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Hasse Principle for the Chow Groups of Zero-cycles on Quadric Fibrations

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Abstract. We give a sufficient condition for the injectivity of the global-to-local map of the relative Chow group of zero-cycles on a quadric fibration of dimension ≤ 3 defined over a number field.

1. Introduction

Let k be a number field and Ω the set of its places. For any variety X over k, $\mathrm{CH}_0(X)$ denotes the Chow group of zero-cycles on X modulo rational equivalence. Then we have the global-to-local map

$$\mathrm{CH}_0(X) \longrightarrow \prod_{v\in\Omega} \mathrm{CH}_0(X \otimes_k k_v)\,,$$

where for each place $v \in \Omega$, k_v denotes the completion of k at v. If there exists a proper morphism $X \to C$ from X to another variety C, we also have the relative version of the global-to-local map

$$\Phi : \mathrm{CH}_0(X/C) \longrightarrow \prod_{v\in\Omega} \mathrm{CH}_0(X \otimes_k k_v/C \otimes_k k_v)\,,$$

where $\mathrm{CH}_0(X/C)$ denotes the kernel of the push-forward map $\mathrm{CH}_0(X) \to \mathrm{CH}_0(C)$. In this paper, we study the injectivity of Φ for quadric fibrations over curves.

First, let us recall some known results for a surface X. In the case where X is a conic bundle surface over the projective line $\mathbf{P}^1_k$, Salberger proved that there exists the following exact sequence of finite abelian groups

$$0 \longrightarrow Ш^1(k, T) \longrightarrow A_0(X) \stackrel{\Phi}{\longrightarrow} \bigoplus_{v\in\Omega} A_0(X \otimes_k k_v) \longrightarrow \mathrm{Hom}(H^1(k, \hat{T}), \mathbf{Q}/\mathbf{Z})\,,$$

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where $A_0(X)$ is the kernel of the degree homomorphism $\deg : \mathrm{CH}_0(X) \to \mathbf{Z}$ (see [7] for the details).

In the case where X is a quadric fibration of dimension ≥ 4, few results are known. Parimala and Suresh proved that if $X \to C$ is a quadratic fibration over a smooth projective curve C whose generic fiber is defined by a Pfister neighbor of rank ≥ 5, then the global-to-local map restricted to real places

$$\Phi_{\mathrm{real}} : \mathrm{CH}_0(X/C) \longrightarrow \bigoplus_{v:\text{real place}} \mathrm{CH}_0(X \otimes_k k_v / C \otimes_k k_v)$$

is injective [6]. By using this injectivity, they deduced a finiteness result of the torsion subgroup of the Chow group $\mathrm{CH}_0(X)$ of zero-cycles on X.

When $X \to C$ is a quadric fibration of dim ≤ 3, not only that the map Φ_{real} is not injective in general, but also the map Φ is not injective [8]. However, the map Φ can be injective. If the generic fiber of $X \to C$ is defined by a quadratic form over a base field k, the map Φ is injective (Theorem 3.1). The above condition does not imply the injectivity of Φ_{real}, and we give an example of this (Proposition 3.3). Note that we do not assume that quadric fibrations are admissible (for the definition of admissibility, see [6]).

NOTATION AND CONVENTIONS. In section 2, k denotes a field of characteristic different from 2. In section 3, k denotes a number field (i.e. a finite extension field of $\mathbf{Q}$). For a variety X, $|X|$ denotes the set of closed points on X. We denote by $\mathrm{CH}_i(X)$ the Chow group of cycles of dimension i on X modulo rational equivalence [3]. For a geometrically integral variety X over k, $k(X)$ denotes the function field of X. For any extension L/k of fields, $L(X)$ denotes the function field of $X \otimes_k L$. If x is a point in X, $k(x)$ denotes the residue field at x.

2. Definition of the map δ

In this section, for a quadric fibration $X \to C$ (see definition 2.2), we define a quotient group $k(C)^*/k^* N_q(k(C))$ and a homomorphism

$$\delta : \mathrm{CH}_0(X/C) \longrightarrow k(C)^*/k^* N_q(k(C))\,.$$

We will see that the homomorphism δ is injective, and our main theorem is based on this property.

First, we recall definitions and some properties about a quadratic space.

By a *quadratic space* over k, we mean a nonsingular quadratic form over k. We denote by $W(k)$ the Witt group of quadratic spaces over k and by Ik the fundamental ideal of $W(k)$ consisting of classes of even rank quadratic spaces. We represent quadratic spaces over k by diagonal matrices $\langle a_1, \ldots, a_n \rangle (a_i \in k^*)$ with respect to the choice of an orthogonal basis. By an n-fold *Pfister form* over k, we mean a quadratic space of the type $\langle 1, a_1 \rangle \otimes \langle 1, a_2 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$. The set of nonzero values of a Pfister form is a subgroup of the multiplicative group k^* of k [5, Theorem 1.8, p. 319]. By a *Pfister neighbor* of an n-fold Pfister form q, we mean a quadratic space of rank at least $2^{n-1} + 1$ which is a subform of q [4, Example 4.1].

For any quadratic space q over k, let $N_q(k)$ be the subgroup of k^* generated by norms from finite extensions E of k such that q is isotropic over E. If q is isotropic, then clearly $N_q(k) = k^*$. For any $a \in k^*$, q is isotropic if and only if $\langle a\rangle \otimes q$ is isotropic. Therefore $N_q(k) = N_{\langle a\rangle\otimes q}(k)$. By Knebusch's norm principle, $N_q(k)$ is generated by elements of the form xy, with $x, y \in k^*$ which are values of q over k [2, Lemme 2.2]. In particular, if a quadratic form q is of the form $q = \langle 1, a\rangle \otimes \langle 1, b\rangle$, then $N_q(k)$ is equal to the group $\mathrm{Nrd}_{D/k}(D^*)$ of reduced norms of the quaternion algebra $D = (-a, -b)_k$. Suppose that q' is a Pfister neighbor of a Pfister form q. Then, for any extension E/k, q' is isotropic over E if and only if q is isotropic over E [4, Example 4.1]. So we have $N_q(k) = N_{q'}(k)$.

The following lemma is elementary and well-known, but the proof does not seem to be written explicitly in the literature.

LEMMA 2.1. *Let q be a Pfister form over k. Then*

$$N_q(k) = \{x \in k^* \mid q \otimes \langle 1, -x\rangle \text{ is isotropic}\}\,.$$

In particular, x belongs to $N_q(k)$ if and only if $q \otimes \langle 1, -x\rangle = 0$ in $W(k)$.

PROOF. Since q is a Pfister form, $N_q(k)$ is the set of non-zero values of q. If q is isotropic, the assertion is clear. We suppose that q is anisotropic. If $q \otimes \langle 1, -x\rangle$ is isotropic, then there exist two vectors v_1, v_2 in the underlying vector space of q, not both zero, such that $q(v_1) - xq(v_2) = 0$. Therefore

$$x = q(v_1)/q(v_2) \in N_q(k)\,.$$

The other implication follows from the fact that a Pfister form represents 1.

The last assertion results from the basic fact on Pfister forms [5, Theorem 1.7, p. 319]. □

DEFINITION 2.2. Let C be a smooth projective geometrically integral curve over k. A *quadric fibration* (X, π) over C is a geometrically integral variety X over k, together with a proper flat k-morphism $\pi : X \to C$ such that each point P of C has an affine neighborhood $\operatorname{Spec} A(P)$, with $X \times_C \operatorname{Spec} A(P)$ isomorphic to a quadric in $\mathbf{P}^n_{A(P)}$ and such that the generic fiber of π is a smooth quadric.

Given a quadratic space q over the function field $k(C)$ of C of rank $n + 1 \geq 3$, we can easily construct a quadric fibration $\pi : X \to C$, whose generic fiber is given by the quadratic space q. Indeed, for the choice of an orthogonal basis the quadratic space q corresponds to a homogeneous polynomial $q(t_0, t_1, \ldots, t_n)$ of degree two with coefficients in $k(C)$. We define the smooth quadric hypersurface Q by

$$Q = \{q(t_0, t_1, \ldots, t_n) = 0\} \subset \operatorname{Proj}(k(C)[t_0, t_1, \ldots, t_n]) = \mathbf{P}^n_{k(C)}\,.$$

Let X be the scheme-theoretic image of the morphism $Q \hookrightarrow \mathbf{P}^n_{k(C)} \to \mathbf{P}^n_k \times_{\operatorname{Spec} k} C$. By composition with the second projection $\mathbf{P}^n_k \times_{\operatorname{Spec} k} C \to C$, we get a quadric fibration $X \to C$, whose generic fiber is Q. Conversely, for any quadric fibration $\pi : X \to C$, the generic fiber of π is a smooth quadric hypersurface over $k(C)$. Its defining equation gives a quadric space q over $k(C)$, which is well-defined up to multiplication by a non-zero element of $k(C)$. However, as mentioned above, the group $N_q(k(C))$ is independent of the choice of q.

Let $\pi : X \to C$ be a quadric fibration with the generic fiber given by a quadratic space q, and $\mathrm{CH}_0(X/C)$ denote the kernel of the map

$$\pi_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(C)\,.$$

We have the following commutative diagram with exact rows (see [2, Proposition 1.1])

$$\begin{array}{ccccccccc}
 & & \bigoplus_{x\in|X_\eta|} k(x)^* & \longrightarrow & \bigoplus_{P\in|C|} \mathrm{CH}_0(X_P) & \longrightarrow & \mathrm{CH}_0(X) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \oplus N_{k(x)/k(C)}} & & \Big\downarrow{\scriptstyle \oplus \deg_{X_P/k(P)}} & & \Big\downarrow{\scriptstyle \pi_*} & & \\
0 & \longrightarrow & k(C)^*/k^* & \longrightarrow & \bigoplus_{P\in|C|} \mathbf{Z} & \longrightarrow & \mathrm{CH}_0(C) & \longrightarrow & 0\,,
\end{array}$$

where X_η is the generic fiber of $\pi : X \to C$ and $\deg_{X_P/k(P)} : \mathrm{CH}_0(X_P) \to \mathbf{Z}$ is the degree map. Since the map $\bigoplus_{x\in|X_\eta|} k(x)^* \to k(C)^*$ is induced by norms, the image is precisely $N_q(k(C))$. By the snake lemma and the fact that $A_0(X_P) = 0$ [9], we have an exact sequence

$$0 \longrightarrow \mathrm{CH}_0(X/C) \xrightarrow{\delta} k(C)^*/k^*N_q(k(C)) \longrightarrow \bigoplus_{P\in C^{(1)}} \mathbf{Z}/\deg_{X_P/k(P)}(\mathrm{CH}_0(X_P))\,.$$

REMARK 2.3. We denote by $k(C)^*_{\mathrm{dn}}(q)$ the subgroup of $k(C)^*$ consisting of functions which, at each closed point $P \in C$, can be written as a product of a unit at P and an element of $N_q(k(C))$. Colliot-Thélène and Skorobogatov [2, Théorème 1.4] proved that the above homomorphism δ induces an isomorphism

$$\delta : \mathrm{CH}_0(X/C) \xrightarrow{\sim} k(C)^*_{\mathrm{dn}}(q)/k^*N_q(k(C))$$

for an *admissible* quadric fibration $\pi : X \to C$. However, we do not have to assume the admissibility for our main results. Indeed, in this paper, we consider only the injectivity of the global-to-local map for the relative Chow group $\mathrm{CH}_0(X/C)$. On the other hand, the map $\delta : \mathrm{CH}_0(X/C) \to k(C)^*/k^*N_q(k(C))$ is injective whether $\pi : X \to C$ is admissible or not. Therefore we can reduce the injectivity of the global-to-local map for the relative chow group $\mathrm{CH}_0(X/C)$ to the injectivity of the global-to-local map for the group $k(C)^*/k^*N_q(k(C))$.

3. Injectivity of the global-to-local map

Let k be a number field and Ω be the set of places of k. For any $v \in \Omega$, k_v denotes the completion of k at v.

In [6, Theorem 5.4], Parimala and Suresh proved that if $X \to C$ is an admissible quadric fibration whose generic fiber is given by a Pfister neighbor over $k(C)$ of rank ≥ 5, then the map

$$\Phi_{\mathrm{real}} : \mathrm{CH}_0(X/C) \longrightarrow \bigoplus_{v:\text{real places}} \mathrm{CH}_0(X \otimes_k k_v / C \otimes_k k_v)$$

is injective, where v runs over all real places of k. In the case where the rank of the quadratic form defining the generic fiber is less than 5, we give a sufficient condition for the injectivity of the global-to-local map Φ.

THEOREM 3.1. *Let $\pi : X \to C$ be a quadric fibration over a number field k. Assume that* $\dim X = 2$ *or* 3, *and the generic fiber of π is isomorphic to a quadric defined over k (i.e. there exists a quadric $Q \subset \mathbf{P}_k^N$ such that the generic fiber is isomorphic to $Q \otimes_k k(C)$ over $k(C)$). Then, the natural map*

$$\Phi : \mathrm{CH}_0(X/C) \longrightarrow \bigoplus_{v \in \Omega} \mathrm{CH}_0(X \otimes_k k_v / C \otimes_k k_v)$$

is injective.

PROOF. Let q be a quadratic form defining the generic fiber of the quadric fibration $\pi : X \to C$. In order to prove the theorem, it is sufficient to show that the natural map

$$k(C)^*/k^* N_q(k(C)) \longrightarrow \prod_{v \in \Omega} k_v(C)^*/k_v^* N_q(k_v(C))$$

is injective (see Remark 2.3).

First, when $\dim X = 3$, that is, $\operatorname{rank} q = 4$, we show that the global-to-local map Φ is injective and for almost all v the v-component $\mathrm{CH}_0(X \otimes_k k_v / C \otimes_k k_v)$ is 0.

We may assume that $q = \langle 1, a, b, abd \rangle$, $a, b, d \in k^*$. Put $L := k(\sqrt{d})$. Note that q is isometric to $\langle 1, a \rangle \otimes \langle 1, b \rangle$ over $L(C)$. Let $f \in k(C)^*$ be an element which belongs to $k_v^* N_q(k_v(C))$ for all places v of k. For any real place w of L, denote by w' the restriction of w to k. Since $f \in k_{w'}^* N_q(k_{w'}(C))$, there exists $\mu_{w'} \in k_{w'}^*$ such that $\mu_{w'} f \in N_q(k_{w'}(C))$. We can choose $\mu \in k^*$ such that the sign of μ is the same as that of $\mu_{w'}$ for each real place w of L. Thus we have $\mu f \in N_q(k_{w'}(C)) \subset N_q(L_w(C))$. Therefore $q \otimes \langle 1, -\mu f \rangle$ is hyperbolic over $L_w(C)$ for each real place w of L. For a complex place w of L, it is clear that $q \otimes \langle 1, -\mu f \rangle$ is hyperbolic over $L_w(C)$. Further, for a finite place w of L, we have $f \in k_v^* N_q(k_v(C))$, where v is the place of k below w. Since $k_v^* \subset N_q(k_v(C))$ (this follows from the fact that $k_v^* = N_q(k_v)$ for any quadratic form q of rank at least 3 over a p-adic field k_v),

$$\mu f \in k_v^* N_q(k_v(C)) = N_q(k_v(C)) \subset N_q(L_w(C)).$$

Therefore $q \otimes \langle 1, -\mu f \rangle$ is hyperbolic over $L_w(C)$ for all places w of L. By [1, Theorem 4], the natural map

$$I^3L(C) \longrightarrow \prod_w I^3L_w(C)$$

is injective, where w runs over all places of L. Hence $q \otimes \langle 1, -\mu f\rangle$ is hyperbolic over $L(C)$. By [2, Proposition 2.3], we have

$$\mu f \in N_q(L(C)) \cap k(C)^* = N_q(k(C))\,.$$

This proves the required injectivity.

The image of the global-to-local map Φ lies in the direct sum $\bigoplus_v \mathrm{CH}_0(X \otimes_k k_v / C \otimes_k k_v)$. Indeed, a quadratic form of rank 4 defined over a number field k is isotropic over k_v for all but finitely many places v of k. Therefore for all but finitely many v, $\mathrm{CH}_0(X \otimes_k k_v / C \otimes_k k_v) = 0$.

Finally, we consider the case $\dim X = 2$, that is, $\operatorname{rank} q = 3$.

We can assume $q = \langle 1, a, b\rangle$, $a, b \in k^*$. Since q is a Pfister neighbor of the Pfister form $q' = \langle 1, a\rangle \otimes \langle 1, b\rangle$, we have $N_q(k(C)) = N_{q'}(k(C))$ [4, Example 4.1]. Therefore by the above case the map

$$k(C)^*/k^*N_q(k(C)) \longrightarrow \prod_{v \in \Omega} k_v(C)^*/k_v^*N_q(k_v(C))$$

is injective. This completes the proof. □

REMARK 3.2. Without the assumption that the generic fiber is defined over k, the natural map

$$\Phi : \mathrm{CH}_0(X/C) \longrightarrow \prod_{v \in \Omega} \mathrm{CH}_0(X \otimes_k k_v / C \otimes_k k_v)$$

is not injective [8].

Finally, we consider the restricted global-to-local map Φ_{real}. Parimala and Suresh's result [6, Theorem 5.4], which is concerned with quadratic forms of rank at least 5, does not hold for forms of smaller rank. We give the following example, which is a variant of [6, Proposition 6.1].

PROPOSITION 3.3. *Let C be the elliptic curve over $\mathbf{Q}$ defined by*

$$y^2 = -x(x+2)(x+3).$$

Assume that the generic fiber of a quadric fibration $\pi : X \to C$ is isomorphic to the quadric defined by the quadratic form $q = \langle 1, -2, 3, -6\rangle$. Then the natural map

$$\Phi_{\mathrm{real}} : \mathrm{CH}_0(X/C) \longrightarrow \mathrm{CH}_0(X \otimes_{\mathbf{Q}} \mathbf{R} / C \otimes_{\mathbf{Q}} \mathbf{R})$$

is not injective.

PROOF. Since q is isotropic over $\mathbf{R}$, $N_q(\mathbf{R}(C)) = \mathbf{R}(C)^*$. So we have $\mathrm{CH}_0(X \otimes_{\mathbf{Q}} \mathbf{R}/C \otimes_{\mathbf{Q}} \mathbf{R}) = 0$.

Since we have $\mathrm{div}_C(x) = 2D$ for some divisor D on C, $x \in \mathbf{Q}(C)^*/\mathbf{Q}^* N_q(\mathbf{Q}(C))$ is contained in $\mathrm{Im}\,\delta$. On the other hand, q is isometric to $\langle 1, 1, 3, 3\rangle$ over $\mathbf{Q}_3$. Thus $x \notin \mathbf{Q}_3^* N_q(\mathbf{Q}_3(C))$ by [6, Proposition 6.1]. Therefore we have $\mathrm{CH}_0(X/C) \neq 0$. □

REMARK 3.4. In the above case the map

$$\Phi : \mathrm{CH}_0(X/C) \longrightarrow \bigoplus_{v \in \Omega} \mathrm{CH}_0(X \otimes_{\mathbf{Q}} \mathbf{Q}_v / C \otimes_{\mathbf{Q}} \mathbf{Q}_v)$$

is injective by Theorem 3.1. The summands of the right hand side vanish except for $\mathrm{CH}_0(X \otimes_{\mathbf{Q}} \mathbf{Q}_2/C \otimes_{\mathbf{Q}} \mathbf{Q}_2)$ and $\mathrm{CH}_0(X \otimes_{\mathbf{Q}} \mathbf{Q}_3/C \otimes_{\mathbf{Q}} \mathbf{Q}_3)$, since the quadratic form $\langle 1, -2, 3, -6\rangle$ is isotropic over $\mathbf{R}$ and over $\mathbf{Q}_p$ for all primes p except 2 and 3.

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