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On a Generalization of the Mukai Conjecture for Fano Fourfolds

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Abstract. Let X be a complex n-dimensional Fano manifold. Let $s(X)$ be the sum of $l(R) - 1$ for all the extremal rays R of X, the edges of the cone $\mathrm{NE}(X)$ of curves of X, where $l(R)$ denotes the minimum of $(-K_X \cdot C)$ for all rational curves C whose classes $[C]$ belong to R. We show that $s(X) \le n$ if $n \le 4$. And for $n \le 4$, we completely classify the case the equality holds. This is a refinement of the Mukai conjecture on Fano fourfolds.

1. Introduction

Let X be an arbitrary n-dimensional Fano manifold with the Picard number ρ_X. In 1988, Mukai [16] made the following conjecture.

CONJECTURE 1.1. *One has*

$$\rho_X(r_X - 1) \le n\,,$$

and the equality holds if and only if $X \simeq (\mathbf{P}^{r_X-1})^{\rho_X}$, *where*

$$r_X := \max\{m \in \mathbf{Z}_{>0} \mid -K_X \sim mL \text{ for some Cartier divisor } L\}\,.$$

There are several approaches and refinements of Conjecture 1.1. See for example [1, 4, 6, 17, 23]. Nowadays, the following conjecture due to Tsukioka [22] (cf. [21]) is the most generalized version of Conjecture 1.1.

CONJECTURE 1.2. *One has*

$$\rho_X(l_X - 1) \le n\,,$$

and the equality holds if and only if $X \simeq (\mathbf{P}^{l_X-1})^{\rho_X}$, *where* l_X *denotes the minimum of the length* $l(R)$ *of all the extremal rays* R *of* X, *and*

$$l(R) := \min\{(-K_X \cdot C) \mid C \subset X \text{ is a rational curve with } [C] \in R\}\,.$$

We think that it is more natural to consider *all* the extremal rays to study a Fano manifold since each extremal ray has various geometric information. We set up the following question.

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QUESTION 1.3. Give a bound of

$$s(X) := \sum_{R \subset \mathrm{NE}(X) \text{ extremal ray}} (l(R) - 1)$$

for arbitrary n-dimensional Fano manifolds X.

This question is a refinement of Conjectures 1.1 and 1.2 since the invariant $s(X)$ satisfies the inequality $\rho_X(r_X - 1) \le \rho_X(l_X - 1) \le s(X)$. We note that the invariant $s(X)$ is a natural invariant. For example, let $X := \prod_{i=1}^m \mathbf{P}^{d_i}$ with $\sum_{i=1}^m d_i = n$. Then $s(X) = n$ holds despite $\rho_X(l_X - 1) = m \cdot \min\{d_i\}$ is less than n unless $d_1 = \cdots = d_m$.

In this paper, we identify the bound of $s(X)$ when $n \le 4$.

THEOREM 1.4 (Main Theorem). *Let X be an n-dimensional Fano manifold.*

(i) *If $n \le 3$, then $s(X) \le n$ holds. Moreover, the equality holds if and only if*

$$X \simeq \prod_{R \subset \mathrm{NE}(X) \text{ extremal ray}} \mathbf{P}^{l(R)-1}\,.$$

(ii) *If $n = 4$, then $s(X) \le n$ holds. Moreover, the equality holds if and only if*

$$X \simeq \prod_{R \subset \mathrm{NE}(X) \text{ extremal ray}} \mathbf{P}^{l(R)-1}$$

or

$$X \simeq \mathrm{Bl}_{p,q}(\mathbf{Q}^4)\,,$$

the blowing up of $\mathbf{Q}^4$ along p and q, where $\mathbf{Q}^4 \subset \mathbf{P}^5$ is a smooth hyperquadric and p, q are distinct points in $\mathbf{Q}^4$ with $\overline{pq} \not\subset \mathbf{Q}^4$, where $\overline{pq} \subset \mathbf{P}^5$ is the line through p and q.

REMARK 1.5. If $n \ge 5$, then there exists an n-dimensional Fano manifold X such that $s(X)$ is strictly larger than n (see Remark 3.5 (iii)). However, such X is very special as far as we know. We think that all such X should be classified.

As an immediate consequence of Theorem 1.4, we can give the affirmative answer to Conjecture 1.2 in the case $n \le 4$. (Tsukioka [22] proved the inequality in the case $n = 4$ but did not settle the assertion on the equality case.)

COROLLARY 1.6 (cf. [22]). *Conjecture 1.2 is true if $n \le 4$.*

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NOTATION AND TERMINOLOGY. We always work over the complex number field $\mathbf{C}$. For a proper variety X, let $\mathrm{N}_1(X)_{\mathbf{Q}}$ (rep. $\mathrm{N}^1(X)_{\mathbf{Q}}$) be the vector space of one-cycles (resp. Cartier divisors) on X, with rational coefficients, modulo numerical equivalence. Let $\mathrm{N}_1(X) := \mathrm{N}_1(X)_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$ and $\mathrm{N}^1(X) := \mathrm{N}^1(X)_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$. The *Picard number* of X, denoted by ρ_X, is defined to be the dimension of the vector space $\mathrm{N}_1(X)$.

For an n-dimensional normal projective variety X, we denote the normalization of the parameterizing space of irreducible and reduced rational curves on X by $\mathrm{RatCurves}^n(X)$ (see [13, Definition II.2.11]). For the theory of extremal contraction, we refer the readers to [12]. A projective surjective morphism $f\colon X \to Z$ is called a *contraction morphism* if Z is normal projective and any fiber of f is connected. For an extremal ray $R \subset \overline{\mathrm{NE}}(X)$, we say that *$R$ defines the contraction morphism* $\mathrm{cont}_R\colon X \to Y$ if cont_R is a contraction morphism and the kernel of the surjection $\mathrm{N}_1(X) \to \mathrm{N}_1(Y)$ is equal to $\mathbf{R}R(= R + (-R))$. The morphism cont_R is called the *associated contraction morphism*. For example, if X is smooth and R is K_X-negative, then R defines the contraction morphism. For an extremal ray $R \subset \overline{\mathrm{NE}}(X)$, we say that R is *of fiber type* (resp. *divisorial, small*) if R defines the contraction morphism $\mathrm{cont}_R\colon X \to Y$ and the morphism is of fiber type (resp. divisorial, small). We define

$$\mathrm{Exc}(R) := \{x \in X \mid \mathrm{cont}_R\colon X \to Y \text{ is not isomorphism at } x\}.$$

For example, if R is of fiber type, then $\mathrm{Exc}(R) = X$. We say that R is *of type* (a, b) if $\dim(\mathrm{Exc}(R)) = a$ and $\dim(\mathrm{cont}_R(\mathrm{Exc}(R))) = b$, and we say that R is *of type* $(n-1, b)^{\mathrm{sm}}$ if the associated contraction morphism is the blowing up morphism of a smooth projective variety along a smooth subvariety of dimension b (in particular, X must be smooth). For an extremal ray $R \subset \overline{\mathrm{NE}}(X)$ and a Cartier divisor E on X, the notation $(E \cdot R) > 0$ (resp. $(E \cdot R) < 0$, $(E \cdot R) = 0$) means that the property $(E \cdot C) > 0$ (resp. $(E \cdot C) < 0$, $(E \cdot C) = 0$) holds for a curve $C \subset X$ with $[C] \in R$.

For an algebraic variety X and a closed subscheme $Y \subset X$, the morphism $\mathrm{Bl}_Y(X) \to X$ denotes the blowing up of X along Y. The symbol $\mathbf{Q}^n$ denotes a smooth hyperquadric in $\mathbf{P}^{n+1}$. We say that X is a *Fano manifold* if X is a smooth projective variety such that the anticanonical divisor $-K_X$ is ample.

2. Preliminaries

2.1. A family of rational curves.

We observe the definition and a property of a family of rational curves for a fixed normal projective variety.

DEFINITION 2.1 (see for example [1]). Let X be a normal projective variety. We define a *family of rational curves* to be an irreducible component $H \subset \mathrm{RatCurves}^n(X)$. For any $x \in X$, let H_x be the subvariety of H parameterizing rational curves passing through

x, and $\tilde{H}_x$ the normalization of the image of H_x in the Chow variety $\mathrm{Chow}(X)$. We define $\mathrm{Locus}(H)$ (resp. $\mathrm{Locus}(H_x)$) to be the union of rational curves parameterized by H (resp. H_x). For a family H of rational curves on X, the family H is said to be *dominating* if the closure $\overline{\mathrm{Locus}(H)}$ is equal to X, *unsplit* if H is projective, and *locally unsplit* if H_x is projective for general $x \in \mathrm{Locus}(H)$.

The following proposition may be familiar.

PROPOSITION 2.2 ([17, Proposition 2.5(b)]). *Let X be a smooth projective variety, H be a family of rational curves on X, and $x \in \mathrm{Locus}(H)$ be a point such that H_x is projective. Then one has*

$$\dim \mathrm{Locus}(H) + \dim \mathrm{Locus}(H_x) \geq \dim X + (-K_X \cdot \mathrm{Fam}\, H) - 1\,,$$

where $\mathrm{Fam}\, H$ *is the numerical class of the curves in X parametrized by H.*

2.2. Properties of extremal contractions. We show some properties of extremal contractions associate to extremal rays that we need to prove Theorem 1.4.

PROPOSITION 2.3. *Let X be an n-dimensional smooth projective variety. Assume that there exist distinct K_X-negative extremal rays $R_1, R_2 \subset \overline{\mathrm{NE}}(X)$ such that R_1 is of type $(n-1, 0)$, $l(R_2) \geq 2$ and* $\mathrm{Exc}(R_1) \cap \mathrm{Exc}(R_2) \neq \emptyset$*. Then R_2 is of fiber type and $\rho_X = 2$.*

PROOF. Let $E_i := \mathrm{Exc}(R_i)$ for $i = 1, 2$ and fix $x \in E_1 \cap E_2$. Let $C \subset X$ be a rational curve such that
(1) $x \in C$ and $[C] \in R_2$,
(2) $(-K_X \cdot C)$ is minimal among satisfying (1)

Let H be a family of rational curves containing $[C] \in \mathrm{RatCurves}^n(X)$. Then H_x is projective by construction. If there exists an irreducible curve $l \subset E_1 \cap \mathrm{Locus}(H_x)$ then $[l] \in R_1 \cap R_2 = \{0\}$, which leads to a contradiction. Hence $\dim(E_1 \cap \mathrm{Locus}(H_x)) = 0$. Thus $\dim \mathrm{Locus}(H_x) \leq 1$ since $\dim E_1 = n-1$. Therefore,

$$\begin{aligned} 1 \geq \dim \mathrm{Locus}(H_x) &\geq (n - \dim \mathrm{Locus}(H)) + (-K_X \cdot \mathrm{Fam}\, H) - 1 \\ &\geq l(R_2) - 1 \geq 1 \end{aligned}$$

by Proposition 2.2. Thus $\dim \mathrm{Locus}(H) = n$ and $l(R_2) = (-K_X \cdot \mathrm{Fam}\, H) = 2$. In particular, H is dominating and unsplit. Hence R_2 is of fiber type. Let $\varphi_2 \colon X \to Y_2$ be the contraction morphism associated to R_2. Since the restriction $\varphi_2|_{E_1} \colon E_1 \to Y_2$ is a finite morphism, $\dim Y_2 = n-1$. We note that all curves in E_1 are numerically proportional. Thus $\rho_{Y_2} = 1$. This implies that $\rho_X = 2$. □

PROPOSITION 2.4. *Let X be an n-dimensional normal projective variety which satisfies that* $\mathrm{Pic}(X) \otimes \mathbf{Q} = \mathrm{N}^1(X)_{\mathbf{Q}}$.

(1) *Assume that $\rho_X \geq 3$. Pick any extremal ray $R \subset \overline{\mathrm{NE}}(X)$ which defines the contraction morphism $\varphi \colon X \to Y$. Then the ray R is neither of type $(n, 0)$ nor of type $(n, 1)$.*

(2) *Set $m \geq 2$. Let $R_i \subset \overline{\mathrm{NE}}(X)$ be an extremal ray which defines the contraction morphism $\varphi_i \colon X \to Y_i$, $C_i \subset X$ be an irreducible curve with $[C_i] \in R_i$, and $E_i := \mathrm{Exc}(R_i)$ for any $1 \leq i \leq m$. We assume that $E_i \cap E_j = \emptyset$ for any $1 \leq i < j \leq m$. Then we can construct the morphism $\varphi \colon X \to Y$ contracting all of $E_1, \dots, E_m$.* (*Glue $\varphi_1, \dots, \varphi_m$ together. We note that Y is a normal proper variety but not necessary projective.*) *Then there is an exact sequence*

$$0 \longrightarrow \sum_{i=1}^m \mathbf{Q}[C_i] \longrightarrow \mathrm{N}_1(X)_{\mathbf{Q}} \xrightarrow{\varphi_*} \mathrm{N}_1(Y)_{\mathbf{Q}} \longrightarrow 0\,.$$

Furthermore, *if X is $\mathbf{Q}$-factorial and R_i is divisorial for any $1 \leq i \leq m$*, *then Y is also $\mathbf{Q}$-factorial and hence $\rho_Y \geq 1$.*

PROOF. (1) is obvious. We prove (2). For $1 \leq i \leq m$, let $\psi_i \colon X \to Z_i$ be the morphism contracting $E_1, \dots, E_i$ obtained by gluing $\varphi_1, \dots, \varphi_i$ together (for construction, see [10, Exercise 2.12]). We note that Z_i is a normal proper variety, $Y = Z_m$ and $\varphi = \psi_m$. Set $Z_0 := X$ and $\psi_0 := id_X$ (the identity morphism). For $1 \leq i \leq m$, let $\pi_i \colon Z_{i-1} \to Z_i$ be the morphism contracting (the image of) E_i such that $\pi_i \circ \psi_{i-1} = \psi_i$. We remark that $\varphi_1 = \psi_1 = \pi_1$. Note that $\mathrm{Pic}(Z_i) \otimes \mathbf{Q} = \mathrm{N}^1(Z_i)_{\mathbf{Q}}$ by Remark 2.5. It is enough to show the exactness of

$$0 \longrightarrow \mathrm{N}^1(Z_i)_{\mathbf{Q}} \xrightarrow{\pi_i{}^*} \mathrm{N}^1(Z_{i-1})_{\mathbf{Q}} \xrightarrow{(\bullet \cdot C_i)} \mathbf{Q}$$

for any $1 \leq i \leq m$ to prove the exactness of the sequence in (2). We can assume that $2 \leq i \leq m$ since the case $i = 1$ follows from the definition of the contraction morphism. The injectivity of $\pi_i{}^* \colon \mathrm{N}^1(Z_i)_{\mathbf{Q}} \to \mathrm{N}^1(Z_{i-1})_{\mathbf{Q}}$ is obvious. Let $\tau_i \colon Y_i \to Z_i$ be the morphism contracting $E_1, \dots, E_{i-1}$ which satisfies that the diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\psi_{i-1}} & Z_{i-1} \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ Y_i & \xrightarrow[\tau_i]{} & Z_i\,. \end{array}$$

Let $V_i := Z_i \setminus (\tau_i \circ \varphi_i(E_1 \sqcup \dots \sqcup E_{i-1}))$ and $U_i := Z_i \setminus (\tau_i \circ \varphi_i(E_i))$. Pick any invertible sheaf $M \in \mathrm{Pic}(Z_{i-1})$ satisfying $(M \cdot C_i) = 0$. Then $0 = (M \cdot C_i) = (\psi_{i-1}{}^* M \cdot C_i)$. There exists an invertible sheaf $L_1 \in \mathrm{Pic}(Y_i)$ and a positive integer t such that $\varphi_i{}^* L_1 \simeq \psi_{i-1}{}^* M^{\otimes t}$ by the property of the ray R_i and the fact $\mathrm{Pic}(X) \otimes \mathbf{Q} = \mathrm{N}^1(X)_{\mathbf{Q}}$. Thus

$$M^{\otimes t} \simeq \psi_{i-1*}\psi_{i-1}{}^* M^{\otimes t} \simeq \psi_{i-1*}\varphi_i{}^* L_1 \simeq \pi_i{}^* \tau_{i*} L_1\,.$$

Indeed, φ_i and π_i are isomorphisms over U_i, and ψ_{i-1} and τ_i are isomorphisms over V_i, respectively. We note that $\tau_{i*}L_1$ is an invertible sheaf since $\tau_{i*}L_1|_{U_i} \simeq M^{\otimes t}|_{\pi_i{}^{-1}(U_i)}$ and $\tau_{i*}L_1|_{V_i} \simeq L_1|_{\tau_i{}^{-1}(V_i)}$. Therefore we have $M^{\otimes t} \in \pi_i{}^*(\mathrm{Pic}(Z_i))$. For the remaining part, see [12, Corollary 3.18] for example. □

REMARK 2.5. For a surjective morphism $\varphi\colon X \to Y$ between normal proper varieties with connected fibers, if $\mathrm{Pic}(X) \otimes \mathbf{Q} = \mathrm{N}^1(X)_{\mathbf{Q}}$ then $\mathrm{Pic}(Y) \otimes \mathbf{Q} = \mathrm{N}^1(Y)_{\mathbf{Q}}$. Indeed, for a numerically trivial invertible sheaf $L \in \mathrm{Pic}(Y)$, since φ^*L is numerically trivial, there exists a positive integer t such that $\varphi^*L^{\otimes t} \simeq \mathcal{O}_X$. Thus $L^{\otimes t} \simeq \mathcal{O}_Y$.

COROLLARY 2.6. *Let X be an n-dimensional normal $\mathbf{Q}$-factorial projective variety such that* $\mathrm{Pic}(X) \otimes \mathbf{Q} = \mathrm{N}^1(X)_{\mathbf{Q}}$. *Assume that there exist distinct divisorial extremal rays $R_1, \dots, R_m \subset \overline{\mathrm{NE}}(X)$ which define the contraction morphisms $\varphi_i\colon X \to Y_i$ for all $1 \le i \le m$ and $\mathrm{Exc}(R_i) \cap \mathrm{Exc}(R_j) = \emptyset$ for any $1 \le i < j \le m$.*

(1) *If $m \ge 3$, then $\rho_X \ge 4$.*
(2) *If X is smooth and R_i is of type $(n-1, b_i)^{\mathrm{sm}}$ (for some $b_i \in \mathbf{Z}_{\ge 0}$) for any $1 \le i \le m$, then $\rho_X \ge m+1$.*

PROOF. Let $\varphi\colon X \to Y$ be the morphism which is the gluing morphism of $\varphi_1, \dots, \varphi_m$ contracting $\mathrm{Exc}(R_1), \dots, \mathrm{Exc}(R_m)$ as in Proposition 2.4 (2). Let $C_i \subset X$ be an irreducible and reduced curve with $[C_i] \in R_i$ for $1 \le i \le m$.

(1) We can assume that the classes $[C_1]$, $[C_2]$, $[C_3]$ are linearly independent in $\mathrm{N}_1(X)$. By Proposition 2.4 (2), Y is $\mathbf{Q}$-factorial and $1 \le \rho_Y \le \rho_X - 3$.

(2) In this case, Y is a smooth proper variety and $\rho_X = m + \rho_Y \ge m+1$. □

We recall Wiśniewski's theorem on the bounds of the length of extremal rays.

THEOREM 2.7 ([24, Theorem 1.1]). *Let X be a smooth projective variety, $R \in \overline{\mathrm{NE}}(X)$ be a K_X-negative extremal ray and $\mathrm{cont}_R\colon X \to Y$ be the associated contraction morphism. Then for every irreducible component $E \subset \mathrm{Exc}(R)$, we have*

$$l(R) \le \dim X + 1 - 2\mathrm{codim}_X E - \dim(\mathrm{cont}_R(E))\,.$$

2.3. Characterizations of the products of projective spaces. We give several criteria so that a given smooth projective variety is isomorphic to the products of projective spaces.

THEOREM 2.8 ([11, Theorem 2.16]). *Let X be a normal projective variety and H be a dominating and locally unsplit family of rational curves on X. For general $x \in X$, consider the rational map*

$$\tau_x\colon \tilde{H}_x \dashrightarrow \mathbf{P}(T_X|_x^{\vee})$$

defined by

$$[l] \mapsto \mathbf{P}(T_l|_x^{\vee})\,.$$

Then the rational map τ_x is a finite morphism.

DEFINITION 2.9 (Variety of Minimal Rational Tangents). Under the assumption in Theorem 2.8, the finite morphism τ_x is called the *tangent morphism*; its image $\mathcal{C}_x := \tau_x(\tilde{H}_x) \subset \mathbf{P}(T_X|_x^{\vee})$ is called the *variety of minimal rational tangents*, or shortly *VMRT*, of H at x.

Araujo [3] showed a criterion for varieties being isomorphic to the products of projective spaces in terms of VMRT.

THEOREM 2.10 ([3, Theorem 1.3]). *Let X be an n-dimensional smooth projective variety with k distinct dominating and unsplit family of rational curves $H_1, \dots, H_k$ on X. Suppose that, for a general $x \in X$, the associated VMRT of H_i at x are linear subspaces of dimension $d_i - 1$ in $\mathbf{P}(T_X|_x^\vee)$ such that $\sum_{i=1}^k d_i = n$. Then $X \simeq \prod_{i=1}^k \mathbf{P}^{d_i}$.*

We give another criterion for varieties being isomorphic to the products of projective spaces in terms of length of extremal rays.

THEOREM 2.11. *Let X be an n-dimensional smooth projective variety with $n = \sum_{i=1}^k d_i$, where $d_1, \dots, d_k \in \mathbf{Z}_{>0}$. Assume that there exist distinct K_X-negative extremal rays $R_1, \dots, R_k \subset \overline{\mathrm{NE}}(X)$ such that R_i are of fiber type and $l(R_i) \geq d_i + 1$ for all $1 \leq i \leq k$. Then $X \simeq \prod_{i=1}^k \mathbf{P}^{d_i}$.*

PROOF. Let $\varphi_i \colon X \to Y_i$ be the contraction morphism associated to R_i and $e_i := \dim X - \dim Y_i$ for $1 \leq i \leq k$. We have $\sum_{i=1}^k e_i \leq n$ and $e_i \geq l(R_i) - 1$ for any i by [24, Theorem 2.2] and Theorem 2.7. Hence we obtain the inequality

$$n \geq \sum_{i=1}^k e_i \geq \sum_{i=1}^k (l(R_i) - 1) \geq \sum_{i=1}^k d_i = n\,.$$

Therefore $e_i = l(R_i) - 1 = d_i$ for any i. Let F_i be a general fiber of φ_i. Then F_i is a d_i-dimensional Fano manifold such that any rational curve l_i in F_i satisfies that $(-K_{F_i} \cdot l_i) \geq d_i + 1$. Hence $F_i \simeq \mathbf{P}^{d_i}$ by [9]. Let H_i be the family of rational curves on X containing points parameterizing *lines* in $F_i \simeq \mathbf{P}^{d_i}$. Then H_i is a dominating and unsplit family since $(-K_X \cdot \mathrm{Fam}\, H_i) = d_i + 1 = l(R_i)$. We consider $\mathcal{C}_x^i \subset \mathbf{P}(T_X|_x^\vee)$ for $x \in F_i$, which is a VMRT of H_i at x. We have $\mathcal{C}_x^i = \mathbf{P}(T_{F_i}|_x^\vee) \subset \mathbf{P}(T_X|_x^\vee)$; a linear subspace of dimension $d_i - 1$. By Theorem 2.10, $X \simeq \prod_{i=1}^k \mathbf{P}^{d_i}$. □

We also give a criterion for varieties being isomorphic to the product of two projective spaces in terms of extremal rays.

PROPOSITION 2.12. *Let X be an n-dimensional smooth projective variety. If there exist distinct K_X-negative extremal rays $R_1, R_2 \subset \overline{\mathrm{NE}}(X)$ such that the intersection* $\mathrm{Exc}(R_1) \cap \mathrm{Exc}(R_2)$ *is not empty. Then we have*

$$(l(R_1) - 1) + (l(R_2) - 1) \leq n\,,$$

and the equality holds if and only if $X \simeq \mathbf{P}^{l(R_1)-1} \times \mathbf{P}^{l(R_2)-1}$.

PROOF. We fix an arbitrary point $x \in \mathrm{Exc}(R_1) \cap \mathrm{Exc}(R_2)$. For $i = 1, 2$, let $\varphi_i \colon X \to Y_i$ be the contraction morphism associated to R_i and set $y_i := \varphi_i(x) \in Y_i$. Let $C_i \subset X$ be a rational curve which satisfies that

(1) $x \in C_i$ and $[C_i] \in R_i$,
(2) $(-K_X \cdot C_i)$ is minimal among satisfying (1).

Let H_i be a family of rational curves on X containing $[C_i] \in \mathrm{RatCurves}^n(X)$. Then $(H_i)_x$ is projective by construction. Hence we have

$$\begin{aligned}\dim \varphi_i^{-1}(y_i) &\geq \dim \mathrm{Locus}((H_i)_x) \\ &\geq (n - \dim \mathrm{Locus}(H_i)) + (-K_X \cdot \mathrm{Fam}\, H_i) - 1 \\ &\geq (-K_X \cdot \mathrm{Fam}\, H_i) - 1 \geq l(R_i) - 1\end{aligned}$$

by Proposition 2.2. We note that the intersection $\varphi_1^{-1}(y_1) \cap \varphi_2^{-1}(y_2)$ does not contain curves since the rays R_1 and R_2 are distinct. Hence $\dim(\varphi_1^{-1}(y_1) \cap \varphi_2^{-1}(y_2)) = 0$. Thus $n \geq \dim \varphi_1^{-1}(y_1) + \dim \varphi_2^{-1}(y_2)$. Hence $n \geq (l(R_1)-1)+(l(R_2)-1)$. If $n = (l(R_1)-1)+(l(R_2)-1)$, then H_i is dominating and unsplit for each $i = 1, 2$ since $(-K_X \cdot \mathrm{Fam}\, H_i) = l(R_i)$ and $\dim \mathrm{Locus}(H_i) = n$. Therefore one has $X \simeq \mathbf{P}^{l(R_1)-1} \times \mathbf{P}^{l(R_2)-1}$ by [18, Theorem 1.1]. □

COROLLARY 2.13. *Let X be an n-dimensional Fano manifold with $\rho_X = 2$. Then* $\mathrm{NE}(X)$ *is spanned by two extremal rays, say R_1 and R_2. If, at least, one of R_1 and R_2 is not small, then we have*

$$(l(R_1) - 1) + (l(R_2) - 1) \leq n\,,$$

and the equality holds if and only if $X \simeq \mathbf{P}^{l(R_1)-1} \times \mathbf{P}^{l(R_2)-1}$.

PROOF. For $i = 1, 2$, let $\varphi_i : X \to Y_i$ be the contraction morphism associated to R_i and $E_i := \mathrm{Exc}(R_i)$. It is enough to show that $E_1 \cap E_2 \neq \emptyset$ by Proposition 2.12. We can assume that R_1 is divisorial. Then we have $(E_1 \cdot R_1) < 0$. Thus $(E_1 \cdot R_2) > 0$ holds since E_1 is a prime divisor and since R_1 and R_2 span the cone $\mathrm{NE}(X)$. Hence $E_1 \cap E_2 \neq \emptyset$. □

3. Fano manifolds having special extremal rays

In this section, we see several classification results of Fano manifolds having special extremal rays and calculate $s(X)$ for such Fano manifolds X.

THEOREM 3.1 ([8, Proposition 3.1, Theorem 1.1]). *Let X be an n-dimensional Fano manifold and $R \subset \mathrm{NE}(X)$ be an extremal ray.*
(1) *If $n \geq 3$ and R is of type $(n-1, 0)$, then $\rho_X \leq 3$.*
(2) *If $n \geq 4$ and R is of type $(n-1, 1)$, then $\rho_X \leq 5$.*

THEOREM 3.2 ([2, Theorem 5.1]). *Let X be an n-dimensional smooth projective variety and $R \subset \overline{\mathrm{NE}}(X)$ be a K_X-negative extremal ray of type $(n-1, m)$ which satisfies that $l(R) = n - 1 - m$ and all nontrivial fibers of the associated contraction morphism of R are of equi-dimensional. Then R is of type $(n-1, m)^{\mathrm{sm}}$.*

PROPOSITION 3.3 ([20, Proposition 5] (and [2, Theorem 5.1])). *Let X be an n-dimensional Fano manifold with $n \geq 4$. Assume that there exist distinct extremal rays $R_1, R_2 \subset \mathrm{NE}(X)$ such that R_i is of type $(n-1, 1)$ and $l(R_i) = n-2$ for each $i = 1$, 2. Then $\mathrm{Exc}(R_1) \cap \mathrm{Exc}(R_2) = \emptyset$.*

THEOREM 3.4 ([5, Theorem 1.1]). *Let Y be an n-dimensional smooth projective variety with $n \geq 3$ and $a \in Y$ be a (closed) point. Then $X := \mathrm{Bl}_a(Y)$ is a Fano manifold if and only if one of the following holds:*

(i) *$Y \simeq \mathbf{P}^n$ and $a \in Y$ is an arbitrary point.*

(ii) *$Y \simeq \mathbf{Q}^n$ and $a \in Y$ is an arbitrary point.*

(iii) *$Y \simeq V_d$ with $1 \leq d \leq n$ and $a \notin H'$ (the strict transform of H) with $V_d := \mathrm{Bl}_Z(\mathbf{P}^n)$, where $H \subset \mathbf{P}^n$ is a hyperplane and $Z \subset H$ is a smooth subvariety of dimension $n-2$ and degree d.*

REMARK 3.5. We have the following properties by easy calculations.

(i) If $X = \mathrm{Bl}_a(Y)$ is in Theorem 3.4 (i), then

$$\begin{aligned} \mathrm{NE}(X) &= \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g], \\ (-K_X \cdot f) &= 2, \\ (-K_X \cdot g) &= n-1 \end{aligned}$$

hold, where f is the strict transform of a line on $Y = \mathbf{P}^n$ passing through a and g is a line in the exceptional divisor ($\simeq \mathbf{P}^{n-1}$) of $X \to Y$. Thus $s(X) = n-1$.

(ii) If $X = \mathrm{Bl}_a(Y)$ is in Theorem 3.4 (ii), then

$$\begin{aligned} \mathrm{NE}(X) &= \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g], \\ (-K_X \cdot f) &= 1, \\ (-K_X \cdot g) &= n-1 \end{aligned}$$

hold, where f is the strict transform of a line on $Y = \mathbf{Q}^n$ passing through a and g is a line in the exceptional divisor ($\simeq \mathbf{P}^{n-1}$) of $X \to Y$. Thus $s(X) = n-2$.

(iii) If $X = \mathrm{Bl}_a(Y)$ is in Theorem 3.4 (iii), then

$$\begin{aligned} \mathrm{NE}(X) &= \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g] + \mathbf{R}_{\geq 0}[l] + \mathbf{R}_{\geq 0}[m]\,, \\ l &\equiv m + g + (1-d)f \text{ in } \mathrm{N}_1(X)\,, \\ (-K_X \cdot f) = 1, &\quad (-K_X \cdot g) = 1\,, \\ (-K_X \cdot l) = n+1-d, &\quad (-K_X \cdot m) = 1 \end{aligned}$$

hold, where $f \subset X$ is a fiber over Z, $g \subset X$ is a line in a fiber over a, $l \subset X$ is a line in H', and $m \subset X$ is a strict transform of a line passing through a and a point in Z. Thus if $d = 1$ then $s(X) = n-2$, but if $d > 1$ then $s(X) = 2n-2-d$. We note that if $d = 2$, then X is isomorphic to $\mathrm{Bl}_{p,q}(\mathbf{Q}^n)$ with $\overline{pq} \not\subset \mathbf{Q}^n (\subset \mathbf{P}^{n+1})$ (see [5, Corollaire 1.2]) and $s(X) = 2n-4$.

THEOREM 3.6 ([7, 19, 22]). *Let Y be an n-dimensional smooth projective variety with $n \geq 4$, $C \subset Y$ be a smooth curve, $X := \mathrm{Bl}_C(Y)$, and E be the exceptional divisor of the morphism $X \to Y$. We assume that X is a Fano manifold.*

(1) *If $\rho_X = 5$, then one of the following holds:*

(i) *$Y \simeq \mathrm{Bl}_{\{p\}\cup\{q\}\cup\mathbf{P}^{n-2}}(\mathbf{P}^n)$ with $\mathbf{P}^{n-2} \cap \overline{pq} = \emptyset$ and C is the strict transform of $\overline{pq}$.*

(ii) *$Y \simeq \mathrm{Bl}_{\{p\}\cup\{q\}\cup\mathbf{Q}^{n-2}}(\mathbf{P}^n)$ with $\mathbf{Q}^{n-2} \cap \overline{pq} = \emptyset$ and C is the strict transform of $\overline{pq}$.*

(2) *Assume that there exists an extremal ray $R \subset \mathrm{NE}(X)$ of fiber type with $l(R) \geq 2$ and $(E \cdot R) > 0$.*

- *If R is of type $(n, n-2)$, then $\rho_X = 2$.*
- *If R is of type $(n, n-1)$, then the pair of (Y, C) is one of the following:*

(i) *$Y \simeq \mathbf{Q}^n$ and C is a line in $\mathbf{Q}^n \subset \mathbf{P}^{n+1}$.*

(ii) *$Y \simeq \mathbf{P}^1 \times \mathbf{P}^{n-1}$ and C is a fiber of the second projection.*

(iii) *$Y \simeq \mathrm{Bl}_{\mathbf{P}^{n-2}}(\mathbf{P}^n)$ and C is the strict transform of a line in $\mathbf{P}^n$ disjoint from $\mathbf{P}^{n-2}$.*

(iv) *$Y \simeq \mathrm{Bl}_{\mathbf{P}^{n-2}}(\mathbf{P}^n)$ and C is a fiber of the blowing up.*

(v) *$Y \simeq \mathbf{P}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1})$ and C is the section of $\mathbf{P}^{n-1}$-bundle over $\mathbf{P}^1$ whose normal bundle $\mathcal{N}_{C/Y}$ is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus n-1}$.*

(3) *Assume that there exists an extremal ray $R \subset \mathrm{NE}(X)$ of fiber type with $(E \cdot R) = 0$. Let $\varphi\colon X \to Z$ be the contraction morphism associated to R. Then R is of type $(n, n-1)$, $C \simeq \mathbf{P}^1$, $E \simeq \mathbf{P}^1 \times \mathbf{P}^{n-2}$, $E = \varphi^* D$ and Z is factorial, where $D := \varphi(E)$ with the reduced structure. Furthermore, if $n = 4$, then there exists an extremal ray $R_Z \subset \mathrm{NE}(Z)$ with the associated contraction morphism $\varphi_Z\colon Z \to W$ such that φ_Z maps D to a point.*

PROOF. (1) and (2) follow from [19, Theorem 1] and [22, Propositions 3, 4]. We prove (3). The ray R is of type $(n, n-1)$, $E = \varphi^* D$ and Z is factorial by the fact $\dim D \geq n-2$ and by [7, Lemmas 3.9 (i), 3.10 (i)]. Moreover, $C \simeq \mathbf{P}^1$ since a one-dimensional fiber of φ in E maps $X \to Y$ onto C. We know that $E \simeq \mathbf{P}^1 \times \mathbf{P}^{n-2}$ since $E \simeq \mathbf{P}_C(\mathcal{N}^\vee_{C/Y})$ and $\dim E > \dim D$, where $\mathcal{N}_{C/Y}$ is the normal bundle of C in Y. The cone $\mathrm{NE}(Z)$ is closed since $\mathrm{NE}(X)$ is so. Assume that $n = 4$. Then the existence of the ray $R_Z \subset \mathrm{NE}(Z)$ follows from [7, Theorem 4.1 (ii)]. □

REMARK 3.7. We have the following properties by easy calculations.

(1) (i) If $X = \mathrm{Bl}_C(Y)$ is in Theorem 3.6 (1) (i), then

$$\begin{aligned}\mathrm{NE}(X) = {} & \mathbf{R}_{\geq 0}[e] + \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g] + \mathbf{R}_{\geq 0}[h] \\ & + \mathbf{R}_{\geq 0}[k] + \mathbf{R}_{\geq 0}[l] + \mathbf{R}_{\geq 0}[m]\,,\end{aligned}$$

$$(-K_X \cdot e) = n-2, \quad (-K_X \cdot f) = 1, \quad (-K_X \cdot g) = 1\,,$$
$$(-K_X \cdot h) = 1,\ (-K_X \cdot k) = 1,\ (-K_X \cdot l) = 1,\ (-K_X \cdot m) = 1\,,$$

and $\mathrm{NE}(X)$ is exactly spanned by the above seven rays, where

- e is a nontrivial fiber of the morphism $X \to Y$,
- f is the strict transform of a line in the exceptional divisor over p,
- g is the strict transform of a line in the exceptional divisor over q,
- h is a fiber over $\mathbf{P}^{n-2}$,
- k is a fiber of $E \simeq C \times \mathbf{P}^{n-2} \to \mathbf{P}^{n-2}$, where E is the exceptional divisor of $X \to Y$,
- l is the strict transform of a line in $\mathbf{P}^n$ passing through p and $\mathbf{P}^{n-2}$,
- m is the strict transform of a line in $\mathbf{P}^n$ passing through q and $\mathbf{P}^{n-2}$.

Thus $s(X) = n-3$.

(ii) If $X = \mathrm{Bl}_C(Y)$ is in Theorem 3.6 (1) (ii), then

$$\mathrm{NE}(X) = \mathbf{R}_{\geq 0}[e] + \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g] + \mathbf{R}_{\geq 0}[h] + \mathbf{R}_{\geq 0}[j] + \mathbf{R}_{\geq 0}[k] + \mathbf{R}_{\geq 0}[l] + \mathbf{R}_{\geq 0}[m]\,,$$
$$(-K_X \cdot e) = n-2,\ (-K_X \cdot f) = 1,\ (-K_X \cdot g) = 1, (-K_X \cdot h) = 1\,,$$
$$(-K_X \cdot j) = 1,\ (-K_X \cdot k) = 1,\ (-K_X \cdot l) = 1,\ (-K_X \cdot m) = 1\,,$$

and $\mathrm{NE}(X)$ is exactly spanned by the above eight rays, where

- e is a nontrivial fiber of the morphism $X \to Y$,
- f is the strict transform of a line in the exceptional divisor over p,
- g is the strict transform of a line in the exceptional divisor over q,
- h is a fiber over $\mathbf{Q}^{n-2}$,
- j is the strict transform of a line in $\mathbf{P}^n$ intersects $\overline{pq}$ with each other and is contained in a unique hyperplane in $\mathbf{P}^n$ which contains $\mathbf{Q}^{n-2}$,
- k is a fiber of $E \simeq C \times \mathbf{P}^{n-2} \to \mathbf{P}^{n-2}$, where E is the exceptional divisor of $X \to Y$,
- l is the strict transform of a line in $\mathbf{P}^n$ passing through p and $\mathbf{Q}^{n-2}$,
- m is the strict transform of a line in $\mathbf{P}^n$ passing through q and $\mathbf{Q}^{n-2}$.

Thus $s(X) = n-3$.

(2) (i) If $X = \mathrm{Bl}_C(Y)$ is in Theorem 3.6 (2) (i), then $\rho_X = 2$. Thus $s(X) < n$ by Corollary 2.13.

(ii) If $X = \mathrm{Bl}_C(Y)$ is in Theorem 3.6 (2) (ii), then

$$\mathrm{NE}(X) = \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g] + \mathbf{R}_{\geq 0}[h],$$
$$(-K_X \cdot f) = n-2, \quad (-K_X \cdot g) = 2, \quad (-K_X \cdot h) = 2$$

hold, where f is a nontrivial fiber of $X \to Y$, g is the strict transform of a general fiber of the first projection $Y = \mathbf{P}^1 \times \mathbf{P}^{n-1} \to \mathbf{P}^{n-1}$ and h is the strict transform of a line in the second projection $Y = \mathbf{P}^1 \times \mathbf{P}^{n-1} \to \mathbf{P}^1$ passing through C. Thus $s(X) = n-1$.

(iii) If $X = \mathrm{Bl}_C(Y)$ is in Theorem 3.6 (2) (iii), then

$$\mathrm{NE}(X) = \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g] + \mathbf{R}_{\geq 0}[h],$$
$$(-K_X \cdot f) = n-2, \quad (-K_X \cdot g) = 1, \quad (-K_X \cdot h) = 2$$

hold, where f is a nontrivial fiber of $X \to Y$, g is a fiber over $\mathbf{P}^{n-2}$ and h is the strict transform of a line in $\mathbf{P}^n$ passing through C and $\mathbf{P}^{n-2}$. Thus $s(X) = n-2$.

(iv) If $X = \mathrm{Bl}_C(Y)$ is in Theorem 3.6 (2) (iv), then

$$\mathrm{NE}(X) = \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g] + \mathbf{R}_{\geq 0}[h],$$
$$(-K_X \cdot f) = n-2, \quad (-K_X \cdot g) = 1, \quad (-K_X \cdot h) = 2$$

hold, where f is a nontrivial fiber of $X \to Y$, g is a general fiber over $\mathbf{P}^{n-2}$ and h is the strict transform of a line in $\mathbf{P}^n$ passing through $\mathbf{P}^{n-2}$ and the image of C in $\mathbf{P}^n$. Thus $s(X) = n-2$.

(v) If $X = \mathrm{Bl}_C(Y)$ is in Theorem 3.6 (2) (v), then

$$\mathrm{NE}(X) = \mathbf{R}_{\geq 0}[f] + \mathbf{R}_{\geq 0}[g] + \mathbf{R}_{\geq 0}[h],$$
$$(-K_X \cdot f) = n-2, \quad (-K_X \cdot g) = 1, \quad (-K_X \cdot h) = 2$$

hold, where f is a nontrivial fiber of $X \to Y$, g is a fiber of $E \simeq C \times \mathbf{P}^{n-2} \to \mathbf{P}^{n-2}$, where E is the exceptional divisor of $X \to Y$, and h is the strict transform of a line in a fiber of $Y \to \mathbf{P}^1$ passing through C. Thus $s(X) = n-2$.

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. If an n-dimensional Fano manifold X satisfies that $s(X) \geq n$ and $\rho_X = 1$, then $s(X) = n$ and $X \simeq \mathbf{P}^n$ by [9]. Hence we can consider only the Fano manifolds X with $\rho_X \geq 2$.

4.1. Proof of Theorem 1.4 (i). We can assume that $n = 3$ since the case $n \leq 2$ is trivial. We prove the assertion without using the result [14] of complete classification of 3-dimensional Fano manifolds X with $\rho_X \geq 2$. Let X be a 3-dimensional Fano manifold with $s(X) \geq 3$. We can assume that $\rho_X \geq 3$ by Corollary 2.13. By Theorem 2.7, Proposition 2.4 (1) and Theorem 3.2, any extremal ray $R \subset \mathrm{NE}(X)$ with $l(R) \geq 2$ satisfies one of the following:

(A) R is of type $(2, 0)^{\mathrm{sm}}$ and $l(R) = 2$.
(B) R is of type $(3, 2)$ and $l(R) = 2$.

(We note that this result directly follows from [15, Theorems 3.3, 3.5].) If there exists an extremal ray $R \subset \mathrm{NE}(X)$ of type (A), then $X \simeq \mathrm{Bl}_a(V_d)$ with $1 \leq d \leq 3$ by Theorem 3.4, thus $s(X) < 3$ by Remark 3.5 (iii). If there exist distinct extremal rays R_1, R_2 and $R_3 \subset \mathrm{NE}(X)$ such that all of them are of type (B), then $X \simeq \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ by Theorem 2.11. Therefore we have completed the proof of Theorem 1.4 (i).

4.2. Proof of Theorem 1.4 (ii). Let X be a 4-dimensional Fano manifold with $s(X) \geq 4$. We can assume that $\rho_X \geq 3$ by Corollary 2.13. (We note that if $\rho_X = 2$ and both extremal rays are small, then $s(X) = 0$.) By Theorem 2.7, Proposition 2.4 (1) and Theorem 3.2, any extremal ray $R \subset \mathrm{NE}(X)$ with $l(R) \geq 2$ satisfies one of the following:

(A) R is of type $(3, 0)^{\mathrm{sm}}$ and $l(R) = 3$.
(B) R is of type $(3, 0)$ and $l(R) = 2$.
(C) R is of type $(3, 1)^{\mathrm{sm}}$ and $l(R) = 2$.
(D) R is of type $(4, 3)$ and $l(R) = 2$.
(E) R is of type $(4, 2)$ and $l(R) = 3$.
(F) R is of type $(4, 2)$ and $l(R) = 2$.

We note that all two distinct divisorial extremal rays R_1, R_2 with $l(R_1), l(R_2) \geq 2$ satisfy that $\mathrm{Exc}(R_1) \cap \mathrm{Exc}(R_2) = \emptyset$ by Propositions 2.3 and 3.3.

Assume that there exists an extremal ray R of type (A). Then $X \simeq \mathrm{Bl}_a(V_2) \simeq \mathrm{Bl}_{p,q}(\mathbf{Q}^4)$ and $s(X) = 4$ by Theorem 3.4 and Remark 3.5 (iii). Assume that there exists an extremal ray R of type (B) and there is no extremal ray of type (A). Then $\rho_X = 3$ and any other extremal ray R' with $l(R') \geq 2$ is of type (B) or (C) by Proposition 2.3 and Theorem 3.1 (1). Since $s(X) \geq 4$, there exist distinct extremal rays R_1, R_2, R_3 apart from R such that each of them is of type (B) or (C). This contradicts to Corollary 2.6 (1). Hence we can assume that any extremal ray R with $l(R) \geq 2$ is of type (C), (D), (E), or (F).

Assume that there exists an extremal ray R_1 of type (C). We have $\rho_X \leq 4$ by Theorems 3.1 (2), 3.6 (1) and Remark 3.7 (1). By Corollary 2.6 (1), the number of extremal rays of type (C) is at most three. Since $s(X) \geq 4$, there exists an extremal ray R_0 of fiber type and $l(R_0) \geq 2$. Then $(\mathrm{Exc}(R_1) \cdot R_0) = 0$ and R_0 is of type (D) by Theorem 3.6 (2), (3) and Remark 3.7 (2). Moreover, any extremal ray R' of fiber type apart from R_0 satisfies that $(\mathrm{Exc}(R_1) \cdot R') > 0$. Indeed, by Theorem 3.6 (3), if $(\mathrm{Exc}(R_1) \cdot R') = 0$ then R' contains the class of a fiber of the morphism $\mathrm{Exc}(R_1) \simeq \mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^2$. This implies that $R' = R_0$, which leads to a contradiction. Thus $l(R') = 1$ by Theorem 3.6 (2) and Remark 3.7 (2). Since $s(X) \geq 4$, there exist distinct extremal rays R_2, R_3 apart from R_1 such that R_2, R_3 are of type (C). We note that $\rho_X = 4$ by Corollary 2.6. Let $\varphi\colon X \to Y$ be the contraction morphism associated to R_0 and set $D_i := \varphi(\mathrm{Exc}(R_i))$ for $1 \leq i \leq 3$. Since $\mathrm{Exc}(R_i) = \varphi^* D_i$, $D_i \cap D_j = \emptyset$ for $1 \leq i < j \leq 3$. By Theorem 3.6 (3), for any $1 \leq i \leq 3$, there exists a contraction morphism $\psi_i\colon Y \to Z_i$ associated to an extremal ray $R_Z^i \subset \mathrm{NE}(Y)$ such that $\psi_i(D_i)$ is a point. Since $\rho_Y = 3$, each ray R_Z^i is divisorial by Proposition 2.4 (1). However, this contradicts to Corollary 2.6 (1).

Therefore, we can assume that any extremal ray R with $l(R) \geq 2$ is of fiber type. Since $s(X) \geq 4$, there exist distinct extremal rays $R_1, \dots, R_m$ of fiber type such that $\sum_{i=1}^{m}(l(R_i) - 1) \geq 4$. By Theorem 2.11, $\sum_{i=1}^{m}(l(R_i) - 1) = 4$ and $X \simeq \prod_{i=1}^{m} \mathbf{P}^{l(R_i)-1}$.

As a consequence, we have completed the proof of Theorem 1.4 (ii).

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