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A Diffusion Process with a Brownian Potential Including a Zero Potential Part

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Abstract. A one-dimensional diffusion process with a Brownian potential including a zero potential part is studied. The maximum process and the minimum process of the diffusion process are also investigated.

1. Model and results

Denote by $\mathbb{W}$ the space of real-valued continuous functions on $\mathbb{R}$ vanishing at 0, and $\widetilde{\mathbb{W}}$ the space of real-valued right-continuous functions on $\mathbb{R}$ with left limits. Let $a \in (0, 1/2)$ be fixed. For $w \in \mathbb{W}$ and $\lambda > 0$, define $T_\lambda w \in \widetilde{\mathbb{W}}$ by

$$(T_\lambda w)(x) = \begin{cases} 0 & \text{for } 0 < x < e^{a\lambda}, \\ w(x) & \text{otherwise.} \end{cases}$$

We denote by Ω the space of real-valued continuous functions on $[0, \infty)$, and for $\omega \in \Omega$ and $t \geq 0$ we write $X(t) = X(t, \omega) = \omega(t)$, the value of ω at t. For $w \in \mathbb{W}$, $\lambda > 0$ and $x_0 \in \mathbb{R}$, $P_{T_\lambda w}^{x_0}$ denotes the probability measure on Ω such that $\{X(t), t \geq 0, P_{T_\lambda w}^{x_0}\}$ is a diffusion process with generator

$$\mathcal{L}_{T_\lambda w} = \frac{1}{2} e^{(T_\lambda w)(x)} \frac{d}{dx}\left(e^{-(T_\lambda w)(x)} \frac{d}{dx}\right)$$

starting from x_0. Let P be the Wiener measure on $\mathbb{W}$, and $\mathcal{P}_\lambda^{x_0}$ be the probability measure on $\mathbb{W} \times \Omega$ defined by

$$\mathcal{P}_\lambda^{x_0}(dw d\omega) = P(dw) P_{T_\lambda w}^{x_0}(d\omega).$$

For each $\lambda > 0$ we regard the process $\{X(t), t \geq 0, \mathcal{P}_\lambda^{x_0}\}$ as one defined on the probability space $(\mathbb{W} \times \Omega, \mathcal{P}_\lambda^{x_0})$, which we call a diffusion process with a Brownian potential including a zero potential part. We study the behavior of the process $\{X(t), t \geq 0, \mathcal{P}_\lambda^{0}\}$ at $t = e^\lambda$ $(\lambda \to \infty)$.

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Brox ([1]) and Schumacher ([7]) studied a one-dimensional diffusion process with a Brownian potential, and Kawazu, Tamura and Tanaka ([5], [6]) investigated a one-dimensional diffusion process in an asymptotically self-similar random environment. Moreover, in [4] and [3] a one-dimensional diffusion process with a one-sided Brownian potential was studied, and in [8] a one-dimensional diffusion process with a random potential consisting of two self-similar processes with different indices for the right and the left hand sides of the origin was investigated.

Our present model is a variant of the diffusion in [1], [7] and also that in [4], [3]. To study the behavior of our process $\{X(t), t \geq 0, \mathcal{P}_\lambda^0\}$ at $t = e^\lambda$, we regard $\mathbb{W}$ as a disjoint union of three subsets $\mathbb{A}_\lambda$, $\mathbb{B}_\lambda$ and $\mathbb{C}_\lambda$; for the definition, see (1.3)–(1.5). We show that $X(e^\lambda)$ exhibits quite different behavior depending on whether it is conditioned on $\mathbb{A}_\lambda$, $\mathbb{B}_\lambda$ or $\mathbb{C}_\lambda$ $(\lambda \to \infty)$. The behavior of $X(e^\lambda)$ conditioned on $\mathbb{B}_\lambda$ is much different from the result in [1], [7]. Roughly speaking, in this case with high probability $X(e^\lambda)$ is not at the bottom of the "valley" but in the interval where the potential is identically zero; for the precise meaning of this, see Theorem 1.1 (ii).

Hereafter we restrict $\mathbb{W}$ to a suitable subset of $\mathbb{W}$ that has P-measure 1 to avoid unpleasant cases. For $w \in \mathbb{W}$ and $\rho \in \mathbb{R}$, we set

$$\begin{aligned}
&\sigma(\rho) = \sigma(\rho, w) = \sup\{x < 0 : w(x) = \rho\}\,,\\
&\zeta = \zeta(w) = \sup\Big\{x < 0 : w(x) - \min_{x \leq y \leq 0} w(y) = 1\Big\}\,,\\
&V = V(w) = \min_{\zeta \leq x \leq 0} w(x)\,.
\end{aligned}$$

We define $b = b(w) \in (\zeta, 0)$ by $w(b) = V$. We note that b is determined uniquely by w (P-a.s.).

To study the behavior of our process, we regard $\mathbb{W}$ as a disjoint union of three subsets $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ defined by

$$\begin{aligned}
&\mathbb{A} = \mathbb{A}' \cup \mathbb{A}''\,,\\
&\mathbb{A}' = \{w \in \mathbb{W} : \sigma(a) < \sigma(-1 + a)\}\,,\\
&\mathbb{A}'' = \{w \in \mathbb{W} : w(1) > 0,\ \sigma(1 - a) < \sigma(-a)\}\,,\\
&\mathbb{B} = \{w \in \mathbb{W} : w(1) > 0,\ \sigma(-a) < \sigma(1 - a)\}\,,\\
&\mathbb{C} = \{w \in \mathbb{W} : w(1) < 0,\ \sigma(-1 + a) < \sigma(a)\}.
\end{aligned}$$

(See Fig. 1–Fig. 4 in the next page.) We note $\mathbb{A}' \cap \mathbb{A}'' = \mathbb{A}' \cap \{w \in \mathbb{W} : w(1) > 0\}$ and have $P\{\mathbb{A}'\} = a$, $P\{\mathbb{A}''\} = (1 - a)/2$, $P\{\mathbb{A}\} = 1/2$, $P\{\mathbb{B}\} = a/2$ and $P\{\mathbb{C}\} = (1 - a)/2$. Moreover, we remark that

$$V(w) < -a \quad \text{if } w \in \mathbb{A}\,. \tag{1.1}$$

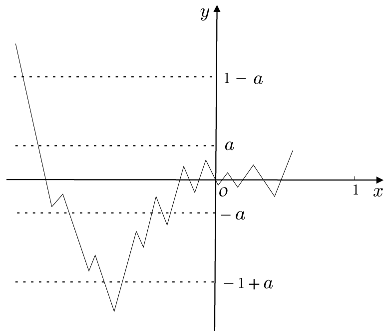

Fig. 1. $w \in \mathbb{A}'$

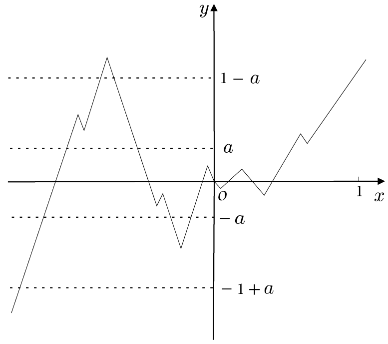

Fig. 2. $w \in \mathbb{A}'' \cap (\mathbb{A}')^c$

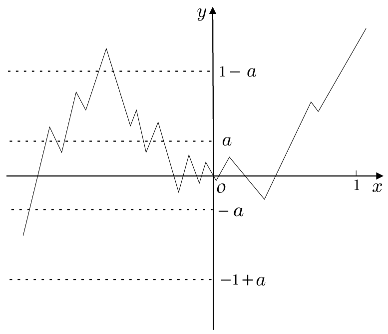

Fig. 3. $w \in \mathbb{B}$

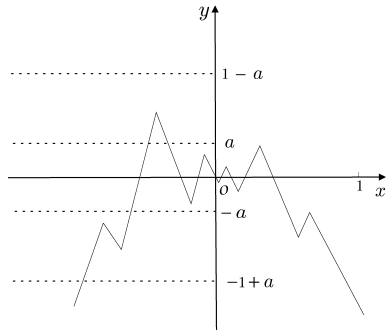

Fig. 4. $w \in \mathbb{C}$

For $w \in \mathbb{W}$ and $\lambda > 0$, we define $\tau_\lambda w \in \mathbb{W}$ by

$$(\tau_\lambda w)\,(x) = \begin{cases} \lambda^{-1} w(\lambda^2 x) & \text{for } x \le 0\,, \\ e^{-a\lambda/2} w(e^{a\lambda} x) & \text{for } x > 0\,. \end{cases}$$

Note that

$$\{\tau_\lambda w,\, P\} \stackrel{\mathrm{d}}{=} \{w,\, P\}\,, \tag{1.2}$$

where $\stackrel{\mathrm{d}}{=}$ means the equality in distribution. To state our result, for each $\lambda > 0$ we regard $\mathbb{W}$ as a disjoint union of three subsets $\mathbb{A}_\lambda$, $\mathbb{B}_\lambda$ and $\mathbb{C}_\lambda$ defined by

$$\mathbb{A}_\lambda = \{w \in \mathbb{W} : \tau_\lambda w \in \mathbb{A}\}\,, \tag{1.3}$$

$$\mathbb{B}_\lambda = \{w \in \mathbb{W} : \tau_\lambda w \in \mathbb{B}\}\,, \tag{1.4}$$

$$\mathbb{C}_\lambda = \{w \in \mathbb{W} : \tau_\lambda w \in \mathbb{C}\}\,. \tag{1.5}$$

(See [3], [8].) We notice that each P-measure of $\mathbb{A}_\lambda$, $\mathbb{B}_\lambda$ and $\mathbb{C}_\lambda$ is equal to that of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, respectively.

In the following theorems, we denote by $P\{\cdots|\cdot\}$ the conditional probability.

THEOREM 1.1. *For any* $\varepsilon > 0$ *the following* (i)–(iii) *hold.*

(i) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{1,\lambda,\varepsilon}|\mathbb{A}_\lambda\} = 1$,

where

$$\mathbb{E}_{1,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{1,\lambda,\varepsilon}(w) > 1-\varepsilon\},$$
$$p_{1,\lambda,\varepsilon}(w) = P^0_{T_\lambda w}\big\{b(\tau_\lambda w) - \varepsilon < \lambda^{-2}X(e^\lambda) < (b(\tau_\lambda w)+\varepsilon)\wedge 0\big\}.$$

(ii) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{2,\lambda,\varepsilon}|\mathbb{B}_\lambda\} = 1$,

where

$$\mathbb{E}_{2,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{2,\lambda}(w) > 1-\varepsilon\},$$
$$p_{2,\lambda}(w) = P^0_{T_\lambda w}\big\{0 < e^{-a\lambda}X(e^\lambda) < 1\big\}.$$

(iii) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{3,\lambda,\varepsilon}|\mathbb{C}_\lambda\} = 1$,

where

$$\mathbb{E}_{3,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{3,\lambda,\varepsilon}(w) > 1-\varepsilon\},$$
$$p_{3,\lambda,\varepsilon}(w) = P^0_{T_\lambda w}\big\{-(\exp\{w(e^{a\lambda}) + e^{a\lambda/2}\varepsilon\}\wedge\varepsilon) < e^{-a\lambda}X(e^\lambda) - 1 < \varepsilon\big\}.$$

The following corollary concerning the occupation time is obtained from the proof of Theorem 1.1 (cf. [3]). In the following, $\mathbf{1}_E$ denotes the indicator function of the (generic) set E.

COROLLARY 1.2. *For any* $\varepsilon > 0$ *the following* (i)–(iii) *hold.*

(i) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{4,\lambda,\varepsilon}|\mathbb{A}_\lambda\} = 1$,

where

$$\mathbb{E}_{4,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{4,\lambda,\varepsilon}(w) > 1-\varepsilon\},$$
$$p_{4,\lambda,\varepsilon}(w) = P^0_{T_\lambda w}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{(b(\tau_\lambda w)-\varepsilon,(b(\tau_\lambda w)+\varepsilon)\wedge 0)}(\lambda^{-2}X(t))dt > 1-\varepsilon\right\}.$$

(ii) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{5,\lambda,\varepsilon}|\mathbb{B}_\lambda\} = 1$,

where

$$\mathbb{E}_{5,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{5,\lambda,\varepsilon}(w) > 1-\varepsilon\},$$
$$p_{5,\lambda,\varepsilon}(w) = P^0_{T_\lambda w}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{(0,1)}(e^{-a\lambda}X(t))dt > 1-\varepsilon\right\}.$$

(iii) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{6,\lambda,\varepsilon}|\mathbb{C}_\lambda\} = 1$, *where*

$$\mathbb{E}_{6,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{6,\lambda,\varepsilon}(w) > 1 - \varepsilon\},$$

$$p_{6,\lambda,\varepsilon}(w) = P^0_{T_\lambda w}\left\{e^{-\lambda}\int_0^{e^\lambda} \mathbf{1}_{(1-\exp\{w(e^{a\lambda})+e^{a\lambda/2}\varepsilon\}\wedge\varepsilon,1+\varepsilon)}(e^{-a\lambda}X(t))dt > 1-\varepsilon\right\}.$$

Next we consider the maximum process and the minimum process of $\{X(t), t \ge 0, \mathcal{P}^0_\lambda\}$. For $\omega \in \Omega$, we set $\overline{X}(t) = \overline{X}(t,\omega) = \max_{0\le s\le t} X(s,\omega)$ and $\underline{X}(t) = \underline{X}(t,\omega) = \min_{0\le s\le t} X(s,\omega)$. We study the behaviors of the processes $\{\overline{X}(t), t \ge 0, \mathcal{P}^0_\lambda\}$ and $\{\underline{X}(t), t \ge 0, \mathcal{P}^0_\lambda\}$ at $t = e^\lambda$.

To study the behavior of the maximum process, we set, for $w \in \mathbb{W}$,

$$H = H(w) = \max_{\zeta\le x\le 0} w(x),$$
$$M = M(w) = \max_{b\le x\le 0} w(x).$$

Note that

$$H(w) < a \quad \text{if } w \in \mathbb{A}'. \tag{1.6}$$

We divide $\mathbb{A}'$ into two subsets $\mathbb{A}'_I$ and $\mathbb{A}'_{II}$ as follows:

$$\mathbb{A}'_I = \{w \in \mathbb{A}' : M \le V + 1\},$$
$$\mathbb{A}'_{II} = \{w \in \mathbb{A}' : M > V + 1\}.$$

(See Fig. 5 and Fig. 6 below.) Moreover, we set

$$\mathbb{D} = (\mathbb{A}'' \cap (\mathbb{A}')^c) \oplus \mathbb{B}.$$

(See Fig. 2 and Fig. 3.) We have $P\{\mathbb{D}\} = (1-a)/2$.

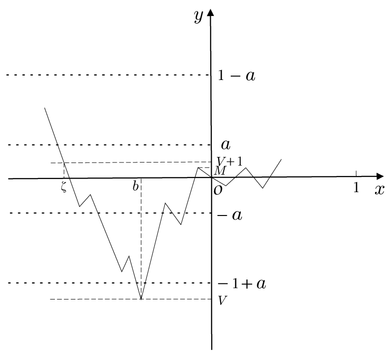

Fig. 5. $w \in \mathbb{A}'_I$

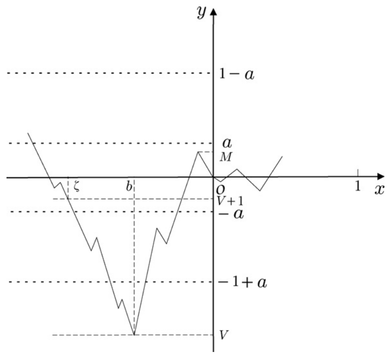

Fig. 6. $w \in \mathbb{A}'_{II}$

To state our result on the maximum process, for each $\lambda > 0$ we regard $\mathbb{W}$ as a disjoint union of four subsets $\mathbb{A}'_{I,\lambda}$, $\mathbb{A}'_{II,\lambda}$, $\mathbb{D}_\lambda$ and $\mathbb{C}_\lambda$, where

$$\begin{aligned}\mathbb{A}'_{I,\lambda} &= \{w \in \mathbb{W} : \tau_\lambda w \in \mathbb{A}'_I\}\,,\\ \mathbb{A}'_{II,\lambda} &= \{w \in \mathbb{W} : \tau_\lambda w \in \mathbb{A}'_{II}\}\,,\\ \mathbb{D}_\lambda &= \{w \in \mathbb{W} : \tau_\lambda w \in \mathbb{D}\}\,.\end{aligned}$$

Note that each P-measure of $\mathbb{A}'_{I,\lambda}$, $\mathbb{A}'_{II,\lambda}$ and $\mathbb{D}_\lambda$ is equal to that of $\mathbb{A}'_I$, $\mathbb{A}'_{II}$ and $\mathbb{D}$, respectively.

THEOREM 1.3. *For any* $\varepsilon > 0$ *the following* (i)–(iv) *hold.*

(i) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{7,\lambda,\varepsilon} | \mathbb{A}'_{I,\lambda}\} = 1$,

where

$$\begin{aligned}\mathbb{E}_{7,\lambda,\varepsilon} &= \{w \in \mathbb{W} : p_{7,\lambda,\varepsilon}(w) > 1-\varepsilon\}\,,\\ p_{7,\lambda,\varepsilon}(w) &= P^0_{T_\lambda w}\big\{e^{\lambda(H(\tau_\lambda w)-\varepsilon)} < \overline{X}(e^\lambda) < e^{\lambda(H(\tau_\lambda w)+\varepsilon)} \wedge e^{a\lambda}\varepsilon\big\}\,.\end{aligned}$$

(ii) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{8,\lambda,\varepsilon} | \mathbb{A}'_{II,\lambda}\} = 1$,

where

$$\begin{aligned}\mathbb{E}_{8,\lambda,\varepsilon} &= \{w \in \mathbb{W} : p_{8,\lambda,\varepsilon}(w) > 1-\varepsilon\}\,,\\ p_{8,\lambda,\varepsilon}(w) &= P^0_{T_\lambda w}\big\{e^{\lambda(H(\tau_\lambda w)-\varepsilon)} < \overline{X}(e^\lambda) < e^{\lambda(H(\tau_\lambda w)+\varepsilon(\lambda))} \wedge e^{a\lambda}\varepsilon\big\}\,,\end{aligned}$$

and $\varepsilon(\lambda) > 0$, $\lambda > 0$, *is assumed to satisfy* $\lim_{\lambda\to\infty}\varepsilon(\lambda) = 0$ *and* $\liminf_{\lambda\to\infty}\lambda(\log\lambda)^{-1}\varepsilon(\lambda) > 2$.

(iii) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{9,\lambda,\varepsilon} | \mathbb{D}_\lambda\} = 1$,

where

$$\begin{aligned}&\mathbb{E}_{9,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{9,\lambda,\varepsilon}(w) > 1-\varepsilon\}\,,\\ &p_{9,\lambda,\varepsilon}(w)\\ &= P^0_{T_\lambda w}\big\{\exp\{-w(e^{a\lambda}) - e^{a\lambda/2}\varepsilon\} < e^{-a\lambda}\overline{X}(e^\lambda) - 1 < \exp\{-w(e^{a\lambda}) + e^{a\lambda/2}\varepsilon\} \wedge \varepsilon\big\}\,.\end{aligned}$$

(iv) $\lim_{\lambda\to\infty} P\{\mathbb{E}_{10,\lambda,\varepsilon} | \mathbb{C}_\lambda\} = 1$,

where

$$\begin{aligned}\mathbb{E}_{10,\lambda,\varepsilon} &= \{w \in \mathbb{W} : p_{10,\lambda,\varepsilon}(w) > 1-\varepsilon\}\,,\\ p_{10,\lambda,\varepsilon}(w) &= P^0_{T_\lambda w}\big\{\exp\{-e^{a\lambda/2}\varepsilon\} < e^{-a\lambda}\overline{X}(e^\lambda) - 1 < \varepsilon\big\}\,.\end{aligned}$$

To study the behavior of the minimum process, we set, for $w \in \mathbb{W}$ and $\gamma > 0$,

$$\zeta_\gamma = \zeta_\gamma(w) = \sup\Big\{x < b(w) : w(x) - \min_{x\le y\le 0} w(y) = \gamma\Big\}\,. \tag{1.7}$$

Notice that $\zeta_1 = \zeta$. The following theorem is concerning the minimum process, where we have more precise upper bound of $\underline{X}(e^\lambda)$ than the corresponding result in [4].

THEOREM 1.4. *Let* $\varepsilon > 0$ *and* $\varepsilon(\lambda) > 0$, $\lambda > 0$, *satisfy* $\lim_{\lambda\to\infty} \varepsilon(\lambda) = 0$.

(i) $\lim\limits_{\lambda\to\infty} P\{\mathbb{E}_{11,\lambda,\varepsilon} | \mathbb{A}_\lambda\} = 1$,

where

$$\mathbb{E}_{11,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{11,\lambda,\varepsilon}(w) > 1 - \varepsilon\},$$
$$p_{11,\lambda,\varepsilon}(w) = P^0_{T_\lambda w}\big\{\zeta_{1+\varepsilon}(\tau_\lambda w) < \lambda^{-2}\underline{X}(e^\lambda) < \zeta_{1-\varepsilon(\lambda)}(\tau_\lambda w)\big\},$$

and $\varepsilon(\lambda)$, $\lambda > 0$, *is assumed additionally to satisfy* $\liminf_{\lambda\to\infty} \lambda(\log\lambda)^{-1}\varepsilon(\lambda) > 4$.

(ii) $\lim\limits_{\lambda\to\infty} P\{\mathbb{E}_{12,\lambda,\varepsilon} | \mathbb{B}_\lambda\} = 1$,

where

$$\mathbb{E}_{12,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{12,\lambda,\varepsilon}(w) > 1 - \varepsilon\},$$
$$p_{12,\lambda,\varepsilon}(w) = P^0_{T_\lambda w}\big\{\sigma(1 - a + \varepsilon, \tau_\lambda w) < \lambda^{-2}\underline{X}(e^\lambda) < \sigma(1 - a - \varepsilon(\lambda), \tau_\lambda w)\big\},$$

and $\varepsilon(\lambda)$, $\lambda > 0$, *is assumed additionally to satisfy* $\liminf_{\lambda\to\infty} \lambda(\log\lambda)^{-1}\varepsilon(\lambda) > 2$.

(iii) $\lim\limits_{\lambda\to\infty} P\{\mathbb{E}_{13,\lambda,\varepsilon} | \mathbb{C}_\lambda\} = 1$,

where

$$\mathbb{E}_{13,\lambda,\varepsilon} = \{w \in \mathbb{W} : p_{13,\lambda,\varepsilon}(w) > 1 - \varepsilon\},$$
$$p_{13,\lambda,\varepsilon}(w) = P^0_{T_\lambda w}\big\{\sigma(a, \tau_\lambda w) < \lambda^{-2}\underline{X}(e^\lambda) < \sigma(a - \varepsilon(\lambda), \tau_\lambda w)\big\},$$

and $\varepsilon(\lambda)$, $\lambda > 0$, *is assumed additionally to satisfy* $\liminf_{\lambda\to\infty} \lambda(\log\lambda)^{-1}\varepsilon(\lambda) > 2$.

2. Preliminaries

For $w \in \mathbb{W}$ and $\lambda > 0$, define $G_\lambda w \in \widetilde{\mathbb{W}}$ by

$$(G_\lambda w)(x) = \begin{cases} \lambda w(\lambda^{-2}e^{a\lambda}x) & \text{for } x \le 0, \\ 0 & \text{for } 0 < x < 1, \\ e^{a\lambda/2}w(x) & \text{for } x \ge 1. \end{cases}$$

For $x_0 \in \mathbb{R}$, $P^{x_0}_{G_\lambda w}$ denotes the probability measure on Ω such that $\{X(t), t \ge 0, P^{x_0}_{G_\lambda w}\}$ is a diffusion process with generator

$$\mathcal{L}_{G_\lambda w} = \frac{1}{2}e^{(G_\lambda w)(x)}\frac{d}{dx}\left(e^{-(G_\lambda w)(x)}\frac{d}{dx}\right)$$

starting from x_0. We can construct such a diffusion process on a probability space $(\widetilde{\Omega}, \widetilde{P})$ as follows ([2], see also [4], [8]). Let $\{B(t), t \ge 0\}$ be a one-dimensional Brownian motion starting from 0 defined on $(\widetilde{\Omega}, \widetilde{P})$, and set

$$L(t,x)=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[x,x+\varepsilon)}(B(s))ds\,,\quad t\ge 0,\ x\in\mathbb{R}\quad\text{(local time)}.$$

We also set

$$S_{G_\lambda w}(x)=\int_0^x e^{(G_\lambda w)(y)}dy,\quad x\in\mathbb{R}\,,$$

$$A_{G_\lambda w}(t)=\int_0^t e^{-2(G_\lambda w)(S_{G_\lambda w}{}^{-1}(B(s)))}ds$$

$$=\int_{-\infty}^{\infty} e^{-2(G_\lambda w)(S_{G_\lambda w}{}^{-1}(x))}L(t,x)dx\,,\quad t\ge 0\,, \tag{2.1}$$

$$X(t;0,G_\lambda w)=S_{G_\lambda w}{}^{-1}(B(A_{G_\lambda w}{}^{-1}(t)))\,,\quad t\ge 0\,. \tag{2.2}$$

Here $S_{G_\lambda w}{}^{-1}$ and $A_{G_\lambda w}{}^{-1}$ denote the inverse functions of $S_{G_\lambda w}$ and $A_{G_\lambda w}$, respectively. For $x_0\in\mathbb{R}$, define $(G_\lambda w)^{x_0}\in\widetilde{\mathbb{W}}$ by $(G_\lambda w)^{x_0}(x)=(G_\lambda w)(x+x_0)$, $x\in\mathbb{R}$, and set

$$X(t;x_0,G_\lambda w)=x_0+X(t;0,(G_\lambda w)^{x_0})\,,\quad t\ge 0\,.$$

Then, on $(\widetilde{\Omega}, \widetilde{P})$, we get a diffusion process $\{X(t;x_0,G_\lambda w), t\ge 0\}$ starting from x_0 whose generator is $\mathcal{L}_{G_\lambda w}$.

LEMMA 2.1. *For any $w\in\mathbb{W}$ and $\lambda>0$*

$$\{X(t), t\ge 0, P^0_{G_\lambda(\tau_\lambda w)}\}\overset{\mathrm{d}}{=}\{e^{-a\lambda}X(e^{2a\lambda}t), t\ge 0, P^0_{T_\lambda w}\}\,.$$

PROOF. We can prove the lemma in the same way as in [6] (see also [1]) by using the equality

$$(G_\lambda(\tau_\lambda w))(x)=(T_\lambda w)(e^{a\lambda}x)\,,\quad x\in\mathbb{R}\,. \qquad \square$$

Owing to Lemma 2.1 and (1.2), we obtain Theorem 1.1 from the following proposition by the same argument as in [8, p. 531] (see also [3]).

PROPOSITION 2.2. (i) *There exists a subset $\mathbb{A}^\#$ of $\mathbb{A}$ with $P\{\mathbb{A}\backslash\mathbb{A}^\#\}=0$ such that, for any $w\in\mathbb{A}^\#$ and $\varepsilon>0$*

$$\lim_{\lambda\to\infty}P^0_{G_\lambda w}\{b-\varepsilon<\lambda^{-2}e^{a\lambda}X(e^{\lambda(1-2a)})<(b+\varepsilon)\wedge 0\}=1\,. \tag{2.3}$$

(ii) *There exists a subset $\mathbb{B}^\#$ of $\mathbb{B}$ with $P\{\mathbb{B}\backslash\mathbb{B}^\#\}=0$ such that, for any $w\in\mathbb{B}^\#$*

$$\lim_{\lambda\to\infty}P^0_{G_\lambda w}\{0<X(e^{\lambda(1-2a)})<1\}=1\,. \tag{2.4}$$

(iii) *There exists a subset* $\mathbb{C}^{\#}$ *of* $\mathbb{C}$ *with* $P\{\mathbb{C}\backslash\mathbb{C}^{\#}\} = 0$ *such that, for any* $w \in \mathbb{C}^{\#}$ *and* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\big\{-(\exp\{e^{a\lambda/2}(w(1)+\varepsilon)\} \wedge \varepsilon) < X(e^{\lambda(1-2a)}) - 1 < \varepsilon\big\} = 1\,. \tag{2.5}$$

In Section 3 we present lemmas on hitting times of the diffusion process constructed in Section 2. In Section 4 we prove Proposition 2.2. In Section 5 we show Theorem 1.3 and in Section 6 we show Theorem 1.4.

3. Estimation on hitting times

In this section we estimate hitting times of the diffusion process introduced in Section 2 by improving (or using) the method in [8] ([1], [4]). For $\omega \in \Omega$, we set

$$\tau(q) = \tau(q,\omega) = \inf\{t > 0 : X(t) = q\}\,, \quad q \in \mathbb{R}\,.$$

In the following lemma which is used to prove Theorem 1.4, we have more precise estimation on a hitting time from above than that in [8] ([4]).

LEMMA 3.1. *Let* $w \in \mathbb{W}$ *and* $p \le p_\lambda < x_0 \le 0$ *for all sufficiently large* $\lambda > 0$. *Assume* $w(p_\lambda) \ge w(x)$ *for all* $x \in [p_\lambda, x_0]$ *for all sufficiently large* $\lambda > 0$. *In addition, assume* (i) $\max_{p\le x\le 0} w(x) < a$ *or* (ii) $w(1) > 0$. *Then for some* $C > 0$

$$\lim_{\lambda\to\infty} P^{\lambda^2 e^{-a\lambda}x_0}_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda}p_\lambda) < Ce^{\lambda J(\lambda)}\} = 1\,, \tag{3.1}$$

where

$$J(\lambda) = \max\big\{w(p_\lambda) - V_\lambda - 2a + 4\lambda^{-1}\log\lambda + \lambda^{-1}\log\log\lambda\,,$$
$$w(p_\lambda) - a + 2\lambda^{-1}\log\lambda + \lambda^{-1}\log\log\lambda\big\}\,,$$
$$V_\lambda = \min_{p_\lambda\le x\le 0} w(x)\,.$$

PROOF. We set

$$\tau(q; x_0, G_\lambda w) = \inf\{t > 0 : X(t; x_0, G_\lambda w) = q\}\,, \quad q \in \mathbb{R}\,,$$
$$T(q) = \inf\{t > 0 : B(t) = q\}\,, \quad q \in \mathbb{R}\,,$$

which are defined on the probability space $(\widetilde{\Omega}, \widetilde{P})$. The assertion (3.1) is equivalent to

$$\lim_{\lambda\to\infty} \widetilde{P}\{\tau(\lambda^2 e^{-a\lambda}p_\lambda; \lambda^2 e^{-a\lambda}x_0, G_\lambda w) < Ce^{\lambda J(\lambda)}\} = 1\,. \tag{3.2}$$

We just prove (3.2) in the case $x_0 = 0$ and the assumption (i) is satisfied. We set

$$E_\lambda = \{\tau(\lambda^2 e^{-a\lambda}p_\lambda; 0, G_\lambda w) < \tau(1; 0, G_\lambda w)\}\,.$$

By the assumption (i), we have

$$\lim_{\lambda\to\infty} \widetilde{P}\{\tau(\lambda^2 e^{-a\lambda}p; 0, G_\lambda w) < \tau(1; 0, G_\lambda w)\} = 1$$

and therefore

$$\lim_{\lambda\to\infty} \widetilde{P}\{E_\lambda\} = 1\,. \tag{3.3}$$

By (2.2) and (2.1), we observe

$$\begin{aligned}\tau(\lambda^2 e^{-a\lambda} p_\lambda; 0, G_\lambda w) &= A_{G_\lambda w}(T(S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p_\lambda))) \\ &= \int_{\lambda^2 e^{-a\lambda} p_\lambda}^{\infty} e^{-(G_\lambda w)(x)} L(T(S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p_\lambda)), S_{G_\lambda w}(x))dx. \end{aligned} \tag{3.4}$$

On the set E_λ, the right-hand side of (3.4) equals

$$\begin{aligned}&\int_{\lambda^2 e^{-a\lambda} p_\lambda}^{1} e^{-(G_\lambda w)(x)} L(T(S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p_\lambda)), S_{G_\lambda w}(x))dx \\ &\overset{\mathrm{d}}{=} |S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p_\lambda)| \int_{\lambda^2 e^{-a\lambda} p_\lambda}^{1} e^{-(G_\lambda w)(x)} L(T(-1), \frac{S_{G_\lambda w}(x)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p_\lambda)|})dx \\ &= \lambda^4 e^{-2a\lambda} \int_{p_\lambda}^{0} e^{\lambda w(y)}dy \int_{p_\lambda}^{0} e^{-\lambda w(z)} L(T(-1), \frac{S_{G_\lambda w}(\lambda^2 e^{-a\lambda} z)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p_\lambda)|})dz \\ &\quad + \lambda^2 e^{-a\lambda} \int_{p_\lambda}^{0} e^{\lambda w(y)}dy \int_{0}^{1} L(T(-1), \frac{S_{G_\lambda w}(z)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p_\lambda)|})dz \\ &\equiv I_\lambda + II_\lambda\,. \end{aligned} \tag{3.5}$$

For $t > 0$, we set $K(t) = \sup_{x\in\mathbb{R}} L(t, x)$. Since $0 < K(T(-1)) < \infty$ ($\widetilde{P}$-a.s.), we have for all sufficiently large $\lambda > 0$

$$\begin{aligned} I_\lambda &\le |p|^2 K(T(-1)) e^{\lambda(w(p_\lambda) - V_\lambda - 2a + 4\lambda^{-1}\log\lambda)}, \quad \widetilde{P}\text{-a.s.}, \\ II_\lambda &\le |p| K(T(-1)) e^{\lambda(w(p_\lambda) - a + 2\lambda^{-1}\log\lambda)}, \quad \widetilde{P}\text{-a.s.} \end{aligned}$$

Set $E'_\lambda = \{K(T(-1)) < \log\lambda\}$. Then we have $\lim_{\lambda\to\infty} \widetilde{P}\{E'_\lambda\} = 1$, and for all sufficiently large $\lambda > 0$ the following holds on E'_λ :

$$I_\lambda + II_\lambda < (|p|^2 + |p|)e^{\lambda J(\lambda)}\,.$$

Therefore we have

$$\lim_{\lambda\to\infty} \widetilde{P}\{I_\lambda + II_\lambda < (|p|^2 + |p|)e^{\lambda J(\lambda)}\} = 1\,. \tag{3.6}$$

By (3.3)–(3.6), we obtain (3.2) in the case $x_0 = 0$. □

The following lemma is easily obtained from Lemma 3.1.

LEMMA 3.2. *Let $w \in \mathbb{W}$ and $p < x_0 \le 0$. Assume $w(p) \ge w(x)$ for all $x \in [p, x_0]$. In addition, assume* (i) $\max_{p\le x\le 0} w(x) < a$ *or* (ii) $w(1) > 0$. *Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} P_{G_\lambda w}^{\lambda^2 e^{-a\lambda} x_0}\{\tau(\lambda^2 e^{-a\lambda} p) < e^{\lambda(J_I + \varepsilon)}\} = 1\,, \tag{3.7}$$

where

$$\begin{aligned} J_I &= \max\{J_0 - 2a, w(p) - a\} \\ &= \begin{cases} J_0 - 2a\,, & \textit{if } \min_{p \le x \le 0} w(x) \le -a\,, \\ w(p) - a\,, & \textit{if } \min_{p \le x \le 0} w(x) \ge -a\,, \end{cases} \\ J_0 &= w(p) - \min_{p \le x \le 0} w(x)\,. \end{aligned} \tag{3.8}$$

LEMMA 3.3. *Let $w \in \mathbb{W}$ and $p < x_0 \le 0$.*

(i) *Assume $w(p) > w(x)$ for all $x \in (p, x_0]$. Then for any $\varepsilon > 0$*

$$\lim_{\lambda \to \infty} P^{\lambda^2 e^{-a\lambda} x_0}_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda} p) > e^{\lambda(J_{x_0} - 2a - \varepsilon)}\} = 1\,,$$

where $J_{x_0} = w(p) - \min_{p \le x \le x_0} w(x)$.

(ii) *Assume $w(p) > w(x)$ for all $x \in (p, 0]$ and $w(p) > a$. Then for any $\varepsilon > 0$*

$$\lim_{\lambda \to \infty} P^0_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda} p) > e^{\lambda(J_I - \varepsilon)}\} = 1\,, \tag{3.9}$$

where J_I is defined in (3.8).

PROOF. We just prove (ii). We show

$$\lim_{\lambda \to \infty} \widetilde{P}\{\tau(\lambda^2 e^{-a\lambda} p; 0, G_\lambda w) > e^{\lambda(J_I - \varepsilon)}\} = 1\,, \tag{3.10}$$

which is equivalent to (3.9). As in the proof of Lemma 3.1, we have

$$\begin{aligned} &\tau(\lambda^2 e^{-a\lambda} p; 0, G_\lambda w) \\ &\quad \overset{\mathrm{d}}{=} |S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)| \int_{\lambda^2 e^{-a\lambda} p}^{\infty} e^{-(G_\lambda w)(x)} L(T(-1), \frac{S_{G_\lambda w}(x)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)|}) dx \\ &\quad \ge |S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)| \int_{\lambda^2 e^{-a\lambda} p}^{1} e^{-(G_\lambda w)(x)} L(T(-1), \frac{S_{G_\lambda w}(x)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)|}) dx \\ &\quad = \lambda^4 e^{-2a\lambda} \int_p^0 e^{\lambda w(y)} dy \int_p^0 e^{-\lambda w(z)} L(T(-1), \frac{S_{G_\lambda w}(\lambda^2 e^{-a\lambda} z)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)|}) dz \\ &\qquad + \lambda^2 e^{-a\lambda} \int_p^0 e^{\lambda w(y)} dy \int_0^1 L(T(-1), \frac{S_{G_\lambda w}(z)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)|}) dz \\ &\quad \equiv III_\lambda + IV_\lambda\,. \end{aligned} \tag{3.11}$$

First we estimate III_λ. We observe that

$$\frac{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} z)|}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)|} = \frac{\int_z^0 e^{\lambda w(u)} du}{\int_p^0 e^{\lambda w(u)} du} \to 0$$

as $\lambda \to \infty$ uniformly on any closed interval contained in $(p, 0]$. From this, it follows that

$$L(T(-1), \frac{S_{G_\lambda w}(\lambda^2 e^{-a\lambda} z)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)|}) \to L(T(-1), 0) > 0 \quad (\widetilde{P}\text{-a.s.})$$

as $\lambda \to \infty$ uniformly on any closed interval contained in $(p, 0]$. Therefore, by virtue of the classical Laplace method, we have

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log III_\lambda = J_0 - 2a\,, \quad \widetilde{P}\text{-a.s.} \tag{3.12}$$

Next we estimate IV_λ. Since

$$\frac{S_{G_\lambda w}(z)}{|S_{G_\lambda w}(\lambda^2 e^{-a\lambda} p)|} = \frac{e^{a\lambda} z}{\lambda^2 \int_p^0 e^{\lambda w(u)} du} \to 0$$

as $\lambda \to \infty$ uniformly on any closed interval contained in $(0, 1)$, we get

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log IV_\lambda = w(p) - a\,, \quad \widetilde{P}\text{-a.s.} \tag{3.13}$$

in the same way as above. By (3.12) and (3.13), we obtain

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log(III_\lambda + IV_\lambda) = J_I$$

in probability with respect to $\widetilde{P}$. Therefore, by (3.11), we arrive at (3.10). □

LEMMA 3.4. *Let $w \in \mathbb{W}$ and assume $\sigma(a) > -\infty$. Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(1) < e^{\lambda(J_{II}+\varepsilon)}\} = 1\,,$$

where $J_{II} = \max\{-\min_{\sigma(a)\le x\le 0} w(x) - a, 0\}$.

PROOF. We can prove the lemma by following the proof of Lemma 3.1 and using

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(1) < \tau(\lambda^2 e^{-a\lambda}\sigma(a))\} = 1 \tag{3.14}$$

instead of (3.3). □

The following lemma can be shown in the same way as Lemma 3.3.

LEMMA 3.5. *Let $w \in \mathbb{W}$ and $p > 1$. Assume $w(p) > w(x)$ for all $x \in [1, p)$. Then for any $\varepsilon > 0$*

$$\lim_{\lambda\to\infty} P^1_{G_\lambda w}\big\{\tau(p) > \exp\{e^{a\lambda/2}(J_{III} - \varepsilon)\}\big\} = 1\,,$$

where $J_{III} = w(p) - \min_{1\le x\le p} w(x)$.

4. Proof of Proposition 2.2

First we prepare two lemmas which are used to prove Proposition 2.2 (i). We show these lemmas by using the method in [8] (see also [3], [4]).

LEMMA 4.1. *There exists a subset* $\mathbb{A}^{\#}$ *of* $\mathbb{A}$ *with* $P\{\mathbb{A}\backslash\mathbb{A}^{\#}\} = 0$ *such that for any* $w \in \mathbb{A}^{\#}$ *the following holds: for any sufficiently small* $u > 0$ *there exists* $\delta_0 > 0$ *such that for any* $\delta \in (0, \delta_0)$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda}\zeta_{1-u}) < e^{\lambda(1-2a-\delta)}\} = 1 . \tag{4.1}$$

PROOF. Let $w \in \mathbb{A}$. In the case $M < V + 1$, we let $u > 0$ satisfy $w(\zeta_{1-u}) > w(x)$ for all $x \in (\zeta_{1-u}, 0]$. Then we can apply Lemma 3.2 to $p = \zeta_{1-u}$ and $x_0 = 0$ because of (1.6) or the definition of $\mathbb{A}''$. In this case the assertion (3.7) holds for $J_I = w(\zeta_{1-u}) - \min_{\zeta_{1-u}\le x\le 0} w(x) - 2a = 1 - u - 2a$ because of (1.1). As a result, we get (4.1) for any $\delta \in (0, u)$.

In the case $M \ge V+1$, we let $u \in (0, 1)$ and set $c_0 = 0$. For some integer $n \ge 2$ we take $\ell_k < c_k < 0$, $k \in \{1, 2, \dots, n\}$, satisfying $\sigma(-a) = c_1 > c_2 > \cdots > c_{n-1} > b > c_n = \zeta_{1-u}$ and

$$\begin{cases} w_k(\ell_k) \ge w_k(x) \quad \text{for all } x \in [\ell_k, c_{k-1}] , \\ w_k(\ell_k) < a \quad \text{if } w \in \mathbb{A}' , \\ w_k(\ell_k) - \min_{c_k\le x\le 0} w_k(x) < 1 , \end{cases} \tag{4.2}$$

for any $k \in \{1, 2, \dots, n\}$, where $w_k \in \mathbb{W}$ is defined by

$$w_k(x) = \begin{cases} w(x) & \text{for } x \ge c_k , \\ -x + w(c_k) + c_k & \text{for } x < c_k . \end{cases}$$

Note that we can take c_k, $k \in \{1, 2, \dots, n-1\}$, independent of u. By (4.2) and Lemma 3.2, we have, for any $k \in \{1, 2, \dots, n\}$ and $\varepsilon_k > 0$

$$\lim_{\lambda\to\infty} P^{\lambda^2 e^{-a\lambda} c_{k-1}}_{G_\lambda w_k}\{\tau(\lambda^2 e^{-a\lambda}\ell_k) < e^{\lambda(\overline{J}_k - 2a + \varepsilon_k)}\} = 1 ,$$

where $\overline{J}_k = w_k(\ell_k) - \min_{\ell_k\le x\le 0} w_k(x) < 1$. Therefore, for any $k \in \{1, 2, \dots, n\}$ and $\delta_k \in (0, 1 - \overline{J}_k)$, we have

$$\lim_{\lambda\to\infty} P^{\lambda^2 e^{-a\lambda} c_{k-1}}_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda} c_k) < e^{\lambda(1-2a-\delta_k)}\} = 1 .$$

Using the strong Markov property, we obtain the lemma in this case, too. □

LEMMA 4.2. *There exists a subset* $\mathbb{A}^{\#}$ *of* $\mathbb{A}$ *with* $P\{\mathbb{A}\backslash\mathbb{A}^{\#}\} = 0$ *such that for any* $w \in \mathbb{A}^{\#}$ *the following holds: for any* $v > 0$ *satisfying* $\min_{\zeta_{1+v}\le x\le \zeta} w(x) > V$ *and any* $\delta \in (0, v)$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda}\zeta_{1+v}) > e^{\lambda(1-2a+\delta)}\} = 1 . \tag{4.3}$$

PROOF. Let $w \in \mathbb{A}$ and $v > 0$ satisfy $\min_{\zeta_{1+v} \le x \le \zeta} w(x) > V$. In the case $M < V+1$, we can apply Lemma 3.3 (i) to $p = \zeta_{1+v}$ and $x_0 = 0$, and because of

$$\overline{J} \equiv w(\zeta_{1+v}) - \min_{\zeta_{1+v} \le x \le 0} w(x) = 1 + v \,, \tag{4.4}$$

we get (4.3) for any $\delta \in (0, v)$. We show (4.3) in the case $M \ge V + 1$. For $c_{n-1} < 0$ defined in the proof of Lemma 4.1, we have, for any sufficiently small $\overline{\delta} > 0$

$$\lim_{\lambda \to \infty} P^0_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda} c_{n-1}) < e^{\lambda(1-2a-\overline{\delta})}\} = 1 \,. \tag{4.5}$$

We note that $w(\zeta_{1+v}) > w(x)$ for all $x \in (\zeta_{1+v}, c_{n-1}]$. Therefore, by Lemma 3.3 (i), we have, for any $\varepsilon > 0$

$$\lim_{\lambda \to \infty} P^{\lambda^2 e^{-a\lambda} c_{n-1}}_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda} \zeta_{1+v}) > e^{\lambda(\overline{J}-2a-\varepsilon)}\} = 1 \,. \tag{4.6}$$

By (4.5), (4.6), (4.4) and the strong Markov property, we obtain (4.3) for any $\delta \in (0, v)$ in this case, too. □

Let us now prove Proposition 2.2. To prove (i) and (ii), we use the coupling method in [6] (see also [8]).

PROOF OF PROPOSITION 2.2. First we show (i). Let $w \in \mathbb{A}$ and set

$$r = \begin{cases} 1 & \text{if } w \in \mathbb{A}' \,, \\ 1 + \eta_1 & \text{if } w \in \mathbb{A}'' \cap (\mathbb{A}')^c \,. \end{cases}$$

In the case $w \in \mathbb{A}'' \cap (\mathbb{A}')^c$, $\eta_1 > 0$ is chosen to be small enough that $C(\eta_1) \equiv \min_{1 \le x \le 1+\eta_1} w(x) > 0$. Let $v > 0$ satisfy

$$\min_{\zeta_{1+v} \le x \le \zeta} w(x) > V \tag{4.7}$$

and

$$w(\zeta_{1+v}) < a \quad \text{if } w \in \mathbb{A}' \,. \tag{4.8}$$

Then we have

$$\lim_{\lambda \to \infty} P^0_{G_\lambda w}\{\tau(r) > \tau(\lambda^2 e^{-a\lambda} \zeta_{1+v})\} = 1 \,, \tag{4.9}$$

because of (1.6) and (4.8) (in the case $w \in \mathbb{A}'$). We set $K_\lambda = [\lambda^2 e^{-a\lambda} \zeta_{1+v}, r]$ and define the probability measure m_λ on K_λ by

$$m_\lambda(dx) = \frac{e^{-(G_\lambda w)(x)} dx}{\int_{K_\lambda} e^{-(G_\lambda w)(y)} dy} \,.$$

This is the invariant probability measure for the reflecting $\mathcal{L}_{G_\lambda w}$-diffusion process on K_λ. Let $\varepsilon \in (0, b-\zeta)$. In the case $w \in \mathbb{A}'' \cap (\mathbb{A}')^c$ we have

$$m_\lambda\{(\lambda^2 e^{-a\lambda}(b-\varepsilon), \lambda^2 e^{-a\lambda}(b+\varepsilon)\wedge 0)\}$$
$$= \frac{\int_{b-\varepsilon}^{(b+\varepsilon)\wedge 0} e^{-\lambda w(y)}dy}{\int_{\zeta_{1+v}}^{0} e^{-\lambda w(y)}dy + \lambda^{-2}e^{a\lambda} + \lambda^{-2}e^{a\lambda}\int_1^{1+\eta_1}\exp\{-e^{a\lambda/2}w(y)\}dy}\,.$$

By virtue of (1.1) and (4.7), we have

$$\lim_{\lambda\to\infty}\lambda^{-1}\log\int_{b-\varepsilon}^{(b+\varepsilon)\wedge 0} e^{-\lambda w(y)}dy = -V > a\,,$$
$$\lim_{\lambda\to\infty}\lambda^{-1}\log\int_{(\zeta_{1+v},0)\setminus(b-\varepsilon,(b+\varepsilon)\wedge 0)} e^{-\lambda w(y)}dy < -V\,,$$
$$\lim_{\lambda\to\infty} e^{-a\lambda/2}\log\int_1^{1+\eta_1}\exp\{-e^{a\lambda/2}w(y)\}dy = -C(\eta_1) < 0\,.$$

Therefore we get

$$\lim_{\lambda\to\infty} m_\lambda\{(\lambda^2 e^{-a\lambda}(b-\varepsilon), \lambda^2 e^{-a\lambda}(b+\varepsilon)\wedge 0)\} = 1 \tag{4.10}$$

in the case $w \in \mathbb{A}'' \cap (\mathbb{A}')^c$. As easily seen from above, we get (4.10) in the case $w \in \mathbb{A}'$, too.

We introduce $\{X_\lambda^{(R)}(t), t \ge 0\}$, the reflecting $\mathcal{L}_{G_\lambda w}$-diffusion process on K_λ with initial distribution m_λ defined on the probability space $(\widetilde{\Omega}, \widetilde{P})$. Since this is a stationary process, we have, by (4.10), for any $t \ge 0$

$$\lim_{\lambda\to\infty}\widetilde{P}\{X_\lambda^{(R)}(t) \in (\lambda^2 e^{-a\lambda}(b-\varepsilon), \lambda^2 e^{-a\lambda}(b+\varepsilon)\wedge 0)\} = 1\,. \tag{4.11}$$

We couple the processes $\{X(t; 0, G_\lambda w), t \ge 0\}$ and $\{X_\lambda^{(R)}(t), t \ge 0\}$ as follows: two processes move independently until they first meet each other; then they move together until they go out from $(\lambda^2 e^{-a\lambda}\zeta_{1+v}, r)$; after going out from the interval they again move independently. Let

$$\sigma_\lambda = \inf\{t > 0 : X(t; 0, G_\lambda w) = X_\lambda^{(R)}(t)\}\,,$$
$$\sigma'_\lambda = \inf\{t > \sigma_\lambda : X(t; 0, G_\lambda w) \notin (\lambda^2 e^{-a\lambda}\zeta_{1+v}, r)\}\,.$$

By (4.11), it follows that

$$\lim_{\lambda\to\infty}\widetilde{P}\{\sigma_\lambda < \tau(\lambda^2 e^{-a\lambda}(b-\varepsilon); 0, G_\lambda w)\} = 1\,. \tag{4.12}$$

By using (4.12) and Lemma 4.1 for sufficiently small $u > 0$ satisfying $\zeta_{1-u} < b-\varepsilon$, we have, for any sufficiently small $\delta > 0$

$$\lim_{\lambda\to\infty}\widetilde{P}\{\sigma_\lambda < e^{\lambda(1-2a-\delta)}\} = 1\,. \tag{4.13}$$

Moreover, by virtue of (4.9) and Lemma 4.2, we have, for any $\delta' \in (0, v)$

$$\lim_{\lambda\to\infty} \widetilde{P}\{\sigma'_\lambda > e^{\lambda(1-2a+\delta')}\} = 1\,. \tag{4.14}$$

Using (4.13), (4.14) and (4.11), we obtain, for any $\varepsilon \in (0, b-\zeta)$

$$\lim_{\lambda\to\infty} \widetilde{P}\{X(e^{\lambda(1-2a)}; 0, G_\lambda w) \in (\lambda^2 e^{-a\lambda}(b-\varepsilon), \lambda^2 e^{-a\lambda}(b+\varepsilon)\wedge 0)\} = 1$$

by the same argument as in [6] (see also [8]). Therefore we get (i).

Next we prove (ii). Let $w \in \mathbb{B}$. In this case, in Lemma 3.4 we have $J_{II} = 0$ and therefore for any $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(1) < e^{\lambda\varepsilon}\} = 1\,. \tag{4.15}$$

We set $q = \sigma(1-a+\xi)$, where $\xi > 0$ is chosen to be small enough that $\min_{q\le x\le\sigma(1-a)} w(x) > -a$. Then we have $\widetilde{V} \equiv \min_{q\le x\le 0} w(x) > -a$. Applying Lemma 3.3 (ii) to $p = q$, we have, for any $\widetilde{\delta} \in (0, \xi)$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda} q) > e^{\lambda(1-2a+\widetilde{\delta})}\} = 1\,. \tag{4.16}$$

Moreover, for any $\eta > 0$, we see that

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(1+\eta) > \tau(\lambda^2 e^{-a\lambda} q)\} = 1\,. \tag{4.17}$$

Choose $\eta_2 > 0$ satisfying $C(\eta_2) \equiv \min_{1\le x\le 1+\eta_2} w(x) > 0$, and set $\widetilde{K}_\lambda = [\lambda^2 e^{-a\lambda} q, 1+\eta_2]$. We define $\widetilde{m}_\lambda$, a probability measure on $\widetilde{K}_\lambda$, by

$$\widetilde{m}_\lambda(dx) = \frac{e^{-(G_\lambda w)(x)}dx}{\int_{\widetilde{K}_\lambda} e^{-(G_\lambda w)(y)}dy}\,.$$

We observe

$$\begin{aligned}\widetilde{m}_\lambda\{(0,1)\} &= \frac{1}{\lambda^2 e^{-a\lambda}\int_q^0 e^{-\lambda w(y)}dy + 1 + \int_1^{1+\eta_2}\exp\{-e^{a\lambda/2}w(y)\}dy}\\ &\to 1\,, \quad \text{as } \lambda\to\infty\,,\end{aligned} \tag{4.18}$$

since

$$\lim_{\lambda\to\infty} \lambda^{-1}\log\int_q^0 e^{-\lambda w(y)}dy = -\widetilde{V} < a\,,$$

$$\lim_{\lambda\to\infty} e^{-a\lambda/2}\log\int_1^{1+\eta_2}\exp\{-e^{a\lambda/2}w(y)\}dy = -C(\eta_2) < 0\,.$$

Let $\{\widetilde{X}^{(R)}_\lambda(t), t\ge 0\}$ be the reflecting $\mathcal{L}_{G_\lambda w}$-diffusion process on $\widetilde{K}_\lambda$ with initial distribution $\widetilde{m}_\lambda$ defined on $(\widetilde{\Omega}, \widetilde{P})$. We couple $\{X(t; 0, G_\lambda w), t\ge 0\}$ and $\{\widetilde{X}^{(R)}_\lambda(t), t\ge 0\}$ in the same way as we coupled $\{X(t; 0, G_\lambda w), t \geq 0\}$ and $\{X_\lambda^{(R)}(t), t \geq 0\}$ in the proof of (i). By the same argument as there and using (4.15)–(4.18), we obtain (ii).

Finally we prove (iii). Let $w \in \mathbb{C}$. In this case, in Lemma 3.4 we have $0 \leq J_{II} < 1 - 2a$ and therefore for any $\delta \in (0, 1 - 2a - J_{II})$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(1) < e^{\lambda(1-2a-\delta)}\} = 1 . \tag{4.19}$$

By Lemma 3.5, for any $\varepsilon > 0$ there exists $C' > 0$ such that

$$\lim_{\lambda\to\infty} P^1_{G_\lambda w}\big\{\tau(1+\varepsilon) > \exp\{e^{a\lambda/2}C'\}\big\} = 1 . \tag{4.20}$$

By (4.19) and (4.20), we get, for any $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{X(e^{\lambda(1-2a)}) < 1 + \varepsilon\} = 1 . \tag{4.21}$$

On the other hand, we let $\varepsilon \in (0, -w(1))$ and choose $\eta_3 > 0$ satisfying $M(\eta_3) \equiv \max_{1\leq x\leq 1+\eta_3} w(x) < w(1) + \varepsilon (< 0)$. We observe

$$\begin{aligned} &P^1_{G_\lambda w}\big\{\tau(1 - \exp\{e^{a\lambda/2}(w(1) + \varepsilon)\}) > \tau(1 + \eta_3)\big\} \\ &\quad = \frac{\exp\{e^{a\lambda/2}(w(1) + \varepsilon)\}}{\int_1^{1+\eta_3} \exp\{e^{a\lambda/2}w(x)\}dx + \exp\{e^{a\lambda/2}(w(1) + \varepsilon)\}} \\ &\quad \to 1 , \quad \text{as } \lambda \to \infty , \end{aligned} \tag{4.22}$$

since

$$\lim_{\lambda\to\infty} e^{-a\lambda/2} \log \int_1^{1+\eta_3} \exp\{e^{a\lambda/2}w(x)\}dx = M(\eta_3) < w(1) + \varepsilon .$$

Using (4.22) and Lemma 3.5, we have for some $C'' > 0$

$$\lim_{\lambda\to\infty} P^1_{G_\lambda w}\big\{\tau(1 - \exp\{e^{a\lambda/2}(w(1) + \varepsilon)\}) > \exp\{e^{a\lambda/2}C''\}\big\} = 1 . \tag{4.23}$$

By (4.19) and (4.23), we get

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\big\{X(e^{\lambda(1-2a)}) > 1 - \exp\{e^{a\lambda/2}(w(1) + \varepsilon)\}\big\} = 1 . \tag{4.24}$$

Combining (4.21) and (4.24), we obtain, for any $\varepsilon \in (0, -w(1))$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\big\{-\exp\{e^{a\lambda/2}(w(1) + \varepsilon)\} < X(e^{\lambda(1-2a)}) - 1 < \varepsilon\big\} = 1 .$$

Therefore we get (iii). □

5. Proof of Theorem 1.3

We obtain Theorem 1.3 from the following proposition in the same way as obtaining Theorem 1.1 from Proposition 2.2.

PROPOSITION 5.1. (i) *There exists a subset* $(\mathbb{A}'_I)^\#$ *of* $\mathbb{A}'_I$ *with* $P\{\mathbb{A}'_I\backslash(\mathbb{A}'_I)^\#\} = 0$ *such that, for any* $w \in (\mathbb{A}'_I)^\#$ *and* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{e^{\lambda(H-a-\varepsilon)} < \overline{X}(e^{\lambda(1-2a)}) < e^{\lambda(H-a+\varepsilon)} \wedge \varepsilon\} = 1\,. \tag{5.1}$$

(ii) *There exists a subset* $(\mathbb{A}'_{II})^\#$ *of* $\mathbb{A}'_{II}$ *with* $P\{\mathbb{A}'_{II}\backslash(\mathbb{A}'_{II})^\#\} = 0$ *such that, for any* $w \in (\mathbb{A}'_{II})^\#$, $\varepsilon > 0$ *and* $\varepsilon(\lambda) > 0, \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty}\varepsilon(\lambda) = 0$ *and* $\liminf_{\lambda\to\infty}\lambda(\log\lambda)^{-1}\varepsilon(\lambda) > 2$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{e^{\lambda(H-a-\varepsilon)} < \overline{X}(e^{\lambda(1-2a)}) < e^{\lambda(H-a+\varepsilon(\lambda))} \wedge \varepsilon\} = 1\,. \tag{5.2}$$

(iii) *There exists a subset* $\mathbb{D}^\#$ *of* $\mathbb{D}$ *with* $P\{\mathbb{D}\backslash\mathbb{D}^\#\} = 0$ *such that, for any* $w \in \mathbb{D}^\#$ *and* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{f_\lambda(\varepsilon) < \overline{X}(e^{\lambda(1-2a)}) - 1 < g_\lambda(\varepsilon) \wedge \varepsilon\} = 1\,,$$

where

$$f_\lambda(\varepsilon) = f_\lambda(\varepsilon, w) = \exp\{-e^{a\lambda/2}(w(1)+\varepsilon)\}\,,$$
$$g_\lambda(\varepsilon) = g_\lambda(\varepsilon, w) = \exp\{-e^{a\lambda/2}(w(1)-\varepsilon)\}\,.$$

(iv) *There exists a subset* $\mathbb{C}^\#$ *of* $\mathbb{C}$ *with* $P\{\mathbb{C}\backslash\mathbb{C}^\#\} = 0$ *such that, for any* $w \in \mathbb{C}^\#$ *and* $\varepsilon > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{h_\lambda(\varepsilon) < \overline{X}(e^{\lambda(1-2a)}) - 1 < \varepsilon\} = 1\,, \tag{5.3}$$

where

$$h_\lambda(\varepsilon) = \exp\{-e^{a\lambda/2}\varepsilon\}\,.$$

To prove Proposition 5.1, we prepare three lemmas.

LEMMA 5.2. *Let* $w \in \mathbb{W}$ *and assume* $w(1) > 0$. *Then for any* $\varepsilon > 0$ *and* $\xi \in (0, \varepsilon)$

$$\lim_{\lambda\to\infty} P^1_{G_\lambda w}\big\{\tau(1+f_\lambda(\varepsilon)) < \exp\{-e^{a\lambda/2}\xi\}\big\} = 1\,. \tag{5.4}$$

PROOF. Assume $w(1) > 0$ and let $\varepsilon > 0$. Then we have $f_\lambda(\varepsilon) \downarrow 0$ as $\lambda \to \infty$. Note that

$$\tau(1+f_\lambda(\varepsilon); 1, G_\lambda w) = \tau(f_\lambda(\varepsilon); 0, (G_\lambda w)^1)\,. \tag{5.5}$$

For $\eta \in (0, 1)$, we set

$$\widetilde{E}_\lambda = \{\tau(f_\lambda(\varepsilon); 0, (G_\lambda w)^1) < \tau(-\eta; 0, (G_\lambda w)^1)\}$$

and observe

$$\widetilde{P}\{\widetilde{E}_\lambda\} = \frac{\eta}{\int_1^{1+f_\lambda(\varepsilon)}\exp\{e^{a\lambda/2}w(x)\}dx + \eta}\,.$$

By the definition of $f_\lambda(\varepsilon)$, we see that

$$\int_1^{1+f_\lambda(\varepsilon)} \exp\{e^{a\lambda/2}w(x)\}dx \le \exp\{e^{a\lambda/2}(M_\lambda - w(1) - \varepsilon)\}\,, \tag{5.6}$$

where $M_\lambda = \max_{1\le x\le 1+f_\lambda(\varepsilon)} w(x)$. Since $M_\lambda \downarrow w(1)$ as $\lambda \to \infty$, the right-hand side of (5.6) converges to 0 as $\lambda \to \infty$. Therefore we get

$$\lim_{\lambda\to\infty} \widetilde{P}\{\widetilde{E}_\lambda\} = 1\,. \tag{5.7}$$

On $\widetilde{E}_\lambda$, by the same argument as in the proof of Lemma 3.1, the right-hand side of (5.5) is equal to

$$\begin{aligned}
&\int_{-\eta}^{f_\lambda(\varepsilon)} e^{-(G_\lambda w)^1(x)} L(T(S_{(G_\lambda w)^1}(f_\lambda(\varepsilon))), S_{(G_\lambda w)^1}(x))dx \\
&\quad \overset{\mathrm{d}}{=} S_{(G_\lambda w)^1}(f_\lambda(\varepsilon)) \int_{-\eta}^{f_\lambda(\varepsilon)} e^{-(G_\lambda w)^1(x)} L\left(T(1), \frac{S_{(G_\lambda w)^1}(x)}{S_{(G_\lambda w)^1}(f_\lambda(\varepsilon))}\right)dx \\
&\quad = I'_\lambda + II'_\lambda\,,
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
S_{(G_\lambda w)^1}(x) &= \int_0^x e^{(G_\lambda w)^1(y)}dy\,, \quad x \in \mathbb{R}\,, \\
I'_\lambda &= \int_1^{1+f_\lambda(\varepsilon)} \exp\{e^{a\lambda/2}w(y)\}dy \int_{1-\eta}^1 L\left(T(1), \frac{S_{(G_\lambda w)^1}(z-1)}{S_{(G_\lambda w)^1}(f_\lambda(\varepsilon))}\right)dz\,, \\
II'_\lambda &= III'_\lambda \times IV'_\lambda\,, \\
III'_\lambda &= \int_1^{1+f_\lambda(\varepsilon)} \exp\{e^{a\lambda/2}w(y)\}dy\,, \\
IV'_\lambda &= \int_1^{1+f_\lambda(\varepsilon)} \exp\{-e^{a\lambda/2}w(z)\} L\left(T(1), \frac{S_{(G_\lambda w)^1}(z-1)}{S_{(G_\lambda w)^1}(f_\lambda(\varepsilon))}\right)dz\,.
\end{aligned}$$

We note that $0 < K(T(1)) < \infty$ ($\widetilde{P}$-a.s.), where $K(\cdot)$ is defined in the proof of Lemma 3.1. From this, we can estimate I'_λ and II'_λ as follows:

$$\begin{aligned}
I'_\lambda &\le \eta K(T(1)) \exp\{e^{a\lambda/2}(M_\lambda - w(1) - \varepsilon)\}\,, \quad \widetilde{P}\text{-a.s.}\,, \\
II'_\lambda &\le K(T(1)) \exp\{e^{a\lambda/2}\{M_\lambda - C_\lambda - 2(w(1)+\varepsilon)\}\}\,, \quad \widetilde{P}\text{-a.s.}\,,
\end{aligned}$$

where $C_\lambda = \min_{1\le x\le 1+f_\lambda(\varepsilon)} w(x)$. Using $M_\lambda \downarrow w(1)$ (as $\lambda \to \infty$) and $C_\lambda \uparrow w(1)$ (as $\lambda \to \infty$), we get

$$\begin{aligned}
&\limsup_{\lambda\to\infty} e^{-a\lambda/2} \log I'_\lambda \le -\varepsilon\,, \quad \widetilde{P}\text{-a.s.}\,, \\
&\limsup_{\lambda\to\infty} e^{-a\lambda/2} \log II'_\lambda \le -2(w(1)+\varepsilon) < -\varepsilon\,, \quad \widetilde{P}\text{-a.s.}
\end{aligned}$$

Therefore, for any $\xi \in (0, \varepsilon)$ we have

$$\lim_{\lambda \to \infty} \widetilde{P}\big\{I'_\lambda + II'_\lambda < \exp\{-e^{a\lambda/2}\xi\}\big\} = 1\,. \tag{5.9}$$

By (5.5) and (5.7)–(5.9), we obtain

$$\lim_{\lambda \to \infty} \widetilde{P}\big\{\tau(1 + f_\lambda(\varepsilon); 1, G_\lambda w) < \exp\{-e^{a\lambda/2}\xi\}\big\} = 1\,,$$

which is equivalent to (5.4). □

LEMMA 5.3. *Let $w \in \mathbb{W}$ and assume $w(1) > 0$. Then for any $\varepsilon \in (0, w(1))$ and $p < 0$*

$$\lim_{\lambda \to \infty} P^0_{G_\lambda w}\{\tau(1 + g_\lambda(\varepsilon)) > \tau(\lambda^2 e^{-a\lambda} p)\} = 1\,. \tag{5.10}$$

PROOF. Assume $w(1) > 0$ and let $\varepsilon \in (0, w(1))$. Then we have $g_\lambda(\varepsilon) \downarrow 0$ as $\lambda \to \infty$. For any $p < 0$ we observe

$$P^0_{G_\lambda w}\{\tau(1 + g_\lambda(\varepsilon)) > \tau(\lambda^2 e^{-a\lambda} p)\} = \frac{1 + \int_1^{1+g_\lambda(\varepsilon)} \exp\{e^{a\lambda/2} w(x)\}dx}{\lambda^2 e^{-a\lambda} \int_p^0 e^{\lambda w(x)}dx + 1 + \int_1^{1+g_\lambda(\varepsilon)} \exp\{e^{a\lambda/2} w(x)\}dx}\,.$$

Setting $C'_\lambda = \min_{1 \le x \le 1+g_\lambda(\varepsilon)} w(x)$, we have

$$\int_1^{1+g_\lambda(\varepsilon)} \exp\{e^{a\lambda/2} w(x)\}dx \ge \exp\{e^{a\lambda/2}(C'_\lambda - w(1) + \varepsilon)\}\,. \tag{5.11}$$

Since $C'_\lambda \uparrow w(1)$ as $\lambda \to \infty$, we see that the right-hand side of (5.11) tends to ∞ as $\lambda \to \infty$ and we obtain (5.10). □

The following lemma can be shown by the same argument as in the proof of Lemma 5.2.

LEMMA 5.4. *Let $w \in \mathbb{W}$ and assume $w(1) < 0$, and let $\varepsilon > 0$ and $J = \max\{w(1) - \varepsilon, -2\varepsilon\}$. Then for any $\xi \in (0, -J)$*

$$\lim_{\lambda \to \infty} P^1_{G_\lambda w}\big\{\tau(1 + h_\lambda(\varepsilon)) < \exp\{-e^{a\lambda/2}\xi\}\big\} = 1\,.$$

Let us now prove Proposition 5.1.

PROOF OF PROPOSITION 5.1. First we show (i) by employing the method in [3, Lemma 6.1]. Let $w \in \mathbb{A}'$. We observe, for any sufficiently small $\varepsilon > 0$

$$P^0_{G_\lambda w}\{\tau(e^{\lambda(H-a-\varepsilon)}) < \tau(\lambda^2 e^{-a\lambda} \zeta_{1-\varepsilon/2})\} = \frac{\lambda^2 \int_{\zeta_{1-\varepsilon/2}}^0 e^{\lambda w(x)}dx}{e^{\lambda(H-\varepsilon)} + \lambda^2 \int_{\zeta_{1-\varepsilon/2}}^0 e^{\lambda w(x)}dx} \to 1\,, \quad \text{as } \lambda \to \infty\,,$$

since $\lim_{\lambda\to\infty}\lambda^{-1}\log\int_{\zeta_{1-\varepsilon/2}}^{0}e^{\lambda w(x)}dx = \max_{\zeta_{1-\varepsilon/2}\le x\le 0} w(x) \ge H-\varepsilon/2 > H-\varepsilon$. Combining this with Lemma 4.1, we have

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{e^{\lambda(H-a-\varepsilon)} < \overline{X}(e^{\lambda(1-2a)})\} = 1\,. \tag{5.12}$$

Moreover, in the case $w \in \mathbb{A}'_I$ we have, for any $\varepsilon \in (0, a-H)$ satisfying $\min_{\zeta_{1+\varepsilon/2}\le x\le\zeta} w(x) > V$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(e^{\lambda(H-a+\varepsilon)}) > \tau(\lambda^2 e^{-a\lambda}\zeta_{1+\varepsilon/2})\} = 1\,,$$

since $\max_{\zeta_{1+\varepsilon/2}\le x\le 0} w(x) = H+\varepsilon/2 < H+\varepsilon$. Combining this with Lemma 4.2, we have, for any $\varepsilon\in(0,a-H)$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\overline{X}(e^{\lambda(1-2a)}) < e^{\lambda(H-a+\varepsilon)}\} = 1\,. \tag{5.13}$$

By (5.12) and (5.13), we obtain (i).

In the case $w \in \mathbb{A}'_{II}$ we show (ii) by improving the method in [3, Lemma 6.1]. In this case we have, for any sufficiently small $\varepsilon > 0$ $\max_{\zeta_{1+\varepsilon}\le x\le 0} w(x) = H$ and therefore for all sufficiently large $\lambda > 0$ satisfying $\varepsilon(\lambda) < a-H$

$$\begin{aligned} P^0_{G_\lambda w}\{\tau(e^{\lambda(H-a+\varepsilon(\lambda))}) > \tau(\lambda^2 e^{-a\lambda}\zeta_{1+\varepsilon})\} &= \frac{e^{\lambda(H+\varepsilon(\lambda))}}{\lambda^2\int_{\zeta_{1+\varepsilon}}^{0} e^{\lambda w(x)}dx + e^{\lambda(H+\varepsilon(\lambda))}} \\ &\ge \frac{e^{\lambda(H+\varepsilon(\lambda))}}{\lambda^2|\zeta_{1+\varepsilon}|e^{\lambda H} + e^{\lambda(H+\varepsilon(\lambda))}} \\ &= \frac{1}{\lambda^2|\zeta_{1+\varepsilon}|e^{-\lambda\varepsilon(\lambda)}+1}\,. \end{aligned} \tag{5.14}$$

We notice that there exists $\xi_0 > 0$ such that for all sufficiently large $\lambda > 0$ $\varepsilon(\lambda) > (2+\xi_0)\lambda^{-1}\log\lambda$. Therefore the right-hand side of (5.14) converges to 1 as $\lambda\to\infty$. Combining this with Lemma 4.2, we get

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\overline{X}(e^{\lambda(1-2a)}) < e^{\lambda(H-a+\varepsilon(\lambda))}\} = 1\,. \tag{5.15}$$

By (5.12) and (5.15), we obtain (ii).

Next we prove (iii). Let $w \in \mathbb{D}$ and $\varepsilon > 0$. In this case, in Lemma 3.4 we notice $0 \le J_{II} < 1-2a$. Therefore, by combining Lemma 3.4 with Lemma 5.2, we get, for any sufficiently small $\delta > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(1+f_\lambda(\varepsilon)) < e^{\lambda(1-2a-\delta)}\} = 1\,. \tag{5.16}$$

On the other hand, we let $\varepsilon \in (0, w(1))$. By combining Lemma 5.3 with Lemma 4.2 in the case $w \in \mathbb{A}'' \cap (\mathbb{A}')^c$ and with (4.16) in the case $w \in \mathbb{B}$, we have, for any sufficiently small

$\delta' > 0$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(1+g_\lambda(\varepsilon)) > e^{\lambda(1-2a+\delta')}\} = 1 . \tag{5.17}$$

By (5.16) and (5.17), we obtain, for any $\varepsilon \in (0, w(1))$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{f_\lambda(\varepsilon) < \overline{X}(e^{\lambda(1-2a)}) - 1 < g_\lambda(\varepsilon)\} = 1 .$$

Therefore we obtain (iii).

As to (iv), we get (5.3) by using (4.19), Lemma 5.4 and (4.20). □

6. Proof of Theorem 1.4

Theorem 1.4 is obtained from the following proposition.

PROPOSITION 6.1. (i) *There exists a subset* $\mathbb{A}^\#$ *of* $\mathbb{A}$ *with* $P\{\mathbb{A}\backslash\mathbb{A}^\#\} = 0$ *such that, for any* $w \in \mathbb{A}^\#$, $\varepsilon > 0$ *and* $\varepsilon(\lambda) > 0, \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty}\varepsilon(\lambda) = 0$ *and* $\liminf_{\lambda\to\infty}\lambda(\log\lambda)^{-1}\varepsilon(\lambda) > 4$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\lambda^2 e^{-a\lambda}\zeta_{1+\varepsilon} < \underline{X}(e^{\lambda(1-2a)}) < \lambda^2 e^{-a\lambda}\zeta_{1-\varepsilon(\lambda)}\} = 1 . \tag{6.1}$$

(ii) *There exists a subset* $\mathbb{B}^\#$ *of* $\mathbb{B}$ *with* $P\{\mathbb{B}\backslash\mathbb{B}^\#\} = 0$ *such that, for any* $w \in \mathbb{B}^\#$, $\varepsilon > 0$ *and* $\varepsilon(\lambda) > 0$, $\lambda > 0$, *satisfying* $\lim_{\lambda\to\infty}\varepsilon(\lambda) = 0$ *and* $\liminf_{\lambda\to\infty}\lambda(\log\lambda)^{-1}\varepsilon(\lambda) > 2$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\lambda^2 e^{-a\lambda}\sigma(1-a+\varepsilon) < \underline{X}(e^{\lambda(1-2a)}) < \lambda^2 e^{-a\lambda}\sigma(1-a-\varepsilon(\lambda))\} = 1 . \tag{6.2}$$

(iii) *There exists a subset* $\mathbb{C}^\#$ *of* $\mathbb{C}$ *with* $P\{\mathbb{C}\backslash\mathbb{C}^\#\} = 0$ *such that, for any* $w \in \mathbb{C}^\#$ *and* $\varepsilon(\lambda) > 0, \lambda > 0$, *satisfying* $\lim_{\lambda\to\infty}\varepsilon(\lambda) = 0$ *and* $\liminf_{\lambda\to\infty}\lambda(\log\lambda)^{-1}\varepsilon(\lambda) > 2$

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\lambda^2 e^{-a\lambda}\sigma(a) < \underline{X}(e^{\lambda(1-2a)}) < \lambda^2 e^{-a\lambda}\sigma(a-\varepsilon(\lambda))\} = 1 . \tag{6.3}$$

PROOF. First we prove (i) by improving Lemma 4.1. Let $w \in \mathbb{A}$. In the case $M < V+1$, we can apply Lemma 3.1 to $p = \zeta$, $p_\lambda = \zeta_{1-\varepsilon(\lambda)}$ and $x_0 = 0$ because of (1.6) or the definition of $\mathbb{A}''$. In this case the assertion (3.1) holds for

$$J(\lambda) = 1 - 2a - \varepsilon(\lambda) + 4\lambda^{-1}\log\lambda + \lambda^{-1}\log\log\lambda , \tag{6.4}$$

since V_λ in Lemma 3.1 is equal to V for all sufficiently large $\lambda > 0$ and (1.1) holds. We notice that there exists $\xi_1 > 0$ such that for all sufficiently large $\lambda > 0$

$$\varepsilon(\lambda) > (4+\xi_1)\lambda^{-1}\log\lambda . \tag{6.5}$$

By (6.4) and (6.5), the assertion (3.1) yields

$$\lim_{\lambda\to\infty} P^0_{G_\lambda w}\{\tau(\lambda^2 e^{-a\lambda}\zeta_{1-\varepsilon(\lambda)}) < e^{\lambda(1-2a-\xi\lambda^{-1}\log\lambda)}\} = 1 \tag{6.6}$$

for any $\xi \in (0, \xi_1)$. By (6.6) and Lemma 4.2, we obtain (6.1).

In the case $M \geq V+1$, we can apply Lemma 3.1 to $p = \zeta$, $p_\lambda = \zeta_{1-\varepsilon(\lambda)}$ and $x_0 = c_{n-1}$ defined in the proof of Lemma 4.1. By the same argument as above, we get, for any $\xi \in (0, \xi_1)$

$$\lim_{\lambda \to \infty} P_{G_\lambda w}^{\lambda^2 e^{-a\lambda} c_{n-1}} \{\tau(\lambda^2 e^{-a\lambda} \zeta_{1-\varepsilon(\lambda)}) < e^{\lambda(1-2a-\xi\lambda^{-1}\log\lambda)}\} = 1\,. \tag{6.7}$$

By (4.5), (6.7) and the strong Markov property, we obtain (6.6) for any $\xi \in (0, \xi_1)$ and therefore (6.1) in this case, too.

Next we prove (ii). Let $w \in \mathbb{B}$. We can apply Lemma 3.1 to $p = \sigma(1-a)$, $p_\lambda = \sigma(1-a-\varepsilon(\lambda))$ and $x_0 = 0$. In this case, in the lemma, for all sufficiently large $\lambda > 0$ $V_\lambda = \min_{\sigma(1-a)\leq x \leq 0} w(x) > -a$. Therefore the assertion (3.1) holds for

$$J(\lambda) = 1 - 2a - \varepsilon(\lambda) + 2\lambda^{-1}\log\lambda + \lambda^{-1}\log\log\lambda\,. \tag{6.8}$$

We notice that there exists $\xi_2 > 0$ such that for all sufficiently large $\lambda > 0$

$$\varepsilon(\lambda) > (2+\xi_2)\lambda^{-1}\log\lambda\,. \tag{6.9}$$

By (6.8) and (6.9), the assertion (3.1) yields

$$\lim_{\lambda \to \infty} P_{G_\lambda w}^{0} \{\tau(\lambda^2 e^{-a\lambda} \sigma(1-a-\varepsilon(\lambda))) < e^{\lambda(1-2a-\xi\lambda^{-1}\log\lambda)}\} = 1$$

for any $\xi \in (0, \xi_2)$. From this and (4.16), we get (6.2).

Finally we prove (iii). Let $w \in \mathbb{C}$. By (3.14), (4.19) and (4.23), we have, for some $C'' > 0$

$$\lim_{\lambda \to \infty} P_{G_\lambda w}^{0} \big\{\tau(\lambda^2 e^{-a\lambda} \sigma(a)) > \exp\{e^{a\lambda/2} C''\}\big\} = 1\,. \tag{6.10}$$

On the other hand, we set $q_\lambda = \sigma(a - \varepsilon(\lambda))$. Then we observe

$$P_{G_\lambda w}^{0} \{\tau(\lambda^2 e^{-a\lambda} q_\lambda) < \tau(1)\} = \frac{e^{a\lambda}}{e^{a\lambda} + \lambda^2 \int_{q_\lambda}^0 e^{\lambda w(x)} dx}$$

and

$$\begin{aligned} e^{-a\lambda}\lambda^2 \int_{q_\lambda}^0 e^{\lambda w(x)} dx \; &\leq |q_\lambda| \exp\{\lambda(w(q_\lambda) - a + 2\lambda^{-1}\log\lambda)\} \\ &\leq |\sigma(a)| \exp\{\lambda(-\varepsilon(\lambda) + 2\lambda^{-1}\log\lambda)\}\,. \end{aligned} \tag{6.11}$$

The right-hand side of (6.11) converges to 0 as $\lambda \to \infty$, since (6.9) holds in this case, too. As a result, we have

$$\lim_{\lambda \to \infty} P_{G_\lambda w}^{0} \{\tau(\lambda^2 e^{-a\lambda} q_\lambda) < \tau(1)\} = 1\,. \tag{6.12}$$

By (4.19), (6.12) and (6.10), we get (6.3). □

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