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Non-orientable Genus of a Knot in Punctured $\mathbf{C}P^2$

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Abstract. For a closed 4-manifold X, any knot K in the boundary of punctured X bounds a non-orientable and null-homologous embedded surface in punctured X. Thus we can define an invariant $\gamma_X^0(K)$ to be the smallest first Betti number of such surfaces. Note that $\gamma_{S^4}^0$ is equal to the non-orientable 4-ball genus. While it is very likely that for a given X, γ_X^0 has no upper bound, it is difficult to show it. Recently, Batson showed that $\gamma_{S^4}^0$ has no upper bound. In this paper we show that for any positive integer n, $\gamma_{n\mathbf{C}P^2}^0$ has no upper bound.

1. Introduction

Throughout this paper, we assume that all manifolds and embedding dealt in this paper are smooth. Moreover, we assume that all 4-manifolds are orientable, oriented and simply-connected, and all surfaces are compact. If X is a closed 4-manifold, punc X denotes X with an open 4-ball deleted.

Let X be a closed 4-manifold and K a knot in $\partial(\text{punc}\,X)$. We say that K *bounds* F *in* $\partial(\text{punc}\,X)$ if F is a surface embedded in punc X with boundary K. For a given 4-manifold X and a second homology class of punc X, the set which consists of the diffeomorphism types of embedding surfaces representing the class and that K bounds, is a significant invariant of the isotopy type of K. In the simplest case that $X = S^4$ and the embedded surfaces are all restricted to orientable surfaces, such an invariant has been studied as *4-ball genus* g_4 by many topologists. For a knot K in $\partial(\text{punc}\,X) \cong S^3$, it is natural to ask which kinds of surfaces K can bound.

In this paper, we focus on non-orientable surfaces embedded in punc X with boundary K. It is known that for any homology class $\xi \in H_2(\text{punc}\,X, \partial(\text{punc}\,X); \mathbf{Z}_2)$ and any knot K in $\partial(\text{punc}\,X)$, K bounds a non-orientable surface which represents ξ. Hence we can define $\gamma_X(K, \xi)$ to be the smallest first Betti number of any non-orientable surface embedded in punc X with boundary K which represents ξ. In particular, we investigate the smallest number

$$\gamma_X(K) := \min\{\gamma_X(K, \xi) | \xi \in H_2(\text{punc}\,X, \partial(\text{punc}\,X); \mathbf{Z}_2)\}$$

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and $\gamma^0(K) := \gamma_X(K, 0)$ in this paper, since they can be defined for any 4-manifold X and characterize X from the viewpoint of knot theory. Moreover, both $\gamma_{S^4}(K)$ and $\gamma^0_{S^4}(K)$ are equal to the *non-orientable 4-ball genus* $\gamma_4(K)$, which is the smallest first Betti number of any non-orientable surface embedded in B^4 with boundary K. Hence $\gamma_X(K)$ and $\gamma^0_X(K)$ are generalizations of $\gamma_4(K)$.

While γ_4 has been investigated since 1975 [9], it is still a difficult problem to evaluate γ_4. In fact, it had been unknown whether or not γ_4 has an upper bound until recently. An excellent reference for related studies is [2]. In 2012, Batson proved that γ_4 has no upper bound by establishing the following inequality.

THEOREM 1 ([1]). *Let $K \subset S^3$ be a knot. Then*

$$\gamma_4(K) \geq \frac{-\sigma(K)}{2} + d(S^3_1(K)),$$

where $\sigma(K)$ denotes the signature of K and $d(S^3_1(K))$ the Heegaard-Floer d-invariant of the 1-surgery along K.

The definition of the (p/q)-surgery $S^3_{p/q}(K)$ along K will be given at the last of this section.

In particular, Batson showed that $\gamma_4(T_{2k,2k-1}) = \gamma_4(T_{-2k,2k-1}) = k - 1$ for any positive integer k, where $T_{p,q}$ denotes the right handed (p, q)-torus knot.

On the other hand, we can see that $T_{-2k,2k-1}$ bounds a null-homologous embedded Möbius band in punc $\mathbf{C}P^2$ as follows. We first consider a Möbius band properly embedded in B^4 with boundary the unknot. Then the boundary of this surface can be deformed to $T_{2k,1}$ by an isotopy. After attaching the $(+1)$-framed 2-handle and handle sliding as in Figure 1, we have the desired surface in punc $\mathbf{C}P^2$. Note that $T_{2k,2k-1}$ bounds a null-homologous embedded Möbius band in punc $\overline{\mathbf{C}P^2}$. This fact implies that γ_X and γ^0_X largely depend on the choice of X.

In this paper, we consider the existence problem of upper bounds of non-orientable X-genera for a general 4-manifold X.

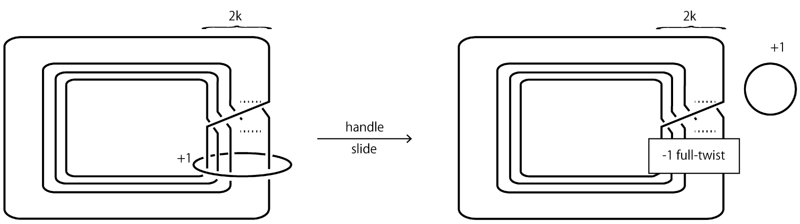

FIGURE 1. $T_{-2k,2k-1}$ bounds a null-homologous embedded Möbius band in punc $\mathbf{C}P^2$.

PROBLEM 1. *For a given* 4*-manifold* X, *do* γ_X *and* γ_X^0 *have upper bounds*?

In the case of γ_X, it is known that there exist infinitely many 4-manifolds which give the affirmative answer of Problem 1. In fact, Suzuki [8] and Norman [3] proved that if X is diffeomorphic to $S^2 \times S^2$ or $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$, then any knot bounds an embedded disk D in punc X. Moreover, by taking the connected sum of (punc X, D), $(S^4, \mathbf{R}P^2)$ and the pair $(N, \emptyset)$ of any closed 4-manifold N and the empty set, we see that any knot which bounds an embedded disk in punc X also bounds an embedded Möbius band in punc$(X \# N)$. This implies that if X is diffeomorphic to $(S^2 \times S^2) \# N$ or $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2} \# N$ for a closed 4-manifold N, then the value of γ_X is always 1. We note that these results have been restricted to indefinite 4-manifolds (i.e., 4-manifolds with indefinite intersection forms).

Now, if X is a definite 4-manifold, then do γ_X and γ_X^0 have upper bounds? In this paper, we give the negative answer for Problem 1 in the case of $\gamma^0_{n\mathbf{C}P^2}$.

THEOREM 2. *For any* $n, k \in \mathbf{N}$, *there exists a knot* K *such that* $\gamma^0_{n\mathbf{C}P^2}(K) = k$.

We prove Theorem 2 by extending Batson's inequality to the case of $\gamma^0_{n\mathbf{C}P^2}$ as follows.

THEOREM 3. *Let* $K \subset S^3$ *be a knot and* $n \in \mathbf{N}$. *Then we have*

$$\gamma^0_{n\mathbf{C}P^2}(K) \geq \frac{-\sigma(K)}{2} + d(S^3_1(K)) - n\,.$$

Moreover, since $-\sigma(T_{2k,2k-1}) = \sigma(T_{-2k,2k-1}) = 2k^2 - 2$, $d(S^3_1(T_{2k,2k-1})) = -k^2 + k$ and $d(S^3_1(T_{-2k,2k-1})) = 0$, it follows that

$$\gamma^0_{\mathbf{C}P^2}(T_{2k,2k-1}) \geq k - 2 \text{ and } \gamma^0_{\mathbf{C}P^2}(T_{-2k,2k-1}) = 1\,.$$

On the other hand, one should compare these conditions with the equalities

$$\gamma_4(T_{2k,2k-1}) = \gamma_4(T_{-2k,2k-1}) = k - 1\,.$$

Here we define *(Dehn)* (p/q)*-surgery* along a knot K in S^3. Let $\nu(A)$ denote the tubular neighborhood of a submanifold A. We define the p/q*-surgery* along a knot K in S^3 to be the following 3-manifold

$$S^3_{p/q}(K) := [S^3 - \nu(K)] \cup (D^2 \times S^1)$$

which is obtained by removing the neighborhood of K and by gluing one solid torus along the both boundaries. The gluing map is defined to be

$$\partial D^2 \times \{\text{pt}\} \to q \cdot \lambda + p \cdot \mu\,,$$

where μ and λ are the meridian and longitude of K. The notation $q \cdot \lambda + p \cdot \mu$ stands for a simple closed curve on the torus whose homology class is $q[\lambda] + p[\mu]$.

2. A short review of the Heegaard Floer theory

In this section we give a brief review of the definition of Heegaard-Floer d-invariant, and some results which are employed in the present paper.

Let $(Y, \mathfrak{s})$ be an oriented closed 3-manifold associated with a Spin^c structure $\mathfrak{s}$. We call such a pair $(Y, \mathfrak{s})$ a *Spin^c 3-manifold*. Ozsváth and Szabó in [4] defined the *Heegaard Floer homologies* $HF^*(Y, \mathfrak{s})$ $(* = +, -, \infty)$ for any Spin^c 3-manifold $(Y, \mathfrak{s})$. If the 1st Chern class of $\mathfrak{s}$ is a torsion element, then the Heegaard Floer homologies become absolutely $\mathbf{Q}$-graded $\mathbf{F}[U] \otimes (H_1(Y)/\mathrm{Tors})$-modules, where U is the action decreasing the grading by 2. Throughout this paper, we consider the coefficient field $\mathbf{F}$ of all homologies as the field with $\mathrm{char}(\mathbf{F}) = 2$. These homology groups are related to one another by the following exact sequence:

$$\cdots \to HF^-(Y, \mathfrak{s}) \to HF^\infty(Y, \mathfrak{s}) \xrightarrow{\pi} HF^+(Y, \mathfrak{s}) \to HF^-(Y, \mathfrak{s}) \to \cdots$$

Let Y be a rational homology 3-sphere. *The* (*Heegaard-Floer*) *d-invariant* $d(Y, \mathfrak{s})$ is defined to be the minimal grading of the image of the map π and the value is a rational number. The one component of the map π is isomorphic to $\mathbf{F}[U, U^{-1}]/(U \cdot \mathbf{F}[U])$ and we denote it by $\mathcal{T}^+_{(d)}$, where d is the minimal grading of the component. If Y is an integer homology 3-sphere, then Y has a unique Spin^c structure. In such a case, we denote the d-invariant simply by $d(Y)$ and the value of the invariant becomes an even integer.

Let $(Y, \mathfrak{s})$ be a Spin^c 3-manifold with a torsion Spin^c structure. Let $d_b(Y, \mathfrak{s})$ denote the *bottom-most d-invariant*, i.e., the minimal grading of the image of π in the kernel of the $(H_1(Y)/\mathrm{Tors})$-action. Then, the following theorem follows. Here β_i is the i-th Betti number and $\beta_2^+(X)$ (or $\beta_2^-(X)$) is the number of positive (or negative respectively) eigenvalues of the intersection form on $H_2(X)$. A Spin^c 3-manifold $(Y, \mathfrak{s})$ with a torsion Spin^c structure is said to have *standard HF^∞* if $HF^\infty(Y, \mathfrak{s})$ is isomorphic to $\mathcal{T}^\infty := (\Lambda^* H^1(Y, \mathbf{Z})) \otimes \mathbf{F}[U, U^{-1}]$.

THEOREM 4 ([4]). *Let $(Y, \mathfrak{t})$ be a Spin^c 3-manifold* (*not necessarily connected*) *with a torsion Spin^c structure which has standard HF^∞. If X is a negative semi-definite 4-manifold with boundary Y such that the restriction map $H^1(X; \mathbf{Z}) \to H^1(Y; \mathbf{Z})$ is trivial, and $\mathfrak{s}$ is a Spin^c structure on X restricting to $\mathfrak{t}$ on Y, then*

$$c_1(\mathfrak{s})^2 + \beta_2^-(X) \le 4d_b(Y, \mathfrak{t}) + 2\beta_1(Y)\,.$$

Let K be a knot in S^3. In [5] the double complex $(CFK^\infty(S^3, K), \partial^\infty)$ with coordinates i, j is defined to be a filtered chain complex of $CF^\infty(S^3)$ associated with K in S^3. It is called *knot Floer chain complex*, and its homology group is called *knot Floer homology*. In this paper, we often omit S^3 in $CFK^\infty(S^3, K)$. The filtered chain homotopy type is a knot isotopy invariant. For the knot Floer homology, we use the same notations as the ones in [5]. We also use the notation ∂^∞ for the differentials even for restricted or quotient complexes of CFK^∞.

We introduce the following proposition and formula for a sufficiently large integer p:

PROPOSITION 1 ([5]). *For a sufficiently large integer p, we have the following isomorphism*

$$HF_{\ell}^{+}(S_p^3(K), [0]) \cong H_k(CFK^{\infty}\{i \geq 0 \text{ or } j \geq 0\}),$$

where $\ell = k + (p-1)/4$.

In particular, we have

$$d(S_1^3(K)) = \tilde{d}(S_p^3(K), [0]), \tag{1}$$

where $\tilde{d}$ is *the unshifted d-invariant*, i.e., $\tilde{d}(S_p^3(K), [0]) = d(S_p^3(K), [0]) - (p-1)/4$.

3. Extension of Batson's inequality

In order to prove Theorem 3, we first prove the following proposition.

PROPOSITION 2. *Let $K \subset \partial(\mathrm{punc}(n\overline{\mathbf{C}P^2}))$ be a knot and $F \subset \mathrm{punc}(n\overline{\mathbf{C}P^2})$ a non-orientable embedded surface with boundary K. Then*

$$\beta_1(F) \geq \frac{e(F)}{2} - 2d(S_{-1}^3(K)).$$

Batson showed in [1, Theorem 4] that this inequality holds for the case where $n = 0$; that is, $F \subset B^4$. Hence this proposition is an extension of [1, Theorem 4].

In Proposition 2, $e(F)$ is the *normal Euler number* of F defined as follows. Let X be a closed 4-manifold and F a properly embedded surface in $\mathrm{punc}\, X$ with $\partial F \cong S^1$. Take an orientation of ∂F and a section $\tilde{F}$ of the normal bundle of F that does not intersect F. Let $e(F) = -\,\mathrm{lk}(\partial F, \partial \tilde{F})$, where the orientation of $\partial \tilde{F}$ is induced from ∂F. Note that $e(F)$ does not depend on the choice of the orientation for ∂F. We call $e(F)$ the *normal Euler number* of F (see [10]). We also note that if F is orientable, then $e(F)$ is equal to the self-intersection number of F.

In order to prove Proposition 2, we need the following lemma. Let X be a closed 4-manifold and $K \subset \partial(\mathrm{punc}\, X)$ a knot. We identify $H_2(X; \mathbf{Z}) \oplus H_2(S^2 \times S^2; \mathbf{Z})$ with $H_2(\mathrm{punc}(X\#(S^2 \times S^2)), \partial(\mathrm{punc}(X\#(S^2 \times S^2))); \mathbf{Z})$.

LEMMA 1. *For any non-orientable embedded surface $F \subset \mathrm{punc}\, X$ with boundary K and odd β_1, there exists an orientable embedded surface $F' \subset \mathrm{punc}(X\#(S^2 \times S^2))$ with boundary K which satisfies*

1. $\beta_1(F') = \beta_1(F) - 1$,
2. $e(F') = e(F) + 2$, *and*
3. $[F', \partial F'] = v \oplus (2\alpha + b\beta)$ *for some* $v \in H_2(X; \mathbf{Z})$ *and* $b \in \mathbf{Z}$.

Here α and β are standard generators of $H_2(S^2 \times S^2; \mathbf{Z})$ such that $\alpha \cdot \alpha = \beta \cdot \beta = 0$, and $\alpha \cdot \beta = 1$.

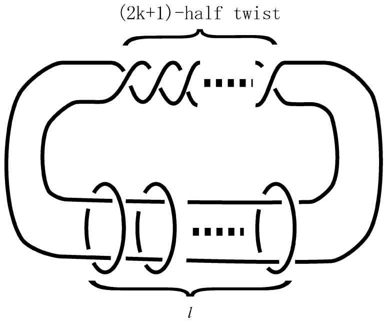

FIGURE 2. Our link L.

PROOF. Since $\beta_1(F)$ is odd, there exists a simple closed curve C in F whose tubular neighborhood in F is diffeomorphic to the Möbius band and $F \setminus C$ is orientable. Since punc X is simply-connected, C is null-homotopic in X. Moreover, every homotopy may be replaced with an isotopy in these dimensions, and hence C is isotopic to a trivial circle. This implies that C bounds an embedded disk D in punc X. We can assume that D is transverse to F in the interior of D. Then $F \cap D$ consists of C and finitely many points $\{p_i\}$ $(i = 1, 2, \ldots, l)$. Moreover, $\nu(D)$ is diffeomorphic to $D \times D^2$, and $F \cap \nu(D)$ consists of a Möbius band properly embedded in $\partial D \times D^2$ and l 2-disks $p_i \times D^2$. This implies that $L := \partial(F \cap \nu(D)) \subset \partial\nu(D)$ is a link as in Figure 2. In the same way as Step 3 and Step 4 in the proof of [1, Proposition 1.4], we can verify that L bounds $l + 1$ embedded disks E in punc$(S^2 \times S^2)$ which satisfy $e(E) = e(F \cap \nu(D)) + 2$. Finally, by removing $\nu(D)$ from punc X and gluing (punc$(S^2 \times S^2)$, E) along $(\partial\nu(D), L)$, we obtain a new orientable embedded surface F' in punc$(X\#(S^2 \times S^2))$. It is easy to check that F' satisfies the above conditions from (1) to (3). □

We next prove the following lemma, which is a generalization of a discussion in [1, Section 4].

LEMMA 2. *Let M be an integer homology 3-sphere, X a simply-connected 4-manifold such that $\partial X = M$ and $\beta_2^+(X) = 1$ and Σ an orientable closed surface embedded in X with genus g and self-intersection m. Then for any Spin^c structure $\mathfrak{s}$ on X which satisfies $\langle c_1(\mathfrak{s}), [\Sigma]\rangle = m - 2g > 0$, the following inequality holds*:

$$c_1(\mathfrak{s})^2 + \beta_2^-(X) \leq 1 + 4d(M)\,.$$

PROOF. Let X' be the complement $X \setminus \nu(\Sigma)$. Then X' is a negative semi-definite 4-manifold with disconnected boundaries $Y_{g,-m} \amalg M$, where $Y_{g,-m}$ denotes the Euler number $-m$ circle bundle over Σ.

We prove that X' is negative semi-definite. Let $n(X')$ denote the number of zero eigenvectors in the intersection form $Q_{X'}$. We can verify from elementary homology theory that

$\beta_2(X') = \beta_2(X) + 2g - 1 = \beta_2^-(X) + 2g$. Furthermore, by Novikov's additivity formula, $\sigma(X') = \sigma(X) - \sigma(\nu(\Sigma)) = -\beta_2^-(X)$. Thus we have $2\beta_2^+(X') + n(X') = 2g$, and $\beta_2^+(X') \le 0$ is equivalent to $n(X') \ge 2g$. The homology exact sequence of the pair $(X', Y_{g,-m})$ shows the following exact sequence:

$$H_2(X'; \mathbf{Q}) \to H_2(X', Y_{g,-m}; \mathbf{Q}) \overset{surj.}{\to} H_1(Y_{g,-m}; \mathbf{Q}) \cong \mathbf{Q}^{2g}$$

and it implies $n(X') \ge 2g$.

We apply Theorem 4 to the tuple $(X', Y_{g,-m} \amalg M, \mathfrak{s}|_{X'})$. The standard-ness of $Y_{g,-m} \amalg M$ is described in the proof of [1, Theorem 1.5]. Moreover, we can verify in the same way as [1, Section 4] that $H^1(X'; \mathbf{Z}) = 0$ and the image of the restriction map $H^2(\nu(\Sigma); \mathbf{Z}) \to H^2(Y_{g,-m}; \mathbf{Z})$ is a torsion group. This implies that for any Spin^c structure on X, the restricted Spin^c structure on $Y_{g,-m}$ is a torsion Spin^c structure. Thus, it follows that the tuple $(X', Y_{g,-m} \amalg M, \mathfrak{s}|_{X'})$ satisfies all conditions of Theorem 4.

By Theorem 4, we have

$$c_1(\mathfrak{s}|_{X'})^2 + \beta_2^-(X') \le 4d_b(Y_{g,-m}, \mathfrak{s}|_{Y_{g,-m}}) + 4d(M) + 2\beta_1(Y_{g,-m})\,. \tag{2}$$

Let us compute each term in the inequality (2). In order to compute $c_1(\mathfrak{s}|_{X'})^2$, we decompose the intersection form of X in terms of the $\mathbf{Q}$-valued intersection forms on $\nu(\Sigma)$ and X'; if $c \in H^2(X)$, then

$$Q_X(c) = Q_{\nu(\Sigma)}(c|_{\nu(\Sigma)}) + Q_{X'}(c|_{X'})\,.$$

This gives $c_1(\mathfrak{s})^2 = c_1(\mathfrak{s}|_{\nu(\Sigma)})^2 + c_1(\mathfrak{s}|_{X'})^2$. Hence we have

$$c_1(\mathfrak{s}|_{X'})^2 = c_1(\mathfrak{s})^2 - c_1(\mathfrak{s}|_{\nu(\Sigma)})^2 = c_1(\mathfrak{s})^2 - \frac{(m-2g)^2}{m}\,.$$

For the above Spin^c structure $\mathfrak{s}|_{Y_{g,-m}}$, the d-invariant of $Y_{g,-m}$ is computed in [4, Section 9]. If $\langle c_1(\mathfrak{s}), [\Sigma]\rangle = m - 2g > 0$, then

$$d_b(Y_{g,-m}, \mathfrak{s}|_{Y_{g,-m}}) = \frac{1}{4} - \frac{g^2}{m} - \frac{m}{4}\,.$$

The substitution of all the values computed above reduces (2) to

$$c_1(\mathfrak{s})^2 - \frac{(m-2g)^2}{m} + \beta_2^-(X') \le 4\left(\frac{1}{4} - \frac{g^2}{m} - \frac{m}{4}\right) + 4d(M) + 2(2g)\,. \tag{3}$$

Since $\beta_2^-(X') = \beta_2^-(X)$, (3) gives the inequality

$$c_1(\mathfrak{s})^2 + \beta_2^-(X) \le 1 + 4d(M)\,.$$

□

PROOF OF PROPOSITION 2. Note that for any knot K, $d(S^3_{-1}(K)) \geq 0$. Hence in the case that $e(F) \leq \beta_1(F)$, it is clear that this proposition holds. Therefore we assume that $e(F) > \beta_1(F)$.

We first give the proof for the case where $\beta_1(F)$ is odd. By applying Lemma 1 to $F \subset \mathrm{punc}(n\mathbf{C}P^2)$, we obtain an orientable embedded surface $F' \subset \mathrm{punc}(n\overline{\mathbf{C}P^2} \# (S^2 \times S^2))$ with boundary K whose homology class is

$$[F', \partial F'] = \sum_{i=1}^{j} 2a_i\overline{\gamma}_i + \sum_{i=j+1}^{n} (2a_i+1)\overline{\gamma}_i + 2\alpha + b\beta \quad (a_i, j \in \mathbf{Z}, 0 \leq j \leq n),$$

where $\overline{\gamma}_i$ $(i = 1, ..., n)$ are standard generators of $H_2(n\overline{\mathbf{C}P^2}; \mathbf{Z})$ such that $\overline{\gamma}_i \cdot \overline{\gamma}_j = -\delta_{ij}$ (Kronecker's delta). Without loss of generality, we may permutate the order of $\overline{\gamma}_i$.

Since F' is orientable, we have

$$e(F') = [F', \partial F'] \cdot [F', \partial F'] = -\sum_{i=1}^{j} 4a_i^2 - \sum_{i=j+1}^{n} (2a_i+1)^2 + 4b.$$

Attaching a (-1)-framed 2-handle along K, we have a 4-manifold $\overline{W}$ with boundary $S^3_{-1}(K)$ and the intersection form

$$Q_{\overline{W}} = \left(\begin{array}{c|ccc|cc} -1 & 0 & \dots & 0 & 0 & 0 \\ \hline 0 & -1 & & O & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & O & & -1 & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{array}\right).$$

We may cap off F' with the core of the 2-handle to form a closed surface Σ embedded in $\overline{W}$ with genus $g = (b_1(F) - 1)/2$, homology class $\overline{\gamma}_0 + \sum_{i=1}^{j} 2a_i\overline{\gamma}_i + \sum_{i=j+1}^{n}(2a_i+1)\overline{\gamma}_i + 2\alpha + b\beta$, and the self-intersection number

$$m = -1 - \sum_{i=1}^{j} 4a_i^2 - \sum_{i=j+1}^{n} (2a_i+1)^2 + 4b = e(F) + 1 > 0.$$

We next choose a Spin^c structure on W. Since $H^2(\overline{W}; \mathbf{Z}) \cong \mathbf{Z}^{n+3}$ has no 2-torsion, Spin^c structures on $\overline{W}$ are distinguished by their first Chern classes. Fix a Spin^c structure $\mathfrak{s}_t$ on $\overline{W}$ satisfying

$$PD(c_1(\mathfrak{s}_t)) = \varepsilon\overline{\gamma}_0 + \sum_{i=1}^{n}(2a_i+1)\overline{\gamma}_i + 2\alpha + 2x\beta,$$

where

$$x = \frac{\sum_{i=1}^{j} 2a_i + 2(b-g) - 1 + \varepsilon}{4}$$

and $\varepsilon \in \{1, -1\}$ is chosen so as to make x an integer. Since the given vector is characteristic for $Q_{\overline{W}}$, it corresponds to a Spin^c structure. Furthermore, $\langle c_1(\mathfrak{s}_t), [\Sigma]\rangle = m - 2g = e(F) - \beta_1(F) + 2 > 0$. Applying Lemma 2 to the pair $(\overline{W}, S^3_{-1}(K))$, we have

$$c_1(\mathfrak{s}_t)^2 + \beta_2^-(\overline{W}) \leq 1 + 4d(S^3_{-1}(K))\,. \tag{4}$$

Since $c_1(\mathfrak{s}_t)^2 = -1 - \sum_{i=1}^{n}(2a_i+1)^2 + 8x = e(F) - j - 1 + 2\varepsilon - 4g$, the inequality (4) implies

$$(e(F) - j - 1 + 2\varepsilon - 4g) + (n+2) \leq 1 + 4d(S^3_{-1}(K))\,. \tag{5}$$

By using $-1 \leq \varepsilon$, $j \leq n$, and $2g = \beta_1(F) - 1$, the inequality (5) reduces to the following inequality

$$\frac{e(F)}{2} - 2d(S^3_{-1}(K)) \leq \beta_1(F)\,. \tag{6}$$

Finally, we consider the case where $\beta_1(F)$ is even. Taking the connected sum of $F \subset \mathrm{punc}(n\overline{\mathbf{C}P^2})$ and the standard embedding of $\mathbf{R}P^2 \subset S^4$ whose normal Euler number is $+2$, we have a non-orientable embedded surface $\hat{F} \subset \mathrm{punc}(n\overline{\mathbf{C}P^2})$ with boundary K such that $\beta_1(\hat{F}) = \beta_1(F) + 1$ and $e(\hat{F}) = e(F) + 2$. Since $\beta_1(\hat{F})$ is odd, $\hat{F}$ satisfies the inequality (6). Hence we have

$$\frac{(e(F)+2)}{2} - 2d(S^3_{-1}(K)) \leq \beta_1(F) + 1\,, \tag{7}$$

and this inequality (7) is equivalent to the inequality claimed in Proposition 2.

This completes the proof of Proposition 2. □

4. Proof of Theorem 3

In this section, we prove Theorem 3 by using Proposition 2 and the following theorem.

THEOREM 5 ([10]). *Let X be a closed 4-manifold and $K \subset \partial(\mathrm{punc}\,X)$ a knot. If K bounds a non-orientable embedded surface F in $\mathrm{punc}\,X$ that represents zero in $H_2(\mathrm{punc}\,X, \partial(\mathrm{punc}\,X); \mathbf{Z}_2)$, then*

$$\left|\sigma(K) + \sigma(X) - \frac{e(F)}{2}\right| \leq \beta_2(X) + \beta_1(F)\,,$$

where $\sigma(X)$ is the signature of X.

By reversing the orientation of X, we obtain the following lemma. We also use this lemma to prove Theorem 3.

LEMMA 3. *For any* 4*-manifold* X *and any knot* K, *the following equality holds*:

$$\gamma^0_X(K) = \gamma^0_{-X}(K^*)\,,$$

where K^* *denotes the mirror image of* K.

Let us prove Theorem 3.

PROOF OF THEOREM 3. Let $F \subset \mathrm{punc}(n\overline{\mathbf{C}P^2})$ be a non-orientable surface with boundary K which represents zero in $H_2(\mathrm{punc}(n\overline{\mathbf{C}P^2}), \partial(\mathrm{punc}(n\overline{\mathbf{C}P^2})); \mathbf{Z}_2)$. It follows from Theorem 5 that

$$\left|\sigma(K) + (-n) - \frac{e(F)}{2}\right| \le n + \beta_1(F)\,.$$

Hence we have

$$\beta_1(F) \ge \sigma(K) - \frac{e(F)}{2} - 2n\,.$$

Combining this inequality with Proposition 2, we have

$$\gamma^0_{n\overline{\mathbf{C}P^2}}(K) \ge \frac{\sigma(K)}{2} - d(S^3_{-1}(K)) - n\,.$$

By using this inequality and Lemma 3, it follows that for any knot $K \subset \partial(\mathrm{punc}(n\mathbf{C}P^2))$,

$$\gamma^0_{n\mathbf{C}P^2}(K) = \gamma^0_{n\overline{\mathbf{C}P^2}}(K^*) \ge \frac{\sigma(K^*)}{2} - d(S^3_{-1}(K^*)) - n = \frac{-\sigma(K)}{2} + d(S^3_1(K)) - n\,.$$

This proves Theorem 3. □

5. Proof of Theorem 2

To prove the existence of K with $\gamma^0_{n\mathbf{C}P^2}(K) = k$, we take the connected sum of $n + k$ copies of 9_{42} for any positive integer k, where 9_{42} is the knot defined in the Rolfsen knot table [7]. Notice that $d(S^3_1(\cdot))$ is a knot concordance invariant but not a homomorphism from the knot concordance group to integers as mentioned in [6].

PROPOSITION 3. *We have* $d(S^3_1(\overset{m}{\#} 9_{42})) = 0$ *for any positive integer* m.

PROOF. Let p be a sufficiently large integer. From the formula (1) and Proposition 1, we obtain the d-invariant $d(S^3_1(\overset{m}{\#} 9_{42}))$ by calculating the homology of $CFK^\infty(\overset{m}{\#} 9_{42})\{i \ge 0 \text{ or } j \ge 0\}$.

First, we consider the $m = 1$ case. For the generators $\{x_i\}_{1\le i\le 9}$ of $CFK^\infty(9_{42})$, we use the same generators as those in Fig.14 in [5] (see Figure 3). Let S_1 denote $\{x_i \mid 1 \le i \le 9\}$. Let

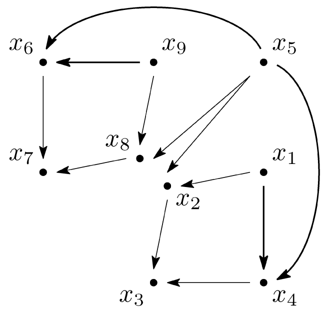

FIGURE 3. The differential maps of G in $CFK^\infty(9_{42})$. See Fig. 14 in [5].

G be the differential $\mathbf{F}$-module generated by S_1, namely, $\mathbf{F}\langle x | x \in S_1\rangle$ with $\mathrm{gr}(x_5) = 0$, where gr is the absolute grading on the chain complex CFK^∞. The chain complex $CFK^\infty(9_{42})$ consists of a differential $\mathbf{F}[U]$-module $G[U, U^{-1}] := G \otimes_{\mathbf{F}} \mathbf{F}[U, U^{-1}]$. The generators of $H_*(CFK^\infty(9_{42}))$ are $\{U^{-i} \cdot \alpha\}_{i\in\mathbf{Z}}$, where $\alpha = x_1 + x_5 + x_9$. The homology of the quotient complex $CFK^\infty(9_{42})\{i \geq 0 \text{ or } j \geq 0\}$ is as follows:

$$H_*(CFK^\infty(9_{42})\{i \geq 0 \text{ or } j \geq 0\}) \cong \mathbf{F}\langle U^{-i}\alpha | i \geq 0\rangle \cong \mathcal{T}^+_{(0)}\,.$$

In fact, it follows from the grading of α that the minimal grading of this homology is zero. Hence, in particular, $d(S^3_1(9_{42})) = \tilde{d}(S^3_p(9_{42}, [0])) = 0$.

Next, we compute $d(S^3_1(\overset{m}{\#}9_{42}))$. From the Künneth type formula of the Heegaard Floer homology we have $CFK^\infty(\overset{m}{\#}9_{42}) \cong \bigotimes^m CFK^\infty(9_{42})$. We denote the set of generators by $S_m = \{x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_m} \,|\, 1 \leq i_k \leq 9\}$ and the vector space generated by S_m by $G_m = \mathbf{F}\langle x \,|\, x \in S_m\rangle$, where $\mathrm{gr}(x_5^{\otimes m}) = 0$. Here let $y^{\otimes m}$ denote the m-th tensor product $y \otimes \cdots \otimes y$.

The chain complex $CFK^\infty(\overset{m}{\#}9_{42})$ is the summation

$$\bigoplus_{i\in\mathbf{Z}}(U^{-i} \cdot G_m) = \mathbf{F}[U, U^{-1}] \otimes_{\mathbf{F}} G_m =: G_m[U, U^{-1}]\,.$$

Hence, we may consider each homology $H_*(U^{-i} \cdot G_m\{i \geq 0 \text{ or } j \geq 0\})$.

The differential ∂^∞ in $\bigotimes^m CFK^\infty(9_{42})$ is computed as follows:

$$\partial^\infty(z_1 \otimes \cdots \otimes z_m) = \sum_{k=1}^{m} z_1 \otimes \cdots \otimes \partial^\infty z_k \otimes \cdots \otimes z_m.$$

By using this definition, we have $\partial^\infty(U^{-l} \cdot \alpha^{\otimes m}) = 0$. Since $U^{-l} \cdot \alpha^{\otimes m}$ has the unique top grading in $U^{-l} \cdot G_m$, we have $\alpha^{\otimes m} \notin \mathrm{Im}(\partial^\infty)$. Hence the generator of $H_*(U^{-l} \cdot G_m)$ is

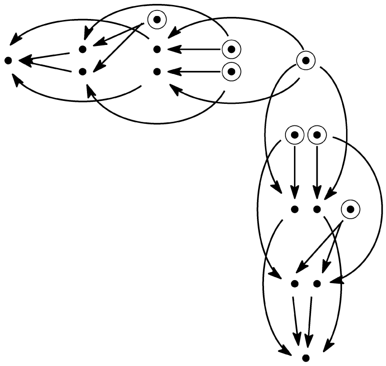

FIGURE 4. The chain complex $G_2\{i \geq 0 \text{ or } j \geq 0\}$ and the homological generator $(x_5 + x_9)^{\otimes 2} + (x_5 + x_1)^{\otimes 2} + x_5^{\otimes 2}$.

$U^{-l} \cdot \alpha^{\otimes m}$. □

For the case where $l < 0$, since the generators in $U^{-l} \cdot G_m$ are in $CFK^{\infty}(\overset{m}{\#}9_{42})\{i < 0 \text{ and } j < 0\}$, the minimal degree of $\mathcal{T}^+$-component in $CFK^{\infty}(\overset{m}{\#}9_{42})\{i \geq 0 \text{ or } j \geq 0\}$ is non-negative.

We consider the component of $l = 0$. Let φ denote the natural isomorphism:

$$\varphi : G_m/(G_m\{i < 0 \text{ and } j < 0\}) \cong G_m\{i \geq 0 \text{ or } j \geq 0\}.$$

LEMMA 4. *The map φ satisfies the following*:

$$\varphi(\alpha^{\otimes m}) = (x_5 + x_9)^{\otimes m} + (x_5 + x_1)^{\otimes m} + x_5^{\otimes m} .$$

PROOF OF LEMMA 4. Expanding $\alpha^{\otimes m}$, we have

$$\alpha^{\otimes m} = \sum_{i_j \in \{1,5,9\}} x_{i_1} \otimes \cdots \otimes x_{i_m} .$$

If the set $\{i_1, \ldots, i_m\}$ of the suffixes of each term in the summation above contains $\{1, 9\}$, then the (i, j)-coordinate must have $i < 0$ and $j < 0$. Conversely, if the (i, j)-coordinate of $x_{i_1} \otimes \cdots \otimes x_{i_m}$ has $i < 0$ and $j < 0$, then the set $\{i_1, \ldots, i_m\}$ must contain $\{1, 9\}$. The whole sum of the terms $x_{i_1} \otimes \ldots \otimes x_{i_m}$ satisfying $\{1, 9\} \not\subset \{i_1, \ldots, i_m\}$ is $(x_5 + x_9)^{\otimes m} + (x_5 + x_1)^{\otimes m} + x_5^{\otimes m}$. Therefore, the assertion claimed in Lemma 4 follows. □

Here, as an example, we describe the boundary maps in $G_2\{i \geq 0 \text{ or } j \geq 0\}$ in Figure 4.

The term $\varphi(\alpha^{\otimes m})$ is a generator in $H_*(G_m\{i \geq 0 \text{ or } j \geq 0\})$, because $\partial^{\infty}(\varphi(\alpha^{\otimes m})) = \varphi(\partial^{\infty}(\alpha^{\otimes m})) = 0$ and the element $\varphi(\alpha^{\otimes m})$ has the top grading in $G_m\{i \geq 0 \text{ or } j \geq 0\}$. The image $\varphi(\alpha^{\otimes m})$ is in the $\mathcal{T}^+$-component with the minimal grading, because the whole

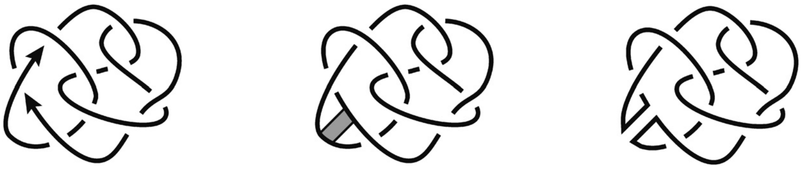

FIGURE 5. The knot 9_{42} bounds an embedded Möbius band in B^4.

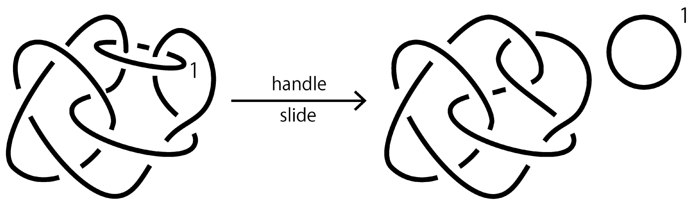

FIGURE 6. The knot 9_{42} bounds an embedded disk in punc $\mathbf{C}P^2$.

chain complex $CFK^\infty(\overset{m}{\#}9_{42})$ is generated by $\{U^{-l} \cdot \alpha^{\otimes m} | l \in \mathbf{Z}\}$. The minimal degree of $G_m\{i \geq 0 \text{ or } j \geq 0\}$ is $\mathrm{gr}(\varphi(\alpha^{\otimes m})) = \mathrm{gr}(\alpha^{\otimes m}) = 0$. This means

$$d(S^3_1(\overset{m}{\#}9_{42})) = \tilde{d}(S^3_p(\overset{m}{\#}9_{42}, [0])) = 0\,.$$

Actually, $(x_5 + x_9)^{\otimes m} + (x_5 + x_1)^{\otimes m} + x_5^{\otimes m}$ is the unique generator in $H_*(G_m\{i \geq 0 \text{ and } j \geq 0\})$. This fact is not needed here, and we skip the proof.

PROOF OF THEOREM 2. Since $\sigma(9_{42}) = -2$ and the knot signature is additive, we have $\sigma(\overset{n+k}{\#}\, 9_{42}) = -2(n+k)$. Thus, by using Theorem 3 and Proposition 3, we have

$$\gamma^0_{n\mathbf{C}P^2}(\overset{n+k}{\#}\, 9_{42}) \geq \frac{-(-2(n+k))}{2} + 0 - n = k\,.$$

We next construct a non-orientable embedded surface $F_{n,k} \subset \mathrm{punc}(n\mathbf{C}P^2)$ satisfying the following:

1. $\partial F_{n,k} = \overset{n+k}{\#}\, 9_{42}$,
2. $\beta_1(F_{n,k}) = k$, and
3. $F_{n,k}$ represents zero in $H_2(\mathrm{punc}(n\mathbf{C}P^2), \partial(\mathrm{punc}(n\mathbf{C}P^2)); \mathbf{Z}_2)$.

The cobordisms in Figure 5 and 6 give a properly embedded Möbius band M in B^4 with boundary 9_{42}, and a properly embedded disk D in punc $\mathbf{C}P^2$ with boundary 9_{42} which represents zero in $H_2(\mathrm{punc}\,\mathbf{C}P^2, \partial(\mathrm{punc}\,\mathbf{C}P^2); \mathbf{Z}_2)$. Taking the boundary connected sum of n

copies of (punc $\mathbf{C}P^2$, D) and k copies of (B^4, M), we have a new non-orientable embedded surface $F_{n,k}$ satisfying the above properties from (1) to (3). This completes the proof. □

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