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The Structure Theorem for the Cut Locus of a Certain Class of Cylinders of Revolution I

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Abstract. The aim of this paper is to determine the structure of the cut locus for a class of surfaces of revolution homeomorphic to a cylinder. Let M denote a cylinder of revolution which admits a reflective symmetry fixing a parallel called the equator of M. It will be proved that the cut locus of a point p of M is a subset of the union of the meridian and the parallel opposite to p respectively, if the Gaussian curvature of M is decreasing on each upper half meridian.

1. Introduction

It is a very difficult problem to determine the structure of the cut locus of a Riemannian manifold and it was difficult even for a quadric surface.

Since Elerath ([E]) succeeded in specifying the structure of the cut locus for paraboloids of revolution and (2-sheeted) hyperboloids of revolution, the structures of the cut locus for quadric surfaces of revolution have been studied. After his work, Sinclair and Tanaka ([ST]) determined the structure of the cut locus for a class of surfaces of revolution containing the ellipsoids. Notice that the structures of the cut locus for triaxial ellipsoids with unequal axes were also determined by Itoh and Kiyohara ([IK]).

On the structure of the cut locus for a cylinder of revolution $(R^1 \times S^1, dt^2 + m(t)^2 d\theta^2)$, Tsuji ([Ts]) first determined the cut locus of a point on the equator $t = 0$ if the cylinder is symmetric with respect to the equator and the Gaussian curvature is decreasing on the upper half meridian $t > 0, \theta = 0$. In 2003, Tamura ([Ta]) determined the structure of the cut locus by adding an assumption $m' \neq 0$ except $t = 0$. In this paper, we determine the structure of the cut locus without this assumption.

Here, let us review the notion of a cut point and the cut locus of a point. Let $\gamma : [0, a] \to M$ be a minimal geodesic segment in a complete Riemannian manifold M. The end point of $\gamma(a)$ is called a *cut point* of $\gamma(0)$ along γ, if any geodesic extension of γ is not minimal anymore. The *cut locus* C_p of a point p of M is by definition the set of the cut points along all minimal geodesic segments emanating from p.

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In this paper we will prove the following theorem.

MAIN THEOREM. *Let (M, ds^2) be a complete Riemannian manifold $R^1 \times S^1$ with a warped product metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$. Suppose that the warping function m is a positive-valued even function and the Gaussian curvature of M is decreasing along the half meridian $t^{-1}[0,\infty) \cap \theta^{-1}(0)$. If the Gaussian curvature of M is positive on $t = 0$, then the structure of the cut locus C_q of a point $q \in \theta^{-1}(0)$ in M is given as follows*:

1. *The cut locus C_q is the union of a subarc of the parallel $t = -t(q)$ opposite to q and the meridian opposite to q if $|t(q)| < t_0 := \sup\{t > 0 \mid m'(t) < 0\}$ and $\varphi(m(t(q))) < \pi$. More precisely,*

$$C_q = \theta^{-1}(\pi) \cup \Big(t^{-1}(-t(q)) \cap \theta^{-1}[\varphi(m(t(q))), 2\pi - \varphi(m(t(q)))]\Big).$$

2. *The cut locus C_q is the meridian $\theta^{-1}(\pi)$ opposite to q if $\varphi(m(t(q))) \geq \pi$ or if $|t(q)| \geq t_0$.*

Here, the function $\varphi(\nu)$ on $(\inf m, m(0))$ is defined as

$$\varphi(\nu) := 2\int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = 2\int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt\,,$$

where $\xi(\nu) := \min\{t > 0 \mid m(t) = \nu\}$. Notice that the point q is an arbitrarily given point if the coordinates (t, θ) are chosen so as to satisfy $\theta(q) = 0$.

REMARK 1.1. If the Gaussian curvature of a cylinder of revolution is nonpositive everywhere, then any geodesic has no conjugate point. Therefore, it is clear to see that the cut locus of a point on the manifold is the meridian opposite to the point.

2. Preliminaries

Let f be the solution of the differential equation

$$f'' + Kf = 0 \tag{2.1}$$

with initial conditions $f(0) = c$ and $f'(0) = 0$. Here c denotes a fixed positive number and $K : [0,\infty) \to R$ denotes a continuous function.

LEMMA 2.1. *If $K(0) > 0$ and $f'(t) \neq 0$ for any $t > 0$, then $f'(t) < 0$ on $(0,\infty)$. Furthermore, if $f > 0$ on $[0,\infty)$, then $K(t) < 0$ for some $t > 0$.*

PROOF. Since $f''(0) = -K(0)f(0) < 0$ by (2.1), $f'(t)$ is strictly decreasing on $(0,\delta)$ for some $\delta > 0$. This implies that $0 = f'(0) > f'(t)$ for any $t \in (0,\delta)$. Since $f' \neq 0$ on $[0,\infty)$, $f'(t) < 0$ on $(0,\infty)$. Furthermore, we assume that $f > 0$ on $[0,\infty)$. Supposing that $K \geq 0$ on $[0,\infty)$, we will get a contradiction. By (2.1),

$$f''(t) = -K(t)f(t) \leq 0$$

on $[0,\infty)$. Hence $f'(t)$ is decreasing on $[0,\infty)$. In particular, $0 = f'(0) > f'(\delta) \geq f'(t)$ for any $t \geq \delta$. This contradicts the assumption $f > 0$. □

LEMMA 2.2. *Suppose that* $K(0) > 0$ *and* $f > 0$ *on* $[0,\infty)$. *If* $f'(t) = 0$ *for some* $t > 0$ *and* K *is decreasing, then there exist a unique solution* $t = t_0 \in (0,\infty)$ *of* $f'(t) = 0$ *such that* $f'(t) < 0$ *on* $(0, t_0)$ *and* $f'(t) > 0$ *on* (t_0,∞) *and there exists* $t_1 \in (0, t_0)$ *satisfying* $K(t_1) = 0$. *Hence* $K \geq 0$ *on* $[0, t_1]$ *and* $K \leq 0$ *on* $[t_1,\infty)$.

PROOF. Let $a > 0$ denote the minimum positive solution $t = a$ of $f'(t) = 0$. Suppose that there exist another solution $b(> a)$ satisfying $f'(b) = 0$. By the mean value theorem, there exist $t_1 \in (0, a)$ and $s_1 \in (a, b)$ satisfying $f''(t_1) = f''(s_1) = 0$. Hence $K(t_1) = K(s_1) = 0$ by (2.1). Since K is decreasing, $K = 0$ on $[t_1, s_1]$. Therefore, by (2.1), $f''(t) = 0$ on $[t_1, s_1]$. In particular, $f'(a) = f'(t_1) = 0$. Since $0 < t_1 < a$, t_1 is a positive solution t of $f'(t) = 0$, which is less than a. This is a contradiction. Therefore, there exists a unique positive solution $t = t_0$ of $f'(t) = 0$. From the mean value theorem and (2.1), there exists $t_1 \in (0, t_0)$ satisfying $K(t_1) = 0$. Since $K(t)$ is decreasing, $K \geq 0$ on $[0, t_1]$ and $K \leq 0$ on $[t_1,\infty)$. Hence by (2.1), $f''(t) = -K(t)f(t) \geq 0$ on $[t_1,\infty)$ and $f'(t) \geq f'(t_0) = 0$ for any $t > t_0$. Since f' has a unique positive zero, $f' > 0$ on (t_0,∞). It is clear from the proof of Lemma 2.1 that $f' < 0$ on $(0, t_0)$. □

3. Review of the behavior of geodesics

From now on, M denotes a complete Riemannian manifold $R^1 \times S^1$ with a warped product Riemannian metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$. Let us review the behavior of a geodesic $\gamma(s) = (t(s), \theta(s))$ on the manifold M. For each unit speed geodesic $\gamma(s) = (t(s), \theta(s))$, there exists a constant ν satisfying

$$m(t(s))^2\theta'(s) = \nu\,. \tag{3.1}$$

Hence, if $\eta(s)$ denotes the angle made by the velocity vector $\gamma'(s)$ of the geodesic $\gamma(s)$ and the tangent vector $(\partial/\partial\theta)_{\gamma(s)}$, then

$$m(t(s))\cos\eta(s) = \nu \tag{3.2}$$

for any s. The constant ν is called the *Clairaut constant* of γ. The reader should refer to Chapter 7 in [SST] for the Clairaut relation. Since $\gamma(s)$ is unit speed,

$$t'(s)^2 + m(t(s))^2\theta'(s)^2 = 1 \tag{3.3}$$

holds. By (3.1) and (3.3), it follows that

$$t'(s) = \pm\frac{\sqrt{m(t(s))^2 - \nu^2}}{m(t(s))} \tag{3.4}$$

$$\theta(s_2) - \theta(s_1) = \varepsilon(t'(s))\int_{t(s_1)}^{t(s_2)} \frac{\nu}{m\sqrt{m^2 - \nu^2}}dt \tag{3.5}$$

holds, if $t'(s) \neq 0$ on (s_1, s_2) and $\varepsilon(t'(s))$ denotes the sign of $t'(s)$.

The length $L(\gamma)$ of a geodesic segment $\gamma(s) = (t(s), \theta(s))$, $s_1 \leq s \leq s_2$ is

$$L(\gamma) = \varepsilon(t'(s)) \int_{t(s_1)}^{t(s_2)} \frac{m(t)}{\sqrt{m(t)^2 - \nu^2}} dt \tag{3.6}$$

if $t'(s) \neq 0$ on (s_1, s_2).

From a direct computation, the Gaussian curvature G of M is given by

$$G(q) = -\frac{m''}{m}(t(q))$$

at each point $q \in M$. Since G is constant on $t^{-1}(a)$ for each $a \in R$, a smooth function K on R is defined by

$$K(u) := G(q)$$

for $q \in t^{-1}(u)$. Therefore m satisfies the following differential equation

$$m'' + Km = 0$$

with $m'(0) = 0$.

From now on, we assume that the Gaussian curvature G of M is positive on $t^{-1}(0)$, and $m(t) = m(-t)$ holds for any $t \in R$. Hence, M is symmetric with respect to the equator $t = 0$ and if K is decreasing on $[0, \infty)$, then by Lemma 2.2, $m'(t) < 0$ for all $t > 0$ or there exists a unique positive solution $t = t_0$ of $m'(t) = 0$ such that $m' < 0$ on $(0, t_0)$ and $m' > 0$ on (t_0, ∞). Furthermore, if the latter case happens, there exists $t_1 \in (0, t_0)$ such that $K \geq 0$ on $[0, t_1]$ and $K \leq 0$ on $[t_1, \infty)$.

For technical reasons, we treat both geodesics on M and its universal covering space $\pi : \widetilde{M} \to M$, where $\widetilde{M} := (R^1 \times R^1, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2)$.

Choose any point p on the equator $t = 0$. We may assume that $\theta(p) = 0$ without loss of generality. Let $\gamma : [0, \infty) \to M$ denote a geodesic emanating from $p = \gamma(0)$ with Clairaut constant $\nu \in (\inf m, m(0))$. Notice that γ is uniquely determined up to the reflection with respect to $t = 0$. The geodesic $\gamma(s) = (t(s), \theta(s))$ is tangent to the parallel $t = \xi(\nu)$ (if $(t \circ \gamma)'(0) > 0$) or $t = -\xi(\nu)$ (if $(t \circ \gamma)'(0) < 0$), where $\xi(\nu) > 0$ denotes the least positive solution of $m(\xi(\nu)) = \nu$, that is,

$$\xi(\nu) := \min\{u > 0 \mid m(u) = \nu\}.$$

After γ is tangent to the parallel $t = \xi(\nu)$ or $-\xi(\nu)$, γ intersects the equator $t = 0$ again. Thus, after $\tilde{\gamma}$ is tangent to the *parallel arc* $\tilde{t} = \xi(\nu)$ or $-\xi(\nu)$, $\tilde{\gamma}$ intersect $\tilde{t} = 0$ again. Here $\tilde{\gamma}$ denotes a geodesic on $\widetilde{M}$ satisfying $\gamma = \pi \circ \tilde{\gamma}$.

From (3.5), we obtain,

$$\tilde{\theta}(s_0) - \tilde{\theta}(0) = \int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = \int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt,$$

and

$$\tilde{\theta}(s_1) - \tilde{\theta}(s_0) = \int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = \int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt ,$$

where $s_0 := \min\{s > 0 \mid m(\tilde{t}(s)) = \nu\}$, $s_1 := \min\{s > 0 \mid \tilde{t}(s) = 0\}$.

By summing up the argument above, we have,

LEMMA 3.1. *Let* $\tilde{\gamma}(s) = (\tilde{t}(s), \tilde{\theta}(s))$ *denote a geodesic emanating from the point* $\tilde{p} := (\tilde{t}, \tilde{\theta})^{-1}(0, 0)$ *with Clairaut constant* $\nu \in (\inf m, m(0))$. *Then* $\tilde{\gamma}$ *intersects* $\tilde{t} = 0$ *again at the point* $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$. *Here*,

$$\varphi(\nu) := 2\int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = 2\int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt . \tag{3.7}$$

LEMMA 3.2. *The length* $l(\nu)$ *of the subarc* $(\tilde{t}(s), \tilde{\theta}(s)), 0 \le \tilde{\theta}(s) \le \varphi(\nu)$, *of* $\tilde{\gamma}(s)$ *is given by*

$$l(\nu) = 2\int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2 - \nu^2}} dt = 2\int_{-\xi(\nu)}^{0} \frac{\sqrt{m^2 - \nu^2}}{m} dt + \nu\varphi(\nu) , \tag{3.8}$$

and

$$\frac{\partial l}{\partial \nu}(\nu) = \nu\varphi'(\nu) . \tag{3.9}$$

PROOF. From (3.6), we obtain,

$$l(\nu) = 2\int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2 - \nu^2}} dt .$$

Since

$$\frac{m}{\sqrt{m^2 - \nu^2}} = \frac{\sqrt{m^2 - \nu^2}}{m} + \frac{\nu^2}{m\sqrt{m^2 - \nu^2}}$$

holds, we get

$$l(\nu) = 2\int_{-\xi(\nu)}^{0} \frac{\sqrt{m^2 - \nu^2}}{m} dt + 2\int_{-\xi(\nu)}^{0} \frac{\nu^2}{m\sqrt{m^2 - \nu^2}} dt .$$

Hence, by (3.7), we get (3.8). By differentiating $l(\nu)$ with respect to ν, we get,

$$l'(\nu) = 2\int_{-\xi(\nu)}^{0} \frac{\partial}{\partial \nu} \frac{\sqrt{m^2 - \nu^2}}{m} dt + \varphi(\nu) + \nu\varphi'(\nu) = \nu\varphi'(\nu) .$$

□

4. The decline of the function $\varphi(\nu)$

Let $\pi : \widetilde{M} = (R^1 \times R^1, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2) \to M$ denote the universal covering space of M. We choose an arbitrary point $\tilde{p}$ of $\tilde{t}^{-1}(-\infty, 0]$, and we denote the cut locus of $\tilde{p}$ by $C_{\tilde{p}}$. Before proving some lemmas on the cut locus, let us review the structure of the cut locus of $\widetilde{M}$. We refer to [ShT] or [SST] on the structure of the cut locus of a 2-dimensional complete Riemannian manifold.

It is known that the cut locus has a local tree structure. Since $\widetilde{M}$ is simply connected, the cut locus has no circle. If two cut points x and y are in a common connected component of the cut locus, then x and y are connected by a unique rectifiable arc in the cut locus.

Since $\widetilde{M}$ is homeomorphic to R^2, we may define a global sector at each cut point. For general surfaces, only local sectors are defined (see [ShT], or [SST]). A global *sector* at each cut point x of the point $\tilde{p}$ is by definition a connected component of $\widetilde{M} \setminus \Gamma_x$, where Γ_x denotes the set of all points lying on a minimal geodesic segment joining $\tilde{p}$ to x. Let $c : [0, a] \to C_{\tilde{p}}$ denote a rectifiable arc in the cut locus. Then for each cut point $c(t)$, $t \in (0, a)$, c bisects the sector at $c(t)$ containing $c[0, t)$ (respectively $c(t, a]$) . For each sector of the point $\tilde{p}$ on $\widetilde{M}$, there exists an end point of $C_{\tilde{p}}$, since $C_{\tilde{p}}$ has no circle. Here, a cut point q of $\tilde{p}$ is called an *end point* if q admits exactly one sector.

In this section, we assume that the Gaussian curvature G of M is increasing on the half meridian $t^{-1}(-\infty, 0] \cap \theta^{-1}(0)$ and that M has a reflective symmetry with respect to $t = 0$. Hence the Gaussian curvature of $\widetilde{M}$ is increasing on the lower half meridian $\tilde{t}^{-1}(-\infty, 0] \cap \tilde{\theta}^{-1}(0)$ and $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$.

LEMMA 4.1. *Suppose that there exists a cut point of the point $\tilde{p}$ in $\tilde{t}^{-1}(-\infty, 0)$. Then there exist two minimal geodesic segments α and β joining $\tilde{p}$ to a cut point y of $\tilde{p}$ such that the global sector $D(\alpha, \beta)$ bounded by α and β has an end point of $C_{\tilde{p}}$ and $D(\alpha, \beta) \subset \tilde{t}^{-1}(-\infty, 0)$.*

PROOF. Since the subset of cut points admitting at least two minimal geodesics is dense in the cut locus, the existence of two minimal geodesics α and β is clear (see [Bh]). Since $\widetilde{M}$ has a reflective symmetry with respective to $\tilde{t} = 0$, it is trivial that $D(\alpha, \beta) \subset \tilde{t}^{-1}(-\infty, 0)$. Let y denote the end point of α distinct from $\tilde{p}$. Since the proof is complete in the case where the cut point y is not an end point of the cut locus, we assume that y is an end point. Then, we get an arc c in the cut locus emanating from y. Any interior point y_1 on c is not an end point of the cut locus. It is clear that there exist two minimal geodesic segments joining $\tilde{p}$ and y_1 which bound a sector containing y as an end point of the cut locus. □

LEMMA 4.2. *For any unit speed minimal geodesic segment $\gamma : [0, L(\gamma)] \to \widetilde{M}$ joining $\tilde{p}$ to any end point x of $C_{\tilde{p}}$ in the domain $D(\alpha, \beta)$, x is conjugate to $\tilde{p}$ along γ and γ is shorter than α and β.*

PROOF. Note that for any end point x of the cut locus, the set of all minimal geodesic segments joining $\tilde{p}$ to x is connected. Therefore, x is conjugate to $\tilde{p}$ along any minimal geodesic segments joining $\tilde{p}$ to the end point of the cut locus. Let $\gamma : [0, L(\gamma)] \to \widetilde{M}$ denote any minimal geodesic segment $\tilde{p}$ to an end point x of $C_{\tilde{p}} \cap D(\alpha, \beta)$. We will prove that γ is shorter than α and β. It follows from Theorem B in [ShT] or [IT] that there exists a unit speed arc $c : [0, l] \to C_{\tilde{p}}$ joining the end point x to y, where y denotes the end point of α distinct from $\tilde{p}$. Since the function $d(\tilde{p}, c(\tau))$ is a Lipschitz function, it follows from Lemma 7.29 in [WZ] that the function is differentiable for almost all τ and

$$d(\tilde{p}, c(l)) - d(\tilde{p}, y) = \int_0^l \frac{d}{d\tau} d(\tilde{p}, c(\tau)) d\tau \tag{4.1}$$

holds. From the Clairaut relation (3.2), the inner angle $\theta(\tau)$ at $c(\tau)$ of the sector containing $c[0, \tau)$ is less than π. Hence, by the first variation formula, we get

$$\frac{d}{d\tau} d(\tilde{p}, c(\tau)) = \cos \frac{\theta(\tau)}{2} > 0$$

for almost all τ. Notice that for each $\tau \in (0, l)$, the curve c bisects the sector at $c(\tau)$ containing $c[0, \tau)$. Therefore, from (4.1),

$$L(\alpha) = L(\beta) = d(\tilde{p}, c(l)) > d(\tilde{p}, y) = L(\gamma) .$$

□

LEMMA 4.3. *Let q be a point on $\tilde{\theta}^{-1}(0)$ and u_0 any real number. Then $d(q, c(\theta))$ is strictly increasing on $[0, \infty)$. Here $c : [0, \infty) \to \widetilde{M}$ denotes $c(\theta) = (u_0, \theta)$ in the coordinates $(\tilde{t}, \tilde{\theta})$ and $d(\cdot, \cdot)$ denotes the Riemannian distance function on $\widetilde{M}$.*

PROOF. Choose any positive numbers $\theta_1 < \theta_2$. Let $\alpha_i, i = 1, 2$, denote minimal geodesic segments joining the point q to $c(\theta_i)$ respectively. Since $\theta_2 > \theta_1$, there exists an intersection $\alpha_2(t_2)$ of α_2 and the meridian $\tilde{\theta} = \theta_1$. The point $c(\theta_1)$ is the unique nearest point on $\tilde{t} = u_0$ from $\alpha_2(t_2)$. Hence,

$$d(\alpha_2(t_2), c(\theta_1)) < d(\alpha_2(t_2), c(\theta_2)) .$$

Therefore, by the triangle inequality, we get

$$\begin{aligned} d(q, c(\theta_2)) = d(q, \alpha_2(t_2)) + d(\alpha_2(t_2), c(\theta_2)) > d(q, \alpha_2(t_2)) \\ + d(\alpha_2(t_2), c(\theta_1)) \geq d(q, c(\theta_1)) . \end{aligned}$$

This implies that $d(q, c(\theta))$ is strictly increasing on $[0, \infty)$. □

LEMMA 4.4. *Suppose that $\gamma : [0, L(\gamma)] \to \widetilde{M}$ is a minimal geodesic segment joining $\tilde{p}$ to an end point $x \in C_{\tilde{p}}$, which is a point in the sector $D(\alpha, \beta)$ bounded by two minimal geodesic segments α and β emanating from $\tilde{p}$. Then, for any $s \in [0, L(\gamma)]$, $\tilde{t}(\alpha(s)) \geq \tilde{t}(\gamma(s)) \geq \tilde{t}(\beta(s))$ holds. Here we assume that*

$$\angle(\alpha'(0), (\partial/\partial\tilde{t})_{\tilde{p}}) < \angle(\gamma'(0), (\partial/\partial\tilde{t})_{\tilde{p}}) < \angle(\beta'(0), (\partial/\partial\tilde{t})_{\tilde{p}}) ,$$

where $\angle(\cdot,\cdot)$ *denotes the angle made by two tangent vectors.*

PROOF. From (3.4), it follows that for sufficiently small $s > 0$, $\tilde{t}(\alpha(s)) > \tilde{t}(\gamma(s)) > \tilde{t}(\beta(s))$ holds. Hence the set $A := \{s \in (0, L(\gamma)) \mid \tilde{t}(\alpha(s)) > \tilde{t}(\gamma(s)) > \tilde{t}(\beta(s))\}$ is a nonempty open subset of $(0, L(\gamma))$. Let $(0, s_0)$ denote the connected component of A. It is sufficient to prove that $s_0 = L(\gamma)$. Suppose that $s_0 < L(\gamma)$. Thus, $\tilde{t}(\alpha(s_0)) = \tilde{t}(\gamma(s_0))$ or $\tilde{t}(\gamma(s_0)) = \tilde{t}(\beta(s_0))$ holds, since A is open. By applying Lemma 4.3 for $u_0 := \tilde{t}(\alpha(s_0))$ and $\tilde{t}(\beta(s_0))$, we get $\alpha(s_0) = \gamma(s_0)$ or $\gamma(s_0) = \beta(s_0)$, which is a contradiction. □

LEMMA 4.5. *For any point* $\tilde{p} \in \tilde{t}^{-1}(-\infty, 0]$, *there does not exist a cut point of* $\tilde{p}$ *in* $\tilde{t}^{-1}(-\infty, 0)$. *In particular, the cut locus of* $\tilde{p}$ *is a subset of* $\tilde{t}^{-1}(0)$ *if* $\tilde{t}(\tilde{p}) = 0$. *This implies that the cut locus* C_p *of a point* $p \in t^{-1}(0)$ *is a subset of* $\theta^{-1}(\pi) \cup t^{-1}(0)$. *Here the coordinates* (t, θ) *are chosen so as to satisfy* $\theta(p) = 0$.

PROOF. Suppose that there exist a cut point of $\tilde{p}$ in $\tilde{t}^{-1}(-\infty, 0)$. By Lemma 4.1, there exist two minimal geodesic segments α and β joining a cut point y of $\tilde{p}$ which bound a sector $D(\alpha, \beta)$ containing an end point x of $C_{\tilde{p}}$. Let $\gamma : [0, L(\gamma)] \to \widetilde{M}$ be a unit speed geodesic segment joining $\tilde{p}$ to the end point x. From Lemmas 4.1 and 4.4, it follows that for any $s \in [0, L(\gamma)]$,

$$0 \geq \tilde{t}(\alpha(s)) \geq \tilde{t}(\gamma(s)) \geq \tilde{t}(\beta(s))$$

holds. Since the Gaussian curvature G is increasing on each lower half meridian, we obtain

$$G(\alpha(s)) \geq G(\gamma(s)) \geq G(\beta(s))\,.$$

By applying the Rauch comparison theorem for the pair of geodesic segments $\alpha|_{[0,L(\gamma)]}$ and γ, $\tilde{p}$ admits a conjugate point on $\alpha|_{[0,L(\gamma)]}$ along α.

This contradicts the fact that α is minimal. Since $\widetilde{M}$ is symmetric with respect to $\tilde{t} = 0$, the cut locus of $\tilde{p}$ is a subset of $\tilde{t}^{-1}(0)$, if $\tilde{t}(\tilde{p}) = 0$. This implies that $C_p \subset \theta^{-1}(\pi) \cup t^{-1}(0)$ for the point $p = t^{-1}(0) \cap \theta^{-1}(0)$. □

PROPOSITION 4.6. *Let* M *be a complete Riemannian manifold* $R^1 \times S^1$ *with a warped product metric* $ds^2 = dt^2 + m(t)^2 d\theta^2$ *of the real line* (R^1, dt^2) *and the unit circle* $(S^1, d\theta^2)$. *Here the warping function* $m : R \to (0, \infty)$ *is a smooth even function. If the Gaussian curvature is positive on the equator and decreasing on the upper half meridian* $t^{-1}(0, \infty) \cap \theta^{-1}(0)$, *then the function* $\varphi(\nu)$ *is decreasing on* $(\inf m, m(0))$.

PROOF. Let $\widetilde{M} := (R^1 \times R^1, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2)$ denote the universal covering space of M. Choose any point $\tilde{p}$ on $\tilde{t}^{-1}(0)$. For each $\nu \in (\inf m, m(0))$, let $\alpha_\nu : [0, \infty) \to \widetilde{M}$ denote the geodesic emanating from the point $\tilde{p} = \alpha_\nu(0)$ with Clairaut constant ν and with $(\tilde{t} \circ \alpha_\nu)'(0) < 0$. From the Clairaut relation, we get $\angle((\partial/\partial\tilde{\theta})_{\tilde{p}}, \alpha'_\nu(0)) = \cos^{-1} \nu/m(0)$. Choose any $\nu_1 < \nu_2$ with $\nu_1, \nu_2 \in (\inf m, m(0))$. Since

$$\cos^{-1}\frac{\nu_2}{m(0)} < \cos^{-1}\frac{\nu_1}{m(0)}\,,$$

it follows from Lemma 4.5 that α_{ν_1} does not cross the domain bounded by the subarc of α_{ν_2} and $\tilde{t}^{-1}(0) \cap \tilde{\theta}^{-1}[\tilde{\theta}(\tilde{p}), \tilde{\theta}(\tilde{p}) + \varphi(\nu_2)]$. This implies that $\varphi(\nu_1) \geq \varphi(\nu_2)$. Therefore, $\varphi(\nu)$ is decreasing on $(\inf m, m(0))$. □

5. The cut locus of a point on $\widetilde{M}$

Choose any point q on $\widetilde{M}$ with $-t_0 < \tilde{t}(q) < 0$, where $t_0 := \sup\{\, t > 0 \mid m'(t) < 0\}$. Without loss of generality, we may assume that $\tilde{\theta}(q) = 0$. We consider two geodesics α_ν and β_ν emanating from the point $q = \alpha_\nu(0) = \beta_\nu(0)$ with Clairaut constant $\nu > 0$. Here we assume that

$$\angle((\partial/\partial\tilde{t})_q, \alpha'_\nu(0)) > \angle((\partial/\partial\tilde{t})_q, \beta'_\nu(0)) .$$

LEMMA 5.1. *The two geodesics α_ν and β_ν intersect again at the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$ if $\nu \in (\inf m, m(0))$, where $u := -\tilde{t}(q)$.*

PROOF. Suppose that $\nu \in (\inf m, m(0))$. Since α_ν is tangent to the parallel arc $\tilde{t} = -\xi(\nu)$, it follows from (3.5) that

$$\tilde{\theta}(\alpha_\nu(s_1)) - \tilde{\theta}(\alpha_\nu(0)) = \int_{-\xi(\nu)}^{-u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt ,$$

where $s_1 := \min\{s > 0 \mid \tilde{t}(\alpha_\nu(s)) = -\xi(\nu)\}$, and

$$\tilde{\theta}(\alpha_\nu(s_2)) - \tilde{\theta}(\alpha_\nu(s_1)) = \int_{-\xi(\nu)}^{u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt ,$$

where $s_2 := \min\{s > 0 \mid \tilde{t}(\alpha_\nu(s)) = u\}$. Hence, we obtain,

$$\tilde{\theta}(\alpha_\nu(s_2)) - \tilde{\theta}(\alpha_\nu(0)) = \int_{-\xi(\nu)}^{u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt + \int_{-\xi(\nu)}^{-u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt . \tag{5.1}$$

Since m is an even function,

$$\int_{-\xi(\nu)}^{u} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = \int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt + \int_{-u}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt$$

holds. Therefore, by (5.1),

$$\tilde{\theta}(\alpha_\nu(s_2)) - \tilde{\theta}(\alpha_\nu(0)) = 2\int_{-\xi(\nu)}^{0} \frac{\nu}{\sqrt{m^2 - \nu^2}} dt = \varphi(\nu) .$$

This implies that α_ν passes through the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$. On the other hand, after β_ν is tangent to $\tilde{t} = \xi(\nu)$ at $\beta_\nu(s_1^+)$, where $s_1^+ := \min\{s > 0 \mid \tilde{t}(\beta_\nu(s)) = \xi(\nu)\}$, the geodesic intersects $\tilde{t} = u$ again at $\beta_\nu(s_2^+)$, where $s_2^+ := \min\{s > s_1^+ \mid \tilde{t}(\beta_\nu(s)) = u\}$. By the similar computation as above, we get

$$\tilde{\theta}(\beta_\nu(s_2^+)) - \tilde{\theta}(\beta_\nu(0)) = \varphi(\nu)\,.$$

This implies that α_ν and β_ν pass through the common point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$. □

LEMMA 5.2. *The two geodesic segments $\alpha_\nu|_{[0,s_2]}$ and $\beta_\nu|_{[0,s_2^+]}$ have the same length and its length equals $l(\nu)$, which is defined in Lemma 3.2. In particular, $s_2 = s_2^+$. Here, s_2 and s_2^+ denote the numbers defined in the proof of Lemma 5.1.*

PROOF. From (3.6), we have

$$L(\alpha_\nu|_{[0,s_1]}) = \int_{-\xi(\nu)}^{-u} \frac{m}{\sqrt{m^2 - \nu^2}} dt\,, \tag{5.2}$$

and

$$L(\alpha_\nu|_{[s_1,s_2]}) = \int_{-\xi(\nu)}^{u} \frac{m}{\sqrt{m^2 - \nu^2}} dt = \int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2 - \nu^2}} dt + \int_{0}^{u} \frac{m}{\sqrt{m^2 - \nu^2}} dt\,,$$

where s_1 denotes the number defined in the proof of Lemma 5.1. Since m is even

$$L(\alpha_\nu|_{[s_1,s_2]}) = \int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2 - \nu^2}} dt + \int_{-u}^{0} \frac{m}{\sqrt{m^2 - \nu^2}} dt\,. \tag{5.3}$$

Therefore, we get, by (3.8), (5.2) and (5.3),

$$L(\alpha_\nu|_{[0,s_2]}) = 2\int_{-\xi(\nu)}^{0} \frac{m}{\sqrt{m^2 - \nu^2}} dt = l(\nu)\,.$$

Analogously we have,

$$L(\beta_\nu|_{[0,s_2^+]}) = l(\nu)\,.$$

□

LEMMA 5.3. *Let q be a point on $\widetilde{M}$ with $|\tilde{t}(q)| \in (0, t_0)$. Then, for any $\nu \in (\inf m, m(u)]$, where $u = -\tilde{t}(q)$, $\alpha_\nu|_{[0,s_2(\nu)]}$ and $\beta_\nu|_{[0,s_2(\nu)]}$ are minimal geodesic segments joining q to the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \tilde{\theta}(q) + \varphi(\nu))$, and in particular, $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq \varphi(m(u)) + \tilde{\theta}(q)\}$ is a subset of the cut locus of the point q. Here, $s_2(\nu) := \min\{s > 0 \mid \tilde{t}(\alpha_\nu(s)) = u\}$ for each $\nu \in (\inf m, m(0))$.*

PROOF. Without loss of generality, we may assume that $\tilde{\theta}(q) = 0$. We will prove that $\alpha_\nu|_{[0,s_2(\nu)]}$ is a minimal geodesic segment joining q to the point $\alpha_\nu(s_2(\nu)) = (\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$. Suppose that $\alpha_{\nu_0}|_{[0,s_2(\nu_0)]}$ is not minimal for some $\nu_0 \in (\inf m, m(u)]$. Here we assume that ν_0 is the minimum solution $\nu = \nu_0$ of $\varphi(\nu) = \varphi(\nu_0)$.

Let $\alpha : [0, d(q, x)] \to M$ be a minimal geodesic segment joining q to $x := \alpha_{\nu_0}(s_2(\nu_0)) = (\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu_0))$. Hence, $\varphi(\nu_1) = \varphi(\nu_0) = \tilde{\theta}(x)$ and α equals $\alpha_{\nu_1}|_{[0,s_2(\nu_1)]}$ or $\beta_{\nu_1}|_{[0,s_2(\nu_1)]}$, where $\nu_1 \in (\inf m, m(0))$ denotes the Clairaut constant of α. By Proposition 4.6, $\varphi(\nu) = \varphi(\nu_0)$ for any $\nu \in [\nu_0, \nu_1]$. Hence, by Lemmas 3.2 and 5.2 we get,

$$s_2(\nu_1) = L(\alpha) = L(\alpha_{\nu_1}|_{[0,s_2(\nu_1)]}) = L(\alpha_{\nu_0}|_{[0,s_2(\nu_0)]}) = s_2(\nu_0)\,.$$

This implies that $\alpha_{\nu_0}|_{[0,s_2(\nu_0)]}$ is minimal, which is a contradiction, since we assumed that $\alpha_{\nu_0}|_{[0,s_2(\nu_0)]}$ is not minimal. Therefore, by Lemma 5.2, for any $\nu \in (\inf m, m(u)]$, the geodesic segments $\alpha_\nu|_{[0,s_2(\nu)]}$ and $\beta_\nu|_{[0,s_2(\nu)]}$ are minimal geodesic segments joining q to the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu)) = \alpha_\nu(s_2(\nu))$. In particular, the point $\alpha_\nu(s_2(\nu)) = \beta_\nu(s_2(\nu))$ is a cut point of q. □

PROPOSITION 5.4. *The cut locus of the point q in Lemma 5.3 equals the set*

$$\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}\,.$$

Here the coordinates $(\tilde{t}, \tilde{\theta})$ are chosen so as to satisfy $\tilde{\theta}(q) = 0$.

PROOF. By Lemma 5.3, geodesic segments $\alpha_\nu|_{[0,s_2(\nu)]}$ and $\beta_\nu|_{[0,s_2(\nu)]}$ are minimal geodesic segments for any $\nu \in (\inf m, m(u)]$. Hence their limit geodesics $\alpha^- := \alpha_{\inf m}$ and $\beta^+ := \beta_{\inf m}$ are rays, that is, any their subarcs are minimal.

Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{\theta} = 0$, it is trivial from Lemma 5.3 that the set $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}$ is a subset of the cut locus of q. Suppose that there exists a cut point $y \notin \{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}$. Without loss of generality, we may assume that $\tilde{\theta}(y) > 0 = \tilde{\theta}(q)$ and $\tilde{t}(q) = -u < 0$. From Lemma 4.5, $\tilde{t}(y) > 0$ and y is not a point in the unbounded domain cut off by two rays α^- and β^+, and hence the point lies in the domain D^+ cut off by β^+ and the submeridian $\tilde{t} > -u, \tilde{\theta} = \tilde{\theta}(q) = 0$. Since the cut locus of C_q has a tree structure, there exists an end point x of the cut locus in the D^+. Hence, x is conjugate to q for any minimal geodesic segment γ joining q to x. Since such a minimal geodesic γ runs in the domain D^+, the Clairaut constant of the segment is positive and less than $\inf m$. From the Clairaut relation (3.2), any geodesic cannot be tangent to any parallel arc $\tilde{t} = c$, if the Clairaut constant is positive and less than $\inf m$. From Corollary 7.2.1 in [SST], γ has no conjugate point of q, which is a contradiction. □

LEMMA 5.5. *Let q be a point on $\widetilde{M}$ with $|\tilde{t}(q)| \geq t_0$. Then the cut locus of q is empty.*

PROOF. Suppose that the cut locus of a point q with $|\tilde{t}(q)| \geq t_0$ is nonempty. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$, we may assume that $\tilde{t}(q) \leq -t_0$. Hence by Lemma 4.5, there exists an end point x of the cut locus C_q in $\tilde{t}^{-1}(0, \infty)$. Let $\gamma : [0, d(q, x)] \to \widetilde{M}$ denote a minimal geodesic segment joining q to x. Then x is conjugate to q along γ, since x is an end point of C_q. Since $\tilde{\theta}(x) > 0 = \tilde{\theta}(q)$, the Clairaut constant ν of γ is positive, by (3.1). Moreover, from the Clairaut relation (3.2), the Clairaut constant ν

is less than $\inf m = m(t_0)$, since γ intersects $\tilde{t} = -t_0$. Therefore, γ cannot be tangent to any parallel arc $\tilde{t} = c$. From Corollary 7.2.1 in [SST], γ has no conjugate point of q, which is a contradiction. □

Now our Main theorem is clear from Proposition 5.4 and Lemma 5.5.

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