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A Wiener-Tauberian Type Theorem for Arbitrary Normed Algebras with an Approximate Identity

Tobias BLENDEK

Helmut Schmidt University Hamburg

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Abstract. For an arbitrary normed algebra with an approximate identity, we introduce some notions of invertibility of a net in this algebra with respect to the given approximate identity. By means of that, we show a Wiener-Tauberian type theorem in that general situation. An application yields an abstract Wiener-Tauberian theorem for a class of commutative Banach algebras, which implies the classical Wiener-Tauberian theorem for locally compact Abelian groups.

1. Introduction

The purpose of this paper is to formulate *Wiener's Tauberian theorem* from abstract harmonic analysis within the general situation of an arbitrary normed algebra with an approximate identity, see also [2]. In order to repeat the classical case, let G be a locally compact Abelian group with its dual group Γ of G. The set of all Fourier transforms $\hat{f}$ of functions $f \in L^1(G)$ such that, for all $\gamma \in \Gamma$,

$$\hat{f}(\gamma) := \int_G f(s)\overline{\gamma(s)}\, ds$$

is denoted by $A(\Gamma)$, and it is a commutative Banach algebra. Then the following result is known as Wiener's Tauberian theorem, see e.g. [1, Theorem 1.1.3]: If a function $\hat{\psi} \in A(\Gamma)$ never vanishes on Γ, then Γ is σ-compact, and for every $\hat{\vartheta} \in A(\Gamma)$ and $\varepsilon > 0$, there are complex numbers $c_1, \ldots, c_n \in \mathbf{C}$ as well as $x_1, \ldots, x_n \in G$ such that

$$\left\| \hat{\vartheta}(\cdot) - \sum_{i=1}^{n} c_i\, \hat{\psi}(\cdot)(\cdot, x_i) \right\|_{A(\Gamma)} < \varepsilon\,.$$

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In the following, let A be an arbitrary normed algebra, and let $(\Lambda, \leq)$ be a directed index set. Then a net $\{e_\lambda\}_{\lambda\in\Lambda}$ in A is called a *left* (*right*) *approximate identity* if, for all $a \in A$,

$$\|a - e_\lambda a\|_A \to 0 \quad (\|a - ae_\lambda\|_A \to 0).$$

A net $\{e_\lambda\}_{\lambda\in\Lambda}$ is called a *two-sided approximate identity* if $\{e_\lambda\}_{\lambda\in\Lambda}$ is both a left and a right approximate identity. Furthermore, $\{e_\lambda\}_{\lambda\in\Lambda}$ is called *bounded* if there is a constant $K > 0$ such that $\|e_\lambda\|_A \leq K$ for all $\lambda \in \Lambda$. A left (right, two-sided) approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ is called *sequential* if $\Lambda = \mathbf{N}$.

Now, regarding [3, Proposition 12.2], we replace the concept of σ-compactness in [1, Theorem 1.1.3] by the concept of an approximate identity. Whereas in case of $L^1(G)$ such an approximate identity can be constructed, we have to assume its existence in an arbitrary normed algebra. The condition in [1, Theorem 1.1.3], i.e. that a given $\hat{\psi} \in A(\Gamma)$ never vanishes on Γ, leads us to our notions of invertibility of a net in that normed algebra with respect to the given approximate identity, which yields the announced Wiener-Tauberian type theorem. In fact, we show that to each $\hat{\psi} \in A(\Gamma)$ never vanishing on Γ, there corresponds such a net in $A(\Gamma)$.

Moreover, we can apply our Wiener-Tauberian type theorem for arbitrary normed algebras with an approximate identity to the situation of completely regular, semi-simple, commutative Banach algebras A with an approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ in A such that each Gelfand transform $\widehat{e_\lambda}$ of e_λ has compact support in the maximal ideal space Φ_A of A. For definitions, we refer to [4]. More precisely, we get an abstract Wiener-Tauberian theorem by proving that the principal ideal $aA := \{ax : x \in A\}$ generated by $a \in A$ is dense in A if the Gelfand transform $\hat{a}$ of a never vanishes on the maximal ideal space Φ_A of A. As a consequence, we obtain the classical Wiener-Tauberian theorem for $A(\Gamma)$ again.

2. A Wiener-Tauberian Type Theorem

DEFINITION 2.1. Let A be a normed algebra with a left (right) approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ in A.

(*a*) We call a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in A *right* (*left*) *invertible* in A if there is a net $\{b_\lambda\}_{\lambda\in\Lambda}$ in A such that, for all $\lambda \in \Lambda$,

$$a_\lambda b_\lambda = e_\lambda \quad (b_\lambda a_\lambda = e_\lambda)\,.$$

If $\{e_\lambda\}_{\lambda\in\Lambda}$ is a two-sided approximate identity in A, we say that $\{a_\lambda\}_{\lambda\in\Lambda}$ is *invertible* in A if it is both left and right invertible.

(*b*) We call a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in A *right-extended* (*left-extended*) *invertible* in A if there is a net $\{b_\lambda\}_{\lambda\in\Lambda}$ in A such that, for all $\lambda \in \Lambda$,

$$a_\lambda b_\lambda e_\lambda = e_\lambda \quad (e_\lambda b_\lambda a_\lambda = e_\lambda)\,.$$

If $\{e_\lambda\}_{\lambda\in\Lambda}$ is a two-sided approximate identity in A, we say that $\{a_\lambda\}_{\lambda\in\Lambda}$ is *extended invertible* in A if it is both left-extended and right-extended invertible.

(c) We call a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in A *right-centered* (*left-centered*) *invertible* in A if there is a net $\{b_\lambda\}_{\lambda\in\Lambda}$ in A such that, for all $\lambda \in \Lambda$,

$$a_\lambda e_\lambda b_\lambda = e_\lambda \quad (b_\lambda e_\lambda a_\lambda = e_\lambda)\,.$$

If $\{e_\lambda\}_{\lambda\in\Lambda}$ is a two-sided approximate identity in A, we say that $\{a_\lambda\}_{\lambda\in\Lambda}$ is *centered invertible* in A if it is both left-centered and right-centered invertible.

THEOREM 2.2. *Let A be a normed algebra with a left* (*right*) *approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ in A. Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be a net in A such that it is*

(*a*) *right* (*left*) *invertible in A or*

(*b*) *right-extended* (*left-extended*) *invertible in A or*

(*c*) *right-centered* (*left-centered*) *invertible in A.*

Then, in each of these cases, for every $c \in A$, there is a net $\{f_\lambda\}_{\lambda\in\Lambda}$ in A such that

$$\|c - a_\lambda f_\lambda\|_A \to 0 \quad (\|c - f_\lambda a_\lambda\|_A \to 0)\,.$$

PROOF. (*a*) Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be a right (left) invertible net in A. Then there is a net $\{b_\lambda\}_{\lambda\in\Lambda}$ in A such that $a_\lambda b_\lambda = e_\lambda$ ($b_\lambda a_\lambda = e_\lambda$) for all $\lambda \in \Lambda$. Let c be an arbitrary element of A. For each $\lambda \in \Lambda$, we set

$$f_\lambda := b_\lambda c \in A \quad (f_\lambda := c b_\lambda \in A)\ .$$

Hence, we get

$$\|c - a_\lambda f_\lambda\|_A = \|c - a_\lambda b_\lambda c\|_A = \|c - e_\lambda c\|_A \to 0$$

$$(\|c - f_\lambda a_\lambda\|_A = \|c - c b_\lambda a_\lambda\|_A = \|c - c e_\lambda\|_A \to 0)\,.$$

(*b*) Similar to (*a*), the assertion follows by setting, for each $\lambda \in \Lambda$,

$$f_\lambda := b_\lambda e_\lambda c \in A \quad (f_\lambda := c e_\lambda b_\lambda \in A)\ .$$

(*c*) Similar to (*a*), the assertion follows by setting, for each $\lambda \in \Lambda$,

$$f_\lambda := e_\lambda b_\lambda c \in A \quad (f_\lambda := c b_\lambda e_\lambda \in A)\ .$$

□

DEFINITION 2.3. Let A be a normed algebra with a bounded two-sided approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ in A. We call a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in A *quadratically invertible* in A if there is a net $\{b_\lambda\}_{\lambda\in\Lambda}$ in A such that, for all $\lambda \in \Lambda$,

$$a_\lambda e_\lambda b_\lambda e_\lambda = e_\lambda^2\,.$$

THEOREM 2.4. *Let A be a normed algebra with a bounded two-sided approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ in A. If $\{a_\lambda\}_{\lambda\in\Lambda}$ is a quadratically invertible net in A, then for every $c \in A$, there is a net $\{f_\lambda\}_{\lambda\in\Lambda}$ in A such that*

$$\|c - a_\lambda f_\lambda\|_A \to 0\,.$$

PROOF. Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be a quadratically invertible net in A. Then there is a net $\{b_\lambda\}_{\lambda\in\Lambda}$ in A such that $a_\lambda e_\lambda b_\lambda e_\lambda = e_\lambda^2$ for all $\lambda \in \Lambda$. Let c be an arbitrary element of A. For each $\lambda \in \Lambda$, we set

$$f_\lambda := e_\lambda b_\lambda e_\lambda c e_\lambda \in A\,.$$

Hence, since the net $\{e_\lambda^2\}_{\lambda\in\Lambda}$ is a bounded two-sided approximate identity in A, too, we get

$$\begin{aligned}
\|c - a_\lambda f_\lambda\|_A &\le \|c - ce_\lambda\|_A + \|ce_\lambda - a_\lambda f_\lambda\|_A \\
&= \|c - ce_\lambda\|_A + \|ce_\lambda - (a_\lambda e_\lambda b_\lambda e_\lambda)ce_\lambda\|_A \\
&= \|c - ce_\lambda\|_A + \|(c - e_\lambda^2 c)e_\lambda\|_A \\
&\le \|c - ce_\lambda\|_A + \|c - e_\lambda^2 c\|_A \|e_\lambda\|_A \\
&\le \|c - ce_\lambda\|_A + K\|c - e_\lambda^2 c\|_A \\
&\to 0\,.
\end{aligned}$$

□

COROLLARY 2.5. *Let A be a normed algebra with a bounded two-sided approximate identity* (*left approximate identity*) $\{e_\lambda\}_{\lambda\in\Lambda}$ *in A, and let I be a closed two-sided ideal in A. In addition, one of the following two conditions should hold*:

(a) *There is a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in A such that it is quadratically invertible* (*right invertible, right-extended invertible, right-centered invertible*) *in I.*

(b) *There is a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in I such that it is quadratically invertible* (*right invertible, right-extended invertible, right-centered invertible*) *in A.*

Then we have $I = A$.

PROOF. (a) If $\{a_\lambda\}_{\lambda\in\Lambda}$ is a net in A such that it is quadratically invertible (right invertible, right-extended invertible, right-centered invertible) in I, then, according to Theorem 2.4 (Theorem 2.2(a), (b), (c)), for every $c \in A$, there is a net $\{f_\lambda\}_{\lambda\in\Lambda}$ in I such that

$$\|c - a_\lambda f_\lambda\|_A \to 0\,.$$

Since I is a two-sided ideal in A, we have $a_\lambda f_\lambda \in I$ for all $\lambda \in \Lambda$. Since I is closed, we conclude that $c \in I$, i.e. we get $I = A$.

(b) This assertion follows in like manner as (a). □

An application of Theorem 2.2(b), (c) and Theorem 2.4, respectively, yields the following Wiener-Tauberian theorem in the classical case of locally compact Abelian groups.

REMARK 2.6. Let G be a locally compact Abelian group with its dual group Γ of G. If $\hat{\psi} \in A(\Gamma)$ never vanishes on Γ, then there is a sequential, bounded approximate identity $\{\widehat{e_n}\}_{n\in\mathbf{N}}$ in $A(\Gamma)$ such that for every $\hat{\vartheta} \in A(\Gamma)$ and $\varepsilon > 0$, there are an $N(\varepsilon)$ and a sequence

$\{\widehat{f_n}\}_{n\in\mathbf{N}}$ in $A(\Gamma)$ such that, for all $n \geq N(\varepsilon)$,

$$\|\hat{\vartheta} - \hat{\psi}\widehat{f_n}\|_{A(\Gamma)} < \varepsilon\,.$$

Hence, the principal ideal $\hat{\psi}A(\Gamma) := \{\hat{\psi}\hat{\varphi} : \hat{\varphi} \in A(\Gamma)\}$ generated by $\hat{\psi}$ is dense in $A(\Gamma)$.

PROOF. Let $\hat{\psi} \in A(\Gamma)$ never vanish, i.e. let the zero set $Z(\hat{\psi})$ be empty. Thus we have $\operatorname{supp}\hat{\psi} = \Gamma$, where $\operatorname{supp}\hat{\psi}$ denotes the support of $\hat{\psi}$. Then, by [1, Proposition 1.1.2(*b*)], the dual group Γ is σ-compact. Consequently, regarding [1, Theorem 1.2.1], there is a sequential, bounded approximate identity $\{\widehat{e_n}\}_{n\in\mathbf{N}}$ in $A(\Gamma)$ such that, for all $n \in \mathbf{N}$,

$$\|\widehat{e_n}\|_{A(\Gamma)} = 1, \quad \widehat{e_n} \geq 0 \quad \text{and} \quad \widehat{e_n} \in A_c(\Gamma) := A(\Gamma) \cap C_c(\Gamma)\,,$$

where $C_c(\Gamma)$ denotes the algebra of all continuous, complex-valued functions on Γ with compact support. Let $C_n := \operatorname{supp}\widehat{e_n}$ be the compact support of $\widehat{e_n}$ for all $n \in \mathbf{N}$. Since $\hat{\psi}(\gamma) \neq 0$ for all $\gamma \in \Gamma$ and since $C_n \subseteq \Gamma$ is compact for all $n \in \mathbf{N}$, there is a $\widehat{\psi_n}' \in A(\Gamma)$ according to Wiener's inversion theorem, see e.g. [1, Proposition 1.1.5(*b*)], such that, for all $\gamma \in C_n$,

$$\widehat{\psi_n}'(\gamma) = 1/\hat{\psi}(\gamma)\,.$$

Since $A(\Gamma)$ is a commutative Banach algebra, we conclude that, for all $\gamma \in C_n$,

$$\hat{\psi}(\gamma)\widehat{e_n}(\gamma)\widehat{\psi_n}'(\gamma) = \hat{\psi}(\gamma)\widehat{\psi_n}'(\gamma)\widehat{e_n}(\gamma) = \widehat{e_n}(\gamma)\,. \tag{2.1}$$

For each $\gamma \notin C_n$, we have $\widehat{e_n}(\gamma) = 0$, and hence, (2.1) is valid for all $n \in \mathbf{N}$ and for all $\gamma \in \Gamma$. Of course, we also have, for all $n \in \mathbf{N}$ and for all $\gamma \in \Gamma$,

$$\hat{\psi}(\gamma)\widehat{e_n}(\gamma)\widehat{\psi_n}'(\gamma)\widehat{e_n}(\gamma) = \widehat{e_n}^2(\gamma).$$

So, identifying $\hat{\psi} \in A(\Gamma)$ with the sequence $\{\widehat{\psi_n}\}_{n\in\mathbf{N}}$ defined by $\widehat{\psi_n} := \hat{\psi}$ for all $n \in \mathbf{N}$, we see that $\hat{\psi}$ is quadratically invertible (extended invertible, centered invertible) with respect to the sequential, bounded approximate identity $\{\widehat{e_n}\}_{n\in\mathbf{N}}$ in $A(\Gamma)$. Consequently, the assertion follows from Theorem 2.4 (Theorem 2.2(*b*), (*c*)). □

Now, we show an abstract Wiener-Tauberian theorem for a class of commutative Banach algebras by using Theorem 2.2 (*b*), (*c*).

THEOREM 2.7. *Let A be a completely regular, semi-simple, commutative Banach algebra with an approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ in A such that each Gelfand transform $\widehat{e_\lambda}$ of e_λ has compact support in the maximal ideal space Φ_A of A. Let a be an arbitrary element of A. If $\hat{a}$ never vanishes on Φ_A, then the principal ideal $aA := \{ax : x \in A\}$ generated by a is dense in A.*

PROOF. Assume that $\hat{a}$ never vanishes on Φ_A and take $\lambda \in \Lambda$ arbitrarily. Denote by K_λ the compact support of $\widehat{e_\lambda}$. By hypothesis, $|\hat{a}(\varphi)| \geq \delta_\lambda$ holds for all $\varphi \in K_\lambda$ and some $\delta_\lambda > 0$. Then, according to [4, Theorem 3.6.15] and [4, Theorem 3.7.1], we can find a $b_\lambda \in A$ such that $\hat{a}(\varphi)\widehat{b_\lambda}(\varphi) = 1$ for all $\varphi \in K_\lambda$. Therefore, we obtain $\hat{a}\widehat{b_\lambda}\widehat{e_\lambda} = \widehat{e_\lambda}$, and hence $ab_\lambda e_\lambda = e_\lambda$ since A is semi-simple. Put $a_\lambda := a$ for each $\lambda \in \Lambda$. Then $\{a_\lambda\}_{\lambda\in\Lambda}$ is an extended invertible and, of course, a centered invertible net in A with respect to $\{e_\lambda\}_{\lambda\in\Lambda}$. Consequently, it follows from Theorem 2.2(b), (c) that the principal ideal aA is dense in A. □

REMARK 2.8. (a) Of course, we also have the converse of Theorem 2.7: Let A be a commutative Banach algebra with maximal ideal space Φ_A, and let a be an arbitrary element of A. If the principal ideal $aA := \{ax : x \in A\}$ generated by a is dense in A, then the Gelfand transform $\hat{a}$ of a never vanishes on Φ_A.

(b) Let G be a locally compact Abelian group, and let $\psi \in L^1(G)$ such that $\hat{\psi} \in A(\Gamma)$ never vanishes on Γ. Since $L^1(G)$ is a completely regular, semi-simple, commutative Banach algebra with the convolution product $*$ and since, according to [1, Theorem 1.2.1], there is a sequential, bounded approximate identity in $A(\Gamma)$ with compact support in Γ, it follows from Theorem 2.7 that the principal ideal $\psi * L^1(G) := \{\psi * \varphi : \varphi \in L^1(G)\}$ generated by ψ is dense in $L^1(G)$, i.e. we obtain the classical Wiener-Tauberian theorem for locally compact Abelian groups, see Remark 2.6.

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References

[1] J. J. BENEDETTO, *Spectral Synthesis*, Academic Press, Inc., New York, 1975.

[2] T. BLENDEK, *Normen von Projektionen im Dual einer kompakten Kac-Algebra*, Doctoral dissertation, University of Hamburg, Hamburg, 2010.

[3] R. S. DORAN and J. WICHMANN, *Approximate Identities and Factorization in Banach Modules*, Lecture Notes in Math. 768, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[4] C. E. RICKART, *General Theory of Banach Algebras*, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1960.

Present Address:
DEPARTMENT OF MATHEMATICS AND STATISTICS,
HELMUT SCHMIDT UNIVERSITY HAMBURG,
HOLSTENHOFWEG 85, 22043 HAMBURG, GERMANY.
e-mail: tobias.blendek@gmx.de