Quantum Yang-Baxter Equation, Braided Semigroups, and Dynamical Yang-Baxter Maps

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Dedicated to Professor Kimio Ueno on the occasion of his sixtieth birthday

Abstract. We prove that braided semigroups with suitable conditions can produce solutions to the quantum Yang-Baxter equation in every tensor category. As an application, some dynamical Yang-Baxter maps, set-theoretic solutions to a version of the quantum dynamical Yang-Baxter equation, are constructed.

1. Introduction

The quantum Yang-Baxter equation (QYBE for short) has attracted much interest in mathematics and physics since the pioneering works [2, 31]. R-matrices, matrix solutions to this equation, gave the quantum groups [4, 11, 30]. We can similarly define dynamical quantum groups [6, 8] by means of solutions to the quantum dynamical Yang-Baxter equation (QDYBE for short) [10], a generalization of the QYBE.

Drinfel'd [5] suggested to study set-theoretical solutions [7, 9, 19, 20] to the QYBE, which are called Yang-Baxter maps [29]. This Yang-Baxter map plays an important role in defining discrete integrable systems [1]. On the other hand, dynamical Yang-Baxter maps [17, 21, 22, 23] are set-theoretical solutions to a version of the QDYBE. This dynamical Yang-Baxter map yields bialgebroids [25] and discrete integrable systems through 3D compatible ternary systems [14]. Furthermore, suitable homogeneous pre-systems [12], related to reductive homogeneous spaces, can produce the dynamical Yang-Baxter map.

Now we introduce simple examples of the Yang-Baxter map ([3], [22, Remark 6.2 (2)]). Let G be a group, and let e_G denote its unit element. The maps σ_1 and σ_2 on $G \times G$ defined by

$$\sigma_1(a, b) = (e_G, ab) \text{ and } \sigma_2(a, b) = (ab, e_G) \quad (a, b \in G) \tag{1.1}$$

satisfy the QYBE

$$\sigma_i \times \mathrm{id}_G \circ \mathrm{id}_G \times \sigma_i \circ \sigma_i \times \mathrm{id}_G = \mathrm{id}_G \times \sigma_i \circ \sigma_i \times \mathrm{id}_G \circ \mathrm{id}_G \times \sigma_i \quad (i = 1, 2) \,.$$

The group G can produce Yang-Baxter maps.

These pairs (G, σ_i) $(i = 1, 2)$ are not always braided groups [24, 28], which are useful in constructing the Yang-Baxter map. However, they still satisfy several properties in the definition of the braided group.

The aim of this paper is to generalize the examples (1.1) from the viewpoint of category theory; braided semigroups play a role of the above pairs (G, σ_i) $(i = 1, 2)$ in this generalization. As an application, we construct dynamical Yang-Baxter maps by means of left quasigroups [18, 26].

The organization of this paper is as follows. Section 2 presents a brief summary of a tensor category $\mathbf{Set}_H$ before constructing dynamical Yang-Baxter maps, which generalize (1.1). In Section 3, our main results are stated and proved. Braided semigroups with suitable conditions give birth to morphisms satisfying the QYBE in every tensor category. In Section 4, we construct the braided semigroups in the previous section from semigroups with a left or right unit. The final section deals with a proof of Theorem 1 about the construction of dynamical Yang-Baxter maps, by making use of the results in Sections 3 and 4.

2. Tensor category $\mathbf{Set}_H$ and dynamical Yang-Baxter maps

In this section, we construct dynamical Yang-Baxter maps, which generalize (1.1), after a brief summary of a tensor category $\mathbf{Set}_H$ [23, 24]. For category theory, see [13, 15].

Let H be a non-empty set. We denote by $\mathbf{Set}_H$ the following category: its object is a pair $(X, \cdot_X)$ of a set X and a map $\cdot_X : H \times X \ni (\lambda, x) \mapsto \lambda \cdot_X x \in H$; its morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ is a map $f : H \to \mathrm{Map}(X, Y)$ satisfying that $\lambda \cdot_Y f(\lambda)(x) = \lambda \cdot_X x$ $(\forall \lambda \in H, \forall x \in X)$; the identity 1 and the composition $\circ$ are defined by

$$1_X(\lambda)(x) = x \quad (\lambda \in H, x \in X) \quad \text{and} \quad (g \circ f)(\lambda) = g(\lambda) \circ f(\lambda) \quad (\lambda \in H)$$

for objects X, Y, Z and morphisms $f : X \to Y$, $g : Y \to Z$. We will often write $\lambda \cdot_X x$ simply λx.

This $\mathbf{Set}_H$ is a tensor category. In fact, the tensor product $X \otimes Y$ of the objects $X = (X, \cdot_X)$ and $Y = (Y, \cdot_Y)$ is a pair $(X \times Y, \cdot)$ of the Cartesian product $X \times Y$ and the following map $\cdot : H \times (X \times Y) \to H$.

$$\lambda \cdot (x, y) = (\lambda \cdot_X x) \cdot_Y y \quad (\lambda \in H, (x, y) \in X \times Y).$$

The tensor product of the morphisms $f : X \to X'$ and $g : Y \to Y'$ is defined by $(f \otimes g)(\lambda)(x, y) = (f(\lambda)(x), g(\lambda x)(y))$ $(\lambda \in H, (x, y) \in X \times Y)$. The definitions of the associativity constraint a, the unit I, and the left and the right unit constraints l, r are as follows: $a_{XYZ}(\lambda)((x, y), z) = (x, (y, z))$; $I = (\{e\}, \cdot_I)$, a pair of the set $\{e\}$ of one element and the map $\cdot_I$ defined by $\lambda \cdot_I e = \lambda$; $l_X(\lambda)(e, x) = x = r_X(\lambda)(x, e)$.

DEFINITION 1.   A morphism $\sigma : X \otimes X \to X \otimes X$ of $\mathbf{Set}_H$ is a dynamical Yang-Baxter map [17, 21, 23, 24], iff σ satisfies the QYBE (2.1) in $\mathbf{Set}_H$.

$$a \circ \sigma \otimes 1_X \circ a^{-1} \circ 1_X \otimes \sigma \circ a \circ \sigma \otimes 1_X = 1_X \otimes \sigma \circ a \circ \sigma \otimes 1_X \circ a^{-1} \circ 1_X \otimes \sigma \circ a . \quad (2.1)$$

Here, $a = a_{X,X,X}$.

If this dynamical Yang-Baxter map is an automorphism in $\mathbf{Set}_H$, then it is exactly a Yang-Baxter operator [13, Definition XIII.3.1].

REMARK 1. (1) The dynamical Yang-Baxter map in the above definition is called a dynamical braiding map satisfying an invariance condition [22, Definition 2.8].
(2) If H is a set of one element, then the tensor category $\mathbf{Set}_H$ is the tensor category $\mathbf{Set}$ consisting of sets, and the dynamical Yang-Baxter map is exactly a Yang-Baxter map.

Every left quasigroup [18, 26] can produce dynamical Yang-Baxter maps.

DEFINITION 2. A left quasigroup Q is a non-empty set, together with a binary operation $(\cdot)$ on Q such that the left translation map $L(a) : Q \ni b \mapsto a \cdot b \in Q$ is bijective for all $a \in Q$.

For simplicity of notation, we write ab $(a, b \in Q)$ instead of $a \cdot b$, and denote $L(a)^{-1}(b)(\in Q)$ by $a \backslash b$. Here, $L(a)^{-1} : Q \to Q$ is the inverse of $L(a)$. We note that the binary operation of the left quasigroup is not always associative.

EXAMPLE 1. (1) The group G is a left quasigroup.
(2) The set $\mathbb{Z}$ of integers, together with a binary operation $a \cdot b := b - a$ $(a, b \in \mathbb{Z})$, is a left quasigroup. This binary operation is not associative.
(3) Let V denote the vector space $(\mathbb{Z}/3\mathbb{Z})^4$ over the finite field $\mathbb{Z}/3\mathbb{Z}$. We define the binary operation $(\cdot)$ on V by

$$a \cdot b = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + (a_3 - b_3)(a_1 b_2 - a_2 b_1))$$

for $a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4) \in V$. The pair $(V, \cdot)$ is a left quasigroup; moreover, this is a smallest commutative Moufang loop that is not a group [18, Example IV.5.1].

For a left quasigroup Q and $\lambda \in Q$, we define the binary operation $(\cdot_\lambda)$ on Q by

$$a \cdot_\lambda b = \lambda\backslash((\lambda a)b) \quad (a, b \in Q). \tag{2.2}$$

We denote by Q_λ the set Q with the above binary operation $(\cdot_\lambda)$ (2.2).

PROPOSITION 1. Q_λ is a left quasigroup for any $\lambda \in Q$.

REMARK 2. The binary operation $(\cdot_\lambda)$ is called a left derivative of $(\cdot)$ with respect to $\lambda \in Q$ [18, Section III.5].

A relation $\sim$ on Q defined by

$$\lambda \sim \mu \iff a \cdot_\lambda b = a \cdot_\mu b \quad (\forall a, b \in Q) \tag{2.3}$$

is an equivalence relation on Q. We write $H := Q/\sim$.

PROPOSITION 2. If $\lambda \sim \mu$, then $\lambda a \sim \mu a$ for any $a \in Q$.

PROOF.   By the definition (2.2), $a \cdot_\lambda (x \cdot_{\lambda a} y) = (a \cdot_\lambda x) \cdot_\lambda y$ for $x, y \in Q$, and consequently $a \cdot_\lambda (x \cdot_{\lambda a} y) = a \cdot_\mu (x \cdot_{\mu a} y) = a \cdot_\lambda (x \cdot_{\mu a} y)$, since $\lambda \sim \mu$. On account of Proposition 1, $x \cdot_{\lambda a} y = x \cdot_{\mu a} y$ for $x, y \in Q$, which is the desired result.   □

REMARK 3.   Each equivalence class $[\lambda] \in H$ containing $\lambda \in Q$ is the set $\lambda \cdot N_l(Q_\lambda)$. Here, $N_l(Q_\lambda)$ is a left nucleus [18, Definition I.3.2] of the left quasigroup Q_λ: $N_l(Q_\lambda) = \{a \in Q_\lambda \mid (a \cdot_\lambda x) \cdot_\lambda y = a \cdot_\lambda (x \cdot_\lambda y) \quad (\forall x, y \in Q_\lambda)\}$.

We define a map $(\cdot) : H \times Q \to H$ by $[\lambda] \cdot a := [\lambda a]$ $(\lambda, a \in Q)$. Because of Proposition 2, this definition makes sense.

PROPOSITION 3. $Q = (Q, \cdot)$ is an object of $\mathbf{Set}_H$.

Let $s : H \to Q$ be a right inverse of the projection $Q \ni \lambda \mapsto [\lambda] \in H$; that is, $s : H \to Q$ is a map satisfying $s([\lambda]) \sim \lambda$ for all $\lambda \in Q$. We denote by $\sigma_1([\lambda])$ and $\sigma_2([\lambda])$ $(\lambda \in Q)$ the maps on $Q \times Q$ defined by:

$$\sigma_1([\lambda])(a, b) = (s([\lambda]) \backslash s([\lambda]), \lambda \backslash ((\lambda a)b)) ;$$

$$\sigma_2([\lambda])(a, b) = (\lambda \backslash ((\lambda a)b), s([(\lambda a)b]) \backslash s([(\lambda a)b])) \quad (a, b \in Q) . \qquad (2.4)$$

THEOREM 1.   *Both σ_1 and σ_2 are dynamical Yang-Baxter maps.*

We will give a proof of this theorem in Section 5 after clarifying the structure of σ_i $(i = 1, 2)$ in Sections 3 and 4 from the viewpoint of category theory.

EXAMPLE 2.   (1) If the left quasigroup Q is a group G (See Example 1 (1)), then the set H has only one element, and the maps $\sigma_i := \sigma_i([\lambda])$ $(i = 1, 2, \lambda \in G)$ are the same as those in (1.1) for any right inverse s (See also Remark 1 (2)).

(2) If the left quasigroup Q is $\mathbb{Z}$ in Example 1 (2), then $\lambda \sim \mu \Leftrightarrow \lambda = \mu$ $(\lambda, \mu \in \mathbb{Z})$. As a result, the set H is isomorphic to $\mathbb{Z}$ as sets, and every right inverse s satisfies $s([\lambda]) = \lambda$ $(\lambda \in \mathbb{Z})$. The maps $\sigma_i([\lambda])$ $(i = 1, 2, \lambda \in \mathbb{Z})$ are as follows:

$$\sigma_1([\lambda])(a, b) = (2\lambda, b - a + 2\lambda) ;$$

$$\sigma_2([\lambda])(a, b) = (b - a + 2\lambda, 2b - 2a + 2\lambda) \quad (\lambda, a, b \in \mathbb{Z}) .$$

(3) If the left quasigroup Q is V in Example 1 (3), then $1 < \#(H) < \#(V)(= 81)$. The element $1_V := (0, 0, 0, 0)$ is the unit element of $(V, \cdot)$, and the inverse a^{-1} of $a(\in V)$ is $-a$. Because $(V, \cdot)$ is a Moufang loop, $(ba^{-1})a = b$ for any $a, b \in V$ [18, Section IV.1]. By virtue of Proposition 2,

$$a \sim b \Rightarrow aa^{-1} \sim ba^{-1} \Leftrightarrow 1_V \sim ba^{-1} \Rightarrow 1_V a \sim (ba^{-1})a \Leftrightarrow a \sim b ;$$

that is, $a \sim b \Leftrightarrow ba^{-1} \sim 1_V$. A straightforward computation shows that $a \sim 1_V \Leftrightarrow a = (0, 0, 0, a_4)$ $(a_4 \in \mathbb{Z}/3\mathbb{Z})$, and the relation $a \sim b$ is consequently equivalent to that $b = a + (0, 0, 0, r)$ $(\exists r \in \mathbb{Z}/3\mathbb{Z})$. $H(= V/\sim)$ is thus a set of order 27. Since the orders of the sets H and V are different, the method in [22] does not produce

this example directly. Finally, the maps $\sigma_i([\lambda])$ $(i = 1, 2, \lambda \in V)$ for any right inverse s are as follows:

$$\sigma_1([\lambda])(a, b) = (1_V, x); \sigma_2([\lambda])(a, b) = (x, 1_V) \quad (\lambda, a, b \in V).$$

Here, the element $x \in V$ is defined by

$$x = (a_1 + b_1, a_2 + b_2, a_3 + b_3,$$
$$a_4 + b_4 + \lambda_1(a_2 b_3 - a_3 b_2) + \lambda_2(a_3 b_1 - a_1 b_3) + (\lambda_3 + a_3 - b_3)(a_1 b_2 - a_2 b_1))$$

for $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4), a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4) \in V$.

3. QYBE and braided semigroups

This section establishes a relation between the QYBE and braided semigroups in tensor categories, which play an essential role in the proof of Theorem 1.

Let $C = (C, \otimes, a, I, l, r)$ be a tensor category. That is to say, C is a category with a tensor product $\otimes : C \times C \to C$, an associativity constraint $a : \otimes \circ (\otimes \times \text{id}) \to \otimes \circ (\text{id} \times \otimes)$, a unit object I, and left and right unit constraints $l : \otimes(I \times \text{id}) \to \text{id}, r : \otimes(\text{id} \times I) \to \text{id}$ with respect to I, satisfying the pentagon axiom and the triangle axiom. We denote by $1_X : X \to X$ the identity morphism of an object X.

A pair (X, m_X) of an object X and a morphism $m_X : X \otimes X \to X$ is a semigroup, iff m_X satisfies

$$m_X \circ (m_X \otimes 1_X) = m_X \circ (1_X \otimes m_X) \circ a_{X,X,X}. \tag{3.1}$$

This morphism m_X is called a multiplication. A pair (X, Δ_X) of an object X and a morphism $\Delta_X : X \to X \otimes X$ is a co-semigroup, the dual concept of the semigroup, iff Δ_X satisfies

$$a_{X,X,X} \circ (\Delta_X \otimes 1_X) \circ \Delta_X = (1_X \otimes \Delta_X) \circ \Delta_X. \tag{3.2}$$

The morphism Δ_X is said to be a comultiplication.

Let $\sigma_{XY} : X \otimes Y \to Y \otimes X$ be a morphism of the tensor category C.

DEFINITION 3. A matched pair of semigroups $X = (X, m_X)$ and $Y = (Y, m_Y)$ (Cf. [16, 24, 27, 28]) is a triple (X, Y, σ_{XY}) satisfying:

$$(1_Y \otimes m_X) \circ a_{Y,X,X} \circ (\sigma_{XY} \otimes 1_X) \circ a_{X,Y,X}^{-1} \circ (1_X \otimes \sigma_{XY})$$
$$= \sigma_{XY} \circ (m_X \otimes 1_Y) \circ a_{X,X,Y}^{-1}; \tag{3.3}$$
$$(m_Y \otimes 1_X) \circ a_{Y,Y,X}^{-1} \circ (1_Y \otimes \sigma_{XY}) \circ a_{Y,X,Y} \circ (\sigma_{XY} \otimes 1_Y)$$
$$= \sigma_{XY} \circ (1_X \otimes m_Y) \circ a_{X,Y,Y}. \tag{3.4}$$

A pair (X, σ_X) of a semigroup X and a morphism $\sigma_X : X \otimes X \to X \otimes X$ is called a braided semigroup, iff the triple (X, X, σ_X) is a matched pair of semigroups.

REMARK 4.   The matched pair (X, Y, σ_{XY}) of semigroups gives birth to a semigroup. In fact, $(Y \otimes X, m_{Y \otimes X})$ is a semigroup with the morphism $m_{Y \otimes X} : (Y \otimes X) \otimes (Y \otimes X) \to Y \otimes X$ defined by

$$m_{Y \otimes X} = m_Y \otimes m_X \circ a_{Y \otimes Y, X, X} \circ (a_{Y,Y,X}^{-1} \circ 1_Y \otimes \sigma_{XY} \circ a_{Y,X,Y}) \otimes 1_X \circ a_{Y \otimes X, Y, X}^{-1}. \quad (3.5)$$

$(X \otimes X, m_{X \otimes X})$ is hence a semigroup, if (X, σ_X) is a braided semigroup.

Let $X = (X, m_X)$ be a semigroup with a comultiplication $\Delta_X : X \to X \otimes X$. We write $\sigma := \Delta_X \circ m_X : X \otimes X \to X \otimes X$.

THEOREM 2.   *If the pair (X, σ) is a braided semigroup, then σ satisfies the QYBE* (2.1) *in the tensor category C.*

PROOF.   Because $\sigma = \Delta_X \circ m_X$,

(Right-hand-side of (2.1)) $= (1_X \otimes \Delta_X) \circ (1_X \otimes m_X) \circ a \circ (\sigma \otimes 1_X) \circ a^{-1} \circ (1_X \otimes \sigma) \circ a$.

Here, $a = a_{X,X,X}$. On account of (3.3), the right-hand-side of the above equation is $(1_X \otimes \Delta_X) \circ \sigma \circ (m_X \otimes 1_X)$. By making use of $\sigma = \Delta_X \circ m_X$ again, $(1_X \otimes \Delta_X) \circ \sigma \circ (m_X \otimes 1_X) = (1_X \otimes \Delta_X) \circ \Delta_X \circ m_X \circ (m_X \otimes 1_X)$. From (3.4), a similar argument induces that

$$\text{(Left-hand-side of (2.1))} = a \circ (\Delta_X \otimes 1_X) \circ \Delta_X \circ m_X \circ (1_X \otimes m_X) \circ a.$$

This completes the proof in view of (3.1) and (3.2).   □

REMARK 5.   (1) The dual concept of the braided semigroup is a braided co-semigroup. Let $\sigma_{XY} : X \otimes Y \to Y \otimes X$ be a morphism of a tensor category C. A matched pair of co-semigroups (X, Δ_X) and (Y, Δ_Y) is a triple (X, Y, σ_{XY}) satisfying:

$$\sigma_{XY} \otimes 1_X \circ a_{X,Y,X}^{-1} \circ 1_X \otimes \sigma_{XY} \circ a_{X,X,Y} \circ \Delta_X \otimes 1_Y = a_{Y,X,X}^{-1} \circ 1_Y \otimes \Delta_X \circ \sigma_{XY};$$

$$1_Y \otimes \sigma_{XY} \circ a_{Y,X,Y} \circ \sigma_{XY} \otimes 1_Y \circ a_{X,Y,Y}^{-1} \circ 1_X \otimes \Delta_Y = a_{Y,Y,X} \circ \Delta_Y \otimes 1_X \circ \sigma_{XY}.$$

A pair (X, σ_X) of a co-semigroup X and a morphism $\sigma_X : X \otimes X \to X \otimes X$ is a braided co-semigroup, iff the triple (X, X, σ_X) is a matched pair of co-semigroups.
   (2) The matched pair (X, Y, σ_{XY}) of co-semigroups defines a co-semigroup $(X \otimes Y, \Delta_{X \otimes Y})$. Here,

$$\Delta_{X \otimes Y} := a_{X \otimes Y, X, Y} \circ (a_{X,Y,X}^{-1} \circ 1_X \otimes \sigma_{XY} \circ a_{X,X,Y}) \otimes 1_Y \circ a_{X \otimes X, Y, Y}^{-1} \circ \Delta_X \otimes \Delta_Y. \quad (3.6)$$

From this fact, $(X \otimes X, \Delta_{X \otimes X})$ is a co-semigroup, if (X, σ_X) is a braided co-semigroup.
   (3) A dual of Theorem 2 is also true. Let (X, Δ_X) be a co-semigroup with a multiplication m_X. We set $\sigma := \Delta_X \circ m_X$. If (X, σ) is a braided co-semigroup, then σ satisfies the QYBE. The proof is similar to that of Theorem 2.

4. Semigroups with left or right unit

In this section, we construct the braided semigroups in Theorem 2 by means of semigroups with a left or right unit.

Let $C = (C, \otimes, a, I, l, r)$ be a tensor category, S an object of C, and $\eta : I \to S$ a morphism of C. We define the morphisms $\Delta_i : S \to S \otimes S$ $(i = 1, 2)$ by

$$\Delta_1 = (\eta \otimes 1_S) \circ l_S^{-1} \quad \text{and} \quad \Delta_2 = (1_S \otimes \eta) \circ r_S^{-1} . \tag{4.1}$$

PROPOSITION 4. *Both (S, Δ_1) and (S, Δ_2) are co-semigroups.*

PROOF.   From (4.1), $\Delta_1 \circ \eta = \eta \otimes \eta \circ l_I^{-1} = \eta \otimes \eta \circ r_I^{-1} = \Delta_2 \circ \eta$, which induces

$$\Delta_1 \otimes 1_S \circ \Delta_1 = \Delta_2 \otimes 1_S \circ \Delta_1 \quad \text{and} \quad 1_S \otimes \Delta_2 \circ \Delta_2 = 1_S \otimes \Delta_1 \circ \Delta_2 . \tag{4.2}$$

By virtue of the triangle axiom,

$$a_{S,S,S} \circ \Delta_2 \otimes 1_S = 1_S \otimes \Delta_1 . \tag{4.3}$$

It follows immediately from (4.2) and (4.3) that Δ_1 and Δ_2 satisfy (3.2). This proves the proposition.   □

A morphism $\eta : I \to S$ is called a left unit (resp. a right unit) of a semigroup (S, m_S), iff η satisfies $m_S \circ \eta \otimes 1_S = l_S$ (resp. $m_S \circ 1_S \otimes \eta = r_S$).

Let (S, m_S) be a semigroup with a left or right unit η. With the aid of the above proposition, the morphisms Δ_1 and Δ_2 are comultiplications of S. We define the morphisms σ_i $(i = 1, 2)$ by $\sigma_i := \Delta_i \circ m_S$.

PROPOSITION 5. *For $i = 1, 2$, (S, σ_i) is a braided semigroup.*

PROOF.   The following lemma and (3.1) immediately establish (3.3) and (3.4) for the case $i = 1$.

LEMMA 1. Δ_1 *satisfies*:

$$1_S \otimes m_S \circ a_{S,S,S} \circ \Delta_1 \otimes 1_S = \Delta_1 \circ m_S ; \tag{4.4}$$

$$m_S \circ m_S \otimes 1_S \circ a_{S,S,S}^{-1} \circ 1_S \otimes \Delta_1 = m_S ; \tag{4.5}$$

$$m_S \otimes 1_S \circ a_{S,S,S}^{-1} \circ 1_S \otimes \Delta_1 \circ \Delta_1 = \Delta_1 . \tag{4.6}$$

For the proof of the case $i = 2$, we use:

$$m_S \otimes 1_S \circ a_{S,S,S}^{-1} \circ 1_S \otimes \Delta_2 = \Delta_2 \circ m_S ; \tag{4.7}$$

$$m_S \circ 1_S \otimes m_S \circ a_{S,S,S} \circ \Delta_2 \otimes 1_S = m_S ; \tag{4.8}$$

$$1_S \otimes m_S \circ a_{S,S,S} \circ \Delta_2 \otimes 1_S \circ \Delta_2 = \Delta_2 . \tag{4.9}$$

This completes the proof.   □

PROOF OF LEMMA 1.   The naturality of the left unit constraint l, together with the fact that $l_{S\otimes S} \circ a_{I,S,S} = l_S \otimes 1_S$, implies (4.4).

If η is a left unit of the semigroup (S, m_S), then

$$m_S \circ \Delta_1 = 1_S, \tag{4.10}$$

which induces (4.5) by virtue of the associativity (3.1) of m_S.

If η is a right unit of (S, m_S), then

$$m_S \otimes 1_S \circ a_{S,S,S}^{-1} \circ 1_S \otimes \Delta_1 = 1_{S\otimes S}. \tag{4.11}$$

In fact, the left-hand-side of (4.11) is $(m_S \circ 1_S \otimes \eta) \otimes 1_S \circ a_{S,I,S}^{-1} \circ 1_S \otimes l_S^{-1}$. The triangle axiom, together with the fact that η is a right unit, gives (4.11), which consequently yields (4.5).

The proof of (4.6) is immediate by making (3.2), (4.10), and (4.11) into account.   □

Some σ_i are idempotent.

PROPOSITION 6.   *If η is a left unit, then $\sigma_1^2 = \sigma_1$; and, if η is a right unit, then $\sigma_2^2 = \sigma_2$.*

PROOF.   From (4.10) and the fact that $\sigma_1 = \Delta_1 \circ m_S$, $\sigma_1^2 = \sigma_1$, if η is a left unit. A similar argument induces that $\sigma_2^2 = \sigma_2$, if η is a right unit.   □

REMARK 6.   (1) (S, σ_i) $(i = 1, 2)$ in Proposition 5 are also braided co-semigroups because of (3.2) and (4.4)–(4.9). As a result, $S \otimes S$ is a semigroup with respect to $m_{S\otimes S}$, but also a co-semigroup with respect to $(\Delta_i)_{S\otimes S}$. Here, $m_{S\otimes S}$ is the morphism (3.5) for $X = Y = S$, and $(\Delta_i)_{S\otimes S}$ is the morphism (3.6) for $X = Y = S$ and $\Delta_X = \Delta_Y = \Delta_i$.

(2) The quartet $(S, m_S, \Delta_i, \sigma_i)$ $(i = 1, 2)$ is a "bi-semigroup." In fact, it follows from (4.4)–(4.9) that

$$m_{S\otimes S} \circ \Delta_i \otimes \Delta_i = \Delta_i \circ m_S. \tag{4.12}$$

The morphism $\Delta_i : S \to S \otimes S$ hence respects the semigroup structures (See (1)). On the other hand, from (3.5) and (3.6), (4.12) is exactly the same as

$$m_S \otimes m_S \circ (\Delta_i)_{S\otimes S} = \Delta_i \circ m_S,$$

which means that $m_S : S \otimes S \to S$ respects the co-semigroup structures. Therefore, we can regard S as a bi-semigroup.

5.   Proof of Theorem 1

This section is devoted to a proof of Theorem 1. We follow the notation used in Section 2.

Let $Q = (Q, \cdot)$ be a left quasigroup (See Definition 2). We denote by H the set of all equivalence classes of the relation (2.3) on Q.

For any $[\lambda] \in H$ $(\lambda \in Q)$, $m_Q([\lambda])$ is the map from $Q \times Q$ to Q defined by $m_Q([\lambda])(a, b) = a \cdot_\lambda b$ for $a, b \in Q$ (Cf. [23, (3.3)]). For $(\cdot_\lambda)$, see (2.2). On account of (2.3), this definition is unambiguous.

PROPOSITION 7. $m_Q : Q \otimes Q \to Q$ *is a morphism of* $\mathbf{Set}_H$. *Moreover,* $Q = (Q, m_Q)$ *is a semigroup in* $\mathbf{Set}_H$.

Let $s : H \to Q$ be a right inverse of the projection $Q \ni \lambda \mapsto [\lambda] \in H$. For any $[\lambda] \in H$ $(\lambda \in Q)$, $\eta_Q^{(s)}([\lambda])$ is the map from $I = \{e\}$ to Q defined by $\eta_Q^{(s)}([\lambda])(e) = s([\lambda]) \backslash s([\lambda])$.

PROPOSITION 8. (1) $\eta_Q^{(s)} : I \to Q$ *is a morphism of* $\mathbf{Set}_H$.
(2) $\eta_Q^{(s)}$ *is a left unit of the semigroup* (Q, m_Q) *(Cf.* [23, (3.5)]).

PROOF. We give only the proof of (1). For the proof, it is sufficient to show that $[\lambda]\eta_Q^{(s)}([\lambda])(e) = [\lambda]e$ $(\lambda \in Q, I = \{e\})$. Because $s([\lambda]) \sim \lambda$, $[s([\lambda])] = [\lambda]$. Hence,

$$[\lambda]\eta_Q^{(s)}([\lambda])(e) = [s([\lambda])](s([\lambda])\backslash s([\lambda])) = [s([\lambda])] = [\lambda] = [\lambda]e \,.$$

This completes the proof. $\square$

We set $\Delta_1 = (\eta_Q^{(s)} \otimes 1_Q) \circ l_Q^{-1}$, $\Delta_2 = (1_Q \otimes \eta_Q^{(s)}) \circ r_Q^{-1}$, and $\sigma_i = \Delta_i \circ m_Q$ $(i = 1, 2)$. It follows from Propositions 4, 5, 7, and 8 that each (Q, σ_i) $(i = 1, 2)$ is a braided semigroup in $\mathbf{Set}_H$ with the comultiplication Δ_i, and a straightforward calculation shows that the morphisms σ_1 and σ_2 are exactly the same as those in (2.4). Theorem 2 thus proves Theorem 1.

REMARK 7. (1) Remarks 5 (3) and 6 (1) with Propositions 7 and 8 also induce Theorem 1.

(2) The construction [17] of dynamical Yang-Baxter maps using dynamical braces cannot produce the above dynamical Yang-Baxter map σ_1, if $\#(Q) > 1$. Suppose, contrary to our claim, that there exists a dynamical brace that gives birth to σ_1 by the method in [17, Corollary 3.2.1]. Then this dynamical Yang-Baxter map satisfies the unitary condition $\sigma_1^2 = 1_{Q \otimes Q}$. As a result, $\sigma_1 = \sigma_1^2 = 1_{Q \otimes Q}$ from Proposition 6. This contradicts the condition that $\#(Q) > 1$.

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