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On Classification of Quandles of Cyclic Type

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Abstract. In this paper, we study quandles of cyclic type, which form a particular subclass of finite quandles. The main result of this paper describes the set of isomorphism classes of quandles of cyclic type in terms of certain cyclic permutations. By using our description, we give a direct classification of quandles of cyclic type with cardinality up to 12.

1. Introduction

The notion of quandle was introduced by Joyce ([6]) as a set with a binary operator, satisfying three axioms corresponding to Reidemeister moves of a classical knot. In knot theory, quandles play a lot of important roles, and have provided several invariants of knots ([2, 3, 5, 8, 10]). For further information, we refer to [1, 7] and references therein. Among others, Carter, Jelsovsky, the first author, Langford and Saito ([2]) gave strong invariants, called quandle cocycle invariants, defined by quandle cocycles. For example, they gave a 3-cocycle of the dihedral quandle R_3 with cardinality 3, and apply it to prove the non-invertibility of the 2-twist spun trefoil.

Quandles provide several invariants of knots, but on the other hand, it is difficult to calculate these invariants explicitly, especially if the structure of the quandle is complicated. Therefore, it is of importance to study special classes of quandles, whose quandle structures are easy to handle. From this point of view, we study quandles of cyclic type, whose name was introduced in [11]. A quandle with cardinality n is said to be of *cyclic type* if all right multiplications are cyclic permutations of order $n-1$. Since this quandle structure is very tractable, quandles of cyclic type are potentially useful for applications in knot theory.

We here recall some known results on quandles of cyclic type. In [9], Lopes and Roseman essentially studied quandles of cyclic type, which they call quandles with constant profile $(\{1, n-1\}, \dots, \{1, n-1\})$. They studied such quandles in terms of cyclic permutations, and classified those with cardinality up to 8. Subsequently, Hayashi ([4]) studied the structures of quandles of cyclic type, and gave a table of those with cardinality up to 35. Note that his

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table is obtained by using the list of connected quandles with cardinality up to 35 (called Vendramin's list [12]). Independently, the second author ([11]) studied quandles of cyclic type, and classified those with prime cardinality. In particular, for every prime number $p \geq 3$, there exists a quandle of cyclic type with cardinality p. This suggests that the class of quandles of cyclic type is fruitful.

In this paper, we study and describe the set C_n of isomorphism classes of quandles of cyclic type with cardinality n. In fact, our main theorem gives a bijection from C_n onto F_n, where F_n denotes the set of cyclic permutations of order $n-1$ satisfying two conditions. This bijection is useful for studying quandles of cyclic type, since such quandles can be characterized by certain cyclic permutations. We then apply our main theorem to the classification of quandles of cyclic type, and provide a list of those with cardinality up to 12. Our study extends some of the results by Lopes and Roseman ([9]). In fact, they also studied cyclic permutations determined by quandles of cyclic type, which are similar to ours. Our new contribution is to show that it gives a well-defined and bijective map. Furthermore, our argument gives a direct and classification-free proof for a part of the table given by Hayashi ([4]).

This paper is organized as follows. In Section 2 we recall some fundamental notions on quandles. In Section 3, the definition and some properties of quandles of cyclic type are summarized. We state the main theorem in Section 4, and give a table of quandles of cyclic type with cardinality up to 12. Section 5 contains the proof of the main theorem.

2. Preliminaries for quandles

In this section we recall some fundamental notions on quandles.

DEFINITION 2.1. Let X be a set and $* : X \times X \to X$ be a binary operator. The pair $(X, *)$ is called a *quandle* if

(Q1) $\forall x \in X, x * x = x$,
(Q2) $\forall x, y \in X, \exists ! z \in X : z * y = x$, and
(Q3) $\forall x, y, z \in X, (x * y) * z = (x * z) * (y * z)$.

If $(X, *)$ is a quandle, then $*$ is called a *quandle structure* on X. We restate the definition of a quandle as follows.

PROPOSITION 2.2 ([3, 11]). *Let X be a set, and assume that there exists a map $s_x : X \to X$ for every $x \in X$. Then, the binary operator $*$ defined by $y * x := s_x(y)$ is a quandle structure on X if and only if*

(S1) $\forall x \in X, s_x(x) = x$,
(S2) $\forall x \in X$, s_x *is bijective, and*
(S3) $\forall x, y \in X, s_x \circ s_y = s_{s_x(y)} \circ s_x$.

Instead of Definition 2.1, throughout this paper, we denote the quandle by $X = (X, s)$ with the quandle structure

$$s : X \to \mathrm{Map}(X, X) : x \mapsto s_x \, . \qquad (1)$$

Here $\mathrm{Map}(X, X)$ denotes the set of all maps from X to X.

EXAMPLE 2.3. The following (X, s) are quandles:

(1) Let X be any set and $s_x := \mathrm{id}_X$ for every $x \in X$. Then the pair (X, s) is called the *trivial quandle*.
(2) Let $X := \{1, \dots, n\}$ and $s_i(j) := 2i - j \pmod{n}$ for any $i, j \in X$. Then the pair (X, s) is called the *dihedral quandle* with cardinality n.
(3) Let $X := \{1, 2, 3, 4\}$ and

$$s_1 := (234), \quad s_2 := (143), \quad s_3 := (124), \quad s_4 := (132).$$

Then the pair (X, s) is called the *tetrahedron quandle*.

Note that (234), (143), and so on, denote the cyclic permutations. We use this symbol frequently in the later sections.

DEFINITION 2.4. Let (X, s^X), (Y, s^Y) be quandles, and $f : X \to Y$ be a map.

(1) f is called a *homomorphism* if for every $x \in X$, $f \circ s^X_x = s^Y_{f(x)} \circ f$ holds.
(2) f is called an *isomorphism* if f is a bijective homomorphism.

An isomorphism from a quandle (X, s) onto itself is called an *automorphism*. The set of automorphisms of (X, s) forms a group, which is called the *automorphism group* and denoted by $\mathrm{Aut}(X, s)$.

Note that s_x $(x \in X)$ is an automorphism of (X, s). The subgroup of $\mathrm{Aut}(X, s)$ generated by $\{s_x \mid x \in X\}$ is called the *inner automorphism group* of (X, s) and denoted by $\mathrm{Inn}(X, s)$.

DEFINITION 2.5. A quandle (X, s) is said to be *connected* if $\mathrm{Inn}(X, s)$ acts transitively on X.

On the connectedness of the quandles in Example 2.3, the following is well-known. We denote by $\#X$ the cardinality of X.

EXAMPLE 2.6. One has the following:

(1) The trivial quandle (X, s) is connected if and only if $\#X = 1$.
(2) The dihedral quandle (X, s) is connected if and only if $\#X$ is odd.
(3) The tetrahedron quandle is connected.

3. Quandles of cyclic type

From now on we always assume that a quandle $X = (X, s)$ is finite and satisfies $\#X \geq 3$. In this section, we recall the definition and some properties of quandles of cyclic type given in [11].

DEFINITION 3.1 ([11]). A quandle (X, s) with $\#X = n \geq 3$ is said to be of *cyclic type* if for every $x \in X$, s_x acts on $X \setminus \{x\}$ as a cyclic permutation of order $n - 1$.

This notion is closely related to the notion of two-point homogeneous quandle. A quandle (X, s) is said to be *two-point homogeneous* if for any $(x_1, x_2), (y_1, y_2) \in X \times X$ satisfying $x_1 \neq x_2$ and $y_1 \neq y_2$, there exists $f \in \mathrm{Inn}(X, s)$ such that $(f(x_1), f(x_2)) = (y_1, y_2)$. The second author studied quandles of cyclic type in [11] because of the following proposition.

PROPOSITION 3.2 ([11]). *Every quandle of cyclic type is two-point homogeneous.*

The following is a characterization of quandles of cyclic type, which we use in the latter arguments. In particular, quandles of cyclic type must be connected.

PROPOSITION 3.3 ([11]). *Let $X = (X, s)$ be a quandle with $\#X = n \geq 3$. Then, X is of cyclic type if and only if*

(i) *X is connected, and*
(ii) *there exists $x \in X$ such that s_x acts on $X \setminus \{x\}$ as a cyclic permutation of order $n - 1$.*

If the structure of a quandle is given, then one can easily check whether it is of cyclic type or not. We here give some easy examples.

EXAMPLE 3.4. One has the following:

(1) The trivial quandles are not of cyclic type.
(2) The dihedral quandle (X, s) is of cyclic type if and only if $\#X = 3$.
(3) The tetrahedron quandle is of cyclic type.

4. Main Theorem

In this section, we state our main theorem, and give a table of quandles of cyclic type with cardinality up to 12. The following notations will be used throughout the remaining of this paper:

- $X := \{1, 2, \ldots, n\}$ with $n \geq 3$,
- S_n denotes the symmetry group of order n,
- $(S_n)_{n-1} := \{\sigma \in S_n \mid \sigma \text{ is a cyclic permutation of order } n - 1\}$.

DEFINITION 4.1. We denote by $C_n^{\#}$ the set of all quandle structures of cyclic type on X, that is,

$$C_n^{\#} := \{s : X \to (S_n)_{n-1} \mid s \text{ satisfies (S1), (S3)}\}.$$

(Note that every $s \in C_n^{\#}$ automatically satisfies (S2).) We denote by C_n the set of isomorphism classes $[s]$ of $s \in C_n^{\#}$.

Consider the inclusion map from $C_n^{\#}$ into the set of quandles of cyclic type with cardinality n. This induces a bijection from C_n to the set of isomorphism classes of quandles of cyclic type with cardinality n.

DEFINITION 4.2. Let $s_1 := (23\ldots n)$. We denote by F_n the set of $s_2 \in (S_n)_{(n-1)}$ satisfying the following two conditions:

(F1) $s_2(2) = 2$, and
(F2) $\{s_2^m s_1 s_2^{-m} \mid m = 1, 2, \ldots, n-2\} = \{s_1^m s_2 s_1^{-m} \mid m = 1, 2, \ldots, n-2\}$.

Recall that $(23\ldots n)$ denotes the cyclic permutation. The following is the main theorem of this paper, which gives a one-to-one correspondence between C_n and F_n.

THEOREM 4.3. *Let $s_1 := (23\ldots n)$, $s_2 \in F_n$, and define $\varphi(s_2) : X \to \mathrm{Map}(X, X)$ by*

$$(\varphi(s_2))_i := \begin{cases} s_1 & (i = 1)\,, \\ s_2 & (i = 2)\,, \\ s_1^{i-2} \circ s_2 \circ s_1^{-i+2} & (i \in \{3, \ldots, n\})\,. \end{cases}$$

Then one has $\varphi(s_2) \in C_n^{\#}$, and hence give a map $\varphi : F_n \to C_n^{\#}$. This induces a bijection from F_n onto C_n by composing with the natural projection from $C_n^{\#}$ onto C_n.

The proof of this theorem will be given in the next section. In the remaining of this section, we provide a table of quandles of cyclic type with cardinality up to 12. For the classification, we have only to determine the set F_n.

PROPOSITION 4.4. *We have $F_3 = \{(13)\}$ and $F_4 = \{(143)\}$.*

PROOF. The basic strategy is the following. First of all, we list up all elements in $(S_n)_{(n-1)}$ satisfying (F1). These elements are called the *candidates* for simplicity. We then check whether each candidate satisfies (F2) or not.

In the case of $n = 3$, the only candidate is $s_2 = (13)$. One can easily see that

$$s_2 s_1 s_2^{-1} = (12) = s_1 s_2 s_1^{-1}\,. \tag{2}$$

Hence s_2 satisfies (F2). This proves the first assertion.

In the case of $n = 4$, there are two candidates, (143) and (134). For $s_2 := (143)$, we have

$$\begin{aligned} s_1 s_2 s_1^{-1} &= (124)\,, \quad & s_1^2 s_2 s_1^{-2} &= (132)\,, \\ s_2 s_1 s_2^{-1} &= (132)\,, \quad & s_2^2 s_1 s_2^{-2} &= (124)\,. \end{aligned} \tag{3}$$

Thus $s_2 = (143)$ satisfies (F2). On the other hand, $s_2 := (134)$ does not satisfy (F2). In fact, $s_2 s_1 s_2^{-1} = (124)$ is not an element of

$$\{s_1^m s_2 s_1^{-m} \mid m = 1, 2\} = \{(142), (123)\}\,. \tag{4}$$

This completes the proof of the second assertion. □

When $n = 3$, the quandle corresponding to $s_2 = (13)$ is the dihedral quandle with cardinality 3. When $n = 4$, the quandle corresponding to $s_2 = (143)$ is the tetrahedron quandle. When $n \geq 5$, the following lemma is useful to examine whether each candidate satisfies (F2) or not.

LEMMA 4.5. *Let $s_2 \in F_n$, and assume that $m \in \mathbb{Z}$ satisfies $s_2(1) = s_1^m(2)$. Then we have*

$$s_1^m s_2 s_1^{-m} = s_2 s_1 s_2^{-1} .$$

PROOF. Since $s_2 \in F_n$ satisfies (F2), there exists $l \in \{1, 2, \ldots, n-2\}$ such that

$$s_1^m s_2 s_1^{-m} = s_2^l s_1 s_2^{-l} . \tag{5}$$

Note that $s_1^m s_2 s_1^{-m}$ is a cyclic permutation of order $n-1$, having the unique fixed point $s_1^m(2)$. Similarly, $s_2^l s_1 s_2^{-l}$ has the unique fixed point $s_2^l(1)$. Hence, combining with the assumption, one has

$$s_2(1) = s_1^m(2) = s_2^l(1) . \tag{6}$$

Since $s_2 \in (S_n)_{n-1}$ and it satisfies (F1), we conclude

$$l = 1 . \tag{7}$$

This completes the proof. □

The above lemma is useful to determine the set F_n for any n. Here we apply it to the case of $n = 5$.

PROPOSITION 4.6. *We have $F_5 = \{(1354), (1435)\}$.*

PROOF. As for the set F_5, there are six candidates,

$$s_2 = (1345)\,,\quad (1354)\,,\quad (1435)\,,\quad (1453)\,,\quad (1534),\ (1543)\,. \tag{8}$$

One can directly see that (1354) and (1435) satisfy (F2). We omit the proof for these two cases.

We here show that the remaining four candidates do not satisfy (F2). For the proof, we use Lemma 4.5. In fact, we determine m satisfying $s_1^m(2) = s_2(1)$, and show that $s_1^m s_2 s_1^{-m} \neq s_2 s_1 s_2^{-1}$.

Case (1): $s_2 := (1345)$. Let $m = 1$. Then one has $s_1^m(2) = 3 = s_2(1)$ and

$$s_1^m s_2 s_1^{-m} = (1452) \neq (1245) = s_2 s_1 s_2^{-1} . \tag{9}$$

Case (2): $s_2 := (1453)$. Let $m = 2$. Then one has $s_1^m(2) = 4 = s_2(1)$ and

$$s_1^m s_2 s_1^{-m} = (1235) \neq (1532) = s_2 s_1 s_2^{-1} . \tag{10}$$

Case (3): $s_2 := (1543)$. Let $m = 3$. Then one has $s_1^m(2) = 5 = s_2(1)$ and

$$s_1^m s_2 s_1^{-m} = (1432) \neq (1342) = s_2 s_1 s_2^{-1} . \tag{11}$$

Case (4): $s_2 := (1534)$. Let $m = 3$. Then one has $s_1^m(2) = 5 = s_2(1)$ and

$$s_1^m s_2 s_1^{-m} = (1423) \neq (1324) = s_2 s_1 s_2^{-1} . \tag{12}$$

We have thus proved that these four candidates do not satisfy (F2). □

By the same arguments, we have determined F_n with $n \leq 12$. We omit the proof since it is long and the arguments are exactly same as above, and we have used some computer programs for calculations. The results are summarized in Table 1, which gives a classification of quandles of cyclic type with cardinality up to 12. Note that $\#F_n$ denotes the cardinality of F_n.

TABLE 1. Quandles of cyclic type with cardinality up to 12

n	$\#F_n$	F_n
3	1	$\{(1\ 3)\}$
4	1	$\{(1\ 4\ 3)\}$
5	2	$\{(1\ 3\ 5\ 4), (1\ 4\ 3\ 5)\}$
6	0	$\emptyset$
7	2	$\{(1\ 7\ 4\ 6\ 5\ 3), (1\ 7\ 5\ 4\ 6\ 3)\}$
8	2	$\{(1\ 5\ 8\ 3\ 7\ 6\ 4), (1\ 7\ 5\ 4\ 8\ 3\ 6)\}$
9	2	$\{(1\ 4\ 3\ 8\ 6\ 9\ 5\ 7), (1\ 5\ 7\ 3\ 6\ 4\ 9\ 8)\}$
10	0	$\emptyset$
11	4	$\{(1\ 3\ 6\ 8\ 4\ 11\ 5\ 10\ 9\ 7), (1\ 4\ 3\ 7\ 10\ 5\ 11\ 9\ 6\ 8),$ $(1\ 6\ 8\ 5\ 3\ 9\ 4\ 7\ 11\ 10), (1\ 7\ 5\ 4\ 9\ 3\ 10\ 6\ 8\ 11)\}$
12	0	$\emptyset$

We note that Table 1 agrees with some previously known results ([4, 9, 11]) mentioned in Introduction. By looking at these classification lists, we conjecture the following.

CONJECTURE 4.7. *Let $n \geq 3$. Then, there exists a quandle of cyclic type with cardinality n if and only if n is a power of a prime number.*

REMARK 4.8. We here note that, after we submitted the first version of this paper to the arXiv, Conjecture 4.7 is solved affirmatively. This follows from the recent results by Vendramin ([13]) and by the third author ([14]). However, we decided to keep the original formulation, since Conjecture 4.7 has inspired such studies and is referred in several papers.

5. Proof of Theorem 4.3

In this section, we prove Theorem 4.3, which gives a bijection from F_n onto C_n. For the proof, we define auxiliary sets E_n and D_n, and construct bijections

$$g_3 : F_n \to E_n\,, \quad g_2 : E_n \to D_n\,, \quad g_1 : D_n \to C_n\,. \tag{13}$$

5.1. A bijection from D_n onto C_n. In this subsection, we define a set D_n, and construct a bijection from D_n onto C_n. Recall that $X := \{1, \ldots, n\}$, and $(S_n)_{n-1}$ is the subset of S_n consisting of all cyclic permutations of order $n-1$. Two subsets $\Sigma, \Sigma' \subset S_n$ are said to be *conjugate* if there exists $g \in S_n$ such that $g^{-1}\Sigma g = \Sigma'$.

DEFINITION 5.1. We denote by $D_n^{\#}$ the set of $\Sigma \subset (S_n)_{n-1}$ satisfying

(D1) $\forall s \in \Sigma$, $s^{-1}\Sigma s \subset \Sigma$, and
(D2) $\forall x \in X$, $\exists! s \in \Sigma : s(x) = x$.

We also denote by D_n the set of conjugacy classes $[\Sigma]$ of $\Sigma \in D_n^{\#}$.

First of all we study $D_n^{\#}$. Note that Conditions (D1) and (D2) are preserved by conjugation. Namely, if $\Sigma \in D_n^{\#}$ and Σ is conjugate to Σ', then one has $\Sigma' \in D_n^{\#}$. Furthermore, the following lemma yields that every $\Sigma \in D_n^{\#}$ satisfies $\#\Sigma = n$.

LEMMA 5.2. *Let $\Sigma \in D_n^{\#}$. For each $x \in X$, denote by $s_x^{\Sigma} \in \Sigma$ the unique element with $s_x^{\Sigma}(x) = x$. Then, the obtained map $s^{\Sigma} : X \to \Sigma$ is bijective.*

PROOF. We show that s^{Σ} is surjective. Take any $s \in \Sigma$. Since $s \in (S_n)_{n-1}$, there exists $x \in X$ such that $s(x) = x$. Then, the uniqueness in (D2) shows $s = s_x^{\Sigma}$.

We next show that s^{Σ} is injective. Let $x, y \in X$ and assume that $s_x^{\Sigma} = s_y^{\Sigma}$. One knows $s_x^{\Sigma}(x) = x$ by definition. Thus x is the unique fixed point of $s_x^{\Sigma} \in (S_n)_{n-1}$. Similarly, y is the unique fixed point of s_y^{Σ}. This concludes $x = y$. □

The aim of this subsection is to construct a bijection from D_n onto C_n. We here see that s^{Σ} defines a map from $D_n^{\#}$ onto $C_n^{\#}$. Recall that $\Sigma \subset (S_n)_{n-1}$.

LEMMA 5.3. *The above defined map $s^{\Sigma} : X \to (S_n)_{n-1}$ satisfies $s^{\Sigma} \in C_n^{\#}$, that is, (X, s^{Σ}) is a quandle of cyclic type.*

PROOF. By definition, s^{Σ} satisfies (S1). Hence we have only to show (S3). Take any $y, z \in X$. Condition (D1) yields that

$$s_z^{\Sigma} \circ s_y^{\Sigma} \circ (s_z^{\Sigma})^{-1} \in \Sigma . \tag{14}$$

On the other hand, one has

$$s_z^{\Sigma} \circ s_y^{\Sigma} \circ (s_z^{\Sigma})^{-1}(s_z^{\Sigma}(y)) = s_z^{\Sigma} \circ s_y^{\Sigma}(y) = s_z^{\Sigma}(y) . \tag{15}$$

Therefore, from the uniqueness in (D2), one has

$$s_z^{\Sigma} \circ s_y^{\Sigma} \circ (s_z^{\Sigma})^{-1} = s_{s_z^{\Sigma}(y)}^{\Sigma} . \tag{16}$$

This proves (S3), which completes the proof. □

One thus has obtained a map from $D_n^{\#}$ to $C_n^{\#}$. For the later use, we here show that this map is surjective.

LEMMA 5.4. *The following map is surjective*:

$$\bar{g}_1 : D_n^{\#} \to C_n^{\#} : \Sigma \mapsto s^{\Sigma} .$$

PROOF. Take any $s \in C_n^{\#}$. Let us put

$$\Sigma := \{s_x \mid x \in X\} \subset (S_n)_{n-1} . \tag{17}$$

We have only to prove that $\Sigma \in D_n^{\#}$ and $\bar{g}_1(\Sigma) = s$.

We show that Σ satisfies (D1). Take any $s_x, s_y \in \Sigma$. Since s_x^{-1} is an automorphism, one has

$$s_x^{-1} \circ s_y \circ s_x = s_{s_x^{-1}(y)} \in \Sigma . \tag{18}$$

This proves $s_x^{-1} \Sigma s_x \subset \Sigma$.

We next show that Σ satisfies (D2). Take any $x \in X$. Since s satisfies (S1), $s_x \in \Sigma$ satisfies $s_x(x) = x$. This proves the existence. Next assume that $s_y(x) = x$. Since $s \in C_n^{\#}$, one has $s_y \in (S_n)_{n-1}$. Hence x is the unique fixed point of s_y. Thus (S1) yields that $x = y$, which proves the uniqueness.

We have proved $\Sigma \in D_n^{\#}$. Furthermore, by the definition of $\bar{g}_1$, it is easy to see that $\bar{g}_1(\Sigma) = s$. This completes the proof. □

We here define a map from D_n to C_n. Recall that $[\Sigma]$ denotes the conjugacy class of $\Sigma \in D_n^{\#}$, and $[s]$ denotes the isomorphism class of $s \in C_n^{\#}$.

LEMMA 5.5. *The following map is well-defined*:

$$g_1 : D_n \to C_n : [\Sigma] \mapsto [s^{\Sigma}].$$

PROOF. Let $\Sigma, \Sigma' \in D_n^{\#}$, and assume that $[\Sigma] = [\Sigma']$. Hence there exists $g \in S_n$ such that $\Sigma = g^{-1}\Sigma' g$. In order to show $[s^{\Sigma}] = [s^{\Sigma'}]$, it is enough to prove that the following map is a quandle isomorphism:

$$g : (X, s^{\Sigma}) \to (X, s^{\Sigma'}) . \tag{19}$$

This is obviously bijective. We show that g is a quandle homomorphism. Take any $x \in X$. By definition, one has

$$s_{g(x)}^{\Sigma'}(g(x)) = g(x) . \tag{20}$$

This means that

$$g^{-1} \circ s_{g(x)}^{\Sigma'} \circ g(x) = x . \tag{21}$$

On the other hand, one has

$$g^{-1} \circ s_{g(x)}^{\Sigma'} \circ g \in g^{-1}\Sigma' g = \Sigma . \tag{22}$$

Hence, from the uniqueness in (D2), we have

$$g^{-1} \circ s_{g(x)}^{\Sigma'} \circ g = s_x^{\Sigma} . \tag{23}$$

This proves that g is a quandle homomorphism. □

We now show that the above defined map g_1 is bijective. The following is the main result of this subsection.

PROPOSITION 5.6. *The map $g_1 : D_n \to C_n$ is bijective.*

PROOF. One knows that g_1 is surjective, since so is $\bar{g}_1$ from Lemma 5.4. It remains to show that g_1 is injective. Let $[\Sigma], [\Sigma'] \in D_n$, and assume that $g_1([\Sigma]) = g_1([\Sigma'])$. By definition, one has $[s^{\Sigma}] = [s^{\Sigma'}]$, that is, there exists a quandle isomorphism

$$g : (X, s^{\Sigma}) \to (X, s^{\Sigma'}) . \tag{24}$$

Since g is bijective, one has $g \in S_n$. Since g is a homomorphism, one has for any $x \in X$ that

$$s^{\Sigma}_x = g^{-1} \circ s^{\Sigma'}_{g(x)} \circ g \in g^{-1} \Sigma' g . \tag{25}$$

This proves $\Sigma \subset g^{-1}\Sigma' g$. Recall that $\#\Sigma = n = \#\Sigma'$ holds from Lemma 5.2. Therefore, we have $\Sigma = g^{-1}\Sigma' g$, and thus $[\Sigma] = [\Sigma']$. This concludes that g_1 is injective. □

5.2. A bijection from E_n onto D_n. In this subsection, we define a set E_n, and construct a bijection from E_n onto D_n. We denote by

$$S_{n,(1,2)} := \{u \in S_n \mid u(1) = 1,\ u(2) = 2\} . \tag{26}$$

Two elements $(u_1, u_2), (v_1, v_2) \in (S_n)_{n-1} \times (S_n)_{n-1}$ are said to be $S_{n,(1,2)}$*-conjugate* if $(u_1, u_2) = (w^{-1}v_1 w, w^{-1}v_2 w)$ for some $w \in S_{n,(1,2)}$.

DEFINITION 5.7. We denote by $E^{\#}_n$ the set of $(u_1, u_2) \in (S_n)_{n-1} \times (S_n)_{n-1}$ satisfying

(E1) $u_1(1) = 1, u_2(2) = 2$, and
(E2) $\{u_1^m u_2 u_1^{-m} \mid m = 1, 2, \ldots, n-2\} = \{u_2^m u_1 u_2^{-m} \mid m = 1, 2, \ldots, n-2\}$.

We also denote by E_n the set of $S_{n,(1,2)}$-conjugacy classes $[(u_1, u_2)]$ of $(u_1, u_2) \in E^{\#}_n$.

First of all, we construct a map from $E^{\#}_n$ to $D^{\#}_n$.

LEMMA 5.8. *Let $(u_1, u_2) \in E^{\#}_n$. Then one has*

$$\Sigma_{(u_1,u_2)} := \{u_1, u_2\} \cup \{u_1^m u_2 u_1^{-m} \mid m = 1, 2, \ldots, n-2\} \in D^{\#}_n . \tag{27}$$

PROOF. We have only to show that $\Sigma_{(u_1,u_2)}$ satisfies (D1) and (D2). In order to show (D1), it is enough to prove

$$u_1^{-1}\Sigma_{(u_1,u_2)}u_1 \subset \Sigma_{(u_1,u_2)} , \quad u_2^{-1}\Sigma_{(u_1,u_2)}u_2 \subset \Sigma_{(u_1,u_2)} . \tag{28}$$

Note that u_1 has order $n-1$. Then one has

$$\begin{aligned} u_1^{-1}u_1u_1 &= u_1 \in \Sigma_{(u_1,u_2)} , \\ u_1^{-1}u_2u_1 &= u_1^{n-2}u_2u_1^{-(n-2)} \in \Sigma_{(u_1,u_2)} , \\ u_1^{-1}(u_1^m u_2 u_1^{-m})u_1 &= u_1^{m-1}u_2u_1^{-(m-1)} \in \Sigma_{(u_1,u_2)} \quad (\text{for } m = 1, \ldots, n-2) . \end{aligned} \tag{29}$$

This proves the former claim of (28). On the other hand, (E2) yields that

$$\Sigma_{(u_1,u_2)} = \{u_1, u_2\} \cup \{u_2^m u_1 u_2^{-m} \mid m = 1, 2, \dots, n-2\}\,. \tag{30}$$

Hence, a similar calculation proves the latter claim of (28).

We next show (D2). Take any $x \in X$. If $x = 1, 2$, then it is fixed by $u_1, u_2 \in \Sigma_{(u_1,u_2)}$, respectively. Assume that $x \neq 1, 2$. By (E1) and $u_1 \in (S_n)_{n-1}$, there exists $m \in \{1, \dots, n-2\}$ such that $x = u_1^m(2)$. Then one has

$$u_1^m u_2 u_1^{-m}(x) = u_1^m u_2 u_1^{-m}(u_1^m(2)) = u_1^m u_2(2) = u_1^m(2) = x\,. \tag{31}$$

This completes the proof of the existence. On the other hand, by definition one has $\#\Sigma_{(u_1,u_2)} \leq n$. This shows the uniqueness. □

This lemma constructs a map from $E_n^\#$ to $D_n^\#$. We next show that this induces a map from E_n to D_n.

LEMMA 5.9. *The following map is well-defined*:

$$g_2 : E_n \to D_n : [(u_1, u_2)] \mapsto [\Sigma_{(u_1,u_2)}]\,.$$

PROOF. Let $[(u_1, u_2)], [(u_1', u_2')] \in E_n$, and assume that $[(u_1, u_2)] = [(u_1', u_2')]$. Then there exists $w \in S_{n,(1,2)}$ such that

$$u_1 = w^{-1} u_1' w\,, \quad u_2 = w^{-1} u_2' w\,. \tag{32}$$

Furthermore, for every $m \in \{1, \dots, n-2\}$, one has

$$w^{-1}(u_1'^{\,m} u_2' u_1'^{\,-m})w = (w^{-1}u_1'w)^m (w^{-1}u_2'w)(w^{-1}u_1'w)^{-m} = u_1^m u_2 u_1^{-m}\,. \tag{33}$$

We thus have $w^{-1}\Sigma_{(u_1',u_2')} w \subset \Sigma_{(u_1,u_2)}$. This proves

$$w^{-1}\Sigma_{(u_1',u_2')} w = \Sigma_{(u_1,u_2)}\,, \tag{34}$$

since $\Sigma_{(u_1',u_2')}, \Sigma_{(u_1,u_2)} \in D_n^\#$, and hence $\#\Sigma_{(u_1',u_2')} = n = \#\Sigma_{(u_1,u_2)}$ by Lemma 5.2. This completes the proof of $[\Sigma_{(u_1,u_2)}] = [\Sigma_{(u_1',u_2')}]$. □

The aim of this subsection is to prove that g_2 is bijective, by constructing the inverse map. For this purpose, we construct a map from $D_n^\#$ to $E_n^\#$. Recall that we have a map

$$\bar{g}_1 : D_n^\# \to C_n^\# : \Sigma \mapsto s^\Sigma\,. \tag{35}$$

LEMMA 5.10. *Let $\Sigma \in D_n^\#$. Then one has $(s_1^\Sigma, s_2^\Sigma) \in E_n^\#$.*

PROOF. For simplicity of the notations, we put $s_x := s_x^\Sigma$ for each $x \in X$. By definition, (s_1, s_2) obviously satisfies (E1). We have only to show (E2).

First of all, we claim that

$$\{s_1^m s_2 s_1^{-m} \mid m = 1, 2, \dots, n-2\} = \{s_x \mid x = 3, 4, \dots, n\}\,. \tag{36}$$

Let $m \in \{1, 2, \dots n-2\}$. Since Σ satisfies (D1), one has

$$s_1^m s_2 s_1^{-m} \in s_1^m \Sigma s_1^{-m} \subset \Sigma \,. \tag{37}$$

Thus, it follows from $s_1^m s_2 s_1^{-m}(s_1^m(2)) = s_1^m(2)$ and the uniqueness in (D2) that

$$s_1^m s_2 s_1^{-m} = s_{s_1^m(2)} \,. \tag{38}$$

Since $s_1(1) = 1$ and $s_1 \in (S_n)_{n-1}$, one has

$$\{s_1^m(2) \mid m = 1, 2, \dots, n-2\} = \{3, 4, \dots, n\} \,. \tag{39}$$

This completes the proof of the claim.

By the same argument, one can see that

$$\{s_2^m s_1 s_2^{-m} \mid m = 1, 2, \dots, n-2\} = \{s_x \mid x = 3, 4, \dots, n\} \,. \tag{40}$$

This and the above claim prove (D2). □

The above lemma gives a map from $D_n^\#$ to $E_n^\#$. We next show that this map induces a map from D_n to E_n.

LEMMA 5.11. *The following map is well-defined*:

$$f_2 : D_n \to E_n : [\Sigma] \mapsto [(s_1^\Sigma, s_2^\Sigma)] \,. \tag{41}$$

PROOF. Let $[\Sigma], [\Sigma'] \in D_n$, and assume that $[\Sigma] = [\Sigma']$. By definition, there exists $g \in S_n$ such that $\Sigma = g^{-1}\Sigma' g$. It then follows from Lemma 5.5 that

$$g : (X, s^\Sigma) \to (X, s^{\Sigma'}) \tag{42}$$

is a quandle isomorphism. Note that $(X, s^{\Sigma'})$ is of cyclic type, and hence two-point homogeneous. Therefore, since $g(1) \neq g(2)$, there exists $h \in \mathrm{Inn}(X, s^{\Sigma'})$ such that

$$(h \circ g(1), h \circ g(2)) = (1, 2) \,. \tag{43}$$

This yields $h \circ g \in S_{n,(1,2)}$. Note that $h \circ g$ is a quandle isomorphism from (X, s^Σ) onto $(X, s^{\Sigma'})$. Thus one has

$$\begin{aligned} (h \circ g) \circ s_1^\Sigma \circ (h \circ g)^{-1} &= s_{h \circ g(1)}^{\Sigma'} = s_1^{\Sigma'} \,, \\ (h \circ g) \circ s_2^\Sigma \circ (h \circ g)^{-1} &= s_{h \circ g(2)}^{\Sigma'} = s_2^{\Sigma'} \,. \end{aligned} \tag{44}$$

This completes the proof of $[(s_1^\Sigma, s_2^\Sigma)] = [(s_1^{\Sigma'}, s_2^{\Sigma'})]$. □

By showing that f_2 is the inverse map of g_2, we have the following main result of this subsection.

PROPOSITION 5.12. *The map $g_2 : E_n \to D_n$ is bijective.*

PROOF. We show that f_2 is the inverse map of g_2. It is clear that the composition $f_2 \circ g_2$ is the identity mapping. Consider $g_2 \circ f_2 : D_n \to D_n$, and take any $[\Sigma] \in D_n$. Then one has $f_2([\Sigma]) = [(s_1^\Sigma, s_2^\Sigma)]$. One also has $g_2 \circ f_2([\Sigma]) = [\Sigma']$, where

$$\Sigma' := \{s_1^\Sigma, s_2^\Sigma\} \cup \{(s_1^\Sigma)^m s_2^\Sigma (s_1^\Sigma)^{-m} \mid m = 1, \ldots, n-2\}. \tag{45}$$

Since s^Σ is a quandle structure, one can see $\Sigma' \subset \Sigma$. Thus we have $\Sigma' = \Sigma$ for cardinality reason. This shows that $g_2 \circ f_2$ is the identity mapping. □

5.3. A bijection from F_n onto E_n. We lastly construct a bijection from F_n onto E_n. Let $s_1 := (23 \ldots n)$, and recall that F_n is the set of $s_2 \in (S_n)_{n-1}$ satisfying (F1) and (F2).

PROPOSITION 5.13. *The following map is bijective*:

$$g_3 : F_n \to E_n : s_2 \mapsto [(s_1, s_2)].$$

PROOF. We show that g_3 is surjective. Take any $[(u_1, u_2)] \in E_n$. Since $u_1 \in (S_n)_{n-1}$ and $u_1(1) = 1$, we can write $u_1 = (2a_3a_4 \ldots a_n)$. Let us define $g \in S_{n,(1,2)}$ by

$$g := \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & a_3 & \cdots & a_n \end{pmatrix}. \tag{46}$$

An easy computation shows $g^{-1} \circ u_1 \circ g = s_1$. Let $s_2 := g^{-1} \circ u_2 \circ g$. Then s_2 obviously satisfies (F1). Furthermore, since (u_1, u_2) satisfies (E2), one can see that s_2 satisfies (F2). We thus have $s_2 \in F_n$. This concludes that g_3 is surjective, since

$$g_3(s_2) = [(s_1, s_2)] = [(g^{-1} \circ u_1 \circ g, g^{-1} \circ u_2 \circ g)] = [(u_1, u_2)]. \tag{47}$$

We show that g_3 is injective. Let $s_2, s_2' \in F_n$, and suppose that $g_3(s_2) = g_3(s_2')$. Hence there exists $h \in S_{n,(1,2)}$ such that

$$(s_1, s_2') = (h \circ s_1 \circ h^{-1}, h \circ s_2 \circ h^{-1}). \tag{48}$$

By definition one has $h(1) = 1$ and $h(2) = 2$. Then it follows from $h(2) = 2$ that

$$3 = s_1(2) = h \circ s_1 \circ h^{-1}(2) = h \circ s_1(2) = h(3). \tag{49}$$

Similarly, this yields that

$$4 = s_1(3) = h \circ s_1 \circ h^{-1}(3) = h \circ s_1(3) = h(4). \tag{50}$$

One can show inductively that $x = h(x)$ for any $x \in X$. This means that $h = \mathrm{id}$, and thus $s_2' = s_2$. This shows that g_3 is injective. □

5.4. Constructing quandles of cyclic type from F_n. In the previous subsections, we have constructed the following bijections:

$$g_3 : F_n \to E_n, \quad g_2 : E_n \to D_n, \quad g_1 : D_n \to C_n. \tag{51}$$

In this subsection, we describe $g_1 \circ g_2 \circ g_3(s_2)$ for each $s_2 \in F_n$.

Take any $s_2 \in F_n$. Recall that $s_1 := (23 \ldots n)$ and

$$\Sigma_{(s_1,s_2)} := \{s_1, s_2\} \cup \{s_1^m s_2 s_1^{-m} \mid m = 1, 2, \ldots, n-2\} \in D_n^{\#} . \tag{52}$$

Then one has $g_2 \circ g_3(s) = [\Sigma_{(s_1,s_2)}]$. We put

$$\varphi(s_2) := s^{\Sigma_{(s_1,s_2)}} \in C_n^{\#} . \tag{53}$$

This means $g_1 \circ g_2 \circ g_3(s_2) = [\varphi(s_2)]$. Note that $(\varphi(s_2))_i \in \Sigma_{(s_1,s_2)}$ is defined as the unique element fixing $i \in X$. This immediately yields

$$(\varphi(s_2))_1 = s_1 , \quad (\varphi(s_2))_2 = s_2 . \tag{54}$$

Let $i \in \{3, \ldots, n\}$. Then one has $i = s_1^{i-2}(2)$, and hence

$$s_1^{i-2} s_2 s_1^{-(i-2)}(i) = s_1^{i-2} s_2(2) = s_1^{i-2}(2) = i . \tag{55}$$

This concludes that

$$(\varphi(s_2))_i = s_1^{i-2} s_2 s_1^{-(i-2)} \quad (\text{for } i \in \{3, \ldots, n\}) , \tag{56}$$

which completes the proof of Theorem 4.3.

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References

[1] J. S. CARTER, A Survey of Quandle Ideas, *Introductory Lectures on Knot Theory*, eds. L. H. Kauffman et al., Ser. Knots Everything 46, World Sci. Publ., 2012, 22–53.

[2] J. S. CARTER, D. JELSOVSKY, S. KAMADA, L. LANGFORD and M. SAITO, Quandle cohomology and state-sum invariants of knotted curves and surfaces, Trans. Amer. Math. Soc. **355** (2003), 3947–3989.

[3] R. FENN and C. ROURKE, Racks and links in codimension two, J. Knot Theory Ramifications **1** (1992), 343–406.

[4] C. HAYASHI, Canonical forms for operation tables of finite connected quandles, Comm. Algebra **41** (2013), 3340–3349.

[5] A. ISHII, M. IWAKIRI, Y. JANG and K. OSHIRO, A G-family of quandles and handlebody-knots, Illinois J. Math. **57** (2013), 817–838.

[6] D. JOYCE, A classifying invariant of knots, the knot quandle, J. Pure Appl. Algebra **23** (1982), 37–65.

[7] S. KAMADA, *Kyokumen musubime riron* (Surface-knot theory) (in Japanese), Springer Gendai Sugaku Series 16, Maruzen Publishing Co. Ltd., 2012.

[8] S. KAMADA and K. OSHIRO, Homology groups of symmetric quandles and cocycle invariants of links and surface-links, Trans. Amer. Math. Soc. **362** (2010), 5501–5527.

[9] P. LOPES and D. ROSEMAN, On finite racks and quandles, Comm. Algebra **34** (2006), 371–406.

[10] T. NOSAKA, Quandle cocycles from invariant theory, Adv. Math. **245** (2013), 423–438.

[11] H. TAMARU, Two-point homogeneous quandles with prime cardinally, J. Math. Soc. Japan **65** (2013), 1117–1134.

[12] L. VENDRAMIN, On the classification of quandles of low order, J. Knot Theory Ramifications **21** (2012), 1250088, 10 pp.

[13] L. VENDRAMIN, Doubly transitive groups and cyclic quandles, J. Math. Soc. Japan, to appear.

[14] K. WADA, Two-point homogeneous quandles with cardinality of prime power, Hiroshima Math. J. **45** (2015), 165–174.

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