TOKYO J. MATH.
VOL. 39, NO. 1, 2016

Higher-order Convolution Identities for Cauchy Numbers

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(Communicated by H. Aoki)

Abstract. Euler's famous formula written in symbolic notation as $(B_0+B_0)^n=-nB_{n-1}-(n-1)B_n$ was extended to $(B_{l_1}+\cdots+B_{l_m})^n$ for $m\ge 2$ and arbitrary fixed integers $l_1,\dots,l_m\ge 0$. In this paper, we consider the higher-order recurrences for Cauchy numbers $(c_{l_1}+\cdots+c_{l_m})^n$, where the n-th Cauchy number c_n $(n\ge 0)$ is defined by the generating function $x/\ln(1+x)=\sum_{n=0}^\infty c_n x^n/n!$. In special, we give an explicit expression in the case $l_1=\cdots=l_m=0$ for any integers $n\ge 1$ and $m\ge 2$. We also discuss the case for Cauchy numbers of the second kind $\widehat{c}_n$ in similar ways.

1. Introduction

Bernoulli numbers B_n, defined by the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^\infty B_n\frac{x^n}{n!}\quad(|x|<2\pi)\,,$$

have been extensively studied by many authors (e.g. see [1, 2, 5] and including Agoh and Dilcher (citeAD1, AD2, D and references there). Use the classical umbral calculus to write

$$(B_l+B_m)^n=\sum_{j=0}^n\binom{n}{j}B_{l+j}B_{m+n-j}\,.$$

Agoh and Dilcher ([2]) extended the Euler's famous formula

$$(B_0+B_0)^n=-nB_{n-1}-(n-1)B_n\quad(n\ge 1)\,,\tag{1}$$

to the higher order recurrences for Bernoulli numbers

$$(B_{l_1}+\cdots+B_{l_m})^n=\sum_{\substack{k_1+\cdots+k_m=n\\k_1,\dots,k_m\ge 0}}\frac{n!}{k_1!\cdots k_m!}B_{k_1+l_1}\cdots B_{k_m+l_m}\quad(l_1,\dots,l_m\ge 0)\,.$$

Received March 17, 2015; revised September 13, 2015
Mathematics Subject Classification: 05A15, 05A40, 11B37, 11B75
Key words and phrases: Cauchy numbers, Bernoulli numbers, higher order recurrences, convolution identities

It is not easy to express this sum explicitly, but some special cases for $m = 3$ can be derived. For example,

$$(B_0 + B_0 + B_0)^n = \frac{(n-1)(n-2)}{2} B_n + \frac{3n(n-2)}{2} B_{n-1} + n(n-1) B_{n-2}\,,$$

$$(B_0 + B_0 + B_1)^n = \frac{n(n-1)}{6} B_{n+1} + \frac{(n-1)(n+1)}{2} B_n + \frac{n(n+1)}{3} B_{n-1}\,,$$

$$(B_0 + B_1 + B_1)^n = \frac{n(n+3)}{24} B_{n+2} + \frac{n(n+8)}{12} B_{n+1} - \frac{n^2 - 19n - 6}{24} B_n - \frac{n(n-2)}{12} B_{n-1}\,,$$

$$(B_0 + B_0 + B_2)^n = \frac{n(n-1)}{12} B_{n+2} + \frac{n(n-1)}{3} B_{n+1} + \frac{(5n-2)(n-1)}{12} B_n + \frac{n(n-2)}{6} B_{n-1}\,.$$

The Cauchy numbers (of the first kind) c_n ($n \ge 0$) are defined by

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)dx$$

and the generating function of c_n is given by

$$c(x) = \frac{x}{\ln(1+x)} = \sum_{n=0}^\infty c_n \frac{x^n}{n!} \quad (|x| < 1)$$

([4, 10]). The Cauchy numbers are special cases of poly-Cauchy numbers $c_n^{(k)}$ defined in [6], where $c_n = c_n^{(1)}$. Several initial values are

$$c_0 = 1\,,\ \ c_1 = \frac{1}{2}\,,\ \ c_2 = -\frac{1}{6}\,,\ \ c_3 = \frac{1}{4}\,,\ \ c_4 = -\frac{19}{30}\,,\ \ c_5 = \frac{9}{4}\,,\ \ c_6 = -\frac{863}{84}\,,\ \ c_7 = \frac{1375}{24}\,.$$

There are many similarities between Bernoulli numbers and Cauchy numbers. In fact, the numbers $b_n = c_n/n!$ are sometimes called Bernoulli numbers of the second kind (see e.g. [3]). Bernoulli numbers can be expressed explicitly in terms of the Stirling numbers of the second kind $\left\{ {n \atop m} \right\}$ as

$$\sum_{m=0}^n \left\{ {n \atop m} \right\} \frac{(-1)^m m!}{m+1}\,.$$

On the contrary, Cauchy numbers can be expressed explicitly in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m} \right]$ as

$$\sum_{m=0}^n \left[{n \atop m} \right] \frac{(-1)^{n-m}}{m+1}\,.$$

As an analogue of the Euler's formula 1, Zhao ([12]) gave the corresponding formula for the Cauchy numbers c_n by

$$(c_0 + c_0)^n = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \geq 0)\,. \tag{2}$$

In [7], an explicit expression of $(c_l + c_m)^n$ for $l, m, n \geq 0$ was determined, where with the classical umbral calculus notation (see, e.g., [11]), $(c_l + c_m)^n$ is defined by

$$(c_l + c_m)^n = \sum_{j=0}^{n} \binom{n}{j} c_{l+j} c_{m+n-j}\,.$$

As some special cases, we gave explicit formulae:

$$(c_0 + c_1)^n = -\frac{1}{2}(n+1)(n-1)c_n - \frac{1}{2}nc_{n+1}\,, \tag{3}$$

$$(c_0 + c_2)^n = \frac{n!}{6}\sum_{k=0}^{n}\frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6}n(2n+1)c_{n+1} - \frac{1}{3}nc_{n+2}\,, \tag{4}$$

$$(c_1 + c_1)^n = -\frac{n!}{6}\sum_{k=0}^{n}\frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6}n(n+5)c_{n+1} - \frac{1}{6}(n+3)c_{n+2}\,. \tag{5}$$

In this paper, we consider the higher order recurrences for Cauchy numbers

$$(c_{l_1} + \cdots + c_{l_m})^n = \sum_{\substack{k_1+\cdots+k_m=n \\ k_1,\ldots,k_m \geq 0}} \frac{n!}{k_1! \cdots k_m!} c_{k_1+l_1} \cdots c_{k_m+l_m} \quad (l_1, \ldots, l_m \geq 0)$$

and give an explicit expression of $(\underbrace{c_0 + \cdots + c_0}_{m})^n$ for any integers $n \geq 1$ and $m \geq 2$.

2. Main result

Let $g(x) = 1/\ln(1+x)$. Then we have the following formulae.

LEMMA 1. *For $n \geq 1$, we have*

$$g^{(n)}(x) = \frac{(-1)^n}{(1+x)^n}\sum_{k=1}^{n} k! \begin{bmatrix} n \\ k \end{bmatrix} g(x)^{k+1}\,, \tag{6}$$

$$g(x)^n = \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} (1+x)^k g^{(k)}(x)\,, \tag{7}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ denote the (unsigned) Stirling numbers of the first kind and the Stirling numbers of the second kind, respectively.

PROOF. The formulae (6) and (7) are equivalent. It is easy to see that both formulae are valid for $n = 1$. Let $n \geq 2$. If the formula (6) is true, then

$$\begin{aligned}
&\frac{(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}\left\{ {n-1 \atop k} \right\}(1+x)^k g^{(k)}(x)\\
&= \frac{(-1)^{n-1}}{(n-1)!}\sum_{k=1}^{n-1}\left\{ {n-1 \atop k} \right\}(1+x)^k \frac{(-1)^k}{(1+x)^k}\sum_{i=1}^{k} i!\left[{k \atop i} \right] g(x)^{i+1}\\
&= \frac{(-1)^{n-1}}{(n-1)!}\sum_{i=1}^{n-1} i!g(x)^{i+1}\sum_{k=i}^{n-1}(-1)^k\left\{ {n-1 \atop k} \right\}\left[{k \atop i} \right]\\
&= \frac{(-1)^{n-1}}{(n-1)!}\sum_{i=1}^{n-1} i!g(x)^{i+1}(-1)^{n-1}\delta_{n-1,i}\\
&= \frac{(-1)^{n-1}}{(n-1)!}(n-1)!g(x)^n(-1)^{n-1} = g(x)^n\,.
\end{aligned}$$

On the other hand, the formula (6) can be proven by induction and by using the recurrence relation

$$\left[{n+1 \atop k} \right] = n\left[{n \atop k} \right] + \left[{n \atop k-1} \right]$$

with

$$\left[{0 \atop 0} \right] = 1, \quad \left[{n \atop 0} \right] = \left[{0 \atop n} \right] = 0 \quad (n \geq 1)\,.$$

□

Since

$$\begin{aligned}
g^{(k)}(x) &= \frac{d^k}{dx^k}\left(\frac{c(x)}{x}\right)\\
&= \sum_{l=0}^{k}\binom{k}{l}\frac{(-1)^{k-l}(k-l)!}{x^{k-l+1}}c^{(l)}(x)\,,
\end{aligned}$$

by Lemma 1 (7), we obtain for $n \geq 1$

$$\begin{aligned}
c(x)^n &= x^n g(x)^n\\
&= \frac{x^n(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}\left\{ {n-1 \atop k} \right\}(1+x)^k g^{(k)}(x)\\
&= \frac{x^n(-1)^{n-1}}{(n-1)!}\sum_{k=0}^{n-1}\left\{ {n-1 \atop k} \right\}(1+x)^k \sum_{l=0}^{k}\binom{k}{l}\frac{(-1)^{k-l}(k-l)!}{x^{k-l+1}}c^{(l)}(x)
\end{aligned}$$

$$= \frac{(-1)^{n-1}x^n}{(n-1)!}\sum_{l=0}^{n-1}\left(\sum_{k=l}^{n-1}\left\{ {n-1 \atop k} \right\}\frac{(-1)^{k-l}k!(1+x)^k}{x^{k-l+1}}\right)\frac{c^{(l)}(x)}{l!}\,.$$

Therefore, we can obtain the following.

THEOREM 1. *For $m \ge 0$, we have*

$$c(x)^{m+1} = \frac{(-1)^m}{m!}\sum_{l=0}^{m}\left(\sum_{k=l}^{m}(-1)^{k-l}k!\left\{ {m \atop k} \right\}x^{m-k+l}(1+x)^k\right)\frac{c^{(l)}(x)}{l!}\,. \tag{8}$$

Since for $\nu, l \ge 0$ it holds that

$$x^\nu c^{(l)}(x) = \sum_{n=0}^{\infty}\frac{n!}{(n-\nu)!}c_{n+l-\nu}\frac{x^n}{n!}\,, \tag{9}$$

we have

$$\begin{aligned}
&c(x)^{m+1}\\
&= \frac{(-1)^m}{m!}\sum_{l=0}^{m}\left(\sum_{k=l}^{m}(-1)^{k-l}k!\left\{ {m \atop k} \right\}x^{m-k+l}\sum_{i=0}^{k}\binom{k}{i}x^i\right)\frac{c^{(l)}(x)}{l!}\\
&= \frac{(-1)^m}{m!}\sum_{l=0}^{m}\frac{1}{l!}\sum_{k=l}^{m}(-1)^{k-l}k!\left\{ {m \atop k} \right\}\sum_{i=0}^{k}\binom{k}{i}\sum_{n=0}^{\infty}\frac{n!}{(n-m+k-l-i)!}c_{n-m+k-i}\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^m n!}{m!}\sum_{l=0}^{m}\frac{1}{l!}\sum_{k=0}^{m-l}(-1)^k(l+k)!\left\{ {m \atop l+k} \right\}\sum_{i=0}^{l+k}\binom{l+k}{i}\frac{c_{n-m+l+k-i}}{(n-m+k-i)!}\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^m n!}{m!}\sum_{l=0}^{m}\frac{1}{l!}\\
&\quad\times\sum_{k=0}^{m-l}(-1)^{m-l-k}(m-k)!\left\{ {m \atop m-k} \right\}\sum_{i=0}^{m-k}\binom{m-k}{i}\frac{c_{n-k-i}}{(n-l-k-i)!}\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{n!}{m!}\sum_{l=0}^{m}\frac{1}{l!}\sum_{k=0}^{m-l}(-1)^{l+k}(m-k)!\left\{ {m \atop m-k} \right\}\sum_{i=k}^{m}\binom{m-k}{i-k}\frac{c_{n-i}}{(n-l-i)!}\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{n!}{m!}\sum_{i=0}^{m}\left(\sum_{l=0}^{m}\sum_{k=0}^{m-l}\frac{(-1)^{l+k}(m-k)!}{l!(n-l-i)!}\left\{ {m \atop m-k} \right\}\binom{m-k}{i-k}\right)c_{n-i}\frac{x^n}{n!}\,,
\end{aligned} \tag{10}$$

where $\binom{n}{-\nu} = 0$ $(\nu > 0)$. On the other hand, we get

$$c(x)^{m+1} = \sum_{n=0}^{\infty} (\underbrace{c_0 + \cdots + c_0}_{m+1})^n \frac{x^n}{n!} .$$

Comparing the coefficients on the both sides, we get the main theorem.

THEOREM 2. *For any integers $n \ge 1$ and $m \ge 2$, we have*

$$(\underbrace{c_0 + \cdots + c_0}_{m})^n = \frac{n!}{(m-1)!} \times \sum_{i=0}^{m-1} \left(\sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!} \left\{ {m-1 \atop m-k-1} \right\} \binom{m-k-1}{i-k} \right) c_{n-i} .$$

3. Examples

If we put $m = 2$ in Theorem 2, we have (2). If we put $m = 3$, $m = 4$, $m = 5$ and $m = 6$ in Theorem 2, we have

$$(c_0 + c_0 + c_0)^n = \frac{(n-1)(n-2)}{2} c_n + \frac{n(n-2)(2n-5)}{2} c_{n-1} + \frac{n(n-1)(n-3)^2}{2} c_{n-2} ,$$

$$\begin{aligned}(c_0 + c_0 + c_0 + c_0)^n &= -\frac{(n-1)(n-2)(n-3)}{6} c_n - \frac{n(n-2)(n-3)^2}{2} c_{n-1} \\ &\quad - \frac{n(n-1)(n-3)(3n^2-21n+37)}{6} c_{n-2} - \frac{n(n-1)(n-2)(n-4)^3}{6} c_{n-3} ,\end{aligned}$$

$$\begin{aligned}(c_0 + c_0 + c_0 + c_0 + c_0)^n &= \frac{(n-1)(n-2)(n-3)(n-4)}{24} c_n + \frac{n(n-2)(n-3)(n-4)(2n-7)}{12} c_{n-1} \\ &\quad + \frac{n(n-1)(n-3)(n-4)(6n^2-48n+97)}{24} c_{n-2} \\ &\quad + \frac{n(n-1)(n-2)(n-4)(2n-9)(2n^2-18n+41)}{24} c_{n-3} \\ &\quad + \frac{n(n-1)(n-2)(n-3)(n-5)^4}{24} c_{n-4} ,\end{aligned}$$

and

$$
\begin{aligned}
&(c_0+c_0+c_0+c_0+c_0+c_0)^n\\
&\quad=-\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{5!}c_n-\frac{n(n-2)(n-3)(n-4)^2(n-5)}{4!}c_{n-1}\\
&\qquad-\frac{n(n-1)(n-3)(n-4)(n-5)(2n^2-18n+41)}{4!}c_{n-2}\\
&\qquad-\frac{n(n-1)(n-2)(n-4)(n-5)^2(2n^2-20n+51)}{4!}c_{n-3}\\
&\qquad-\frac{n(n-1)(n-2)(n-3)(n-5)(5n^4-110n^3+910n^2-3355n+4651)}{5!}c_{n-4}\\
&\qquad-\frac{n(n-1)(n-2)(n-3)(n-4)(n-6)^5}{5!}c_{n-5}\,,
\end{aligned}
$$

respectively.

4. Applications

By applying Theorem 2, we can obtain some combinations of other expressions of $(c_{l_1}+\cdots+c_{l_m})^n$.

By differentiating both sides of (10) μ times with respect to x, the right-hand side becomes equal to

$$
\sum_{n=0}^\infty\frac{(n+\mu)!}{m!}\sum_{i=0}^m\left(\sum_{l=0}^m\sum_{k=0}^{m-l}\frac{(-1)^{l+k}(m-k)!}{l!(n+\mu-l-i)!}\left\{{m \atop m-k}\right\}\binom{m-k}{i-k}\right)c_{n+\mu-i}\frac{x^n}{n!}\,.
$$

Due to the General Leibniz's rule, the left-hand side becomes equal to

$$
\sum_{n=0}^\infty\sum_{\substack{\kappa_1+\cdots+\kappa_{m+1}=\mu\\ \kappa_1,\dots,\kappa_{m+1}\ge 0}}\frac{\mu!}{\kappa_1!\cdots\kappa_{m+1}!}(c_{\kappa_1}+\cdots+c_{\kappa_{m+1}})^n\frac{x^n}{n!}\,.
$$

Therefore, we have the following.

THEOREM 3. *For any integers $n\ge 1$, $\mu\ge 0$ and $m\ge 2$, we have*

$$
\begin{aligned}
&\sum_{\substack{\kappa_1+\cdots+\kappa_m=\mu\\ \kappa_1,\dots,\kappa_m\ge 0}}\frac{\mu!}{\kappa_1!\cdots\kappa_m!}(c_{\kappa_1}+\cdots+c_{\kappa_m})^n\\
&\quad=\frac{(n+\mu)!}{(m-1)!}\\
&\qquad\times\sum_{i=0}^{m-1}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}\frac{(-1)^{l+k}(m-k-1)!}{l!(n+\mu-l-i)!}\left\{{m-1 \atop m-k-1}\right\}\binom{m-k-1}{i-k}\right)c_{n+\mu-i}\,.
\end{aligned}
$$

When $\mu = 1$ in Theorem 3, we have

$$m(\underbrace{c_0 + \cdots + c_0}_{m-1} + c_1)^n$$
$$= \frac{(n+1)!}{(m-1)!}$$
$$\times \sum_{i=0}^{m-1} \left(\sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i+1)!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{m-k-1}{i-k} \right) c_{n-i+1} .$$

Hence, we have

COROLLARY 1. *For any integers $n \geq 1$ and $m \geq 2$,*

$$(\underbrace{c_0 + \cdots + c_0}_{m-1} + c_1)^n$$
$$= \frac{(n+1)!}{m!}$$
$$\times \sum_{i=0}^{m-1} \left(\sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i+1)!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{m-k-1}{i-k} \right) c_{n-i+1} .$$

When $m = 2$, $m = 3$ and $m = 4$ in Corollary 1, we have

$$(c_0 + c_1)^n = -\frac{1}{2} n c_{n+1} - \frac{1}{2}(n+1)(n-1)c_n ,$$
$$(c_0 + c_0 + c_1)^n$$
$$= \frac{n(n-1)}{6} c_{n+1} + \frac{(n+1)(n-1)(2n-3)}{6} c_n + \frac{n(n+1)(n-2)^2}{6} c_{n-1}$$

and

$$(c_0 + c_0 + c_0 + c_1)^n$$
$$= -\frac{n(n-1)(n-2)}{24} c_{n+1} - \frac{(n+1)(n-1)(n-2)^2}{8} c_n$$
$$- \frac{(n+1)n(n-2)(3n^2 - 15n + 19)}{24} c_{n-1} - \frac{(n+1)n(n-1)(n-3)^3}{24} c_{n-2} ,$$

respectively.

When $\mu = 2$ in Theorem 3, we have

$$m(\underbrace{c_0 + \cdots + c_0}_{m-1} + c_2)^n + 2\binom{m}{2}(\underbrace{c_0 + \cdots + c_0}_{m-2} + c_1 + c_1)^n$$

$$
=\frac{(n+2)!}{(m-1)!}
\times\sum_{i=0}^{m-1}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}\frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i+2)!}\left\{ m-1 \atop m-k-1 \right\}\binom{m-k-1}{i-k}\right)c_{n-i+2}\,.
$$

Hence, we have

COROLLARY 2. *For any integers $n\geq 1$ and $m\geq 2$,*

$$
(\underbrace{c_0+\cdots+c_0}_{m-1}+c_2)^n+(m-1)(\underbrace{c_0+\cdots+c_0}_{m-2}+c_1+c_1)^n
$$
$$
=\frac{(n+2)!}{m!}
\times\sum_{i=0}^{m-1}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}\frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i+2)!}\left\{ m-1 \atop m-k-1 \right\}\binom{m-k-1}{i-k}\right)c_{n-i+2}\,.
$$

When $m=2$, $m=3$ and $m=4$ in Corollary 2, we have

$$
(c_0+c_2)^n+(c_1+c_1)^n=-\frac{n+1}{2}c_{n+2}-\frac{n(n+2)}{2}c_{n+1}\,,
$$
$$
(c_0+c_0+c_2)^n+2(c_0+c_1+c_1)^n
=\frac{n(n+1)}{6}c_{n+2}+\frac{n(n+2)(2n-1)}{6}c_{n+1}+\frac{(n+1)(n+2)(n-1)^2}{6}c_n
$$

and

$$
(c_0+c_0+c_0+c_2)^n+3(c_0+c_0+c_1+c_1)^n
=-\frac{(n+1)n(n-1)}{24}c_{n+2}-\frac{(n+2)n(n-1)^2}{8}c_{n+1}
-\frac{(n+2)(n+1)(n-1)(3n^2-9n+7)}{24}c_n-\frac{(n+2)(n+1)n(n-2)^3}{24}c_{n-1}\,,
$$

respectively.

When $\mu=3$ in Theorem 3, we have

$$
m(\underbrace{c_0+\cdots+c_0}_{m-1}+c_3)^n+6\binom{m}{2}(\underbrace{c_0+\cdots+c_0}_{m-2}+c_1+c_2)^n
+3!\binom{m}{3}(\underbrace{c_0+\cdots+c_0}_{m-3}+c_1+c_1+c_1)^n
=\frac{(n+3)!}{(m-1)!}
$$

$$\times \sum_{i=0}^{m-1}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}\frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i+3)!}\left\{ {m-1 \atop m-k-1} \right\}\binom{m-k-1}{i-k}\right)c_{n-i+3}\,.$$

Hence, we have

COROLLARY 3. *For any integers $n \ge 1$ and $m \ge 3$,*

$$(\underbrace{c_0+\cdots+c_0}_{m-1}+c_3)^n+3(m-1)(\underbrace{c_0+\cdots+c_0}_{m-2}+c_1+c_2)^n$$
$$+(m-1)(m-2)(\underbrace{c_0+\cdots+c_0}_{m-3}+c_1+c_1+c_1)^n$$
$$=\frac{(n+3)!}{m!}$$
$$\times \sum_{i=0}^{m-1}\left(\sum_{l=0}^{m-1}\sum_{k=0}^{m-l-1}\frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i+3)!}\left\{ {m-1 \atop m-k-1} \right\}\binom{m-k-1}{i-k}\right)c_{n-i+3}\,.$$

When $m=3$ and $m=4$ in Corollary 3, we have

$$(c_0+c_0+c_3)^n+6(c_0+c_1+c_2)^n+2(c_1+c_1+c_1)^n$$
$$=\frac{(n+1)(n+2)}{6}c_{n+3}+\frac{(n+1)(n+3)(2n+1)}{6}c_{n+2}+\frac{n^2(n+2)(n+3)}{6}c_{n+1}$$

and

$$(c_0+c_0+c_0+c_3)^n+9(c_0+c_0+c_1+c_2)^n+6(c_0+c_1+c_1+c_1)^n$$
$$=-\frac{(n+2)(n+1)n}{24}c_{n+3}-\frac{(n+3)(n+1)n^2}{8}c_{n+2}$$
$$-\frac{(n+3)(n+2)n(3n^2-3n+1)}{24}c_{n+1}-\frac{(n+3)(n+2)(n+1)(n-1)^3}{24}c_n\,,$$

respectively.

5. Convolution identities for Cauchy numbers of the second kind

Cauchy numbers of the second kind $\widehat{c}_n$ may be defined by the generating function:

$$\widehat{c}(x)=\frac{x}{1+x}g(x)=\frac{x}{(1+x)\ln(1+x)}=\sum_{n=0}^{\infty}\widehat{c}_n\frac{x^n}{n!}\,. \tag{11}$$

([4, 10]). Several initial values are

$$\widehat{c}_0=1\,,\ \widehat{c}_1=-\frac{1}{2}\,,\ \widehat{c}_2=\frac{5}{6}\,,\ \widehat{c}_3=-\frac{9}{4}\,,\ \widehat{c}_4=\frac{251}{30}\,,\ \widehat{c}_5=-\frac{475}{12}\,,\ \widehat{c}_6=\frac{19087}{84}\,.$$

In [8], an explicit expression of $(\widehat{c}_l+\widehat{c}_m)^n$ for $l,m,n\ge 0$ was determined, where with the classical umbral calculus notation (see, e.g., [11]), $(\widehat{c}_l+\widehat{c}_m)^n$ is defined by

$$(\widehat{c}_l+\widehat{c}_m)^n=\sum_{j=0}^n\binom{n}{j}\widehat{c}_{l+j}\widehat{c}_{m+n-j}\,.$$

Since

$$\begin{aligned}g^{(k)}(x)&=\frac{d^k}{dx^k}\left(1+\frac{1}{x}\right)\widehat{c}(x)\\&=\widehat{c}^{(k)}(x)+\sum_{l=0}^k\binom{k}{l}\frac{(-1)^{k-l}(k-l)!}{x^{k-l+1}}\widehat{c}^{(l)}(x)\,,\end{aligned}$$

by Lemma 1 (7), we obtain for $n\ge 1$

$$\begin{aligned}\widehat{c}(x)^n&=\frac{x^n}{(1+x)^n}g(x)^n\\&=\frac{x^n(-1)^{n-1}}{(1+x)^n(n-1)!}\sum_{k=0}^{n-1}\left\{{n-1\atop k}\right\}(1+x)^kg^{(k)}(x)\\&=\frac{(-1)^{n-1}x^n}{(n-1)!(1+x)^n}\sum_{k=0}^{n-1}\left\{{n-1\atop k}\right\}(1+x)^k\sum_{l=0}^k\binom{k}{l}\frac{(-1)^{k-l}(k-l)!}{x^{k-l+1}}\widehat{c}^{(l)}(x)\\&\quad+\frac{(-1)^{n-1}x^n}{(n-1)!(1+x)^n}\sum_{k=0}^{n-1}\left\{{n-1\atop k}\right\}(1+x)^k\widehat{c}^{(k)}(x)\,.\end{aligned}$$

Therefore, we obtain the following.

THEOREM 4. *For $m\ge 0$, we have*

$$\begin{aligned}\widehat{c}(x)^{m+1}=&\frac{(-1)^m}{m!}\sum_{l=0}^m\left(\sum_{k=l}^m(-1)^{k-l}k!\left\{{m\atop k}\right\}\frac{x^{m-k+l}}{(1+x)^{m-k+1}}\right)\frac{\widehat{c}^{(l)}(x)}{l!}\\&+\frac{(-1)^m}{m!}\sum_{k=0}^m\left\{{m\atop k}\right\}\frac{x^{m+1}}{(1+x)^{m-k+1}}\widehat{c}^{(k)}(x)\,.\end{aligned}\tag{12}$$

Now,

$$\begin{aligned}&\frac{(-1)^m}{m!}\sum_{l=0}^m\left(\sum_{k=l}^m(-1)^{k-l}k!\left\{{m\atop k}\right\}\frac{x^{m-k+l}}{(1+x)^{m-k+1}}\right)\frac{\widehat{c}^{(l)}(x)}{l!}\\&=\frac{(-1)^m}{m!}\sum_{l=0}^m\sum_{k=l}^m\frac{(-1)^{k-l}k!}{l!}\left\{{m\atop k}\right\}\sum_{i=0}^\infty(-1)^i\binom{m-k+i}{i}\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{n=0}^\infty \frac{n!}{(n-m+k-l-i)!}\widehat{c}_{n-m+k-i}\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty \frac{n!}{m!}\sum_{l=0}^m\sum_{k=0}^{m-l}\sum_{i=0}^{n-k-l}\frac{(-1)^{l+k+i}(m-k)!}{(n-k-l-i)!l!}\left\{ m \atop m-k \right\}\binom{k+i}{i}\widehat{c}_{n-k-i}\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty \frac{n!}{m!}\sum_{\mu=0}^n\sum_{l=0}^{n-\mu}\sum_{k=0}^{\min\{\mu,m-l\}}\frac{(-1)^{\mu+l}(m-k)!}{(n-\mu-l)!l!}\left\{ m \atop m-k \right\}\binom{\mu}{\mu-k}\widehat{c}_{n-\mu}\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty \frac{n!}{m!}\sum_{\mu=0}^n\sum_{l=0}^{\mu}\sum_{k=0}^{\min\{n-\mu,m-l\}}\frac{(-1)^{n-\mu+l}(m-k)!}{(\mu-l)!l!}\left\{ m \atop m-k \right\}\binom{n-\mu}{k}\widehat{c}_{\mu}\frac{x^n}{n!} \qquad (13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(-1)^m}{m!}\sum_{k=0}^m \left\{ m \atop k \right\}\frac{x^{m+1}}{(1+x)^{m-k+1}}\widehat{c}^{(k)}(x)\\
&=\frac{(-1)^m}{m!}\sum_{k=0}^m \left\{ m \atop k \right\}\sum_{i=0}^\infty(-1)^i\binom{m-k+i}{i}\sum_{n=0}^\infty\frac{n!}{(n-m-i-1)!}\widehat{c}_{n+k-m-i-1}\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty \frac{n!}{m!}\sum_{k=0}^m \left\{ m \atop k \right\}\sum_{i=0}^{n-m-1}\frac{(-1)^{n-i-1}}{i!}\binom{n-k-i-1}{m-k}\widehat{c}_{k+i}\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty \frac{n!}{m!}\sum_{\nu=0}^{n-1}\sum_{k=0}^m \left\{ m \atop k \right\}\binom{n-\nu-1}{m-k}\frac{(-1)^{n-\nu+k-1}}{(\nu-k)!}\widehat{c}_{\nu}\frac{x^n}{n!}\,. \qquad (14)
\end{aligned}
$$

Therefore, we have the following.

THEOREM 5. *For any integers $n \ge 1$ and $m \ge 1$, we have*

$$
\begin{aligned}
&(\underbrace{\widehat{c}_0+\cdots+\widehat{c}_0}_{m})^n\\
&=\frac{n!}{(m-1)!}\\
&\quad\times\sum_{i=0}^{n-1}\left(\sum_{l=0}^{i}\sum_{k=0}^{\min\{n-i,m-l-1\}}\frac{(-1)^{n-i+l}(m-k-1)!}{(i-l)!l!}\left\{ m-1 \atop m-k-1 \right\}\binom{n-i}{k}\right.\\
&\quad\left.+\sum_{k=0}^{m-1}\left\{ m-1 \atop k \right\}\binom{n-i-1}{m-k-1}\frac{(-1)^{n-i+k-1}}{(i-k)!}\right)\widehat{c}_i\\
&\quad+\sum_{l=0}^{\min\{n,m-1\}}(-1)^l\binom{n}{l}\widehat{c}_n\,.
\end{aligned}
$$

When $m=2$, $m=3$, $m=4$ and $m=5$ in Theorem 5, we have

$$(\widehat{c}_0+\widehat{c}_0)^n = n!\sum_{k=0}^{n}(-1)^{n-k}\frac{\widehat{c}_k}{k!} - n\widehat{c}_n\,,$$

$$(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0)^n = \frac{n^2}{2}\widehat{c}_n + \frac{n!}{2}\sum_{i=0}^{n}\frac{(-1)^{n-i}(n-4i+2)}{i!}\widehat{c}_i\,,$$

$$\begin{aligned}&(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_0)^n\\ &\quad= -\frac{n^3}{6}\widehat{c}_n + \frac{n!}{12}\sum_{i=0}^{n}\frac{(-1)^{n-i}(n^2-16in+11n+27i^2-33i+12)}{i!}\widehat{c}_i\,,\end{aligned}$$

$$\begin{aligned}&(\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_0+\widehat{c}_0)^n\\ &\quad= \frac{n^4}{24}\widehat{c}_n + \frac{n!}{144}\sum_{i=0}^{n}\frac{(-1)^{n-i}}{i!}(n^3-48in^2+39n^2+243i^2n\\ &\qquad -393in+176n-256i^3+564i^2-476i+144)\widehat{c}_i\,,\end{aligned}$$

respectively.

By differentiating (13) and (14) μ times with respect to x, we have the following.

THEOREM 6. *For any integers $n\ge 1$ and $m\ge 2$, we have*

$$\begin{aligned}&\sum_{\substack{\kappa_1+\cdots+\kappa_m=\mu\\ \kappa_1,\dots,\kappa_m\ge 0}}\frac{\mu!}{\kappa_1!\cdots\kappa_m!}(\widehat{c}_{\kappa_1}+\cdots+\widehat{c}_{\kappa_m})^n\\ &\quad= \frac{(n+\mu)!}{(m-1)!}\sum_{i=0}^{n+\mu-1}\Bigg(\sum_{l=0}^{i}\sum_{k=0}^{\min\{n+\mu-i,m-l-1\}}\frac{(-1)^{n+\mu-i+l}(m-k-1)!}{(i-l)!l!}\\ &\qquad\qquad \times\left\{ {m-1 \atop m-k-1} \right\}\binom{n+\mu-i}{k}\\ &\qquad +\sum_{k=0}^{m-1}\left\{ {m-1 \atop k} \right\}\binom{n+\mu-i-1}{m-k-1}\frac{(-1)^{n+\mu-i+k-1}}{(i-k)!}\Bigg)\widehat{c}_i\\ &\qquad +\sum_{l=0}^{\min\{n+\mu,m-1\}}(-1)^l\binom{n+\mu}{l}\widehat{c}_{n+\mu}\,.\end{aligned}$$

REMARK 1. The special cases where $m=3,4,5$ were discussed in [9].

If $m = 2$ and $\mu = 1$ in Theorem 6, then

$$2(\widehat{c}_0 + \widehat{c}_1)^n = -n\widehat{c}_{n+1} - (n+1)! \sum_{l=0}^{n} \frac{(-1)^{n-l}}{l!} \widehat{c}_l .$$

If $m = 3$ and $\mu = 1$ in Theorem 6, then

$$\begin{aligned} &3(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1)^n \\ &= \frac{n(n-1)}{2} \widehat{c}_{n+1} - \frac{(n+1)!}{2} \sum_{l=0}^{n} \frac{(-1)^{n-l}}{l!} (n - 4l + 3) \widehat{c}_l . \end{aligned}$$

If $m = 3$ and $\mu = 2$ in Theorem 6, we have

$$\begin{aligned} &3(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_2)^n + 6(\widehat{c}_0 + \widehat{c}_1 + \widehat{c}_1)^n \\ &= \frac{n(n+1)}{2} \widehat{c}_{n+2} + \frac{(n+2)!}{2} \sum_{l=0}^{n+1} \frac{(-1)^{n-l}}{l!} (n - 4l + 4) \widehat{c}_l . \end{aligned}$$

If $m = 4$ and $\mu = 1$ in Theorem 6, we have

$$\begin{aligned} &4(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1)^n \\ &= -\binom{n}{3} \widehat{c}_{n+1} + \frac{(n+1)!}{12} \sum_{l=0}^{n} \frac{(-1)^{n-l+1}}{l!} \big(27l^2 - 16nl - 49l + n^2 + 13n + 24\big) \widehat{c}_l . \end{aligned}$$

If $m = 4$ and $\mu = 2$ in Theorem 6, we have

$$\begin{aligned} &12(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1 + \widehat{c}_1)^n + 4(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_2)^n \\ &= -\binom{n+1}{3} \widehat{c}_{n+2} + \frac{(n+2)!}{12} \sum_{l=0}^{n+1} \frac{(-1)^{n-l}}{l!} \big(27l^2 - 16nl - 65l + n^2 + 15n + 38\big) \widehat{c}_l . \end{aligned}$$

If $m = 4$ and $\mu = 3$ in Theorem 6, we have

$$\begin{aligned} &24(\widehat{c}_0 + \widehat{c}_1 + \widehat{c}_1 + \widehat{c}_1)^n + 36(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1 + \widehat{c}_2)^n + 4(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_3)^n \\ &= -\binom{n+2}{3} \widehat{c}_{n+3} + \frac{(n+3)!}{12} \sum_{l=0}^{n+2} \frac{(-1)^{n-l+1}}{l!} \big(27l^2 - 16nl - 81l + n^2 + 17n + 54\big) \widehat{c}_l . \end{aligned}$$

If $m = 5$ and $\mu = 1$ in Theorem 6, we have

$$\begin{aligned} 5(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1)^n &= \binom{n}{4} \widehat{c}_{n+1} \\ &- \frac{(n+1)!}{144} \sum_{l=0}^{n} \frac{(-1)^{n-l+1}}{l!} \big(256l^3 - (243n + 807)l^2 \\ &+ (48n^2 + 489n + 917)l - (n^3 + 42n^2 + 257n + 360)\big) \widehat{c}_l . \end{aligned}$$

ACKNOWLEDGMENT. The author thanks the anonymous referee for his/her careful reading of the manuscript. This work was supported in part by the grant of Wuhan University and by the grant of Hubei Provincial Experts Program.

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