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Weighted Norm Inequalities for Spectral Multipliers without Gaussian Estimates

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Abstract. Let L be a nonnegative self-adjoint operator on $L^2(\mathbf{R}^n)$ satisfying the full off-diagonal estimates $L^{q_0} - L^2$ for some $q_0 \in [1, 2)$. In this paper, we study the sharp weighted L^p estimates for the spectral multipliers of the operator L and their commutators with BMO functions b. As an application, we study the weighted norm inequalities for spectral multipliers of Schrödinger operators with negative potentials.

1. Introduction

Suppose that L is a nonnegative self-adjoint operator on $L^2(\mathbf{R}^n)$. Let $E(\lambda)$ be the spectral resolution of L. By the spectral theorem, for any bounded Borel function $F : [0, \infty) \to \mathbf{C}$, one can define the operator

$$F(L) = \int_0^\infty F(\lambda) dE(\lambda) ,$$

which is bounded on $L^2(\mathbf{R}^n)$.

The problem concerning the boundedness of $F(L)$ has attracted a lot of attention and has been studied by many authors, see for example [1, 7, 8, 21, 19, 18, 13, 14] and the references therein. In most of these papers, the Gaussian upper bound condition plays an essential role, see for example [1, 13, 14]. Recall that the semigroup $\{e^{-tL}\}_{t>0}$ generated by L has the kernels $p_t(x, y)$ satisfying the Gaussian upper bounds if there exist $c, C > 0$ so that

$$(1) \qquad |p_t(x, y)| \leq \frac{C}{t^{n/2}} \exp\left(- c\frac{|x - y|^2}{t}\right)$$

for all $t > 0$ and $x, y \in \mathbf{R}^n$. It was proved in [14] that if the bounded Borel function $F : [0, \infty) \to \mathbf{C}$ satisfies the following condition for some $s > n/2$

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$$\sup_{t>0} \|\eta\delta_t F\|_{W_s^\infty} < \infty$$

where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W_s^p} = \|(I - d^2/dx^2)^{s/2} F\|_{L^p}$ and η is an auxiliary non-zero cut-off function such that $\eta \in C_c^\infty(\mathbf{R}_+)$, then the spectral multiplier $F(L)$ is of weak type $(1, 1)$ and hence by duality arguments, $F(L)$ is bounded on L^p for all $p \in (1, \infty)$. However, there are many important operators L which do not satisfy (1). It is natural to raise a question of the boundedness of the spectral multipliers $F(L)$ without the Gaussian upper bound condition (1). In [7], S. Blunck replaced the pointwise kernel bounds (1) by the $L^p - L^q$ estimates and obtained the boundedness of $F(L)$ on L^p for an appropriate range of p whenever the bounded Borel function F satisfies $\sup_{t>0} \|\eta\delta_t F\|_{W_s^2} < \infty$ with $s > n/2 + 1/2$. Under the similar condition $\sup_{t>0} \|\eta\delta_t F\|_{W_s^\infty} < \infty$ with $s > n/2$, it was proved in [15] that the spectral multiplier $F(L)$ is bounded on H_L^p which is the Hardy space associated to the operator L. Moreover, it was also proved that if the semigroup e^{-tL} satisfies the $L^{q_0} - L^2$ off-diagonal estimates for some $q_0 \in [1, 2)$, then $F(L)$ is bounded on $L^p(w)$ for $2 < p < q_0'$ and $w \in A_{p/2} \cap RH_{(q_0'/p)'}$ and hence by duality, $F(L)$ is bounded on $L^p(w)$ for $q_0 < p < 2$ and $w \in A_{p/q_0} \cap RH_{(2/p)'}$.

The main aim of this paper is to study the sharp weighted estimates for spectral multipliers $F(L)$ and the commutators $[b, F(L)]$ of $F(L)$ with BMO functions b under the condition that L generates the semigroup e^{-tL} satisfying the $L^{q_0} - L^2$ full off-diagonal estimates (see Section 2 for precise definition). It is important to note that the pointwise Gaussian estimates are not assumed in this paper. Precisely, we prove the following result.

THEOREM 1.1. *Let L be a nonnegative self-adjoint operator on L^2 satisfying the $L^{q_0} - L^2$ full off-diagonal estimates for some $q_0 \in [1, 2)$. Set $r_0 = \max(q_0, \frac{n}{s})$. If a bounded Borel function $F : [0, \infty) \to \mathbf{C}$ satisfies the following condition for some $s > n/2$*

$$\sup_{t>0} \|\eta\delta_t F\|_{W_s^\infty} < \infty$$

where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W_s^p} = \|(I - d^2/dx^2)^{s/2} F\|_{L^p}$ and η is an auxiliary non-zero cut-off function such that $\eta \in C_c^\infty(\mathbf{R}_+)$, then (a) *$F(L)$ is bounded on $L^p(w)$ for all $r_0 < p < q_0'$ and $w \in A_{p/r_0} \cap RH_{(q_0'/r_0)'}$;* (b) *moreover, for $b \in BMO(\mathbf{R}^n)$, the commutator $[b, F(L)]$ is also bounded on $L^p(w)$ for all $r_0 < p < q_0'$ and $w \in A_{p/r_0} \cap RH_{(q_0'/r_0)'}$. Hence by duality $F(L)$ and the commutators $[b, F(L)]$ are bounded on $L^p(w)$ for $q_0 < p \le r_0$ and $w \in A_{p/q_0} \cap RH_{(r_0'/p)'}$.*

Note that since $A_{p/2} \cap RH_{(q_0/p)'} \subset A_{p/r_0} \cap RH_{(q_0'/p)'}$ for $p > 2$, $A_{p/q_0} \cap RH_{(2/p)'} \subset A_{p/r_0} \cap RH_{(q_0'/p)'}$ for $r_0 < p < 2$, and $A_{p/q_0} \cap RH_{(2/p)'} \subset A_{p/q_0} \cap RH_{(r_0'/p)'}$ for $r_0 < p < 2$ and $q_0 < p \le r_0$, the results in Theorem 1.1 are better than those in [15]. Moreover, since $r_0 < 2$, we also obtain the weighted estimates for the spectral multipliers $F(L)$ and their commutators $[b, F(L)]$ with BMO functions b on the weighted L^p spaces when $p = 2$. It seems that the obtained results in [15] did not tell us the weighted estimate of $F(L)$ on the weighted L^2 spaces.

Moreover, under the Gaussian upper bound conditions (1) and the weaker condition on F, Theorem 1.1 gives the same conclusion as [6, Theorem 1.3]. In comparison with the results in [16], it is important to note that in the Euclidian setting, if the condition (1) holds, the obtained results in Theorem 1.1 are in line with those in [16, Theorem 3.1]. Therefore, Theorem 1.1 can be considered to be an extension to [16, Theorem 3.1] and [6, Theorem 1.3] in some sense. More importantly, the results on the commutators $[b, F(L)]$ with BMO functions b are new (even for the unweighted case).

The outline of this paper is as follows. In section 2, we recall some basic properties of Muckenhoupt weights and a criterion on weighted estimates for singular integrals in [4]. The proof of Theorem 1.1 will be given in Section 3. Finally, we give an application to study the weighted norm inequalities for spectral multipliers of Schrödinger operators with negative potentials.

2. Muckenhoupt weights and weighted estimates for singular integrals

2.1. Muckenhoupt weights. We now recall the definition of Muckenhoupt weights and their basic properties. For details, we refer to [12].

Throughout this article, we will often just use B for $B(x_B, r_B) := \{x : |x - x_B| \le r_B\}$. Also given $\lambda > 0$, we will write λB for the λ-dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset \mathbf{R}^n$ we set

$$S_0(B) = B \quad \text{and } S_j(B) = 2^j B \backslash 2^{j-1} B \quad \text{for } j \in \mathbf{N}\,.$$

We shall denote $w(E) := \int_E w(x)dx$ for any measurable set $E \subset \mathbf{R}^n$. For $1 \le p \le \infty$ let p' be the conjugate exponent of p, i.e. $1/p + 1/p' = 1$.

We first introduce some notation. We use the notation

$$⨍_B h(x)dx = \frac{1}{|B|} \int_B h(x)dx\,.$$

A weight w is a non-negative locally integrable function on $\mathbf{R}^n$. We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset \mathbf{R}^n$,

$$\left(⨍_B w(x)dx\right)\left(⨍_B w^{-1/(p-1)}(x)dx\right)^{p-1} \le C\,.$$

For $p = 1$, we say that $w \in A_1$ if there is a constant C such that for every ball $B \subset \mathbf{R}^n$,

$$⨍_B w(y)dy \le Cw(x) \quad \text{for a.e. } x \in B\,.$$

We set $A_\infty = \cup_{p \ge 1} A_p$.

The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for any ball $B \subset \mathbf{R}^n$,

$$\left(\fint_B w^q(x)dx\right)^{1/q} \leq C \fint_B w(x)dx\,.$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, there is a constant C such that for any ball $B \subset \mathbf{R}^n$,

$$w(x) \leq C \fint_B w(y)dy \quad \text{for a.e. } x \in B\,.$$

Let $w \in A_\infty$. For $1 \leq p < \infty$, the weighted spaces $L^p(w)$ can be defined by

$$\left\{f : \int_{\mathbf{R}^n} |f(x)|^p w(x)dx < \infty\right\}$$

with the norm

$$\|f\|_{L^p(w)} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)dx\right)^{1/p}.$$

We sum up some of the properties of A_p classes in the following results, see [12].

LEMMA 2.1. *The following properties hold*:

(i) $A_1 \subset A_p \subset A_q$ *for* $1 < p \leq q < \infty$.
(ii) $RH_\infty \subset RH_q \subset RH_p$ *for* $1 < p \leq q < \infty$.
(iii) *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $1 < q < p$ *such that* $w \in A_q$.
(iv) *If* $w \in RH_q$, $1 < q < \infty$, *then there exists* $q < p < \infty$ *such that* $w \in RH_p$.
(v) $A_\infty = \cup_{1 \leq p < \infty} A_p = \cup_{1 < q \leq \infty} RH_q$.

2.2. Weighted norm inequalities for singular integrals.

THEOREM 2.2. *Let* $1 < p_0 < q_0 \leq \infty$. *Let* T *be a bounded sublinear operator on* $L^{p_0}(\mathbf{R}^n)$, *Let* $\{A_r\}_{r>0}$ *be a family of operators acting on* $L^{p_0}(\mathbf{R}^n)$. *Assume that*

$$\left(\fint_B \left|T(I - A_{r_B})f\right|^{p_0}dx\right)^{1/p_0} \leq C\sum_j \alpha_j\left(\fint_{2^jB} |f|^{p_0}dx\right)^{1/p_0} \tag{2}$$

and

$$\left(\fint_B \left|TA_{r_B}f\right|^{q_0}dx\right)^{1/q_0} \leq C\sum_j \alpha_j\left(\fint_{2^jB} |Tf|^{p_0}dx\right)^{1/p_0} \tag{3}$$

for all $f \in L_c^\infty(\mathbf{R}^n)$, *and all balls* B *with radius* r_B.

If $\sum_j j\alpha_j < \infty$, *then for all* $p_0 < p < q_0$ *and* $w \in A_{p/p_0} \cap RH_{(q_0/p)'}$, *there exists a constant* C *such that*

$$\|Tf\|_{L^p(w)} \le C\|f\|_{L^p(w)}\,. \tag{4}$$

and

$$\|[b,T]f\|_{L^p(w)} \le C\|b\|_{\mathrm{BMO}}\|f\|_{L^p(w)} \tag{5}$$

for all $b \in$ BMO.

PROOF. The proof of this theorem is just a combination of the arguments in Theorems 3.7 and 3.16 in [4] and we omit details here. □

3. Proof of Theorem 1.1

We recall the definition and some basic properties of $L^p - L^q$ full off-diagonal estimates.

DEFINITION 3.1 ([5]). Let $1 \le p \le q \le \infty$. We say that the family $\{T_t\}_{t>0}$ of sublinear operators satisfies $L^p - L^q$ full off-diagonal estimates, in short $T_t \in \mathcal{F}(L^p - L^q)$, if there exists some $c > 0$, for all closed sets E and F, all f with supp $f \subset E$ and all $t > 0$ so that

$$\|T_t f\|_{L^q(F)} \le ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\exp\Big(-c\frac{d^2(E,F)}{t}\Big)\|f\|_{L^p(E)}\,. \tag{6}$$

Let us summarize some basic properties concerning the classes $\mathcal{F}(L^p - L^q)$, see [5].

(i) For $p \le p_1 \le q_1 \le q$, $\mathcal{F}(L^{p_1} - L^{q_1}) \subset \mathcal{F}(L^p - L^q)$.

(ii) $T_t \in \mathcal{F}(L^1 - L^\infty)$ if and only if the associated kernel $p_t(x,y)$ of T_t satisfies the Gaussian upper bound, that is, there exist positive constants c and C so that

$$|p_t(x,y)| \le \frac{C}{t^{n/2}}\exp\Big(-c\frac{|x-y|^2}{t}\Big)$$

for all $x, y \in \mathbf{R}^n$ and $t > 0$.

(iii) If $\{T_t\}_{t>0}$ is a family of linear operators, then

$$T_t \in \mathcal{F}(L^p - L^q) \Longleftrightarrow T_t^* \in \mathcal{F}(L^{q'} - L^{p'})\,.$$

(iv) If $S_t \in \mathcal{F}(L^p - L^r)$ and $T_t \in \mathcal{F}(L^r - L^q)$ for $p \le r \le q$, then $T_t \circ S_t \in \mathcal{F}(L^p - L^q)$.

Full off-diagonal estimates appear when dealing with semigroups of second order elliptic operators (see for example [20, 3]) or semigroups of Shrödinger operators with real potentials [2]. The most studied case is when $p = 1$ and $q = \infty$ which means that the kernel of T_t has pointwise Gaussian upper bounds (1).

Let L be a nonnegative self-adjoint operator. Assume that the semigroup e^{-tL} satisfies $L^{q_0} - L^2$ full off-diagonal estimates for some $q_0 \in [1,2)$. By (iii), $e^{-tL} \in \mathcal{F}(L^2 - L^{q_0'})$.

Since $e^{-tL} = e^{-\frac{t}{2}L} \circ e^{-\frac{t}{2}L}$, by (iv) we have $e^{-tL} \in \mathcal{F}(L^{q_0} - L^{q_0'})$. In this paper, we will work with operators whose associated semigroup $e^{-tL} \in \mathcal{F}(L^{q_0} - L^{q_0'})$ for some $q_0 \in [1, 2)$.

To prove Theorem 1.1, we need some auxiliary lemmas.

LEMMA 3.2. *Let $p \in (q_0, 2)$ and F be a bounded Borel function with* $\operatorname{supp} F \subset [0, R]$. *There exists a constant $C > 0$ such that*

$$\|F(\sqrt{L})f\|_{L^2(B)} \leq CR^{n(1/p-1/2)}\|f\|_{L^p(S_j(B))}\|F\|_{L^\infty}$$

for all balls B, $j \geq 3$ and all $f \in L^p(S_j(B))$.

PROOF. Setting $G(\lambda) = e^{\lambda^2/R^2}F(\lambda)$, then $\|G\|_{L^\infty} \approx \|F\|_{L^\infty}$. Moreover, we have $F(\sqrt{L})f = G(L)e^{-\frac{1}{R^2}L}f$. Therefore,

$$\begin{aligned}\|F(\sqrt{L})f\|_{L^2(B)} &= \|G(L)e^{-\frac{1}{R^2}L}f\|_{L^2(B)} \leq \|G(L)e^{-\frac{1}{R^2}L}\|_{L^p(S_j(B))\to L^2(B)}\|f\|_{L^p(S_j(B))}\\ &\leq \|G(L)\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \times \|e^{-\frac{1}{R^2}L}\|_{L^p(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \times \|f\|_{L^p(S_j(B))}\\ &\leq CR^{n(1/p-1/2)}\|G\|_{L^\infty}\|f\|_{L^p(S_j(B))} \approx CR^{n(1/p-1/2)}\|f\|_{L^p(S_j(B))}\|F\|_{L^\infty}.\end{aligned}$$

□

LEMMA 3.3. *For any $p \in (q_0, 2)$, there exist two constants $C > 0$ and $c > 0$ so that for all closed sets E and F, all f with* $\operatorname{supp} f \subset E$ *and all $z \in \mathbf{C}_+ = \{z \in \mathbf{C} : \Re z > 0\}$, there holds*

$$\|e^{-zL}f\|_{L^2(F)} \leq C(|z|\cos\theta)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\exp\bigg(-c\frac{d^2(E,F)}{|z|}\cos\theta\bigg)\|f\|_{L^p(E)}$$

where $\theta = \arg z$.

To prove Lemma 3.3, we need the following version of Phragmen-Lindelöf Theorem, see for example [10, Lemma 9].

LEMMA 3.4. *Suppose that function G is analytic in $\{z \in \mathbf{C} : \Re z > 0\}$ and that*

$$|G(|z|e^{i\theta})| \leq a_1(|z|\cos\theta)^{-\beta_1},$$

$$|G(|z|)| \leq a_1|z|^{-\beta_1}\exp(-a_2|z|^{-\beta_2})$$

for some $a_1, a_2 > 0$, $\beta_1 \geq 0$, $\beta_2 \in (0, 1]$, all $|z| > 0$ and all $\theta \in (-\pi/2, \pi/2)$. Then

$$|G(|z|e^{i\theta})| \leq 2^{\beta_1}a_1(|z|\cos\theta)^{-\beta_1}\exp\bigg(-\frac{a_2\beta_2}{2}|z|^{-\beta_2}\cos\theta\bigg)$$

for all $|z| > 0$ and all $\theta \in (-\pi/2, \pi/2)$.

PROOF OF LEMMA 3.3. Let $g \in L^2(\mathbf{R}^n)$ so that $\|g\|_{L^2} = 1$ and supp $f \subset F$. We define the holomorphic function $G_f : \mathbf{C}_+ \to \mathbf{C}$ by setting

$$G_f(z) = \int e^{-zL} f(x) g(x) dx \,.$$

For any $z \in \mathbf{C}_+$, we have

$$\begin{aligned} G(z) \le \|e^{-zL} f\|_{L^2(F)} &= \|e^{-i\Im zL} \circ e^{-\Re zL} f\|_{L^2(F)} \\ &\le C \|e^{-i\Im zL}\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \|e^{-\Re zL}\|_{L^p(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \|f\|_{L^p(E)} \\ &\le C(|z|\cos\theta)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \|f\|_{L^p(E)} \,. \end{aligned}$$

In particular when $\theta = 0$, we have

$$G_f(|z|) \le C(|z|)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \exp\biggl(-c\frac{d^2(E,F)}{|z|}\biggr) \|f\|_{L^2(E)} \,.$$

At this stage, applying Lemma 3.4 with $a_1 = C\|f\|_{L^p(E)}$, $a_2 = -cd^2(E,F)$, $\beta_1 = \frac{n}{2}(1/p - 1/2)$ and $\beta_2 = 1$, we obtain the desired estimate. □

LEMMA 3.5. *Let $p \in (q_0, 2)$ and $R > 0, s > 0$. For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, s) > 0$ so that*

$$\|F(\sqrt{L})f\|_{L^2(B)} \le C\frac{R^{n(1/p-1/2)}}{(2^j r_B R)^s} \|\delta_R F\|_{W^\infty_{s+\varepsilon}} \|f\|_{L^p(S_j(B))} \tag{7}$$

for all balls B, all $j \ge 3$, all $f \in L^p(S_j(B))$ and all bounded Borel functions F supported in $[R/4, R]$.

PROOF. Using the Fourier inversion formula, we write

$$G(L/R^2)e^{-\frac{1}{R^2}L} = c\int_{\mathbf{R}} e^{-\frac{1-i\tau}{R^2}L} \widehat{G}(\tau) d\tau \,.$$

Hence,

$$F(\sqrt{L})f = c\int_{\mathbf{R}} \widehat{G}(\tau) e^{-\frac{1-i\tau}{R^2}L} f d\tau$$

where $G(\lambda) = [\delta_R F](\sqrt{\lambda})e^{\lambda}$.

Applying Lemma 3.3, we have, for any f supported in $S_j(B)$,

$$
\begin{aligned}
\|F(\sqrt{L})f\|_{L^2(B)} &\le c\int_{\mathbf{R}} \widehat{G}(\tau)\|e^{-\frac{1-i\tau}{R^2}L}f\|_{L^2(B)}d\tau \\
&\le cR^{n(1/p-1/2)}\int_{\mathbf{R}} \widehat{G}(\tau)\exp\Big(-c\frac{(2^jr_BR)^2}{(1+\tau^2)}\Big)d\tau \times \|f\|_{L^p(S_j(B))} \\
&\le cR^{n(1/p-1/2)}\|f\|_{L^p(S_j(B))}\int_{\mathbf{R}} \widehat{G}(\tau)\frac{(1+\tau^2)^{s/2}}{(2^jr_BR)^s}d\tau \\
&\le c\frac{R^{n(1/p-1/2)}}{(2^jr_BR)^s}\|f\|_{L^p(S_j(B))}\Big(\int_{\mathbf{R}} |\widehat{G}(\tau)|^2(1+\tau^2)^{s+\varepsilon+1/2}d\tau\Big)^{1/2} \\
&\quad \times \Big(\int_{\mathbf{R}}(1+\tau^2)^{-\varepsilon-1/2}d\tau\Big)^{1/2} \\
&\le c\frac{R^{n(1/p-1/2)}}{(2^jr_BR)^s}\|G\|_{W^2_{s+\varepsilon+1/2}}\|f\|_{L^p(S_j(B))}.
\end{aligned}
$$

Note that since $\operatorname{supp} F\subset[\frac{R}{4},R]$,we have$\|G\|_{W^2_{s+\varepsilon+1/2}}\le C\|\delta_RF\|_{W^2_{s+\varepsilon+1/2}}\le C\|\delta_RF\|_{W^\infty_{s+\varepsilon+1/2}}$, and so

$$
\|F(\sqrt{L})f\|_{L^2(B)} \le c\frac{R^{n(1/p-1/2)}}{(2^jr_BR)^s}\|\delta_RF\|_{W^\infty_{s+\varepsilon+1/2}}\|f\|_{L^p(S_j(B))}. \tag{8}
$$

To get rid of $1/2$ on the RHS of (8), we use the interpolation arguments as in [21, 14]. We first note that (7) is equivalent to the following estimate

$$
\|\delta_{1/R}H(\sqrt{L})f\|_{L^2(B)}\times(2^jr_BR)^s\le CR^{n(1/p-1/2)}\|f\|_{L^p(S_j(B))}\|H\|_{W^\infty_{s+\varepsilon}}
$$

for all bounded Borel functions H with $\operatorname{supp} H\subset[1/4,1]$.

Now we define the linear operator $\mathcal{A}_{R,f}: L^\infty([1/4,1])\to L^2(B,dx)$ by setting

$$
\mathcal{A}_{R,f}(H)=\delta_{1/R}H(\sqrt{L})f.
$$

By Lemma 3.2,

$$
\|\mathcal{A}_{R,f}\|_{L^\infty([1/4,1])\to L^2(B,dx)}\le CR^{n(1/2-1/p)}\|f\|_{L^p(S_j(B))}.
$$

Setting $d\mu_{s,R}=(R2^jr_B)^sdx$, then (8) tells us that

$$
\|\mathcal{A}_{R,f}\|_{W^\infty_{s+1/2+\varepsilon}([1/4,1])\to L^2(B,\mu_{s,R})}\le CR^{n(1/p-1/2)}\|f\|_{L^p(S_j(B))}.
$$

By interpolation, for each $\theta\in(0,1)$ there exists a constant C such that

$$
\|\mathcal{A}_{R,f}(H)\|_{L^2(B,\mu_{s\theta,R})}\le CR^{n(1/p-1/2)}\|f\|_{L^p(S_j(B))}\|H\|_{[L^\infty,W^\infty_{s+1/2+\varepsilon}]_{[\theta]}}.
$$

Therefore, for all $s>0$, $\varepsilon'>0$ and $\theta\in(0,1)$,

$$
\|\mathcal{A}_{R,f}(H)\|_{L^2(B,\mu_{s\theta,R})}\le CR^{n(1/p-1/2)}\|f\|_{L^p(S_j(B))}\|H\|_{W^\infty_{s\theta+\varepsilon'+\theta/2}}.
$$

By choosing $s' = s/\theta$ and taking θ small enough we obtain

$$\|\mathcal{A}_{R,f}(H)\|_{L^2(B,\mu_{s'\theta,R})} \leq CR^{n(1/p-1/2)}\|f\|_{L^p(S_j(B))}\|H\|_{W^\infty_{s'+\varepsilon''}}.$$

This completes our proof. □

We are now in position to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Take $p_0 \in (r_0, 2)$. Let $M \in \mathbf{N}$ such that $M > s/2$. We will show that (2) and (3) hold for $T = F(\sqrt{L})$ and $A_{r_B} = I - (I - e^{-r_B^2 L})^M$. To verify (2), we will show that for all balls B,

$$\left(\fint_B \left|F(L)(I - A_{r_B})f(x)\right|^{p_0} dx\right)^{1/p_0} \leq C\sum_j \alpha_j\left(\fint_{2^j B} |f(x)|^{p_0} dx\right)^{1/p_0} \tag{9}$$

for all $f \in L^\infty_c(X)$, where $\alpha_j = 2^{-j(s-\frac{n}{p_0})}$.

Let us prove (9). Since $\sup_{t>0}\|\phi(\cdot)F(t\cdot)\|_{W^\infty_s} \approx \sup_{t>0}\|\phi(\cdot)\widetilde{F}(t\cdot)\|_{W^\infty_s}$ where $\widetilde{F}(\lambda) = F(\sqrt{\lambda})$, instead of proving (9) for $F(L)$, we will prove (9) for $F(\sqrt{L})$.

Let ϕ_ℓ denote the function $\phi(2^{-\ell}\cdot)$. Following the standard arguments, we write

$$F(\lambda) = \sum_{\ell=-\infty}^{\infty} \phi(2^{-\ell}\lambda)F(\lambda) = \sum_{\ell=-\infty}^{\infty} F^\ell(\lambda), \quad \forall \lambda > 0.$$

For every $\ell \in \mathbf{Z}$ and $r > 0$, we set for $\lambda > 0$,

$$F_{r,M}(\lambda) = F(\lambda)(1 - e^{-(r\lambda)^2})^M,$$

$$F^\ell_{r,M}(\lambda) = F^\ell(\lambda)(1 - e^{-(r\lambda)^2})^M.$$

Given a ball $B \subset \mathbf{R}^n$, we write $f = \sum_{j=0}^{\infty} f_j$ in which $f_j = f\chi_{S_j(B)}$. Hence,

$$\begin{aligned} F(\sqrt{L})(1 - e^{-r_B^2 L})^M f &= F_{r_B,M}(\sqrt{L})f \\ &= \sum_{j=0}^{2} F_{r_B,M}(\sqrt{L})f_j + \sum_{j=3}^{\infty}\sum_{l=-\infty}^{\infty} F^l_{r_B,M}(\sqrt{L})f_j. \end{aligned}$$

Since $e^{-tL} \in \mathcal{F}(L^{q_0} - L^{q_0'})$, we have that for any $t > 0$, $\|e^{-tL}f\|_{L^p} \leq C\|f\|_{L^p}$ for all $p \in (q_0, q_0')$. This, in combination with L^p-boundedness of the operator $F(\sqrt{L})$ (see Theorem 5.2, [15]), gives that for all balls $B \ni x$,

$$\left(\fint_B \left|F_{r_B,M}(\sqrt{L})f_j\right|^{p_0} dx\right)^{1/p_0} \leq |B|^{-1/p_0}\left\|F_{r_B,M}(\sqrt{L})f_j\right\|_{L^{p_0}(X)}$$

$$\leq C|B|^{-1/p_0}\big\| f_j \big\|_{L^{p_0}(X)} \tag{10}$$
$$\leq C\Big(\fint_{2^j B} |f|^{p_0} dx\Big)^{1/p_0}$$

for $j = 0, 1, 2$.

Fix $j \geq 3$. Let $p_1 \geq 2$. Since $p_0 < 2$, using Hölder's inequality, we have

$$\Big(\fint_B \big| F^{\ell}_{r_B,M}(\sqrt{L}) f_j \big|^{p_0} dx\Big)^{1/p_0} \leq |B|^{-\frac{1}{2}} \big\| F^{\ell}_{r_B,M}(\sqrt{L}) f_j \big\|_{L^2(B)} . \tag{11}$$

Note that supp $F^{\ell}_{r_B,M} \subset [2^{\ell-2}, 2^{\ell}]$. So, using Lemma 3.5 we obtain, for $s > s' > n/p_0$,

$$\big\| F^{\ell}_{r_B,M}(\sqrt{L}) f_j \big\|_{L^2(B)} \leq 2^{\ell n(1/p_0-1/2)} (2^{\ell} r_B)^{-s'} 2^{-js'} \|\delta_{2^{\ell}} F^{\ell}_{r_B,M}\|_{W^{\infty}_s} \|f_j\|_{L^{p_0}} .$$

Let k be an integer so that $k > s$. Then we have

$$\begin{aligned}\|\delta_{2^{\ell}} F^{\ell}_{r_B,M}\|_{W^{\infty}_s} &\leq C\|\phi\delta_{2^{\ell}} F\|_{W^{\infty}_s} \|(1 - e^{(2^{\ell} r_B \cdot)^2})^M\|_{C^k[1/4,1]} \\ &\leq C \sup_{t>0} \|\phi\delta_t F\|_{W^{\infty}_s} \min\{1, (2^{\ell} r_B)^{2M}\} .\end{aligned}$$

Therefore,

$$\big\| F^{\ell}_{r_B,M}(\sqrt{L}) f_j \big\|_{L^2(B)} \leq C2^{-s'j} \min\{1, (2^{\ell} r_B)^{2M}\} (2^{\ell} r_B)^{-s'} 2^{\ell n(\frac{1}{p_0}-\frac{1}{2})} \sup_{t>0} \|\phi\delta_t F\|_{W^{\infty}_s} \|f_j\|_{L^{p_0}}$$

for all $\ell \in \mathbf{Z}$.

This together with (11) gives

$$\begin{aligned}&\Big(\fint_B \big| F^{\ell}_{r_B,M}(\sqrt{L}) f_j \big|^{p_0} dx\Big)^{1/p_0} \\ &\quad\leq 2^{-j(s'-n/p_0)} |B|^{\frac{1}{p_0}-\frac{1}{2}} \min\{1, (2^{\ell} r_B)^{2M}\} (2^{\ell} r_B)^{-s'} 2^{\ell n(\frac{1}{p_0}-\frac{1}{2})} \\ &\qquad\times \sup_{t>0} \|\phi\delta_t F\|_{W^{\infty}_s} \Big(\fint_{2^j B} |f|^{p_0} dx\Big)^{1/p_0} \\ &\quad\leq c2^{-j(s'-\frac{n}{p_0})} \min\{1, (2^{\ell} r_B)^{2M}\} (2^{\ell} r_B)^{-(s'-\frac{n}{p_0}+\frac{n}{2})} \sup_{t>0} \|\phi\delta_t F\|_{W^{\infty}_s} \Big(\fint_{2^j B} |f|^{p_0} dx\Big)^{\frac{1}{p_0}} .\end{aligned}$$

This gives (9).

Since $e^{-tL} \in \mathcal{F}(L^{q_0} - L^{q_0'})$, $A_{r_B} \in \mathcal{F}(L^{q_0} - L^{q_0'})$. This together with the fact that A_{r_B} and $F(\sqrt{L})$ are commutative gives

$$\Big(\fint_B \big| F(\sqrt{L}) A_{r_B} f \big|^{q_0'} dx\Big)^{1/q_0'} \leq C \sum_j \alpha_j \Big(\fint_{2^j B} |F(\sqrt{L}) f|^{p_0}\Big)^{1/p_0} . \tag{12}$$

Therefore, Theorem 2.2 tells us that $F(\sqrt{L})$ and the commutator $[b, F(\sqrt{L})]$ are bounded on $L^p(w)$ for all $p_0 < p < q_0'$ and $w \in A_{p/p_0} \cap RH_{(q_0'/p_0)'}$. Since $A_{p/r_0} = \cup_{p_0>r_0} A_{p/p_0}$ and $RH_{(q_0'/r_0)'} = \cup_{p_0>r_0} RH_{(q_0'/p_0)'}$, letting $p_0 \to r_0$ we obtain the desired results.

This competes our proof. □

4. Applications

In this section, we consider Schrödinger operators with real potential as in [2].

Let A be the Schrödinger operator with a negative potential on $\mathbf{R}^n$, $n \geq 3$ defined by

$$A := -\Delta - V, \quad V \geq 0\,.$$

We assume that the potential V is strongly subcritical, i.e, there exists some $\varepsilon > 0$ so that for all $u \in W^{1,2}(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} Vu^2 \leq \frac{1}{1+\varepsilon} \int_{\mathbf{R}^n} |\nabla u|^2\,. \tag{13}$$

Putting

$$p_0 = \left[\frac{2n}{(n-2)\left(1 - \sqrt{1 - \frac{1}{1+\varepsilon}}\right)} \right]',$$

we then have $e^{-tA} \in \mathcal{F}(L^{p_0} - L^{p_0'})$, see [2, Theorem 2.1]. Hence, as a direct consequence of Theorem 1.1, we have:

THEOREM 4.1. *Let $A = -\Delta - V$ where V is a negative potential satisfying* (13). *Set $r_0 = \max(p_0, \frac{n}{s})$. If $F : [0, \infty) \to \mathbf{C}$ is a bounded Borel function satisfying the following condition for some $s > n/2$*

$$\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} < \infty$$

then

(a) *$F(A)$ is bounded on $L^p(w)$ for all $r_0 < p < p_0'$ and $w \in A_{p/r_0} \cap RH_{(p_0'/r_0)'}$;*

(b) *moreover, for $b \in BMO(\mathbf{R}^n)$, the commutator $[b, F(A)]$ is also bounded on $L^p(w)$ for all $r_0 < p < p_0'$ and $w \in A_{p/r_0} \cap RH_{(p_0'/r_0)'}$.*

Hence by duality $F(A)$ and the commutators $[b, F(A)]$ are bounded on $L^p(w)$ for $p_0 < p \leq r_0$ and $w \in A_{p/p_0} \cap RH_{(r_0'/p)'}$.

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