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The Best Constant of L^p Sobolev Inequality Including j-th Derivative Corresponding to Periodic and Neumann Boundary Value Problem for $(-1)^M(d/dx)^{2M}$

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Abstract. In this paper, we study the best constant of L^p Sobolev inequality including j-th derivative:

$$\sup_{0\le y\le 1} \left|u^{(j)}(y)\right| \le C\left(\int_0^1 |\, u^{(M)}(x)\,|^p dx\right)^{1/p},$$

where u is an element of Sobolev space with periodic or Neumann boundary condition. The best constant can be expressed by L^q norm of Bernoulli polynomial. In [1, 4], the best constant of the above inequality was obtained for the case of $1 < p < \infty$ and $j = 0$. This paper extends the results of [1, 4] to $j = 1, 2, 3, \dots, M-1$.

1. Introduction

Throughout this paper, we assume that $M = 1, 2, 3, \dots,\ p, q > 1$ and $1/p + 1/q = 1$. We introduce the following notation for L^p norm

$$\|\, u\,\|_p = \left(\int_0^1 |\, u(x)\,|^p\, dx\right)^{1/p}$$

and the series of Sobolev spaces

$$W(\mathrm{X}, M, p) = \left\{ u \;\middle|\; u^{(M)} \in L^p(0,1), \quad u \text{ satisfies } A(\mathrm{X}) \right\},$$

where the condition $A(\mathrm{X})$ assumes

$$A(\mathrm{P}) \;\;:\; u^{(i)}(1) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1)\,, \qquad \int_0^1 u(x)dx = 0,$$
$$A(\mathrm{AP}) \;:\; u^{(i)}(1) + u^{(i)}(0) = 0 \quad (0 \le i \le M-1)\,,$$

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$$
\begin{aligned}
&A(\mathrm{C}) &&: u^{(i)}(0)=u^{(i)}(1)=0 \quad (0\le i\le M-1)\,,\\
&A(\mathrm{D}) &&: u^{(2i)}(0)=u^{(2i)}(1)=0 \quad (0\le i\le [\,(M-1)/2\,])\,,\\
&A(\mathrm{N}) &&: u^{(2i+1)}(0)=u^{(2i+1)}(1)=0 \ (0\le i\le [\,(M-2)/2\,])\,, \qquad \int_0^1 u(x)dx=0\,,\\
&A(\mathrm{DN}) &&: u^{(2i)}(0)=0 \quad (0\le i\le [\,(M-1)/2\,])\,,\\
& && \quad u^{(2i+1)}(1)=0 \quad (0\le i\le [\,(M-2)/2\,])\,,\\
&A(\mathrm{CF}) &&: u^{(i)}(0)=0 \quad (0\le i\le M-1)\,.
\end{aligned}
$$

It should be noted that if $M=1$ the boundary conditions for u in $A(\mathrm{N})$ and for u on $x=1$ in $A(\mathrm{DN})$ are not required. The script X means that P is Periodic, AP is Anti Periodic, C is Clamped, D is Dirichlet, N is Neumann, DN is Dirichlet-Neumann and CF is Clamped-Free boundary conditions. Let us consider L^p Sobolev inequality including j-th derivative, where j satisfies $j=0,1,2,\dots,M-1$. We call this inequality j-th L^p Sobolev inequality for short:

$$\sup_{0\le y\le 1} \left|u^{(j)}(y)\right| \le C \left\| u^{(M)} \right\|_p, \tag{1.1}$$

where u is an element of Sobolev space $W(\mathrm{X},M,p)$. In our previous work, the best constants of j-th L^p Sobolev inequality (1.1) were obtained in various boundary conditions as the following table.

X	$j=0$ and $p=2$	$j=0$ and $1<p<\infty$	$j=1,2,3,\dots$ and $1<p<\infty$
P	[9]	[1]	this paper
AP	[9]	—	—
C	[7]	$M=1,2,3$ [6]	—
D	[9]	$M=2m$ [2], $M=1,3,5$ [3]	—
N	[9]	[4]	this paper ($j=2l$)
DN	[9]	[10]	[10]
CF	[5]	[8]	[8]

From this table, we see that the difficulty in obtaining the best constant seems to increase in the case of $p\ne 2$. Indeed, each result in the case of $p\ne 2$ was obtained through a different method. No unified approach (maximizing the diagonal value of reproducing kernels; see [5, 7, 9]) as in the case of $p=2$ seems to exist for the case of $p\ne 2$.

In this paper, we would like to treat the case of $W(\mathrm{P},M,p)$ and $W(\mathrm{N},M,p)$ and enrich the above table. To state the conclusion, we introduce Bernoulli polynomials $b_k(x)$ defined by the following recurrence relation:

$$\begin{cases} b_0(x)=1\,,\\ b_k'(x)=b_{k-1}(x)\,, \quad \displaystyle\int_0^1 b_k(x)dx=0 \quad (k=1,2,3,\dots)\,. \end{cases} \tag{1.2}$$

Hence, we have

$$b_1(x)=x-\frac{1}{2}\,,\quad b_2(x)=\frac{x^2}{2}-\frac{x}{2}+\frac{1}{12}\,,\quad b_3(x)=\frac{x^3}{6}-\frac{x^2}{4}+\frac{x}{12}\,,\dots$$

and the auxiliary function $b_m(\alpha; x) = b_m(x) - b_m(\alpha)$. These polynomials play an important role in this paper. The main result is as follows:

THEOREM 1.1. *Let $M, m = 1, 2, 3, \ldots,$ $j = 0, 1, \ldots, M-1$ and $l = 0, 1, 2, \ldots$. For any function $u \in W(\mathrm{P}, M, p)$, there exists a positive constant C which is independent of u such that j-th L^p Sobolev inequality* (1.1) *holds. Moreover, the best constant $C(\mathrm{P}, M, j)$ of* (1.1) *is given by*

$$\begin{aligned}
&C(\mathrm{P}, 2m-1, 2l) = \| b_{2(m-1-l)+1} \|_q && (M = 2m-1, \quad j = 2l)\,,\\
&C(\mathrm{P}, 2m-1, 2l+1) = \| b_{2(m-1-l)}(\alpha_{m-1-l}; \cdot) \|_q && (M = 2m-1, \quad j = 2l+1)\,,\\
&C(\mathrm{P}, 2m, 2l) = \| b_{2(m-l)}(\alpha_{m-l}; \cdot) \|_q && (M = 2m, \quad j = 2l)\,,\\
&C(\mathrm{P}, 2m, 2l+1) = \| b_{2(m-1-l)+1} \|_q && (M = 2m, \quad j = 2l+1)\,.
\end{aligned} \tag{1.3}$$

From Lemma 2.1, *we see that $\alpha = \alpha_k$ is the unique solution to the equation*

$$\partial_\alpha \| b_{2k}(\alpha; \cdot) \|_q^q = 0 \quad (0 < \alpha < 1/2, k = 0, 1, 2, \ldots)\,. \tag{1.4}$$

If we replace C by $C(\mathrm{P}, M, j)$ as (1.3), *then the equality holds for $u(x) = c\, U(\mathrm{P}, M, j; x)$ where c is an arbitrary constant. $U(\mathrm{P}, M, j; x)$ $(0 < x < 1)$ is given by*

$$U(\mathrm{P}, 2m-1, 2l; x) = \int_0^1 \mathrm{sgn}(x-y) b_{2m-1}(|x-y|) \mathrm{sgn}\big(b_{2(m-1-l)+1}(y)\big) \big|b_{2(m-1-l)+1}(y)\big|^{q-1} dy\,, \tag{1.5}$$

$$U(\mathrm{P}, 2m-1, 2l+1; x) = \int_0^1 \mathrm{sgn}(x-y) b_{2m-1}(|x-y|) \mathrm{sgn}\big(b_{2(m-1-l)}(\alpha_{m-1-l}; y)\big) \big|b_{2(m-1-l)}(\alpha_{m-1-l}; y)\big|^{q-1} dy\,, \tag{1.6}$$

$$U(\mathrm{P}, 2m, 2l; x) = \int_0^1 b_{2m}(|x-y|) \mathrm{sgn}\big(b_{2(m-l)}(\alpha_{m-l}; y)\big) \big|b_{2(m-l)}(\alpha_{m-l}; y)\big|^{q-1} dy\,, \tag{1.7}$$

$$U(\mathrm{P}, 2m, 2l+1; x) = \int_0^1 b_{2m}(|x-y|) \mathrm{sgn}\big(b_{2(m-1-l)+1}(y)\big) \big|b_{2(m-1-l)+1}(y)\big|^{q-1} dy\,. \tag{1.8}$$

THEOREM 1.2. *Let $M, m = 1, 2, 3, \ldots,$ $j = 0, 1, \ldots, M-1$ and $l = 0, 1, 2, \ldots$. For any function $u \in W(\mathrm{N}, M, p)$, there exists a positive constant C which is independent of*

u such that j-th L^p Sobolev inequality (1.1) *holds. Moreover, the best constant $C(\mathrm{N}, M, j)$ of* (1.1) *is given by*

$$\begin{aligned} C(\mathrm{N}, 2m-1, 2l) &= 2^{2(m-1-l)+1} \| b_{2(m-1-l)+1} \|_q & (M = 2m-1, \quad j = 2l)\,, \\ C(\mathrm{N}, 2m, 2l) &= 2^{2(m-l)} \| b_{2(m-l)}(\alpha_{m-l}; \cdot) \|_q & (M = 2m, \quad j = 2l)\,, \end{aligned} \tag{1.9}$$

where α_{m-l} is the unique solution to the equation (1.4) *for $k = m - l$. If we replace C by $C(\mathrm{N}, M, j)$ as* (1.9), *then the equality holds for $u(x) = c\, V(\mathrm{P2}, M, j; x)$ where c is an arbitrary constant. $V(\mathrm{P2}, M, j; x)$ $(0 < x < 1)$ is given by*

$$\begin{aligned} &V(\mathrm{P2}, 2m-1, 2l; x) \\ &= \int_0^2 \mathrm{sgn}(x-y) b_{2m-1}\left(\frac{|x-y|}{2}\right) \mathrm{sgn}\left(b_{2(m-1-l)+1}\left(\frac{y}{2}\right)\right) \left|b_{2(m-1-l)+1}\left(\frac{y}{2}\right)\right|^{q-1} dy\,, \end{aligned} \tag{1.10}$$

$$\begin{aligned} &V(\mathrm{P2}, 2m, 2l; x) \\ &= \int_0^2 b_{2m}\left(\frac{|x-y|}{2}\right) \mathrm{sgn}\left(b_{2(m-l)}\left(\alpha_{m-l}; \frac{y}{2}\right)\right) \left|b_{2(m-l)}\left(\alpha_{m-l}; \frac{y}{2}\right)\right|^{q-1} dy\,. \end{aligned} \tag{1.11}$$

REMARK 1. We could not obtain the result for the case of $M = 2m-1,\ j = 2l+1$ and $M = 2m,\ j = 2l+1$ for technical reason. The difficulties for these cases are briefly stated at the end of section 3.

2. Basic properties of Bernoulli polynomials

In this section, we present the basic properties of Bernoulli polynomials. Bernoulli polynomials are introduced by the recurrence relation (1.2). Also, Bernoulli polynomials are defined by the generating function

$$\frac{e^{xt}}{t^{-1}(e^t - 1)} = \sum_{k=0}^{\infty} b_k(x) t^k \quad (|t| < 2\pi)\,. \tag{2.1}$$

Bernoulli polynomial $b_k(x)$ is k-th polynomial with respect to x. Replacing x to $1-x$ in (2.1), we have

$$b_k(1-x) = (-1)^k b_k(x) \quad (k = 0, 1, 2, \dots)\,. \tag{2.2}$$

Inserting $x = 0$ and $x = 1$ into (2.1), we have

$$b_k(1) - b_k(0) = \begin{cases} 1 & (k = 1)\,, \\ 0 & (k \neq 1)\,. \end{cases} \tag{2.3}$$

From (2.2) and (2.3), we have

$$b_{2k+1}(0) = \begin{cases} -1/2 & (k=0)\,, \\ 0 & (k=1,2,3,\dots)\,, \end{cases} \qquad b_{2k+1}(1/2) = 0 \quad (k=0,1,2,\dots)\,. \tag{2.4}$$

Let $\{x\}$ be a decimal part of a real number x as

$$\{x\} = x - [x]\,, \quad [x] = \sup\{n \in \mathbf{Z} \,|\, n \le x\}\,.$$

Since $\{x\}$ is a periodic function of x with period 1, we derive Fourier expansion formula of $b_k(\{x\})$ as

$$b_k(\{x\}) = -\sum_{l\neq 0} \left(\sqrt{-1}\,2\pi l\right)^{-k} \exp\!\left(\sqrt{-1}\,2\pi l x\right) \quad (k=1,2,3,\dots)\,.$$

Hence we have

$$b_{2k}(\{x\}) = (-1)^{k+1} 2 \sum_{l=1}^{\infty} (2\pi l)^{-2k} \cos(2\pi l x)\,.$$

Inserting $x=0$ and $x=1/2$ into the above relation, we have

$$\begin{aligned} (-1)^{k+1} b_{2k}(0) &= \frac{2}{(2\pi)^{2k}} \zeta(2k)\,, \\ (-1)^{k+1} b_{2k}(1/2) &= -\left(1 - 2^{-(2k-1)}\right) \frac{2}{(2\pi)^{2k}} \zeta(2k)\,, \end{aligned} \tag{2.5}$$

where $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$ (Re $z > 1$) is Riemann-zeta function. So we have $|b_{2k}(0)| > |b_{2k}(1/2)|$. For $k = 2,3,4,\dots$, the boundary value problem

$$\begin{cases} -\left(\dfrac{d}{dx}\right)^2 \left((-1)^{k+1} b_{2k-1}(x)\right) = (-1)^k b_{2k-3}(x) \quad (0 < x < 1/2)\,, \\ b_{2k-1}(0) = b_{2k-1}(1/2) = 0 \end{cases}$$

has a unique solution

$$(-1)^{k+1} b_{2k-1}(x) = \int_0^{1/2} G(x,y)(-1)^k b_{2k-3}(y)\,dy\,,$$

where $G(x,y)$ is positive-valued Green function given by

$$G(x,y) = x \wedge y - 2xy = \min\{x,y\} - 2xy > 0 \quad (0 < x, y < 1/2)\,.$$

Starting from

$$b_1(x) = x - \frac{1}{2} < 0 \quad (0 < x < 1/2)\,,$$

we can show the following inequalities recurrently

$$(-1)^{k+1} b_{2k-1}(x) < 0 \quad (0 < x < 1/2)\,. \tag{2.6}$$

On the other hand, for any fixed $0 \le \alpha \le 1/2$, we define $b_{2k}(\alpha; x)$ as

$$b_{2k}(\alpha; x) = b_{2k}(x) - b_{2k}(\alpha) \quad (k = 1, 2, 3, \dots,\ 0 < x < 1)\,. \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have

$$(-1)^{k+1} b_{2k}(\alpha; x) = (-1)^{k+1}\Big(b_{2k}(x) - b_{2k}(\alpha)\Big) \begin{cases} > 0 & (0 \le x < \alpha)\,, \\ = 0 & (x = \alpha)\,, \\ < 0 & (\alpha < x \le 1/2)\,, \end{cases} \tag{2.8}$$

$$\partial_x\big((-1)^{k+1} b_{2k}(\alpha; x)\big) = (-1)^{k+1} b_{2k-1}(x) < 0 \quad (0 < x < 1/2)\,, \tag{2.9}$$

$$\partial_\alpha\big((-1)^{k+1} b_{2k}(\alpha; x)\big) = (-1)^{k} b_{2k-1}(\alpha) > 0 \quad (0 < \alpha < 1/2)\,. \tag{2.10}$$

LEMMA 2.1. *The equation* (1.4) *with respect to* α *has a unique solution* $\alpha = \alpha_k$.

PROOF. We define the function $g(\alpha)$ as

$$\begin{aligned} g(\alpha) &= \|\, b_{2k}(\alpha; \cdot)\,\|_q^q = 2\int_0^{1/2} |\, b_{2k}(\alpha; x)\,|^q dx \\ &= 2\int_0^{\alpha} \Big((-1)^{k+1} b_{2k}(\alpha; x)\Big)^q dx + 2\int_\alpha^{1/2} \Big((-1)^{k} b_{2k}(\alpha; x)\Big)^q dx\,, \end{aligned}$$

where we use (2.2) and (2.8). Differentiating $g(\alpha)$ and using (2.10), we have

$$g'(\alpha) = \partial_\alpha \|\, b_{2k}(\alpha; \cdot)\,\|_q^q = 2q(-1)^k b_{2k-1}(\alpha) h(\alpha)\,,$$

where

$$h(\alpha) = \int_0^{\alpha} \Big((-1)^{k+1}\, b_{2k}(\alpha; x)\Big)^{q-1} dx - \int_\alpha^{1/2} \Big((-1)^{k}\, b_{2k}(\alpha; x)\Big)^{q-1} dx\,. \tag{2.11}$$

From $q > 1$ and (2.10), we have

$$\begin{aligned} h'(\alpha) &= (q-1)(-1)^k b_{2k-1}(\alpha) \\ &\quad \times \left[\int_0^{\alpha} \Big((-1)^{k+1} b_{2k}(\alpha; x)\Big)^{q-2} dx + \int_\alpha^{1/2} \Big((-1)^{k} b_{2k}(\alpha; x)\Big)^{q-2} dx\right] > 0 \end{aligned}$$

for $0 < \alpha < 1/2$. Since $h(0) < 0$ and $h(1/2) > 0$, there exists a unique $\alpha_k \in (0, 1/2)$ such that

$$h(\alpha) \begin{cases} < 0 & (0 \le \alpha < \alpha_k)\,, \\ = 0 & (\alpha = \alpha_k)\,, \\ > 0 & (\alpha_k < \alpha \le 1/2)\,. \end{cases}$$

So we have

$$g'(\alpha) = 2q(-1)^k b_{2k-1}(\alpha) h(\alpha) \begin{cases} < 0 & (0 \le \alpha < \alpha_k)\,, \\ = 0 & (\alpha = \alpha_k)\,, \\ > 0 & (\alpha_k < \alpha \le 1/2)\,, \end{cases} \qquad \min_{0 \le \alpha \le 1/2} g(\alpha) = g(\alpha_k)\,.$$

This completes the proof of Lemma 2.1. □

LEMMA 2.2. *For any fixed* y $(0 \le y \le 1)$, *we have the following relations*:
(1) $\|b_k(|\cdot - y|)\|_q^q = \|b_k\|_q^q$.
(2) $\|b_k(\alpha; |\cdot - y|)\|_q^q = \|b_k(\alpha; \cdot)\|_q^q$.

PROOF. (1) follows from

$$\|b_k(|\cdot - y|)\|_q^q = \int_0^1 |b_k(|x-y|)|^q\,dx = \int_0^y \big|b_k(y-x)\big|^q dx + \int_y^1 \big|b_k(x-y)\big|^q dx$$
$$= \int_0^y \big|b_k(\xi)\big|^q d\xi + \int_y^1 \big|b_k(1-\xi)\big|^q d\xi = \int_0^y \big|b_k(\xi)\big|^q d\xi + \int_y^1 \big|(-1)^k b_k(\xi)\big|^q d\xi$$
$$= \int_0^y \big|b_k(\xi)\big|^q d\xi + \int_y^1 \big|b_k(\xi)\big|^q d\xi = \int_0^1 \big|b_k(\xi)\big|^q d\xi = \|b_k\|_q^q\,,$$

where we use (2.2). (2) is shown by the same way. □

LEMMA 2.3.

$$(1)\quad \int_0^1 \mathrm{sgn}\big(b_{2k+1}(x)\big)\big|b_{2k+1}(x)\big|^{q-1}dx = 0\,.$$
$$(2)\quad \int_0^1 \mathrm{sgn}\big(b_{2k}(\alpha_k;x)\big)\big|b_{2k}(\alpha_k;x)\big|^{q-1}dx = 0\,.$$

PROOF. Using (2.2), we have

$$\mathrm{sgn}\big(b_{2k+1}(1-x)\big)\big|b_{2k+1}(1-x)\big|^{q-1} = -\mathrm{sgn}\big(b_{2k+1}(x)\big)\big|b_{2k+1}(x)\big|^{q-1}\,.$$

So we have (1). Using Lemma 2.1, we have (2) since

$$\int_0^1 \mathrm{sgn}\big(b_{2k}(\alpha_k;x)\big)\big|b_{2k}(\alpha_k;x)\big|^{q-1}dx = 2\int_0^{1/2} \mathrm{sgn}\big(b_{2k}(\alpha_k;x)\big)\big|b_{2k}(\alpha_k;x)\big|^{q-1}dx$$
$$= 2\left[\int_0^{\alpha_k} \big((-1)^{k+1}b_{2k}(\alpha_k;x)\big)^{q-1}dx - \int_{\alpha_k}^{1/2} \big((-1)^k b_{2k}(\alpha_k;x)\big)^{q-1}dx\right] = 2h(\alpha_k) = 0\,,$$

where $h(\alpha)$ is defined by (2.11). □

LEMMA 2.4. *For* $f(x) = |b_k(x)|^{q-1}$ *and* $g(x) = |b_k(\alpha;x)|^{q-1}$, *we have*

$$\|f\|_p = \|b_k\|_q^{q-1} \quad \textit{and} \quad \|g\|_p = \|b_k(\alpha;\cdot)\|_q^{q-1}\,.$$

PROOF. Noting the relation $(q-1)p = q\left(1 - \dfrac{1}{q}\right)p = q$, we have

$$\|f\|_p = \left(\int_0^1 |f(x)|^p dx\right)^{1/p} = \left(\int_0^1 |b_k(x)|^{(q-1)p}dx\right)^{1/p} = \left(\int_0^1 |b_k(x)|^q dx\right)^{1/p}$$
$$= \|b_k\|_q^{q/p} = \|b_k\|_q^{q-1}\,.$$

The second relation is shown by the same way. □

PROPOSITION 2.1. *Let N be $N = 1, 2, 3, \ldots$. For any bounded continuous function $f(x)$ satisfying $\int_0^1 f(y)dy = 0$, the periodic boundary value problem*

$$\text{BVP (P, } N)$$

$$\begin{cases} (-1)^{[(N+1)/2]}u^{(N)} = f(x) & (0 < x < 1)\,, \\ u^{(i)}(1) - u^{(i)}(0) = 0 & (0 \le i \le N-1)\,, \\ \int_0^1 u(x)dx = 0 & \end{cases}$$

has a unique classical solution $u(x)$ expressed as

$$u(x) = \int_0^1 G(N; x, y)\, f(y)\, dy \quad (0 < x < 1), \tag{2.12}$$

where Green function $G(N; x, y)$ is given by

$$G(N; x, y) = (-1)^{[(N+3)/2]}\big(\mathrm{sgn}(x-y)\big)^N b_N(\,|x-y|\,) \quad (0 < x, y < 1)\,. \tag{2.13}$$

Especially, for $M = 1, 2, 3, \ldots$, we have

$$G(2M-1; x, y) = (-1)^{M+1}\mathrm{sgn}(x-y)\, b_{2M-1}(\,|x-y|\,)\,,$$

$$G(2M; x, y) = (-1)^{M+1} b_{2M}(\,|x-y|\,)\,.$$

PROOF. See; Kametaka et al. [1]. Here, as an example, we would like to show the case $N = 3$. Assume $N = 3$, then using (2.2)∼(2.4), we have

$$\int_0^1 G(3; x, y) f(y)dy = \int_0^1 G(3; x, y)(-1)^2 u^{(3)}(y)dy$$

$$= -\int_0^x b_3(x-y)u^{(3)}(y)dy + \int_x^1 b_3(y-x)u^{(3)}(y)dy$$

$$= -\big[b_3(x-y)u^{(2)}(y) + b_2(x-y)u^{(1)}(y) + b_1(x-y)u(y)\big]_{y=0}^{y=x}$$

$$+ \big[b_3(y-x)u^{(2)}(y) - b_2(y-x)u^{(1)}(y) + b_1(y-x)u(y)\big]_{y=x}^{y=1} - \int_0^x u(y)dy - \int_x^1 u(y)dy$$

$$= u(x) - b_3(x)\big(u^{(2)}(1) - u^{(2)}(0)\big) - b_2(x)\big(u^{(1)}(1) - u^{(1)}(0)\big) - b_1(x)\big(u(1) - u(0)\big)$$

$$- \int_0^1 u(y)dy\,.$$

Using the boundary condition and orthogonality condition of BVP(P,3), we have (2.12) and (2.13) in the case of $N = 3$. □

LEMMA 2.5. *The following relations hold if $0 < x < 1$, $0 < y < 1$ and $x \neq y$.*

$$\partial_y^k \partial_x^M G(2M; x, y) = (-1)^{M+1+k}\big(\mathrm{sgn}(x-y)\big)^{M+k} b_{M-k}(\,|x-y|\,)$$

$$
= \begin{cases} \operatorname{sgn}(x-y) b_{2(m-1-l)+1}(\,|x-y|\,) & (M=2m-1, \quad k=2l)\,, \\ -b_{2(m-1-l)}(\,|x-y|\,) & (M=2m-1, \quad k=2l+1)\,, \\ -b_{2(m-l)}(\,|x-y|\,) & (M=2m, \quad k=2l)\,, \\ \operatorname{sgn}(x-y) b_{2(m-1-l)+1}(\,|x-y|\,) & (M=2m, \quad k=2l+1)\,. \end{cases}
$$

PROOF. Noting the relations

$$
\partial_x|x-y| = \operatorname{sgn}(x-y)\,, \quad \partial_y|x-y| = -\operatorname{sgn}(x-y)\,,
$$

we have this lemma. □

LEMMA 2.6. *For any $u \in W(\mathrm{P}, M, p)$ and for any fixed y $(0 \le y \le 1)$, the following reproducing relation holds*:

$$
u^{(j)}(y) = \int_0^1 u^{(M)}(x)\, \partial_y^j \partial_x^M G(2M; x, y)\, dx\,.
$$

PROOF. Using Lemma 2.5, we have

$$
\begin{aligned}
I &= \int_0^1 u^{(M)}(x)\, \partial_y^j \partial_x^M G(2M; x, y)\, dx \\
&= \int_0^1 u^{(M)}(x)\, (-1)^{M+1+j} \big(\operatorname{sgn}(x-y)\big)^{M+j} b_{M-j}(\,|x-y|\,)\, dx \\
&= -\int_0^y u^{(M)}(x)\, b_{M-j}(y-x)\, dx - (-1)^{M+j} \int_y^1 u^{(M)}(x)\, b_{M-j}(x-y)\, dx \\
&= -I_1 - I_2\,,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^y u^{(M)}(x)\, b_{M-j}(y-x)\, dx \\
&= \sum_{k=0}^{M-1-j} \big[u^{(M-1-k)}(y)\, b_{M-j-k}(0) - u^{(M-1-k)}(0)\, b_{M-j-k}(y)\big] + \int_0^y u^{(j)}(x) dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= (-1)^{M+j} \int_y^1 u^{(M)}(x)\, b_{M-j}(x-y)\, dx \\
&= \sum_{k=0}^{M-1-j} (-1)^{M+j+k} \big[u^{(M-1-k)}(1)\, b_{M-j-k}(1-y) - u^{(M-1-k)}(y)\, b_{M-j-k}(0)\big] \\
&\quad + \int_y^1 u^{(j)}(x) dx\,.
\end{aligned}
$$

So we have

$$
\begin{aligned}
I &= -I_1 - I_2 \\
&= \sum_{k=0}^{M-1-j} \big[- u^{(M-1-k)}(y)\, b_{M-j-k}(0) + u^{(M-1-k)}(0)\, b_{M-j-k}(y) \\
&\quad - u^{(M-1-k)}(1)\, (-1)^{M+j+k} b_{M-j-k}(1-y) + u^{(M-1-k)}(y)\, (-1)^{M+j+k} b_{M-j-k}(0) \big] \\
&\quad - \int_0^1 u^{(j)}(x) dx.
\end{aligned}
$$

Using (2.2), we have

$$
\begin{aligned}
I = \sum_{k=0}^{M-1-j} \Big[&\big(-1 + (-1)^{M+j+k} \big) b_{M-j-k}(0)\, u^{(M-1-k)}(y) \\
&- \big(u^{(M-1-k)}(1) - u^{(M-1-k)}(0) \big) b_{M-j-k}(y) \Big] - \int_0^1 u^{(j)}(x) dx\,.
\end{aligned}
$$

For

$$
\begin{aligned}
&\big(-1 + (-1)^{M+j+k} \big) b_{M-j-k}(0)\, u^{(M-1-k)}(y) \\
&\qquad = \big(-1 + (-1)^{M-j-k} \big) b_{M-j-k}(0)\, u^{(M-1-k)}(y)\,,
\end{aligned}
$$

setting $l = M - j - k$ $(1 \le l \le M - j)$ and using (2.4), we have

$$
\Big(-1 + (-1)^l \Big)\, b_l(0)\, u^{(l+j-1)}(y) = \begin{cases} u^{(j)}(y) & (l = 1), \\ 0 & (2 \le l \le M - j). \end{cases}
$$

So we have

$$
\begin{aligned}
I &= u^{(j)}(y) - \sum_{k=0}^{M-1-j} \big(u^{(M-1-k)}(1) - u^{(M-1-k)}(0) \big) b_{M-j-k}(y) - \int_0^1 u^{(j)}(x) dx \\
&= \begin{cases} u(y) - \displaystyle\sum_{k=0}^{M-1} \big(u^{(M-1-k)}(1) - u^{(M-1-k)}(0) \big) b_{M-k}(y) - \int_0^1 u(x) dx \\ (j = 0)\,, \\ \\ u^{(j)}(y) - \displaystyle\sum_{k=0}^{M-j} \big(u^{(M-1-k)}(1) - u^{(M-1-k)}(0) \big) b_{M-j-k}(y) \\ (j = 1, 2, 3, \ldots, M-1)\,. \end{cases}
\end{aligned}
$$

If we chose $u \in W(\mathrm{P}, M, p)$, then we have Lemma 2.6. □

3. Proof of Theorems

3.1. Proof of Theorem 1.1. We put the number as

$$
\begin{array}{lll}
\text{(I)} & M = 2m-1\,, & j = 2l\,, \\
\text{(II)} & M = 2m-1\,, & j = 2l+1\,, \\
\text{(III)} & M = 2m\,, & j = 2l\,, \\
\text{(IV)} & M = 2m\,, & j = 2l+1\,.
\end{array}
$$

For any $u \in W(\mathrm{P}, M, p)$ and any fixed y $(0 \le y \le 1)$, using Lemma 2.5, the reproducing relation Lemma 2.6 is given by

$$
u^{(j)}(y) = \int_0^1 u^{(M)}(x) \left\{ \begin{array}{ll}
\operatorname{sgn}(x-y) b_{2(m-1-l)+1}(\,|x-y|\,) & \text{(I)} \\
-b_{2(m-1-l)}(\,|x-y|\,) & \text{(II)} \\
-b_{2(m-l)}(\,|x-y|\,) & \text{(III)} \\
\operatorname{sgn}(x-y) b_{2(m-1-l)+1}(\,|x-y|\,) & \text{(IV)}
\end{array} \right\} dx\,.
$$

For (II) and (III), we add the correction term $b_{2(m-1-l)}(\alpha_{m-1-l})$ and $b_{2(m-l)}(\alpha_{m-l})$, respectively. Noting

$$
\int_0^1 u^{(M)}(x)dx = u^{(M-1)}(1) - u^{(M-1)}(0) = 0\,,
$$

we have

$$
u^{(j)}(y) = \int_0^1 u^{(M)}(x) \left\{ \begin{array}{ll}
\operatorname{sgn}(x-y) b_{2(m-1-l)+1}(\,|x-y|\,) & \text{(I)} \\
-b_{2(m-1-l)}(\,\alpha_{m-1-l};\,|x-y|\,) & \text{(II)} \\
-b_{2(m-l)}(\,\alpha_{m-l};\,|x-y|\,) & \text{(III)} \\
\operatorname{sgn}(x-y) b_{2(m-1-l)+1}(\,|x-y|\,) & \text{(IV)}
\end{array} \right\} dx\,.
$$

Applying Hölder inequality to the above relation, using Lemma 2.2 and taking the supremum of the both sides with respect to y, we have j-th L^p Sobolev inequality:

$$
\sup_{0 \le y \le 1} \left| u^{(j)}(y) \right| \le \left\{ \begin{array}{ll}
\|\, b_{2(m-1-l)+1} \,\|_q & \text{(I)} \\
\|\, b_{2(m-1-l)}(\alpha_{m-1-l};\,\cdot) \,\|_q & \text{(II)} \\
\|\, b_{2(m-l)}(\alpha_{m-l};\,\cdot) \,\|_q & \text{(III)} \\
\|\, b_{2(m-1-l)+1} \,\|_q & \text{(IV)}
\end{array} \right\} \left\| u^{(M)} \right\|_p = C(M, j) \left\| u^{(M)} \right\|_p\,. \tag{3.1}
$$

The equality holds for $u(x) = U(x) = U(M, j; x)$ which satisfy $U^{(M)}(M, j; x) = F(M, j; x)$, where

$$F(M, j; x) = \left\{ \begin{array}{ll} \operatorname{sgn}\big(b_{2(m-1-l)+1}(x)\big)\big|b_{2(m-1-l)+1}(x)\big|^{q-1} & \text{(I)} \\ \operatorname{sgn}\big(b_{2(m-1-l)}(\alpha_{m-1-l}; x)\big)\big|b_{2(m-1-l)}(\alpha_{m-1-l}; x)\big|^{q-1} & \text{(II)} \\ \operatorname{sgn}\big(b_{2(m-l)}(\alpha_{m-l}; x)\big)\big|b_{2(m-l)}(\alpha_{m-l}; x)\big|^{q-1} & \text{(III)} \\ \operatorname{sgn}\big(b_{2(m-1-l)+1}(x)\big)\big|b_{2(m-1-l)+1}(x)\big|^{q-1} & \text{(IV)} \end{array} \right\}.$$

On the other hand, from Lemma 2.3, we see that

$$\int_0^1 F(M, j; x)dx = 0\,.$$

Thus $U(M, j; x)$ given by (1.5)∼(1.8) satisfies BVP(P, M), so $U(M, j; x) \in W(\mathrm{P}, M, p)$ and $U(M, j; x)$ achieves the equality in (3.1). In fact, using Lemma 2.4, we have

$$\big|\, U^{(j)}(0) \,\big| = \left\{ \begin{array}{ll} \| b_{2(m-1-l)+1} \|_q^q & \text{(I)} \\ \| b_{2(m-1-l)}(\alpha_{m-1-l}; \cdot) \|_q^q & \text{(II)} \\ \| b_{2(m-l)}(\alpha_{m-l}; \cdot) \|_q^q & \text{(III)} \\ \| b_{2(m-1-l)+1} \|_q^q & \text{(IV)} \end{array} \right\} = C(M, j) \| F \|_p = C(M, j) \big\| U^{(M)} \big\|_p.$$

Combining this with (3.1), we have

$$C(M, j) \big\| U^{(M)} \big\|_p = \big|\, U^{(j)}(0) \,\big| \le \sup_{0 \le y \le 1} \big|\, U^{(j)}(y) \,\big| \le C(M, j) \big\| U^{(M)} \big\|_p\,.$$

This shows that $C(M, j)$ is the best constant of j-th L^p Sobolev inequality (3.1) and the equality holds for $U(M, j; x)$ defined by (1.5) ∼ (1.8). This completes the proof of Theorem 1.1. □

3.2. Proof of Theorem 1.2. To prove Theorem 1.2, we prepare the following lemma.

LEMMA 3.1. *$V(\mathrm{P2}, M, 2l; x)$ defined by* (1.10) *and* (1.11) *in Theorem* 1.2 *satisfies the following properties.*

$$V(\mathrm{P2}, M, 2l; 2 - x) = V(\mathrm{P2}, M, 2l; x) \quad (0 < x < 1)\,. \tag{3.2}$$

$$V^{(2i+1)}(\mathrm{P2}, M, 2l; 0) = 0 \quad \left(0 \le i \le \left[\frac{M-2}{2}\right]\right). \tag{3.3}$$

$$V^{(2i+1)}(\mathrm{P2}, M, 2l; 1) = 0 \quad \left(0 \le i \le \left[\frac{M-2}{2}\right]\right). \tag{3.4}$$

$$V^{(i)}(\mathrm{P2}, M, 2l; 2) - V^{(i)}(\mathrm{P2}, M, 2l; 0) = 0 \quad (0 \le i \le M - 1)\,. \tag{3.5}$$

$$\int_0^2 V(\mathrm{P2}, M, 2l; x)dx = 0\,. \tag{3.6}$$

$$\int_0^1 V(\mathrm{P2}, M, 2l; x)dx = 0 . \tag{3.7}$$

PROOF. We only prove the case $M = 2m-1$. From (2.2), we have (3.2) since

$$\begin{aligned}
&V(\mathrm{P2}, 2m-1, 2l; 2-x) \\
&= \int_0^2 \mathrm{sgn}(2-x-y) b_{2m-1}\Big(\frac{|2-x-y|}{2}\Big) \mathrm{sgn}\Big(b_{2(m-1-l)+1}\Big(\frac{y}{2}\Big)\Big) \Big|b_{2(m-1-l)+1}\Big(\frac{y}{2}\Big)\Big|^{q-1} dy \\
&= \int_0^2 \mathrm{sgn}(z-x) b_{2m-1}\Big(\frac{|z-x|}{2}\Big) \mathrm{sgn}\Big(b_{2(m-1-l)+1}\Big(\frac{2-z}{2}\Big)\Big) \Big|b_{2(m-1-l)+1}\Big(\frac{2-z}{2}\Big)\Big|^{q-1} dz \\
&= \int_0^2 \mathrm{sgn}(x-z) b_{2m-1}\Big(\frac{|x-z|}{2}\Big) \mathrm{sgn}\Big(b_{2(m-1-l)+1}\Big(\frac{z}{2}\Big)\Big) \Big|b_{2(m-1-l)+1}\Big(\frac{z}{2}\Big)\Big|^{q-1} dz \\
&= V(\mathrm{P2}, 2m-1, 2l; x) .
\end{aligned}$$

Again by (2.2), we obtain (3.3) since

$$\begin{aligned}
&V^{(2i+1)}(\mathrm{P2}, 2m-1, 2l; 0) \\
&= \int_0^2 2^{-(2i+1)} b_{2(m-1-i)}\Big(\frac{y}{2}\Big) \mathrm{sgn}\Big(b_{2(m-1-l)+1}\Big(\frac{y}{2}\Big)\Big) \Big|b_{2(m-1-l)+1}\Big(\frac{y}{2}\Big)\Big|^{q-1} dy \\
&= -\int_0^2 2^{-(2i+1)} b_{2(m-1-i)}\Big(\frac{2-y}{2}\Big) \mathrm{sgn}\Big(b_{2(m-1-l)+1}\Big(\frac{2-y}{2}\Big)\Big) \\
&\quad \times \Big|b_{2(m-1-l)+1}\Big(\frac{2-y}{2}\Big)\Big|^{q-1} dy \\
&= -\int_0^2 2^{-(2i+1)} b_{2(m-1-i)}\Big(\frac{z}{2}\Big) \mathrm{sgn}\Big(b_{2(m-1-l)+1}\Big(\frac{z}{2}\Big)\Big) \Big|b_{2(m-1-l)+1}\Big(\frac{z}{2}\Big)\Big|^{q-1} dy \\
&= -V^{(2i+1)}(\mathrm{P2}, 2m-1, 2l; 0).
\end{aligned}$$

Differentiating (3.2), $2i$ and $2i+1$ times, we have

$$\begin{aligned}
&V^{(2i)}(\mathrm{P2}, 2m-1, 2l; 2-x) = V^{(2i)}(\mathrm{P2}, 2m-1, 2l; x) , \\
&-V^{(2i+1)}(\mathrm{P2}, 2m-1, 2l; 2-x) = V^{(2i+1)}(\mathrm{P2}, 2m-1, 2l; x) .
\end{aligned}$$

Putting $x = 1$, we have

$$-V^{(2i+1)}(\mathrm{P2}, 2m-1, 2l; 1) = V^{(2i+1)}(\mathrm{P2}, 2m-1, 2l; 1).$$

So we have (3.4). Putting $x = 0$, we have

$$\begin{aligned}
&V^{(2i)}(\mathrm{P2}, 2m-1, 2l; 2) = V^{(2i)}(\mathrm{P2}, 2m-1, 2l; 0) , \\
&-V^{(2i+1)}(\mathrm{P2}, 2m-1, 2l; 2) = V^{(2i+1)}(\mathrm{P2}, 2m-1, 2l; 0) = 0 ,
\end{aligned}$$

where we use (3.3). So we have (3.5). From

$$\int_0^2 \operatorname{sgn}(x-y) b_{2m-1}\Big(\frac{|x-y|}{2}\Big)dx = -\int_0^y b_{2m-1}\Big(\frac{y-x}{2}\Big)dx + \int_y^2 b_{2m-1}\Big(\frac{x-y}{2}\Big)dx$$
$$= -2\int_0^{y/2} b_{2m-1}(z)dz + 2\int_{y/2}^1 b_{2m-1}(1-z)dz = -2\int_0^1 b_{2m-1}(z)dz = 0\,,$$

we have (3.6). From (3.2) and (3.6), we have (3.7). □

Now we prove Theorem 1.2.

PROOF OF THEOREM 1.2. We define the space $W(\mathrm{P2}, M, p)$ as

$$W(\mathrm{P2}, M, p) = \left\{\, u \;\middle|\; u^{(M)} \in L^p(0,2),\ u \text{ satisfies } A(\mathrm{P2})\right\},$$

where

$$A(\mathrm{P2}):\quad u^{(i)}(2) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1)\,,\qquad \int_0^2 u(x)dx = 0\,.$$

For any $u \in W(\mathrm{N}, M, p)$, we define $\widetilde{u}(x)$ $(0 \le x \le 2)$ as

$$\widetilde{u}(x) = \begin{cases} u(x) & (0 \le x \le 1)\,, \\ u(2-x) & (1 \le x \le 2)\,. \end{cases}$$

Since $u \in W(\mathrm{N}, M, p)$ satisfies Neumann boundary condition at $x = 0$ and $x = 1$, it is easy to see that $\widetilde{u}$ is an element of $W(\mathrm{P2}, M, p)$. So, we have

$$\sup_{0\le y\le 1} \left|u^{(j)}(y)\right| = \sup_{0\le y\le 2} \left|\widetilde{u}^{(j)}(y)\right|$$
$$\le C(\mathrm{P2}, M, j)\left\|\widetilde{u}^{(M)}\right\|_{L^p(0,2)} = 2^{1/p} C(\mathrm{P2}, M, j)\left\|u^{(M)}\right\|_{L^p(0,1)}\,, \tag{3.8}$$

where $C(\mathrm{P2}, M, j)$ is the best constant of j-th L^p Sobolev inequality:

$$\sup_{0\le y\le 2} \left|u^{(j)}(y)\right| \le C\left\|u^{(M)}\right\|_{L^p(0,2)} \quad (\forall u \in W(\mathrm{P2}, M, p))\,.$$

Note that by simple computation, it holds that $C(\mathrm{P2}, M, j) = 2^{M-j-1/p} C(\mathrm{P}, M, j)$ (See; Appendix). Next, we construct the function which attains the equality in (3.8) when $j = 2l$. Let $\widetilde{u}_0(x) = V(\mathrm{P2}, M, 2l; x)$ $(0 \le x \le 2)$. From Lemma 3.1, $\widetilde{u}_0 \in W(\mathrm{P2}, M, p)$. Substituting $\widetilde{u}_0$ into (3.8), from Theorem 1.1, we have the equality in (3.8). Let $\widetilde{\widetilde{u}}_0$ be the restriction of $\widetilde{u}_0$ on [0, 1]. From Lemma 3.1, we see that $\widetilde{\widetilde{u}}_0 \in W(\mathrm{N}, M, p)$ and attains the equality of the following inequality

$$\sup_{0\le y\le 1} \left|u^{(2l)}\right| \le 2^{1/p} C(\mathrm{P2}, M, 2l)\left\|u^{(M)}\right\|_{L^p(0,1)} = 2^{M-2l} C(\mathrm{P}, M, 2l)\left\|u^{(M)}\right\|_{L^p(0,1)}\,.$$

This proves Theorem 1.2. □

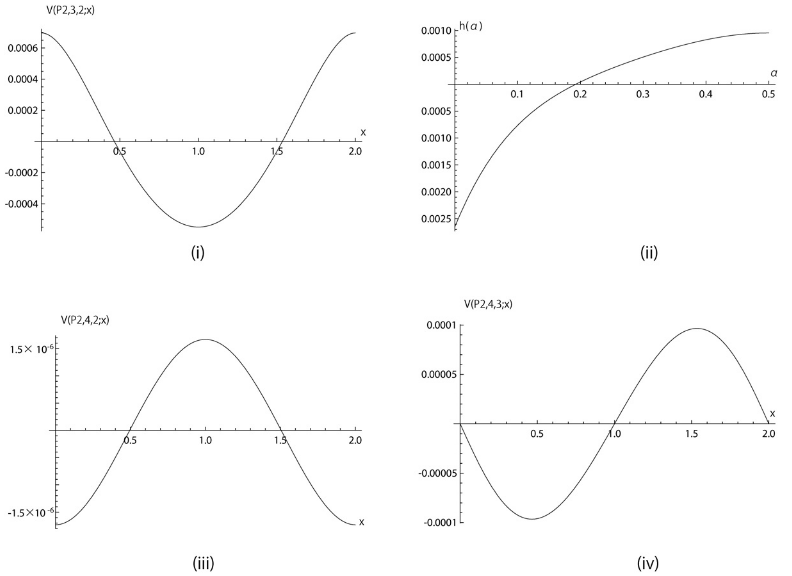

FIGURE 1. The graphs of (i) $V(\mathrm{P2}, 3, 2; x)$, (ii) $h(\alpha)$, (iii) $V(\mathrm{P2}, 4, 2; x)$ for $\alpha_1 = 0.193$ and (iv) $V(\mathrm{P2}, 4, 3; x)$ for $q = 3.2$.

Figure 1 shows the graphs of $V(\mathrm{P2}, M, j; x)$. From Figure (i) ($M = 3,\ j = 2,\ q = 3.2$) and (iii) ($M = 4,\ j = 2,\ q = 3.2$), we see that Neumann boundary condition is satisfied at both $x = 0$ and $x = 1$. Figure (ii) shows the graph of $h(\alpha)$ for $M = 4$, $k = m - l = 2 - 1 = 1$. From (ii), we see the root of $h(\alpha) = 0$ is $\alpha_1 = 0.193$. We used this value for drawing the figure (iii). Figure (iv) shows the graph of $V(\mathrm{P2}, 4, 3; x)$ for $q = 3.2$. From (iv), we see that Neumann boundary condition is not satisfied at both $x = 0$ and $x = 1$. Moreover, $x = 0.5$ and 1.5 are not critical point of $V(\mathrm{P2}, 4, 3; x)$. Thus we can not apply the proof of Theorem 1.2 for the case $M = 2m$ and $j = 2l + 1$. The case $M = 2m - 1$ and $j = 2l + 1$ also remains for the same reason. Therefore we need yet another method for these two cases. But we have not any at this time.

Appendix

For

$$\sup_{0 \le y \le 2} \left| u^{(j)}(y) \right| \le C(\mathrm{P2}, M, j) \left\| u^{(M)} \right\|_{L^p(0,2)} \quad (\forall u \in W(\mathrm{P2}, M, p)), \tag{3.9}$$

we put

$$u(x) = v(\xi)\Big|_{\xi = x/2} \quad (0 < x < 2,\ 0 < \xi < 1).$$

Differentiating this function i times with respect to x, we have

$$u^{(i)}(x) = 2^{-i} v^{(i)}(\xi) \quad (i = 0, 1, 2, \dots, M) .$$

Since $u \in W(\mathrm{P2}, M, p)$, we have

$$2^{-i} v^{(i)}(1) = u^{(i)}(2) = u^{(i)}(0) = 2^{-i} v^{(i)}(0) \quad (0 \le i \le M-1) ,$$

and

$$0 = \int_0^2 u(x)dx = 2\int_0^1 v(\xi)d\xi .$$

So we have $v \in W(\mathrm{P}, M, p)$. Moreover, from

$$\|u^{(M)}\|^p_{L^p(0,2)} = \int_0^2 |u^{(M)}(x)|^p dx = \int_0^1 2^{-Mp}|v^{(M)}(\xi)|^p 2d\xi = 2^{-Mp+1}\|v^{(M)}\|^p_{L^p(0,1)} ,$$

we have

$$\|u^{(M)}\|_{L^p(0,2)} = 2^{-M+1/p}\|v^{(M)}\|_{L^p(0,1)} .$$

Hence, for (3.9), we have

$$\begin{aligned} &\sup_{0\le\eta\le1} |2^{-j} v^{(j)}(\eta)| = \sup_{0\le y\le2} |u^{(j)}(y)| \\ &\le C(\mathrm{P2}, M, j)\|u^{(M)}\|_{L^p(0,2)} = C(\mathrm{P2}, M, j)2^{-M+1/p}\|v^{(M)}\|_{L^p(0,1)} . \end{aligned}$$

So we have

$$\sup_{0\le\eta\le1} |v^{(j)}(\eta)| \le 2^{-M+j+1/p} C(\mathrm{P2}, M, j)\|v^{(M)}\|_{L^p(0,1)}$$
$$(\forall v \in W(\mathrm{P}, M, p)) .$$

Since there is an element which attains the equality above,

$$2^{-M+j+1/p} C(\mathrm{P2}, M, j) = C(\mathrm{P}, M, j) .$$

Thus we have

$$C(\mathrm{P2}, M, j) = 2^{M-j-1/p} C(\mathrm{P}, M, j) .$$

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