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On the Symmetric and Rees Algebras of Certain Determinantal Ideals

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Abstract. The purpose of this paper is to give elementary proofs to the theorems due to Avramov on certain determinantal ideals of linear type.

1. Introduction

Let R be a Noetherian ring and let m, n be integers such that $1 \leq m \leq n$. We denote by $\mathrm{M}(m, n\,;\, R)$ the set of $m \times n$ matrices with entries in R. Let $M = (x_{ij}) \in \mathrm{M}(m, n\,;\, R)$ and S a polynomial ring over R with variables $T_1, T_2, \ldots, T_n$. We regard S as a graded ring by setting deg $T_j = 1$ for all $j = 1, \ldots, n$. For each $i = 1, \ldots, m$, we set

$$f_i = x_{i1}T_1 + x_{i2}T_2 + \cdots + x_{in}T_n\,,$$

and let $\mathrm{I}_k(M)$ be the ideal of R generated by the k-minors of M for $k = 1, \ldots, m$. The purpose of this paper is to give elementary proofs to the following theorems due to Avramov [1].

THEOREM 1.1. (c.f. Proposition 1 in [1]). *The following conditions are equivalent*:
(1) grade $\mathrm{I}_k(M) \geq m - k + 1$ *for* $k = 1, \ldots, m$.
(2) grade $(f_1, f_2, \ldots, f_m)S = m$.

THEOREM 1.2. (c.f. Proposition 4 and 9 in [1]). *Suppose* $n = m + 1$. *We set* $I = \mathrm{I}_m(M)$. *Let* $\mathrm{S}(I)$ *and* $\mathrm{R}(I)$ *be the symmetric algebra and the Rees algebra of* I, *respectively. Then the following conditions are equivalent*:
(1) grade $\mathrm{I}_k(M) \geq m - k + 2$ *for* $k = 1, \ldots, m$.
(2) (i) *The natural map* $\mathrm{S}(I) \to \mathrm{R}(I)$ *is isomorphic*, (ii) $\mathrm{S}(I) \cong S/(f_1, f_2, \ldots, f_m)S$ *as graded* R*-algebras and* (iii) grade $(f_1, f_2, \ldots, f_m)S = m$.

The Rees algebra of I is the subalgebra of the polynomial ring $R[t]$ generated by It over R. If the conditions (i) and (ii) of Theorem 1.2 are satisfied, there exists a surjection $S \to \mathrm{R}(I)$

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of graded R-algebras whose kernel coincides with $(f_1, f_2, \dots, f_m)S$. Moreover, if the condition (iii) of Theorem Theorem 1.2 is satisfied, the Koszul complex of $f_1, f_2, \dots, f_m$ is an acyclic complex of graded free S-modules by [2, 1.6.17]. Therefore, under the condition (2) of 1.2, we get a graded S-free resolution of $\mathrm{R}(I)$, and taking its homogeneous part of degree $r \in \mathbf{Z}$, we get an R-free resolution of I^r, from which we can deduce some homological properties of powers of I. In the subsequent paper [3], using the R-free resolution of I^r constructed in this way, we study the associated prime ideals of R/I^m and compute the saturation of I^m. So, the author thinks that Theorem 1.2 is very convenient and it may have more application. Although we may lose sight of some important meaning of the original proof, however, the existence of an elementary proof is quite helpful for users.

2. Preliminaries

In this section we summarize preliminary results. Although these facts might be well-known, we give the proofs for completeness.

LEMMA 2.1. $\operatorname{Ass} S = \{\, \mathfrak{p}S \mid \mathfrak{p} \in \operatorname{Ass} R \,\}$.

PROOF. It is enough to show in the case where $n = 1$. We put $T = T_1$.

Let us take any $\mathfrak{p} \in \operatorname{Ass} R$. Then there exists $x \in R$ such that $\mathfrak{p} = 0 :_R x$. It is easy to see $\mathfrak{p}S = 0 :_S x$. On the other hand, $S/\mathfrak{p}S \cong (R/\mathfrak{p})[T]$, which is an integral domain. Hence $\mathfrak{p}S \in \operatorname{Ass} S$.

Conversely, we take any $Q \in \operatorname{Ass} S$. Then Q is homogeneous , and so $Q = 0 :_S yT^n$ for some $y \in R$ and $0 \le n \in \mathbf{Z}$. We put $\mathfrak{q} = Q \cap R$. Then $\mathfrak{q} \in \operatorname{Spec} R$ and $\mathfrak{q} = 0 :_R y$, which means $\mathfrak{q} \in \operatorname{Ass} R$. Moreover, we have $\mathfrak{q}S = 0 :_S yT^n = Q$. Thus the proof is complete.

LEMMA 2.2. *f_1 is a non-zerodivisor on S if and only if* $\operatorname{grade}(x_{11}, x_{12}, \dots, x_{1n})R > 0$.

PROOF. Suppose $\operatorname{grade}(x_{11}, x_{12}, \dots, x_{1n})R = 0$. Then there exists $\mathfrak{p} \in \operatorname{Ass} R$ such that $(x_{11}, x_{12}, \dots, x_{1n})R \subseteq \mathfrak{p}$. In this case, we have $f_1 \in \mathfrak{p}S \in \operatorname{Ass} S$ by Lemma 2.1, and so f_1 is a zerodivisor on S.

Conversely, suppose that f_1 is a zerodivisor on S. Then there exists $Q \in \operatorname{Ass} S$ such that $f_1 \in Q$, and $Q = \mathfrak{q}S$ for some $\mathfrak{q} \in \operatorname{Ass} R$. In this case, we have $x_{1j} \in \mathfrak{q}$ for all j, which means $\operatorname{grade}(x_{11}, x_{12}, \dots, x_{1n})R = 0$. Thus the proof is complete.

LEMMA 2.3. *Let $N = (y_{ij}) \in \mathrm{M}(m, n\,; R)$ be the matrix induced from M by some elementary operation. For each $i = 1, \dots, m$, we set*

$$g_i = y_{i1}T_1 + y_{i2}T_2 + \cdots + y_{in}T_n\,.$$

Then there exists an isomorphism $\varphi : S \xrightarrow{\sim} S$ of R-algebras such that $\varphi((f_1, f_2, \dots, f_m)S) = (g_1, g_2, \dots, g_m)S$. In particular, we have $\operatorname{grade}(f_1, f_2, \dots, f_m)S = \operatorname{grade}(g_1, g_2, \dots, g_m)S$.

PROOF. The elementary operation is one of the followings:

(i) Fixing a unit $c \in R$ and $1 \leq i \leq m$, replace x_{ij} with cx_{ij} for all j.
(ii) Fixing an element $c \in R$ and $1 \leq i, k \leq m$ $(i \neq k)$, replace x_{ij} with $x_{ij} + cx_{kj}$ for all j.
(iii) Fixing $1 \leq i < k \leq m$, exchange x_{ij} with x_{kj} for all j.
(iv) Fixing a unit $c \in R$ and $1 \leq j \leq n$, replace x_{ij} with cx_{ij} for all i.
(v) Fixing an element $c \in R$ and $1 \leq j, \ell \leq n$ $(j \neq \ell)$, replace x_{ij} with $x_{ij} + cx_{i\ell}$ for all i.
(vi) Fixing $1 \leq j < \ell \leq n$, exchange x_{ij} with $x_{i\ell}$ for all i.

In each case stated above the following relations between f_i's and g_i's hold:

(i) $g_i = cf_i$ and $g_p = f_p$ for any $p \neq i$.
(ii) $g_i = f_i + cf_k$ and $g_p = f_p$ for any $p \neq i, k$.
(iii) $g_i = f_k$, $g_k = f_i$ and $g_p = f_p$ for any $p \neq i, k$.
(iv) $g_i = f_i(T_1, \ldots, T_{j-1}, cT_j, T_{j+1}, \ldots, T_n)$ for any i.
(v) $g_i = f_i(T_1, \ldots, T_{j-1}, T_j + cT_\ell, T_{j+1}, \ldots, T_n)$ for any i.
(vi) $g_i = f_i(T_1, \ldots, \underbrace{T_\ell}_{j}, \ldots, \underbrace{T_j}_{\ell}, \ldots, T_n)$, for any i.

In the cases of (i), (ii) and (iii), we set $\varphi = \mathrm{id}_S$. In the other cases we define φ by

(iv) $\varphi(T_j) = cT_j$ and $\varphi(T_q) = T_q$ for any $q \neq j$,
(v) $\varphi(T_j) = T_j + cT_\ell$ and $\varphi(T_q) = T_q$ for any $q \neq j$,
(vi) $\varphi(T_j) = T_\ell$, $\varphi(T_\ell) = T_j$ and $\varphi(T_q) = T_q$ for any $q \neq j, \ell$.

Then, φ becomes an isomorphism and the required equality holds.

LEMMA 2.4. *Let $m \geq 2$ and $\mathfrak{p} \in \operatorname{Spec} R$. We assume $\mathrm{I}_1(M) \notin \mathfrak{p}$. Then there exists a matrix $N = (y_{ij}) \in \mathrm{M}(m-1, n-1\,; R_\mathfrak{p})$ satisfying the following conditions*:

(a) $\mathrm{I}_k(N) = \mathrm{I}_{k+1}(M)R_\mathfrak{p}$ *for* $k = 1, \ldots, m-1$.
(b) *Let* $g_i = y_{i1}T_1 + y_{i2}T_2 + \cdots + y_{i,n-1}T_{n-1}$ *for* $i = 1, \ldots, m-1$. *Then, there exists an isomorphism* $\varphi : S_\mathfrak{p} \xrightarrow{\sim} S_\mathfrak{p}$ *of* $R_\mathfrak{p}$*-algebras such that* $\varphi((f_1, f_2, \ldots, f_m)S_\mathfrak{p}) = (g_1, g_2, \ldots, g_{m-1}, T_n)S_\mathfrak{p}$. *In particular, we have*

$$\operatorname{grade}(g_1, g_2, \ldots, g_{m-1})S' = \operatorname{grade}(f_1, f_2, \ldots, f_m)S_\mathfrak{p} - 1\,,$$

where $S' = R_\mathfrak{p}[T_1, T_2, \ldots, T_{n-1}]$.

PROOF. As one of the entries of M is a unit of $R_\mathfrak{p}$, applying elementary operations to M in $\mathrm{M}(m, n\,; R_\mathfrak{p})$, we get a matrix of the form

$$\left(\begin{array}{c|c} & 0 \\ N & \vdots \\ & 0 \\ \hline 0 \ \cdots \ 0 & 1 \end{array}\right),$$

where $N = (y_{ij}) \in \mathrm{M}(m-1, n-1\,; R_\mathfrak{p})$. It is easy to see that the condition (a) is satisfied. Furthermore, by Lemma 2.3 we see that the condition (b) is satisfied. We get the equality on the grades since $S_\mathfrak{p} = S'[T_n]$. Thus the proof is complete.

Let $K_\bullet$ be the graded Koszul complex with respect to $f_1, f_2, \dots, f_m$. By $\partial_\bullet$ we denote its boundary map . Let $e_1, e_2, \dots, e_m$ be an S-free basis of K_1 consisting of homogeneous elements of degree 1 such that $\partial_1(e_i) = f_i$ for $i = 1, \dots, m$. Then, for $r = 1, \dots, m$,

$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} | 1 \le i_1 < \cdots < i_r \le m\}$$

is an S-free basis of K_r consisting of homogeneous elements of degree r, and we have

$$\partial_r(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{p=1}^{r} (-1)^{p-1} f_{ip} \cdot e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_r} .$$

Let $\ell \in \mathbf{Z}$. Taking the homogeneous part of degree ℓ of $K_\bullet$, we get a complex

$$[K_\bullet]_\ell : 0 \longrightarrow [K_m]_\ell \xrightarrow{[\partial_m]_\ell} [K_{m-1}]_\ell \longrightarrow \cdots \longrightarrow [K_1]_\ell \xrightarrow{[\partial_1]_\ell} [K_0]_\ell \longrightarrow 0$$

of finitely generated free R-modules, where $[\partial_r]_\ell$ denotes the restriction of ∂_r to $[K_r]_\ell$ for any r. It is obvious that $[K_r]_\ell = 0$ if $\ell < r$. On the other hand, if $\ell \ge r$, then

$$\{T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_n^{\alpha_n} e_{i_1} \wedge \cdots \wedge e_{i_r} \mid 0 \le \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbf{Z}, \sum_{k=1}^{n} \alpha_k = \ell - r , 1 \le i_1 < i_2 < \cdots < i_r \le m\}$$

is an R-free basis of $[K_r]_\ell$.

LEMMA 2.5. $\operatorname{rank}_R[K_m]_m = 1, \operatorname{rank}_R[K_{m-1}]_m = mn$ *and* $\mathrm{I}_1([\partial_m]_m) = \mathrm{I}_1(M)$.

PROOF. We get $\operatorname{rank}_R[K_m]_m = 1$ as $[K_m]_m$ is generated by $e_1 \wedge e_2 \wedge \cdots \wedge e_m$. Moreover, we get $\operatorname{rank}_R[K_{m-1}]_m = mn$ as $\{T_j \check{e}_i \mid 1 \le i \le m, 1 \le j \le n\}$ is an R-free basis of $[K_{m-1}]_m$, where $\check{e}_i = e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_m$ for all i. The last assertion holds since

$$\begin{aligned} \partial_m(e_1 \wedge e_2 \wedge \cdots \wedge e_m) &= \sum_{i=1}^{m} (-1)^{i-1} f_i \check{e}_i \\ &= \sum_{i=1}^{m} (-1)^{i-1} (\sum_{j=1}^{n} x_{ij} T_j) \check{e}_i \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i-1} x_{ij} T_j \check{e}_i . \end{aligned}$$

LEMMA 2.6. $\operatorname{rank}_R[K_1]_1 = m, \operatorname{rank}_R[K_0]_1 = n$ *and* $\mathrm{I}_m([\partial_1]_1) = \mathrm{I}_m(M)$.

PROOF. We get $\operatorname{rank}_R[K_1]_1 = m$ as $e_1, e_2, \dots, e_m$ is an R-free basis of $[K_1]_1$. Moreover, we get $\operatorname{rank}_R[K_0]_1 = n$ as $T_1, T_2, \dots, T_n$ is an R-free basis of $[K_0]_1$. The last assertion holds since

$$\partial_1(e_i) = f_i = x_{i1}T_1 + x_{i2}T_2 + \cdots + x_{in}T_n$$

for all $i = 1, \ldots, m$.

3. Proofs of Theorem 1.1 and Theorem 1.2

Let us prove Theorem 1.1.

PROOF OF THEOREM 1.1. We prove by induction on m. If $m = 1$, then the assertion holds by Lemma 2.2. So, we may assume $m \geq 2$.

(1) $\Rightarrow$ (2) (The proof of this implication is same as that of [1, Proposition 1]). Put $t = \operatorname{grade}(f_1, f_2, \ldots, f_m)S$ and suppose $t < m$. Take a maximal S-regular sequence $g_1, g_2, \ldots, g_t$ contained in $(f_1, f_2, \ldots, f_m)S$. Then, as any element in $(f_1, f_2, \ldots, f_m)S$ is a zerodivisor on $S/(g_1, g_2, \ldots, g_t)S$, there exists $P \in \operatorname{Ass}_S S/(g_1, g_2, \ldots, g_t)S$ such that $(f_1, f_2, \ldots, f_m)S \subseteq P$. When this is the case, we have $\operatorname{depth} S_P = \operatorname{grade} P = t < m$. Here we put $\mathfrak{p} = P \cap R$. Then $\operatorname{grade}\mathfrak{p} \leq \operatorname{grade} P < m$, and so $\mathrm{I}_1(M) \not\subseteq \mathfrak{p}$ as $\operatorname{grade}\mathrm{I}_1(M) \geq m$. Hence, there exists a matrix $N = (y_{ij}) \in \mathrm{M}(m-1, n-1\,; R_\mathfrak{p})$ satisfying the conditions (a) and (b) of Lemma 2.4. By (a), we have

$$\operatorname{grade}\mathrm{I}_k(N) \geq \operatorname{grade}\mathrm{I}_{k+1}(M) \geq m - (k+1) + 1 = (m-1) - k + 1$$

for $k = 1, \ldots, m-1$. As is defined in (b), we put $S' = R_\mathfrak{p}[T_1, T_2, \ldots, T_{n-1}]$ and

$$g_i = y_{i1}T_1 + y_{i2}T_2 + \cdots + y_{i,n-1}T_{n-1}$$

for $i = 1, \ldots, m-1$. The hypothesis of induction implies $\operatorname{grade}(g_1, g_2, \ldots, g_{m-1})S' = m-1$. Hence we get $\operatorname{grade}(f_1, f_2, \ldots, f_m)S_\mathfrak{p} = m$ by (b) of Lemma 2.4, and so

$$\operatorname{grade}(f_1, f_2, \ldots, f_m)S_P \geq m\,,$$

which contradicts to $\operatorname{depth} S_P < m$. Thus we see $t = m$.

(2) $\Rightarrow$ (1) By[2, 1.6.17], $K_\bullet = K_\bullet(f_1, \ldots, f_m)$ is acyclic, and so

$$0 \longrightarrow [K_m]_m \xrightarrow{[\partial_m]_m} [K_{m-1}]_m \longrightarrow \cdots \longrightarrow [K_1]_m \xrightarrow{[\partial_1]_m} [K_0]_m \longrightarrow 0$$

is acyclic, too. Then, as $[K_m]_m \cong R$ and $\mathrm{I}_1([\partial_m]_m) = \mathrm{I}_1(M)$ by Lemma 2.5, we get $\operatorname{grade}\mathrm{I}_1(M) \geq m$ by [2, 1.4.13]. Suppose that $\ell := \operatorname{grade}\mathrm{I}_k(M) \leq m-k$ for some k with $2 \leq k \leq m$. We take a maximal R-regular sequence $c_1, c_2, \ldots, c_\ell$ contained in $\mathrm{I}_k(M)$. Then, as any element in $\mathrm{I}_k(M)$ is a zerodivisor on $R/(c_1, c_2, \ldots, c_\ell)R$, we have $\mathrm{I}_k(M) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R R/(c_1, c_2, \ldots, c_\ell)R$. When this is the case, we have $\operatorname{depth} R_\mathfrak{p} = \operatorname{grade}\mathfrak{p} = \ell \leq m-k$, which means $\mathrm{I}_1(M) \not\subseteq \mathfrak{p}$. Then, we get a matrix $N = (y_{ij}) \in \mathrm{M}(m-1, n-1\,; R_\mathfrak{p})$ satisfying the conditions (a) and (b) of Lemma 2.4.

As is defined in (b), we put $S' = R_{\mathfrak{p}}[T_1, T_2, \dots, T_{n-1}]$ and

$$g_i = y_{i1}T_1 + y_{i2}T_2 + \cdots + y_{i,n-1}T_{n-1}$$

for each $i = 1, \dots, m-1$. By (b) of Lemma 2.4, we have grade $(g_1, g_2, \dots, g_{m-1})S' = m-1$. Then, the hypothesis of induction implies

$$\operatorname{grade} \mathrm{I}_{k-1}(N) \geq (m-1) - (k-1) + 1 = m - k + 1\,,$$

which contradicts to depth $R_{\mathfrak{p}} \leq m - k$ as $\mathrm{I}_{k-1}(N) = \mathrm{I}_k(M)R_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}}$. Thus we get

$$\operatorname{grade} \mathrm{I}_k(M) \geq m - k + 1$$

for all $k = 1, \dots, m$ and the proof is complete.

Next, we prove Theorem 1.2.

PROOF OF THEOREM 1.2. (1) $\Rightarrow$ (2) By the hypothesis we have grade $I \geq 2$ as $I = \mathrm{I}_m(M)$, and so

$$0 \longrightarrow R^m \xrightarrow{{}^tM} R^{m+1} \xrightarrow{\phi} I \longrightarrow 0$$

is a free resolution of I by [2, 1.4.17]. We can regard S as the symmetric algebra of R^{m+1}, where $T_1, T_2, \dots, T_{m+1}$ corresponds to the standard free basis of R^{m+1}. Then the homomorphism ϕ induces a surjection $\mathrm{S}(\phi) : S \to \mathrm{S}(I)$. It is well known that the kernel of $\mathrm{S}(\phi)$ is generated by linear forms, so we get

$$\operatorname{Ker} \mathrm{S}(\phi) = (f_1, f_2, \dots, f_m)S$$

from the short exact sequence stated above. Hence the condition (ii) is satisfied. Moreover, we see the condition (iii) is satisfied by (1) $\Rightarrow$ (2) of Theorem 1.1. So, we have to show the assertion (i). Let L be the kernel of the natural map $\mathrm{S}(I) \to \mathrm{R}(I)$, and consider the exact sequence

$$0 \longrightarrow L \longrightarrow \mathrm{S}(I) \longrightarrow \mathrm{R}(I) \longrightarrow 0$$

of S-modules. Let us prove $L = 0$ by induction on m.

First, we consider the case where $m = 1$. Suppose $L \neq 0$. Then there exists $P \in \operatorname{Ass}_S L$. Because $L \subseteq \mathrm{S}(I) \cong S/f_1S$ by (ii), we have $P \in \operatorname{Ass}_S S/f_1S$, and so grade $P = 1$ by (iii). We put $\mathfrak{p} = P \cap R$. Then grade $\mathfrak{p} \leq 1$. Because $I = \mathrm{I}_1(M)$ and grade $\mathrm{I}_1(M) \geq 2$, we have $I \nsubseteq \mathfrak{p}$. This means $I_{\mathfrak{p}} = R_{\mathfrak{p}}$, and so the natural map $\mathrm{S}(I_{\mathfrak{p}}) \to \mathrm{R}(I_{\mathfrak{p}})$ is an isomorphism. Then, looking at the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{\mathfrak{p}} & \longrightarrow & \mathrm{S}(I)_{\mathfrak{p}} & \longrightarrow & \mathrm{R}(I)_{\mathfrak{p}} & \longrightarrow & 0 \text{ (ex)} \\ & & & & \downarrow \wr & & \downarrow \wr & & \\ & & & & \mathrm{S}(I_{\mathfrak{p}}) & \xrightarrow{\sim} & \mathrm{R}(I_{\mathfrak{p}})\,, & & \end{array}$$

we get $L_{\mathfrak{p}} = 0$, and so $L_P = 0$, which contradicts to $P \in \operatorname{Ass}_R L$. Therefore we see $L = 0$.

Next, we consider the case where $m \geq 2$. Suppose $L \neq 0$. Then there exists $P \in \mathrm{Ass}_S L$. Because $L \subseteq \mathrm{S}(I) \cong S/(f_1, \dots, f_m)S$ by (ii), we have $P \in \mathrm{Ass}_S S/(f_1, \dots, f_m)S$, and so $\mathrm{grade}\, P = m$ by (iii). We put $\mathfrak{p} = P \cap R$. Then $\mathrm{grade}\, \mathfrak{p} \leq m$, and so $\mathrm{I}_1(M) \nsubseteq \mathfrak{p}$ as $\mathrm{grade}\, \mathrm{I}_1(M) \geq m+1$. Hence, there exists a matrix $N = (y_{ij}) \in \mathrm{M}(m-1, m\,; R_\mathfrak{p})$ satisfying the conditions of Lemma 2.4. When this is the case, for all $k = 1, \dots, m-1$, we have

$$\begin{aligned} \mathrm{grade}\, \mathrm{I}_k(N) &= \mathrm{grade}\, \mathrm{I}_{k+1}(M) R_\mathfrak{p} \\ &\geq m - (k+1) + 2 \\ &= (m-1) - k + 2\,. \end{aligned}$$

We notice that $\mathrm{I}_{m-1}(N) = \mathrm{I}_m(M) R_\mathfrak{p} = I_\mathfrak{p}$, and so the hypothesis of induction implies that the natural map $\mathrm{S}(I_\mathfrak{p}) \to \mathrm{R}(I_\mathfrak{p})$ is isomorphic. Now we look at the commutative diagram above again, and get $L_\mathfrak{p} = 0$. Hence $L_P = 0$, which contradicts to $P \in \mathrm{Ass}_S L$.

(2) $\Rightarrow$ (1) By (i) and (ii), there exists a surjection $\pi : S \to \mathrm{R}(I)$ of graded R-algebras such that $\mathrm{Ker}\, \pi = (f_1, f_2, \dots, f_m)S$. On the other hand, $K_\bullet = K_\bullet(f_1, f_2, \dots, f_m)$ is acyclic. Then,

$$0 \longrightarrow K_m \xrightarrow{\partial_m} K_{m-1} \longrightarrow \cdots \longrightarrow K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\pi'} R[t] \qquad (\natural)$$

is also acyclic, where π' is the composition of π and $\mathrm{R}(I) \hookrightarrow R[t]$. Now we take the homogeneous part of $(\natural)$ of degree m. Then we get an acyclic complex

$$0 \longrightarrow [K_m]_m \xrightarrow{[\partial_m]_m} [K_{m-1}]_m \longrightarrow \cdots \longrightarrow [K_1]_m \xrightarrow{[\partial_1]_m} [K_0]_m \xrightarrow{\varepsilon_m} R$$

of finitely generated free R-modules, where ε_m is the composition of

$$[\pi']_m : S_m = [K_0]_m \to I^m t^m \quad \text{and} \quad I^m t^m \ni a t^m \mapsto a \in R.$$

As a consequence, it follows that $\mathrm{grade}\, \mathrm{I}_1(M) \geq m+1$ by 2.5 and [2, 1.4.13]. On the other hand, by taking the homogeneous part of $(\natural)$ of degree 1, we get an acyclic complex

$$0 \longrightarrow [K_1]_1 \xrightarrow{[\partial_1]_1} [K_0]_1 \longrightarrow R$$

of finitely generated free R-modules, and so it follows that $\mathrm{grade}\, \mathrm{I}_m(M) \geq 2$ by 2.6.

In the rest, by induction on m, we prove $\mathrm{grade}\, \mathrm{I}_k(M) \geq m - k + 2$ for all $k = 1, \dots, m$. This is certainly true if $m = 1$ or 2 by our observation stated above. So we may assume $m \geq 3$. Suppose that we have $\ell := \mathrm{grade}\, \mathrm{I}_k(M) \leq m - k + 1$ for some k with $1 < k < m$. We take a maximal R-regular sequence $c_1, c_2, \dots, c_\ell$ contained in $\mathrm{I}_k(M)$. Then, there exists $\mathfrak{p} \in \mathrm{Ass}_R R/(c_1, c_2, \dots, c_\ell)R$ such that $\mathrm{I}_k(M) \subseteq \mathfrak{p}$. When this is the case, we have $\mathrm{grade}\, \mathfrak{p} = \mathrm{depth}\, R_\mathfrak{p} = \ell \leq m - k + 1$. Hence $\mathrm{I}_1(M) \nsubseteq \mathfrak{p}$ as $\mathrm{grade}\, \mathrm{I}_1(M) \geq m + 1 > m - k + 1$. Then, we get a matrix $N = (y_{ij}) \in \mathrm{M}(m-1, m\,; R_\mathfrak{p})$ satisfying the conditions of Lemma 2.4. Let us notice that $\mathrm{I}_{m-1}(N) = \mathrm{I}_m(M) R_\mathfrak{p} = I_\mathfrak{p}$. As $\mathrm{S}(I)_\mathfrak{p} \xrightarrow{\sim} \mathrm{R}(I)_\mathfrak{p}$ by (i), we have

$\mathrm{S}(I_\mathfrak{p}) \xrightarrow{\sim} \mathrm{R}(I_\mathfrak{p})$. We set $S' = R_\mathfrak{p}[T_1, T_2, \dots, T_m]$ and

$$g_i = y_{i1}T_1 + y_{i2}T_2 + \cdots + y_{im}T_m$$

for each $i = 1, \dots, m-1$. Then, there exists an isomorphism $\varphi : S_\mathfrak{p} \xrightarrow{\sim} S_\mathfrak{p}$ of $R_\mathfrak{p}$-algebras such that $\varphi((f_1, f_2, \dots, f_m)S_\mathfrak{p}) = (g_1, g_2, \dots, g_{m-1}, T_m)S_\mathfrak{p}$, and we have

$$\begin{aligned}\mathrm{S}(I_\mathfrak{p}) &\cong \mathrm{S}(I)_\mathfrak{p} \\ &\cong S_\mathfrak{p}/(f_1, f_2, \dots, f_m)S_\mathfrak{p} \quad \text{(by (ii))} \\ &\cong S_\mathfrak{p}/(g_1, g_2, \dots, g_{m-1}, T_m)S_\mathfrak{p} \quad \text{(isomorphism induced from } \varphi) \\ &\cong S'/(g_1, \dots, g_{m-1})S'.\end{aligned}$$

Moreover, we get $\operatorname{grade}(g_2, \dots, g_m)S' = m - 1$. Therefore the hypothesis of induction implies that

$$\operatorname{grade} \mathrm{I}_{k-1}(N) \geq (m-1) - (k-1) + 2 = m - k + 2,$$

which contradicts to $\operatorname{depth} R_\mathfrak{p} \leq m - k + 1$ as $\mathrm{I}_{k-1}(N) = \mathrm{I}_k(M)R_\mathfrak{p} \subseteq \mathfrak{p}R_\mathfrak{p}$. Thus we see $\operatorname{grade} \mathrm{I}_k(M) \geq m - k + 2$ for all $k = 1, \dots, m$ and the proof is complete.

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