lambda})}, \quad if \ p = 1,$$

where we use the obvious modification when $v = \infty$. The same conclusion holds for $\dot{B}_w^u(L_{p,\lambda}(\ell^v))(\mathbb{R}^n)$.

COROLLARY 11. *Let $p, \lambda, u, v, \Theta, \sigma, \tau$ and w be as in (5.8). Assume that $\sigma + \lambda < 0$. If a singular integral operator T is bounded on $L^p(\ell^v)(\mathbb{R}^n)$ with $p \in (1, \infty)$, then T can be extended to a bounded operator on $B_w^u(L_{p,\lambda}(\ell^v))(\mathbb{R}^n)$. If T is bounded from $L^1(\ell^v)(\mathbb{R}^n)$*

to $WL^1(\ell^v)(\mathbb{R}^n)$, then T can be extended to a bounded operator from $B_w^u(L_{1,\lambda}(\ell^v))(\mathbb{R}^n)$ to $B_w^u(WL_{1,\lambda}(\ell^v))(\mathbb{R}^n)$. The same conclusion holds for $\dot{B}_w^u(L_{p,\lambda}(\ell^v))(\mathbb{R}^n)$.

COROLLARY 12. *Let $\alpha \in (0, n)$, and let $p, q, \lambda, \mu, u, v, \Theta, \sigma, \tau$ and w be as in (5.8). Assume that $\mu = \lambda + \alpha$, $q \leq (\lambda/\mu)p$ and $\sigma + \lambda + \alpha < 0$. Then fractional integral operators I_α are bounded from $B_w^u(L_{p,\lambda}(\ell^v))(\mathbb{R}^n)$ to $B_w^u(L_{q,\mu}(\ell^v))(\mathbb{R}^n)$ if $p \in (1, \infty)$, and from $B_w^u(L_{1,\lambda}(\ell^v))(\mathbb{R}^n)$ to $B_w^u(WL_{q,\mu}(\ell^v))(\mathbb{R}^n)$ if $p = 1$. The same conclusion holds for $\dot{B}_w^u(L_{p,\lambda}(\ell^v))(\mathbb{R}^n)$.*

On fractional maximal operators M_α, $\alpha \in [0, n)$, using Theorem 9 and [25, Theorem 15], we have the following:

COROLLARY 13. *Let $\alpha \in [0, n)$, and let $p, q, \lambda, \mu, u, \Theta, \sigma, \tau$ and w be as in (5.8). Assume that $\mu = \lambda + \alpha$ and $q \leq (\lambda/\mu)p$. Assume also one of the following conditions.*

(i) *$\sigma + \lambda + \alpha < 0$ and $v \in (1, \infty]$,*
(ii) *$\sigma + \lambda + \alpha = 0$ and $v = \infty$.*

Then the operator M_α can be extended to a bounded operator from $B_w^u(L_{p,\lambda}(\ell^v))(\mathbb{R}^n)$ to $B_w^u(L_{q,\mu}(\ell^v))(\mathbb{R}^n)$ if $p \in (1, \infty)$, and from $B_w^u(L_{1,\lambda}(\ell^v))(\mathbb{R}^n)$ to $B_w^u(WL_{q,\mu}(\ell^v))(\mathbb{R}^n)$ if $p = 1$. The same conclusion holds for $\dot{B}_w^u(L_{p,\lambda}(\ell^v))(\mathbb{R}^n)$.

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