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Classification of Continuous Fractional Binary Operations on the Real and Complex Fields

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Abstract. In this paper, we consider a classification problem for continuous fractional binary operations on **K**, where **K** denotes the real field **R** or the complex field **C**. We first show that there exist exactly two continuous fractional binary operations on **R** up to isomorphism. In the complex case, we describe completely all continuous fractional binary operations on **C** in terms of ordinary fraction. Applying this description, we give a partial solution to the classification problem in the complex case. Moreover we show that there exist exactly two homogeneous cancellative binary operations on **K** up to isomorphism.

1. Introduction

Recently S. Saitoh gave the formal identities 100/0=0 and 0/0=0 by the concept of Tikhonov regularization using the theory of reproducing kernels. Also he asked whether there exist some real examples supporting the above results (cf. [1, 2]). Actually take two real numbers a, b arbitrarily. For any positive number t,

$$x_t = \frac{ab}{t + b^2}$$

is a value which minimizes the Tikhonov function $tx^2 + (bx - a)^2$. This is called the fractional in the sense of Tikhonov. Put

$$S(a, b) = \lim_{t \to +0} x_t .$$

Then we have

$$S(a, b) = \begin{cases} a/b & (b \neq 0)\,, \\ 0 & (b = 0)\,. \end{cases}$$

We call $S(a, b)$ Saitoh's fraction. Of course we can consider Saitoh's fraction in the complex case.

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In this paper, inspired by his idea, we investigate the continuous fractional binary operations on **K** (see the next section for the definition). Here **K** denotes the field **R** of real numbers or the field **C** of complex numbers.

In fact our purpose is to classify all continuous fractional binary operations on **K**. We first show that there exist exactly two continuous fractional binary operations on **R** up to isomorphism (see Theorem 1). In the complex case, we completely describe all continuous fractional binary operations on **C** in terms of ordinary fraction (see Theorem 2). Applying this description, we give a partial solution to the classification problem in the complex case (see Theorems 3 and 4). Moreover we show that there exist exactly two homogeneous cancellative binary operations on **K** up to isomorphism (see Theorem 5).

2. Preliminary and main results

Let $*$ be a binary operation on **K**. We say that $*$ is fractional if

$$(a+b)*c = (a*c)+(b*c) \quad \text{(distribution)}$$

and

$$(ax)*(bx) = a*b \quad \text{(cancellation)}$$

for all $a, b, c, x \in \mathbf{K}$ with $x \neq 0$. Also we say that $*$ is continuous if the map : $x \mapsto x*b$ is continuous on **K** for each $b \in \mathbf{K}$. Moreover we say that $*$ is homogeneous if

$$(ab)*c = a(b*c)$$

for all $a, b, c \in \mathbf{K}$. Of course, the binary operation $*$ on **K** defined by $a*b = 0$ $(a, b \in \mathbf{K})$ is continuous, fractional and homogeneous. Such a binary operation is said to be trivial. Let $\mathcal{CF}(\mathbf{K})$ be the set of all continuous fractional binary operations on **K**. For two operations $*, \circ \in \mathcal{CF}(\mathbf{K})$ we say that $*$ is isomorphic to $\circ$ (simply $* \cong \circ$) if there exists a homeomorphism $f : \mathbf{K} \to \mathbf{K}$ such that

$$f(a*b) = f(a) \circ f(b)$$

holds for all $a, b \in \mathbf{K}$. Clearly "$\cong$" is an equivalent relation on $\mathcal{CF}(\mathbf{K})$. We hope to classify all continuous fractional binary operations on **K** modulo "$\cong$".

The first classification result is the following theorem which asserts that there exist exactly two continuous fractional binary operations on **R** up to isomorphism.

THEOREM 1. *All nontrivial continuous fractional binary operations on* **R** *are isomorphic to Saitoh's fraction on* **R**.

For the complex case, we can completely describe all continuous fractional binary operations on **C** in terms of ordinary fraction as follows:

THEOREM 2. *If $*$ is a continuous fractional binary operation on* $\mathbf{C}$, *then there exist two unique complex numbers* α *and* β *such that*

$$z * w = \begin{cases} \alpha \operatorname{Re} \frac{z}{w} + i\beta \operatorname{Im} \frac{z}{w} & (w \neq 0) \\ 0 & (w = 0), \end{cases} \tag{1}$$

for all $z, w \in \mathbf{C}$. *Conversely, the binary operation given by* (1) *is a continuous fractional binary operation on* $\mathbf{C}$.

We denote by $*_{(\alpha,\beta)}$ the binary operation defined by (1). Then the map

$$\Phi : (\alpha, \beta) \mapsto *_{(\alpha,\beta)}$$

is a bijection from $\mathbf{C}^2$ to $\mathcal{CF}(\mathbf{C})$.

REMARK. $\Phi(0, 0)$ is the trivial fractional binary operation on $\mathbf{C}$. Also $\Phi(1, 1)$ is just Saitoh's fraction on $\mathbf{C}$.

Let $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and put

$$L_\gamma = \begin{cases} \{(\alpha, \beta) \in \mathbf{C}^2 \setminus \{(0, 0)\} : \beta = \alpha\gamma\} & (\gamma \in \mathbf{C}) \\ \{(\alpha, \beta) \in \mathbf{C}^2 \setminus \{(0, 0)\} : \alpha = 0\} & (\gamma = \infty). \end{cases}$$

Then we have

$$\mathbf{C}^2 = \{(0, 0)\} \cup \bigcup_{\gamma \in \hat{\mathbf{C}}} L_\gamma \quad \text{(disjoint union)}.$$

The following two theorems give a partial solution to the classification problem in the complex case.

THEOREM 3. *For each* $\gamma \in \hat{\mathbf{C}}$, *it holds that* $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$ *for all* $(\alpha, \beta), (\alpha', \beta') \in L_\gamma$.

THEOREM 4. (i) $\Phi(0, 0)$ *is not isomorphic to any nontrivial continuous fractional binary operation on* $\mathbf{C}$.

(ii) *If* $\gamma = 0$ *or* ∞ *and* $\gamma' \in \mathbf{C} \setminus \{0\}$, *then* $\Phi(\alpha, \beta) \ncong \Phi(\alpha', \beta')$ *for each* $(\alpha, \beta) \in L_\gamma$ *and* $(\alpha', \beta') \in L_{\gamma'}$ *with* $\operatorname{Re} \alpha' \overline{\beta'} \neq 0$.

(iii) $\Phi(\alpha, \beta) \ncong \Phi(\alpha', \beta')$ *for each* $(\alpha, \beta) \in L_0$ *and* $(\alpha', \beta') \in L_\infty$.

(iv) *Let* $\alpha, \beta \in \mathbf{C} \setminus \{0\}$. *Then* $\Phi(\alpha, \beta) \cong \Phi(1, 1)$ *if and only if* $\alpha = \beta$.

REMARK. (a) By (ii) and (iv), we have that if $\gamma \in \hat{\mathbf{C}} \setminus \{1\}$, then $\Phi(\alpha, \beta) \ncong \Phi(\alpha', \beta')$ for each $(\alpha, \beta) \in L_1$ and $(\alpha', \beta') \in L_\gamma$. However we do not know whether $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$ or not for each $(\alpha, \beta) \in L_\gamma$ and $(\alpha', \beta') \in L_{\gamma'}$ when $\gamma, \gamma' \in \hat{\mathbf{C}} \setminus \{1\}$ and $\gamma \neq \gamma'$.

(b) If the function $x \mapsto 1 * x$ is continuous at $x = 0$, then $*$ is trivial.

The last classification result is the following theorem which asserts that there exist exactly two homogeneous cancellative binary operations on $\mathbf{K}$ up to isomorphism.

THEOREM 5. *All nontrivial homogeneous cancellative binary operations on* **K** *are isomorphic to Saitoh's fraction on* **K**.

3. Proof of main results

I. PROOF OF THEOREM 1. Let $*$ be a nontrivial continuous fractional binary operation on **R**. For each $b \in \mathbf{R}$, put

$$R(x) = x * b \quad (x \in \mathbf{R}) .$$

Since R is a continuous additive map from **R** to itself, we can find a unique real number $\varphi(b)$ such that $R(x) = x\varphi(b)$ for all $x \in \mathbf{R}$. Therefore we have

$$a * b = a\varphi(b)$$

for all $a, b \in \mathbf{R}$. So $*$ is necessarily homogeneous. Take $a, b \in \mathbf{R}$ with $b \neq 0$ arbitrarily. Put $e = 1 * 1$. Since $*$ is cancellative, it follows that

$$a * b = a\varphi(b) = \frac{ab\varphi(b)}{b} = \frac{a(b * b)}{b} = \frac{a(1 * 1)}{b} = \frac{ae}{b} .$$

Also since $*$ is cancellative and homogeneous, it follows that

$$a * 0 = (2a) * 0 = 2(a * 0) ,$$

and hence $a * 0 = 0$. Therefore we have

$$a * b = \begin{cases} \frac{ae}{b} & (b \neq 0) \\ 0 & (b = 0) . \end{cases}$$

Since $*$ is nontrivial, it follows that $e \neq 0$. Define

$$f(x) = \frac{x}{e}$$

for each $x \in \mathbf{R}$. Then f is a homeomorphism from **R** to itself. Take $a, b \in \mathbf{R}$ arbitrarily. If $b \neq 0$, then $f(b) \neq 0$, and hence

$$f(a * b) = f\left(\frac{ae}{b}\right) = \frac{1}{e}\frac{ae}{b} = \frac{a}{b} = \frac{a/e}{b/e} = S(f(a), f(b)) .$$

If $b = 0$, then

$$f(a * b) = f(0) = 0 = S(f(a), 0) = S(f(a), f(0)) = S(f(a), f(b)) .$$

Consequently, $*$ is isomorphic to Saitoh's fraction on **R**. □

II. PROOF OF THEOREM 2. We need the following lemma. It seems that this lemma is a known result, but we give a proof for the sake of completeness.

LEMMA 1. *If φ is a continuous additive map from* **C** *to itself, then it is mixed-linear, that is, there exist two unique complex numbers α and β such that $\varphi(z) = \alpha z + \beta\overline{z}$ for all $z \in$* **C**.

PROOF. Let φ be a continuous additive map from **C** to itself. Put

$$u(x) = \operatorname{Re}\varphi(x) \text{ and } v(x) = \operatorname{Im}\varphi(x)$$

for each $x \in \mathbf{R}$. Then

$$\varphi(x + y) = u(x + y) + iv(x + y)$$

and

$$\begin{aligned}\varphi(x) + \varphi(y) &= u(x) + iv(x) + u(y) + iv(y)\\ &= u(x) + u(y) + i(v(x) + v(y))\end{aligned}$$

for all $x, y \in \mathbf{R}$. Since $\varphi(x + y) = \varphi(x) + \varphi(y)$ $(x, y \in \mathbf{R})$, it follows that

$$u(x + y) = u(x) + u(y) \text{ and } v(x + y) = v(x) + v(y)$$

hold for all $x, y \in \mathbf{R}$. Then both u and v are continuous additive real-valued functions on **R**. This implies easily that $u(x) = ax$ and $v(x) = bx$ for all real numbers x and some real numbers a, b. We next put

$$u'(x) = \operatorname{Re}\varphi(ix) \quad \text{and} \quad v'(x) = \operatorname{Im}\varphi(ix)$$

for each $x \in \mathbf{R}$. Then

$$\varphi(i(x + y)) = u'(x + y) + iv'(x + y)$$

and

$$\begin{aligned}\varphi(ix) + \varphi(iy) &= u'(x) + iv'(x) + u'(y) + iv'(y)\\ &= u'(x) + u'(y) + i(v'(x) + v'(y))\end{aligned}$$

for all $x, y \in \mathbf{R}$. Since $\varphi(i(x + y)) = \varphi(ix) + \varphi(iy)$ $(x, y \in \mathbf{R})$, it follows that

$$u'(x + y) = u'(x) + u'(y) \text{ and } v'(x + y) = v'(x) + v'(y)$$

hold for all $x, y \in \mathbf{R}$. Then both u' and v' are also continuous additive real-valued functions on **R**. This implies easily that $u'(x) = cx$ and $v'(x) = dx$ for all real numbers x and some real numbers c, d. Therefore

$$\begin{aligned}\varphi(z) &= \varphi(x + iy)\\ &= \varphi(x) + \varphi(iy)\\ &= ax + ibx + cy + idy\\ &= (a + ib)\operatorname{Re} z + (c + id)\operatorname{Im} z\end{aligned}$$

holds for all $z = x + iy \in \mathbf{C}$. Put

$$\alpha = \frac{a+ib}{2} + \frac{c+id}{2i} \quad \text{and} \quad \beta = \frac{a+ib}{2} - \frac{c+id}{2i} .$$

Then we have from the above equation that $\varphi(z) = \alpha z + \beta \overline{z}$ for all $z \in \mathbf{C}$. Moreover it will be clear that such α and β are unique. □

Let $*$ be a continuous fractional binary operation on $\mathbf{C}$. For each $w \in \mathbf{C}$, put

$$f(z) = z * w \quad (z \in \mathbf{C}) .$$

Since f is a continuous additive map from $\mathbf{C}$ to itself, it follows from Lemma 1 that there exist two unique complex numbers $\varphi(w)$ and $\psi(w)$ such that $f(z) = z\varphi(w) + \overline{z}\psi(w)$ for all $z \in \mathbf{C}$. Therefore

$$z * w = z\varphi(w) + \overline{z}\psi(w) \tag{2}$$

holds for all $z, w \in \mathbf{C}$. Hence we have

$$(rz) * w = r(z * w) \tag{3}$$

for all $r \in \mathbf{R}$ and $z, w \in \mathbf{C}$. Put

$$\alpha = 1 * 1 \quad \text{and} \quad \beta = \frac{i * 1}{i} .$$

If x is a nonzero real number, we have from (2) that

$$\alpha = 1 * 1 = x * x = x\varphi(x) + \overline{x}\psi(x) = x(\varphi(x) + \psi(x))$$

holds because $*$ is cancellative. Then

$$\varphi(x) + \psi(x) = \frac{\alpha}{x} \tag{4}$$

holds for all $x \in \mathbf{R} \setminus \{0\}$. Similarly we have that if x is a nonzero real number, then

$$i\beta = i * 1 = (xi) * x = xi\varphi(x) + \overline{xi}\psi(x) = xi\varphi(x) - xi\psi(x)$$

holds. Then

$$\varphi(x) - \psi(x) = \frac{\beta}{x} \tag{5}$$

holds for all $x \in \mathbf{R} \setminus \{0\}$. Therefore we have from (4) and (5) that

$$\begin{cases} \varphi(x) = \frac{\alpha+\beta}{2x} \\ \psi(x) = \frac{\alpha-\beta}{2x} \end{cases} \tag{6}$$

holds for all $x \in \mathbf{R} \setminus \{0\}$. Then we have from (2) and (6) that

$$x * y = x\varphi(y) + x\psi(y) = \frac{x(\alpha+\beta)}{2y} + \frac{x(\alpha-\beta)}{2y} = \frac{x\alpha}{y} \tag{7}$$

holds for all $x, y \in \mathbf{R}$ with $y \neq 0$. Note that

$$i * x = \frac{i}{x} * \frac{x}{x} = \frac{i}{x} * 1 = \frac{1}{x}(i * 1) = \frac{i\beta}{x} \tag{8}$$

holds for all $x \in \mathbf{R} \setminus \{0\}$. If $a, b \in \mathbf{R}$ with $a^2 + b^2 \neq 0$, then we have from (3), (7) and (8) that

$$\begin{aligned} i * (a + ib) &= ((a - ib)i) * (a^2 + b^2) \\ &= (b + ia) * (a^2 + b^2) \\ &= b * (a^2 + b^2) + (ai) * (a^2 + b^2) \\ &= \frac{b\alpha}{a^2 + b^2} + \frac{ai\beta}{a^2 + b^2} \\ &= \frac{b\alpha + ai\beta}{a^2 + b^2} \end{aligned}$$

and

$$\begin{aligned} 1 * (a + ib) &= (a - ib) * (a^2 + b^2) \\ &= a * (a^2 + b^2) - b(i * (a^2 + b^2)) \\ &= \frac{a\alpha}{a^2 + b^2} - \frac{bi\beta}{a^2 + b^2} \\ &= \frac{a\alpha - bi\beta}{a^2 + b^2}. \end{aligned}$$

Therefore if $z = a + ib$, $w = c + id \neq 0$, then

$$\begin{aligned} z * w &= a(1 * w) + b(i * w) \\ &= \frac{a(c\alpha - di\beta)}{c^2 + d^2} + \frac{b(d\alpha + ci\beta)}{c^2 + d^2} \\ &= \frac{(ac + bd)\alpha + (bc - ad)i\beta}{c^2 + d^2} \\ &= \frac{\alpha \operatorname{Re}(z\overline{w}) + i\beta \operatorname{Im}(z\overline{w})}{|w|^2} \\ &= \alpha \operatorname{Re}\left(\frac{z\overline{w}}{|w|^2}\right) + i\beta \operatorname{Im}\left(\frac{z\overline{w}}{|w|^2}\right) \\ &= \alpha \operatorname{Re}\left(\frac{z}{w}\right) + i\beta \operatorname{Im}\left(\frac{z}{w}\right). \end{aligned}$$

Moreover since $*$ is cancellative, it follows from (3) that

$$z * 0 = 0$$

holds for all $z \in \mathbf{C}$ as observed in the proof of Theorem 1. Hence we have

$$z * w = \begin{cases} \alpha \operatorname{Re}\left(\frac{z}{w}\right) + i\beta \operatorname{Im}\left(\frac{z}{w}\right) & (w \neq 0) \\ 0 & (w = 0)\,. \end{cases}$$

To show the uniqueness of α and β, suppose that

$$\alpha \operatorname{Re}\left(\frac{z}{w}\right) + i\beta \operatorname{Im}\left(\frac{z}{w}\right) = \alpha' \operatorname{Re}\left(\frac{z}{w}\right) + i\beta' \operatorname{Im}\left(\frac{z}{w}\right)$$

holds for all $z, w \in \mathbf{C}$ with $w \neq 0$. Taking $z = w = 1$ in the above equation, we have $\alpha = \alpha'$. Also taking $z = i$ and $w = 1$ in the above equation, we have $i\beta = i\beta'$ and hence $\beta = \beta'$. Then α and β are unique. □

III. PROOF OF THEOREM 3. Let $\gamma \in \hat{\mathbf{C}}$ and $(\alpha, \beta), (\alpha', \beta') \in L_\gamma$. Then we must show that $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$. Note that $\beta = \alpha\gamma$ and $\beta' = \alpha'\gamma$.

(a) The case where $\gamma \in \mathbf{C} \setminus \{0\}$. Note that $\alpha \neq 0$, $\beta \neq 0$, $\alpha' \neq 0$ and $\beta' \neq 0$. Then we have

$$\frac{\beta}{\alpha} = \frac{\beta'}{\alpha'}\,.$$

Put

$$\lambda = \frac{\alpha'}{\alpha} = \frac{\beta'}{\beta}$$

and

$$f(z) = \lambda z$$

for each $z \in \mathbf{C}$. Then f is a homeomorphism from $\mathbf{C}$ to itself and

$$\begin{aligned} f(z *_{(\alpha,\beta)} w) &= \lambda\left(\alpha \operatorname{Re}\frac{z}{w} + i\beta \operatorname{Im}\frac{z}{w}\right) \\ &= \alpha' \operatorname{Re}\frac{z}{w} + i\beta' \operatorname{Im}\frac{z}{w} \\ &= \alpha' \operatorname{Re}\frac{f(z)}{f(w)} + i\beta' \operatorname{Im}\frac{f(z)}{f(w)} \\ &= f(z) *_{(\alpha',\beta')} f(w) \end{aligned}$$

for all $z, w \in \mathbf{C}$ with $w \neq 0$. Moreover we have

$$f(z *_{(\alpha,\beta)} 0) = f(0) = 0 = f(z) *_{(\alpha',\beta')} 0 = f(z) *_{(\alpha',\beta')} f(0)$$

for all $z \in \mathbf{C}$. Consequently we obtain $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$.

(b) The case where $\gamma = 0$. Note that $\beta = \beta' = 0$ and $\alpha \neq 0, \alpha' \neq 0$. Put

$$f(z) = \frac{\alpha}{\alpha'} z$$

for each $z \in \mathbf{C}$. Then f is a homeomorphism from $\mathbf{C}$ to itself and

$$\begin{aligned} f(z *_{(\alpha',\beta')} w) &= \frac{\alpha}{\alpha'}(z *_{(\alpha',\beta')} w) = \frac{\alpha}{\alpha'}\alpha' \mathrm{Re}\,\frac{z}{w} = \alpha\,\mathrm{Re}\,\frac{\frac{\alpha}{\alpha'}z}{\frac{\alpha}{\alpha'}w} \\ &= \alpha\,\mathrm{Re}\,\frac{f(z)}{f(w)} = f(z) *_{(\alpha,\beta)} f(w) \end{aligned}$$

holds for all $z, w \in \mathbf{C}$ with $w \neq 0$. Moreover we have

$$f(z *_{(\alpha',\beta')} 0) = f(0) = 0 = f(z) *_{(\alpha,\beta)} 0 = f(z) *_{(\alpha,\beta)} f(0)$$

holds for all $z \in \mathbf{C}$. Consequently we obtain $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$.

(c) The case where $\gamma = \infty$. Note that $\alpha = \alpha' = 0$ and $\beta \neq 0, \beta' \neq 0$. Put

$$f(z) = \frac{\beta}{\beta'} z$$

for each $z \in \mathbf{C}$. Then f is a homeomorphism from $\mathbf{C}$ to itself and

$$\begin{aligned} f(z *_{(\alpha',\beta')} w) &= \frac{\beta}{\beta'}(z *_{(\alpha',\beta')} w) = \frac{\beta}{\beta'}\beta' i\,\mathrm{Im}\,\frac{z}{w} = \beta i\,\mathrm{Im}\,\frac{\frac{\beta}{\beta'}z}{\frac{\beta}{\beta'}w} \\ &= \beta i\,\mathrm{Im}\,\frac{f(z)}{f(w)} = f(z) *_{(\alpha,\beta)} f(w) \end{aligned}$$

holds for all $z, w \in \mathbf{C}$ with $w \neq 0$. Moreover we have

$$f(z *_{(\alpha',\beta')} 0) = f(0) = 0 = f(z) *_{(\alpha,\beta)} 0 = f(z) *_{(\alpha,\beta)} f(0)$$

holds for all $z \in \mathbf{C}$. Consequently we obtain $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$. □

IV. PROOF OF THEOREM 4. Let $\alpha, \beta, \alpha', \beta' \in \mathbf{C}$ and suppose that $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$. Let f be a corresponding homeomorphism from $\mathbf{C}$ to itself. Then

$$f(z *_{(\alpha,\,\beta)} w) = f(z) *_{(\alpha',\,\beta')} f(w) \tag{9}$$

holds for all $z, w \in \mathbf{C}$. In this case we have

$$f(0) = 0\,. \tag{10}$$

Actually since f is bijective, we can choose a $z_0 \in \mathbf{C}$ with $f(z_0) = 0$. Taking $z = z_0$ and $w = 0$ in (9), we have $f(0) = f(z_0) *_{(\alpha',\,\beta')} f(0) = 0 *_{(\alpha',\,\beta')} f(0) = 0$.

By (9), (10) and Theorem 2, we have

$$f\left(\alpha \operatorname{Re} \frac{z}{w} + i\beta \operatorname{Im} \frac{z}{w}\right) = \alpha' \operatorname{Re} \frac{f(z)}{f(w)} + i\beta' \operatorname{Im} \frac{f(z)}{f(w)} \tag{11}$$

for all $z, w \in \mathbf{C}$ with $w \neq 0$. Also taking $z = w = 1$ in (11), we have

$$f(\alpha) = \alpha' . \tag{12}$$

(i) Let $\alpha, \beta \in \mathbf{C}$ and suppose that $\Phi(\alpha, \beta) \cong \Phi(0, 0)$. If f is a corresponding homeomorphism, then

$$f(\alpha \operatorname{Re} z + i\beta \operatorname{Im} z) = f(z *_{(\alpha,\beta)} 1) = f(z) *_{(0,0)} f(1) = 0$$

for all $z \in \mathbf{C}$. Taking $z = 1$ in the above equation, we obtain that $f(\alpha) = f(0)$, hence $\alpha = 0$ since $f(0) = 0$ by (10). Similarly taking $z = i$ in the same equation, we obtain $i\beta = 0$, namely, $\beta = 0$. Consequently, $\Phi(0, 0)$ is not isomorphic to any nontrivial continuous fractional binary operation on $\mathbf{C}$.

(ii) Let $\gamma = 0$ or ∞, $\gamma' \in \mathbf{C} \setminus \{0\}$, $(\alpha, \beta) \in L_\gamma$ and $(\alpha', \beta') \in L_{\gamma'}$ with $\operatorname{Re} \alpha'\overline{\beta'} \neq 0$. Then we must show that $\Phi(\alpha, \beta) \ncong \Phi(\alpha', \beta')$.

(ii-a) The case where $\gamma = 0$. Note that $\beta = 0$ and $\alpha \neq 0$. Assume that $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$. By Theorem 3,

$$\Phi(\alpha', \beta') \cong \Phi(\alpha, \beta) = \Phi(\alpha, 0) \cong \Phi(1, 0) .$$

Then $\Phi(\alpha', \beta') \cong \Phi(1, 0)$, so let f be its corresponding homeomorphism. Then we have from (11) that

$$f\left(\alpha' \operatorname{Re} \frac{z}{w} + i\beta' \operatorname{Im} \frac{z}{w}\right) = \operatorname{Re} \frac{f(z)}{f(w)}$$

holds for all $z, w \in \mathbf{C}$ with $w \neq 0$. Taking $w = 1$ in the above equation, we obtain that

$$f(\alpha' \operatorname{Re} z + i\beta' \operatorname{Im} z) = \operatorname{Re} \frac{f(z)}{f(1)} \in \mathbf{R}$$

for all $z \in \mathbf{C}$. Since $\operatorname{Re} \alpha'\overline{\beta'} \neq 0$ by hypothesis, we can easily see that

$$\{w \in \mathbf{C} : w = \alpha' \operatorname{Re} z + i\beta' \operatorname{Im} z, z \in \mathbf{C}\} = \mathbf{C}. \tag{13}$$

Therefore we have from (13) that $f(\mathbf{C}) = \mathbf{R}$, a contradiction. Consequently, $\Phi(\alpha, \beta) \ncong \Phi(\alpha', \beta')$.

(ii-b) The case where $\gamma = \infty$. Note that $\alpha = 0$ and $\beta \neq 0$. Assume that $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$. By Theorem 3,

$$\Phi(\alpha', \beta') \cong \Phi(\alpha, \beta) = \Phi(0, \beta) \cong \Phi(0, 1) .$$

Then $\Phi(\alpha', \beta') \cong \Phi(0, 1)$, so let g be its corresponding homeomorphism. Then we have from (11) that

$$g\left(\alpha'\text{Re}\,\frac{z}{w} + i\beta'\text{Im}\,\frac{z}{w}\right) = i\,\text{Im}\,\frac{g(z)}{g(w)}$$

holds for all $z, w \in \mathbf{C}$ with $w \neq 0$. Taking $w = 1$ in the above equation, we obtain that

$$g(\alpha'\text{Re}\,z + i\beta'\,\text{Im}\,z) = i\,\text{Im}\,\frac{g(z)}{g(1)} \in i\,\mathbf{R}$$

for all $z \in \mathbf{C}$. Since $\text{Re}\,\alpha'\overline{\beta'} \neq 0$ by hypothesis, it follows from (13) that $g(\mathbf{C}) = i\,\mathbf{R}$, a contradiction. Consequently, $\Phi(\alpha, \beta) \ncong \Phi(\alpha', \beta')$.

(iii) Let $(\alpha, \beta) \in L_0$ and $(\alpha', \beta') \in L_\infty$. Assume that $\Phi(\alpha, \beta) \cong \Phi(\alpha', \beta')$. Since $\alpha \neq 0, \beta = 0, \alpha' = 0$ and $\beta' \neq 0$, it follows from Theorem 3 that

$$\Phi(0, 1) \cong \Phi(0, \beta') = (\Phi(\alpha', \beta') \cong \Phi(\alpha, \beta) = \Phi(\alpha, 0) \cong \Phi(1, 0)\,.$$

Then $\Phi(0, 1) \cong \Phi(1, 0)$, so let f be its corresponding homeomorphism. Then we have from (11) that

$$f\left(i\,\text{Im}\,\frac{z}{w}\right) = \text{Re}\,\frac{f(z)}{f(w)}$$

holds for all $z, w \in \mathbf{C}$ with $w \neq 0$. Taking $z = w = 1$ in the above equation, we obtain that

$$f(0) = f(i\,\text{Im}\,1) = \text{Re}\,\frac{f(1)}{f(1)} = \text{Re}\,1 = 1$$

Since $f(0) = 0$ by (10), it follows that $0 = 1$, a contradiction. Consequently, $\Phi(\alpha, \beta) \ncong \Phi(\alpha', \beta')$.

(iv) Let $\alpha, \beta \in C \setminus \{0\}$ and suppose that $\Phi(\alpha, \beta) \cong \Phi(1, 1)$. Let f be a corresponding homeomorphism. Then $f(\alpha) = 1$ by (12). Also we have from (11) that

$$f\left(\alpha\,\text{Re}\,\frac{z}{w} + i\beta\,\text{Im}\,\frac{z}{w}\right) = \frac{f(z)}{f(w)}$$

for all $z, w \in \mathbf{C}$ with $w \neq 0$. Taking $w = \alpha$ in the above equation, we obtain

$$f\left(\alpha\,\text{Re}\,\frac{z}{\alpha} + i\beta\,\text{Im}\,\frac{z}{\alpha}\right) = \frac{f(z)}{f(\alpha)} = f(z)$$

for all $z \in \mathbf{C}$. Since f is injective, it follows that

$$\alpha\,\text{Re}\,\frac{z}{\alpha} + i\beta\,\text{Im}\,\frac{z}{\alpha} = z$$

holds for all $z \in \mathbf{C}$. Taking $z = i\alpha$ in the above equation, we obtain $i\beta = i\alpha$, hence $\alpha = \beta$.

The converse follows immediately from Theorem 3. □

IV. PROOF OF THEOREM 5. Let $*$ be a nontrivial homogeneous cancellative binary operation on $\mathbf{K}$. Put

$$\varphi(b) = 1 * b$$

for each $b \in \mathbf{K}$. Then we have $a * b = a\varphi(b)$ for all $a, b \in \mathbf{K}$. Then $*$ must be isomorphic to Saitoh's fraction on $\mathbf{K}$ as observed in the proof of Theorem 1. □

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References

[1] L. P. CASTRO and S. SAITOH, Fractional functions and their representations, *Complex Analysis and Operator Theory* **7** (2013), 1049–1063. DOI: 10.1007/s11785-011-0154-1.

[2] S. SAITOH, A natural interpretation on 100/0=0 and 0/0=0, and an open question, preprint.

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