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Maximal Diameter Sphere Theorem for Manifolds with Nonconstant Radial Curvature

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Abstract. We generalize Toponogov's maximal diameter sphere theorem from the radial curvature geometry's standpoint. As a corollary to our main theorem, we prove that for a complete connected Riemannian n-manifold M having radial sectional curvature at a point bounded from below by the radial curvature function of an ellipsoid of prolate type, the diameter of M does not exceed the diameter of the ellipsoid. Furthermore if the diameter of such an M equals that of the ellipsoid, then M is isometric to the n-dimensional ellipsoid of revolution.

1. Introduction

The maximal diameter sphere theorem proved by Toponogov says as follows:

THEOREM 1.1 ([T]). *Let M be a complete connected Riemannian manifold whose sectional curvature is bounded from below by a positive constant H. Then the diameter of M does not exceed $\pi/\sqrt{H}$. Furthermore if the diameter of M equals $\pi/\sqrt{H}$, then M is isometric to the sphere with radius $\sqrt{H}$.*

This theorem was generalized by Cheng [Ch] for a complete connected Riemannian manifold whose Ricci curvature is bounded from below by a positive constant H.

A natural extension of the maximal diameter sphere theorem by the radial curvature would be that for a complete connected Riemannian manifold M whose radial sectional curvature at a point $p \in M$ is not less than a positive constant H,

(A) is the diameter of M at most $\pi/\sqrt{H}$?
(B) Furthermore, if the diameter of M equals $\pi/\sqrt{H}$, is M isometric to the sphere with the radius $\sqrt{H}$?

Notice that the problem (A) can be affirmatively solved. It is an easy consequence from Theorem **??** (or the Main theorem in [SST]). Here, we define the radial plane and radial curvature from a point p of a complete connected Riemannian manifold M. For each point

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$q \in M$ distinct from the point p, a 2-dimensional linear subspace σ of T_qM is called a *radial plane* at q if there exists a unit speed minimal geodesic segment $\gamma : [0, d(p,q)] \to M$ satisfying $\gamma'(d(p,q)) \in \sigma$. The sectional curvature $K(\sigma)$ of a radial plane $\sigma \subset T_qM$ at q is called a *radial curvature* at p.

The problem (B) is still open, but one can generalize the maximal diameter sphere theorem for a manifold which *has radial curvature at a point bounded from below by the radial curvature function of a* 2*-sphere of revolution,* which will be defined later, if the 2-sphere of revolution belongs to a certain class.

For introducing this class of a 2-sphere of revolution, we start to define a 2-sphere of revolution. Let $\widetilde{M}$ denote a complete Riemannian manifold homeomorphic to a 2-sphere. $\widetilde{M}$ is called a 2*-sphere of revolution* if $\widetilde{M}$ admits a point $\tilde{p}$ such that for any two points $\tilde{q}_1, \tilde{q}_2$ on $\widetilde{M}$ with $d(\tilde{p}, \tilde{q}_1) = d(\tilde{p}, \tilde{q}_2)$, where $d(\ ,\ )$ denotes the Riemannian distance function, there exists an isometry f on $\widetilde{M}$ satisfying $f(\tilde{q}_1) = \tilde{q}_2$ and $f(\tilde{p}) = \tilde{p}$. The point $\tilde{p}$ is called a *pole* of $\widetilde{M}$. It is proved in [ST] that $\widetilde{M}$ has another pole $\tilde{q}$ and the Riemannian metric g of $\widetilde{M}$ is expressed as $g = dr^2 + m(r)^2 d\theta^2$ on $\widetilde{M} \setminus \{\tilde{p}, \tilde{q}\}$, where (r, θ) denote geodesic polar coordinates around $\tilde{p}$ and

$$m(r(x)) := \sqrt{g\left(\left(\frac{\partial}{\partial\theta}\right)_x, \left(\frac{\partial}{\partial\theta}\right)_x\right)}.$$

Hence $\widetilde{M}$ has a pair of poles $\tilde{p}$ and $\tilde{q}$. In what follows, $\tilde{p}$ denotes a pole of $\widetilde{M}$ and we fix it. Each unit speed geodesic emanating from $\tilde{p}$ is called a *meridian*. It is observed in [ST] that each meridian $\mu : [0, 4a] \to \widetilde{M}$, where $a := \frac{1}{2} d(\tilde{p}, \tilde{q})$, passes through $\tilde{q}$ and is periodic, hence, $\mu(0) = \mu(4a) = \tilde{p}$, $\mu'(0) = \mu'(4a)$. The function $G \circ \mu : [0, 2a] \to R$ is called the *radial curvature function* of $\widetilde{M}$, where G denotes the Gaussian curvature of $\widetilde{M}$.

A 2-sphere of revolution $\widetilde{M}$ with a pair of poles $\tilde{p}$ and $\tilde{q}$ is called a *model surface* if $\widetilde{M}$ satisfies the following two properties:

(1.1) $\widetilde{M}$ has a reflective symmetry with respect to the *equator*, $r = a = \frac{1}{2} d(\tilde{p}, \tilde{q})$.

(1.2) The Gaussian curvature G of $\widetilde{M}$ is strictly decreasing along a meridian from the point $\tilde{p}$ to the point on the equator.

A typical example of a model surface is an ellipsoid of prolate type, i.e., the surface defined by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad b > a > 0\,.$$

The points $(0, 0, \pm b)$ are a pair of poles and $z = 0$ is the equator.

The fact that the Gaussian curvature of a model surface is not always positive everywhere is the worthy of note. In [ST], an interesting model surface was introduced. The surface generated by the (x, z)-plane curve $(m(t), 0, z(t))$ is a model surface, where

$$m(t) := \frac{\sqrt{3}}{10}\left(9 \sin \frac{\sqrt{3}}{9} t + 7 \sin \frac{\sqrt{3}}{3} t\right), \quad z(t) := \int_0^t \sqrt{1 - m'(t)^2} dt \,.$$

It is easy to see that the Gaussian curvature of the equator $r = 3\sqrt{3}\pi/2$ is -1.

Let M be a complete connected n-dimensional Riemannian manifold with a base point p. M is said to have *radial sectional curvature at p bounded from below by that of a model surface* $\widetilde{M}$ if for any point $q (\neq p)$ and any radial plane $\sigma \subset T_q M$ at q, the sectional curvature $K(\sigma)$ of M satisfies $K(\sigma) \geq G \circ \mu(d(p, q))$.

For each 2-dimensional model $\widetilde{M}$ with a Riemannian metric $dr^2 + m(r)^2 d\theta^2$, we define an n-dimensional model $\widetilde{M}^n$ homeomorphic to an n-sphere S^n with a Riemannian metric

$$g^* = dr^2 + m(r)^2 d\Theta^2,$$

where $d\Theta^2$ denotes the Riemannian metric of the $(n-1)$-dimensional unit sphere $S^{n-1}(1)$. For example, the n-dimensional model of the ellipsoid above is the n-dimensional ellipsoid defined by

$$\sum_{i=1}^{n} \frac{x_i^2}{a^2} + \frac{x_{n+1}^2}{b^2} = 1 \,.$$

In this paper, we generalize the maximal diameter sphere theorem as follows:

MAIN THEOREM. *Let M be a complete connected n-dimensional Riemannian manifold with a base point $p \in M$ whose radial sectional curvature at p bounded from below by that of a model surface $\widetilde{M}$. Then, the diameter of M does not exceed the diameter of $\widetilde{M}$. Furthermore if the diameter of M equals that of $\widetilde{M}$, then M is isometric to the n-dimensional model $\widetilde{M}^n$.*

As a corollary, we get an interesting result:

COROLLARY TO MAIN THEOREM. *For any complete connected n-dimensional Riemannian manifold M having radial sectional curvature at a point p bounded from below by that of the ellipsoid $\widetilde{M}$ defined by*

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad b > a > 0,$$

the diameter of M does not exceed the diameter of $\widetilde{M}$. Furthermore if the diameter of such an M equals that of $\widetilde{M}$, then M is isometric to the n-dimensional ellipsoid $\sum_{i=1}^{n} \frac{x_i^2}{a^2} + \frac{x_{n+1}^2}{b^2} = 1$.

We refer to [CE] for basic tools in Riemannian Geometry, and [SST] for some properties of geodesics on a surface of revolution.

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2. Preliminaries

Here, we review the notion of a cut point and a cut locus. Let M be a complete Riemannian manifold with a base point p. Let $\gamma : [0, a] \to M$ denote a unit speed minimal geodesic segment emanating from $p = \gamma(0)$ on M. If any extended geodesic segment $\bar{\gamma} : [0, b] \to M$ of γ, where $b > a$, is not minimizing arc joining p to $\bar{\gamma}(b)$ anymore, then the endpoint $\gamma(a)$ of the geodesic segment is called a *cut point* of p along γ. For each point p on M, the *cut locus C_p* is defined by the set of all cut points along the minimal geodesic segments emanating from p.

REMARK 2.1. It is known (for example see [SST]) that the cut locus has a local tree structure for 2-dimensional Riemannian manifolds.

We need the following two theorems, which was proved by Sinclair and Tanaka [ST].

THEOREM 2.2 ([ST]). *Let M be a 2-sphere of revolution with a pair of poles p, q satisfying the following two properties,*

(i) *M is symmetric with respect to the reflection fixing $r = a$, where $2a$ denotes the distance between p and q.*
(ii) *The Gaussian curvature G of M is monotone along a meridian from the point p to the point on $r = a$.*

Then the cut locus of a point $x \in M \setminus \{p, q\}$ with $\theta(x) = 0$ is a single point or a subarc of the opposite half meridian $\theta = \pi$ (resp. the parallel $r = 2a - r(x)$) when G is decreasing (resp. increasing) along a meridian from p to the point on $r = a$. Furthermore, if the cut locus of a point $x \in M \setminus \{p, q\}$ is a single point, then the Gaussian curvature is constant.

THEOREM 2.3 ([ST]). *Let M be a complete connected n-dimensional Riemannian manifold with a base point p such that M has radial sectional curvature at p bounded from below by the radial curvature function of a 2-sphere of revolution $\widetilde{M}$ with a pair of poles $\tilde{p}, \tilde{q}$. Suppose that the cut locus of any point on $\widetilde{M}$ distinct from its two poles is a subset of the half meridian opposite to the point. Then for each geodesic triangle $\triangle(pxy)$ in M, there exists a geodesic triangle $\widetilde{\triangle}(pxy) := \triangle(\tilde{p}\tilde{x}\tilde{y})$ in $\widetilde{M}$ such that*

$$d(p, x) = d(\tilde{p}, \tilde{x}), \quad d(p, y) = d(\tilde{p}, \tilde{y}), \quad d(x, y) = d(\tilde{x}, \tilde{y}), \tag{2.1}$$

and such that

$$\angle(pxy) \geqslant \angle(\tilde{p}\tilde{x}\tilde{y}), \quad \angle(pyx) \geqslant \angle(\tilde{p}\tilde{y}\tilde{x}), \quad \angle(xpy) \geqslant \angle(\tilde{x}\tilde{p}\tilde{y}) . \tag{2.2}$$

Here, $\angle(pxy)$ denotes the angle at the vertex x of the geodesic triangle $\triangle(pxy)$.

3. Proof of Main Theorem

Let M be a complete connected n-dimensional Riemannian manifold with a base point p and $\widetilde{M}$ a 2-sphere of revolution with a pair of poles $\tilde{p}, \tilde{q}$ satisfying (1.1) and (1.2) in the introduction, i.e., a model surface.

From now on, we assume that M has radial sectional curvature at p bounded from below by that of $\widetilde{M}$. By scaling the Riemannian metrics of M and $\widetilde{M}$, we may assume that $2a = \pi$.

LEMMA 3.1. *The perimeter of any geodesic triangle $\widetilde{\triangle}(pxy)$ of $\widetilde{M}$ does not exceed 2π, i.e.,*

$$d(\tilde{p}, \tilde{x}) + d(\tilde{p}, \tilde{y}) + d(\tilde{x}, \tilde{y}) \leqslant 2\pi \,. \tag{3.1}$$

PROOF. Since $d(\tilde{p}, \tilde{q}) = 2a = \pi$, it follows from the triangle inequality that

$$\begin{aligned} d(\tilde{x}, \tilde{y}) &\leqslant d(\tilde{q}, \tilde{x}) + d(\tilde{q}, \tilde{y}) \\ &= (\pi - d(\tilde{p}, \tilde{x})) + (\pi - d(\tilde{p}, \tilde{y})) \\ &= 2\pi - d(\tilde{p}, \tilde{x}) - d(\tilde{p}, \tilde{y}) \,. \end{aligned}$$

Therefore, the inequality (3.1) holds. □

LEMMA 3.2. *The perimeter of a geodesic triangle $\triangle(pxy)$ of M does not exceed 2π.*

PROOF. Let $\triangle(pxy)$ be any geodesic triangle of M. From Theorem **??**, we get a geodesic triangle $\widetilde{\triangle}(pxy)$ of $\widetilde{M}$ satisfying (2.1). Hence, by Lemma 3.1, the perimeter of $\triangle(pxy)$ does not exceed 2π. □

LEMMA 3.3. *The diameter of $\widetilde{M}$ equals π, where the diameter* diam $\widetilde{M}$ *of $\widetilde{M}$ is defined by*

$$\operatorname{diam} \widetilde{M} := \max\{d(\tilde{x}, \tilde{y}) | \tilde{x}, \tilde{y} \in \widetilde{M}\} \,.$$

PROOF. Choose any points $\tilde{x}, \tilde{y}$ on $\widetilde{M}$. By the triangle inequality,

$$d(\tilde{x}, \tilde{y}) \leqslant d(\tilde{p}, \tilde{x}) + d(\tilde{p}, \tilde{y}) \,. \tag{3.2}$$

Thus, by combining (3.1) and (3.2), we obtain

$$d(\tilde{x}, \tilde{y}) \leqslant \pi = d(\tilde{p}, \tilde{q})$$

for any $\tilde{x}, \tilde{y}$ on $\widetilde{M}$. □

LEMMA 3.4. *The diameter* diam M *of M does not exceed the diameter of $\widetilde{M}$.*

PROOF. Choose a pair of points $x, y \in M$ satisfying $d(x, y) = \operatorname{diam} M$. We first consider the case where $x = p$ or $y = p$. By the Rauch comparison theorem, there does not exist a minimal geodesic segment emanating from p whose length exceeds π, since the manifold M has radial curvature at p bounded from below by the radial curvature function of the model surface $\widetilde{M}$. Thus, $\operatorname{diam} M = d(x, y) \leqslant \pi$. Hence we assume $x \neq p$ and $y \neq p$. Then, for the geodesic triangle $\triangle(pxy)$ in M, there exists a geodesic triangle $\widetilde{\triangle}(pxy)$ in $\widetilde{M}$ satisfying (2.1). Therefore, we obtain $\operatorname{diam} M = d(\tilde{x}, \tilde{y}) \leqslant \operatorname{diam} \widetilde{M}$. □

LEMMA 3.5. *If* $\operatorname{diam} M = \operatorname{diam} \widetilde{M}$, *then there exists a point* $q \in M$ *with* $d(p, q) = \operatorname{diam} \widetilde{M}$.

PROOF. Let $x, y \in M$ be points satisfying $\pi = \operatorname{diam} M = d(x, y)$. Supposing that $x \neq p$ and $y \neq p$, we will get a contradiction. Then, there exists a geodesic triangle $\triangle(pxy)$ with $d(x, y) = \pi$. It follows from Theorem **??** that there exists a geodesic triangle $\widetilde{\triangle}(pxy)$ corresponding to $\triangle(pxy)$ satisfying $d(\tilde{x}, \tilde{y}) = d(x, y) = \pi$. By the triangle inequality, $d(\tilde{p}, \tilde{x}) + d(\tilde{p}, \tilde{y}) \geqslant d(\tilde{x}, \tilde{y}) = \pi$, and Lemma 3.1, we get

$$d(\tilde{p}, \tilde{x}) + d(\tilde{p}, \tilde{y}) = \pi = d(\tilde{x}, \tilde{y}) .$$

This means that $\angle(\tilde{x}\tilde{p}\tilde{y}) = \pi$ so that the subarc α (passing through $\tilde{p}$) of the meridian joining $\tilde{x}$ to $\tilde{y}$ is minimal. Hence the complementary subarc of α in the meridian is also a minimal geodesic segment joining $\tilde{x}$ to $\tilde{y}$, since the length of each meridian is 2π. Therefore, by Theorem **??**, $\tilde{y}$ is a unique cut point of $\tilde{x}$ and hence, the Gaussian curvature G of $\widetilde{M}$ is constant. We get a contradiction since G is strictly decreasing along a meridian from p to the point on the equator. This implies the existence of the point q. □

LEMMA 3.6. *If there exists a point* $q \in M$ *with* $d(p, q) = \operatorname{diam} M$, *then* q *is a unique cut point of* p, *and*

$$K(\sigma) = G \circ \mu(d(p, x))$$

holds for any point $x \in M \setminus \{p\}$ *and any radial plane* σ *at* x.

PROOF. It follows from Lemma 3.4 that the point q is the farthest point from p. Hence $q \in C_p$. Choose any point $x \in M \setminus \{p, q\}$. By the triangle inequality,

$$d(p, x) + d(x, q) \geqslant d(p, q) = \pi$$

and by Lemma 3.2,

$$d(p, x) + d(x, q) + d(p, q) \leqslant 2\pi .$$

Hence, we get

$$d(p, x) + d(x, q) = d(p, q) = \pi$$

and it is easy to see that q is a unique cut point of p because $\angle(pxq) = \pi$.

Next, we will prove that $K(\sigma) = G \circ \mu(d(p, x))$ for any $x \in M \setminus \{p, q\}$ and any radial plane σ at x. Suppose that there exist a point $x \in M \setminus \{p, q\}$ and a radial plane σ at x such that $K(\sigma) > G \circ \mu(d(p, x))$. Let $\gamma : [0, \pi] \to M$ denote the minimal geodesic segment emanating from p passing through x. Choose a unit tangent vector $v \in \sigma \subset T_x M$ orthogonal to $\gamma'(d(p, x))$. Let $Y(t)$ denote the Jacobi field along $\gamma(t)$ satisfying $Y(0) = 0$ and $Y(d(p, x)) = v$, and hence σ is spanned by $Y(d(p, x))$ and $\gamma'(d(p, x))$. By the Rauch comparison theorem, there exists a conjugate point $\gamma(t_1)$ of p along γ for some $t_1 \in (0, \pi)$, since $K(\sigma) > G \circ \mu(d(p, x))$ and the sectional curvature of the radial plane spanned by

$Y(t)$ and $\gamma'(t)$ is not less than $G \circ \mu(t)$ for each $t \in (0, \pi)$. This contradicts the fact that the geodesic segment γ is minimal. □

PROOF OF MAIN THEOREM. The first claim is clear from Lemma 3.4. Assume $\operatorname{diam} M = \operatorname{diam} \widetilde{M}$. By Lemmas 3.5 and 3.6, $K(\sigma) = G \circ \mu(d(p, x))$ for any point $x \in M \setminus \{p\}$ and any radial plane σ at x. Thus, it follows from Lemma 1 and Theorem 3 in [KK] that M is isometric to the n-dimensional model of $\widetilde{M}$. Incidentally, the explicit isometry φ between M and the n-dimensional model of $\widetilde{M}$ is given by

$$\varphi(x) := \begin{cases} \exp_{\tilde{p}} \circ I \circ \exp_p^{-1}(x) & \text{if } x \neq q \\ \tilde{q} & \text{if } x = q, \end{cases}$$

where $I : T_p M \to T_{\tilde{p}} \widetilde{M}$ denotes a linear isometry and q denotes the unique cut point of p.

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