TOKYO J. MATH.
VOL. 37, NO. 1, 2014

Mod p Equality Theorem for Seiberg-Witten Invariants under $\mathbf{Z}_p$-actions

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Abstract. When a cyclic group G of prime order acts on a 4-manifold X, we prove a formula which relates the Seiberg-Witten invariants of X to those of X/G.

1. Introduction

The Seiberg-Witten invariants under group actions are investigated by many authors. In several cases, one can relate the Seiberg-Witten invariants of a 4-manifold X with an action of a group G to those of its quotient (V-)manifold X/G. In fact, in the case of free actions of prime order cyclic groups $G = \mathbf{Z}_p$, it is proved that the Seiberg-Witten invariant of X is equal modulo p to a sum of invariants of X/G, by Ruan-Wang [11], Szymik [12] and the author [7]. This *mod p equality theorem* is extended to the case of double branched coverings by Ruan-Wang [11], B. D. Park [9] and Cho-Hong [2]. On the other hand, F. Fang [3] proved a *mod p vanishing theorem* for $\mathbf{Z}_p$-actions. This is extended by the author [8], and in the view point there, the mod p vanishing theorem can be considered as a version of mod p equality theorem: If all the involved invariants of X/G are 0 by reason of negative dimensional moduli, then the invariant of X is divisible by p.

In this paper, we shall prove a mod p equality theorem for $\mathbf{Z}_p$-actions which is a generalization of both the mod p equality theorem for free actions and the mod p vanishing theorem. First, let us fix the notation. For an oriented closed 4-manifold X with a Spin^c-structure c, the Seiberg-Witten invariant of (X, c) is denoted by $\mathrm{SW}(X, c)$, and the virtual dimension of the moduli by $d(c)$. Suppose it is given an orientation-preserving action of a finite group G on X, and the G-action has a lift to c. In general, there are several ways of such lifts, and we use the suffix α to parameterize these lifts as G_α. When $G = \mathbf{Z}_p$, we may assume α to be an integer satisfying $0 \leq \alpha \leq p-1$. (See §3.) When the data (X, c, G_α) of a 4-manifold X with a G-action, a Spin^c-structure c, and a lift G_α of the G-action to c are given, Y. Ruan [10] defined the G-monopole invariant, denoted by $\mathrm{SW}(X, c, G_\alpha)$, which is naturally identified with the Seiberg-Witten invariant of V-manifold X/G with a V-Spin^c-structure c/G_α. (See §2.) The virtual dimension of the moduli of G_α-invariant solutions is denoted by $d(c, G_\alpha)$.

Received November 2, 2012; revised August 8, 2013

For any G-space Y, let Y^G be the fixed point set of the G-action. Let $b_\bullet^G = \dim H_\bullet(X;\mathbf{R})^G$, where $\bullet = 1, 2, +$.

Our main theorem is,

THEOREM 1.1. *Let $G = \mathbf{Z}_p$ be the cyclic group of odd prime order p, and X a closed oriented 4-manifold with $b_1 = 0$ and $b_+ \geq 2$. Suppose G acts on X with $b_+^G \geq 2$, and the G-action has a lift to a* Spin^c*-structure c with $d(c) = 0$. If $d(c, G_\alpha) \leq 0$ for any lift of the G-action, then*

$$\mathrm{SW}(X, c) \equiv \sum_{\alpha=0}^{p-1} \mathfrak{m}_\alpha \, \mathrm{SW}(X, c, G_\alpha) \mod p\,, \tag{1.2}$$

where $\mathfrak{m}_\alpha$ are integers determined by the G-index of the Dirac operator and the G-action on $H^+(X;\mathbf{R})$. (For instance, $\mathfrak{m}_\alpha = 0$ if $d(c, G_\alpha) < 0$. For the other cases, see around (3.2) for the precise definition.)

REMARK 1.3. Theorem 1.1 can be generalized to the case when $p = 2$ or $b_1 > 0$ with appropriate assumptions. To avoid a complicated description, we only give the proof of the case of Theorem 1.1, and the detail of such generalizations will be left to readers. Other possibilities of generalizations will be referred in Remark 3.3 below.

The strategy of the proof of Theorem 1.1 is analogous to those in [7] and [8]. We will work out a G-equivariant perturbation of the monopole map. Under the G-action, the moduli space splits into two parts: the G-fixed part and the G-free part. When the dimension of the moduli is 0, the number of solutions in the G-free part is a multiple of p. On the other hand, the number of G-invariant solutions *is* the G-monopole invariant. However, the transversality is not necessarily achieved on these G-invariant solutions. Then, we give a canonical way of G-equivariant perturbation, which enables us to determine the multiplicities of these solutions.

The organization of the paper is as follows: Section 2 gives a brief review on G-monopole invariants. In Section 3, we prove Theorem 1.1. In Section 4, we discuss several examples.

2. G-monopole invariants

In this section, we give a brief review on the G-monopole invariants defined by Ruan [10].

Let X be a closed oriented 4-manifold, and c a Spin^c-structure on X. Let $\mathcal{G}$ be the gauge transformation group which consists of automorphisms of c covering the identity map of X. Note $\mathcal{G} = \mathrm{Map}(X, S^1)$. We introduce another automorphism group $\tilde{\mathcal{G}}$ consisting of pairs $(f, \tilde{f})$, where $f\colon X \to X$ is an orientation preserving diffeomorphism of X, and $\tilde{f}\colon c \to c$ is an automorphism of c covering f. Then, we have an exact sequence,

$$1 \to \mathcal{G} \to \tilde{\mathcal{G}} \to \mathrm{Diff}^+(X)\,,$$

where $\mathrm{Diff}^+(X)$ is the group of orientation-preserving diffeomorphisms of X.

Let G be a finite group. Note that giving an effective orientation-preserving G-action on X is equivalent to giving a subgroup G of $\mathrm{Diff}^+(X)$. Suppose such a G-action on X is given, and c satisfies $g^*c \cong c$ for any $g \in G$. Then the following group extension exists,

$$1 \to \mathcal{G} \to \hat{\mathcal{G}} \xrightarrow{\theta} G \to 1 . \tag{2.1}$$

Note that giving a lift of the G-action to c is equivalent to giving a splitting of (2.1), that is, giving a subgroup G_α of $\hat{\mathcal{G}}$ which is isomorphic to G via θ.

Suppose we are given data (X, c, G_α) as above. In such a situation, Y. Ruan defined the *G-monopole invariant* [10] as follows. In this case, the Seiberg-Witten equations are G_α-equivariant, and the G_α-invariant moduli space $\mathcal{M}(X, c, G_\alpha)$ is defined as the set of equivalence classes of G_α-invariant solutions modulo G_α-invariant gauge transformations. The virtual dimension of $\mathcal{M}(X, c, G_\alpha)$ is given by

$$d(c, G_\alpha) = 2 \operatorname{ind} D^{G_\alpha} - (1 - b_1^G + b_+^G) ,$$

where $\operatorname{ind} D^{G_\alpha}$ is the virtual dimension of the trivial part of the G_α-equivariant Dirac index. Note that we can orient all of $\mathcal{M}(X, c, G_\alpha)$ at the same time by fixing an orientation of $(H^1(X; \mathbf{R}) \oplus H^+(X; \mathbf{R}))^G$. If $d(c, G_\alpha) = 0$, then the G-monopole invariant $\mathrm{SW}(X, c, G_\alpha)$ is defined as the signed count of the number of elements in $\mathcal{M}(X, c, G_\alpha)$. In general, we need to perturb the equations to avoid reducibles and achieve transversality. The standard argument proves that $\mathrm{SW}(X, c, G_\alpha)$ is well-defined if $b_+^G \geq 2$, and depends on chambers if $b_+^G = 1$.

3. Proof of Theorem 1.1

In this section, we prove our main theorem (Theorem 1.1).

Suppose (X, c) with a $G = \mathbf{Z}_p$-action satisfies the conditions in Theorem 1.1, and a lift of the G-action to c, say G_0, is given. Fix a G-invariant metric and a G_0-invariant connection A_0 on the determinant line bundle of c. Then the monopole map μ is a proper $G \times S^1$-equivariant map. Taking a finite dimensional approximation of μ [1], we have a $G \times S^1$-equivariant map between finite rank representations:

$$f_0 \colon V \oplus R \to W \oplus R \oplus H ,$$

where V and W are complex representations of G on which S^1 acts by multiplication, and R and $H = H^+(X; \mathbf{R})$ are real representations of G on which S^1 acts trivially. More explicitly, when $\mathbf{C}_j$ is the complex 1-dimensional weight j representation of G, V and W can be written as,

$$V = \mathbf{C}_0^{a_0} \oplus \mathbf{C}_1^{a_1} \oplus \cdots \mathbf{C}_{p-1}^{a_{p-1}} ,$$

$$W = \mathbf{C}_0^{b_0} \oplus \mathbf{C}_1^{b_1} \oplus \cdots \mathbf{C}_{p-1}^{b_{p-1}} .$$

The G_0-index of the Dirac operator associated to A_0 is written as

$$\mathrm{ind}_{G_0} D_{A_0} = \sum_{j=0}^{p-1} (a_j - b_j)\mathbf{C}_j \,.$$

Now, the lifts G_α are given as the splittings of the sequence

$$1 \to S^1 \to G \times S^1 \to G \to 1 \,.$$

(*Cf.* [12], Proposition 4.) The original lift G_0 corresponds to $G \times \{1\} \subset G \times S^1$. If we fix a generator g of G, then the other lifts G_α are given by the subgroups generated by the elements $(g, e^{-2\pi\sqrt{-1}j/p}) \in G \times S^1$ for $1 \le j \le p-1$. Therefore all the lifts G_α are parameterized by $\alpha = j$ where $0 \le j \le p-1$. As G_j-representations, V and W become $V \otimes \mathbf{C}_{-j}$ and $W \otimes \mathbf{C}_{-j}$.

First, perturb f_0 $G \times S^1$-equivariantly so that the zero locus does not contain any reducible as follows: Take a nonzero element v in $H^+(X; \mathbf{R})^G$, and perturb f_0 to $f := f_0 + v$. Then $(f^{-1}(0))^{S^1} = \emptyset$. (See [8], Section 2.3.)

Dividing f by S^1, we obtain a section $s \colon B \to E$ of the vector bundle $E \to B$ which is given by

$$E = ((V \setminus \{0\}) \times R) \times_{S^1} (W \oplus R \oplus H) \,,$$
$$B = (V \setminus \{0\})/S^1 \times R \,.$$

When $d(c) = 0$, $\mathrm{SW}(X, c)$ is the signed count of zero points of s if s is transversal to the zero section. Note that $(V \setminus \{0\})/S^1$ is G-equivariantly homeomorphic to $P(V) \times \mathbf{R}_+$, where $P(V)$ is the projective space of V, and $\mathbf{R}_+$ is the space of positive real numbers. The G-fixed point set of $P(V)$ can be written as ([8], Lemma 3.1),

$$P(V)^G = \coprod_{j=0}^{p-1} P(\mathbf{C}_j^{a_j}) \,.$$

Let $B_j = \mathbf{R}_+ \times P(\mathbf{C}_j^{a_j}) \times R_0$, where R_0 is the G-fixed part of R. Then the G-fixed point set of B decomposes into its connected components as $B^G = B_0 \cup B_1 \cup \cdots \cup B_{p-1}$.

Note that each B_j corresponds to the lift G_j, and

$$d(c, G_j) = 2(a_j - b_j) - (1 + b_+^G) \,.$$

When $d(c, G_j) = 0$, $\mathrm{SW}(X, c, G_j)$ is given by the signed count of zero points of $s|_{B_j}$ if $s|_{B_j}$ is transversal to the zero section in E^G.

Now, let us carry out the G-equivariant perturbation of s. When $d(c, G_j) < 0$, we can perturb s G-equivariantly around B_j so that $s^{-1}(0) \cap B_j = \emptyset$. When $d(c, G_j) = 0$, we can perturb s G-equivariantly around B_j so that $s|_{B_j}$ is *transversal along* B_j. Then, the problem is how to count multiplicities of zero points on B_j.

Let x be a point in $s^{-1}(0) \cap B_j$. We would like to describe the differentiation $(Ds)_x$ of s at x. The tangent space of B at x decomposes into the G-invariant direction and its complement: $T_x B = T_x B_j \oplus V'$. Then $T_x B_j$ and V' can be identified as

$$T_x B_j = \mathbf{R} \times \mathbf{C}_0^{a_j - 1} \times R_0 ,$$

$$V' = \sum_{k \neq j} \mathbf{C}_{k-j}^{a_k} \oplus R' ,$$

where R' is the orthogonal complement of R_0 in R. By reordering $\mathbf{C}_j$'s in V', rewrite V' as $V' = \mathbf{C}_1^{a'_1} \oplus \cdots \oplus \mathbf{C}_{p-1}^{a'_{p-1}} \oplus R'$, where $a'_k = a_{k+j}$.

Similarly, the vertical tangent space $V_{s(x)}E$ of E at $s(x)$ decomposes as, $V_{s(x)}E = W_0 \oplus W'$, where W_0 is G-invariant part and W' is its complement. When we decompose $H = H^+(X; \mathbf{R})$ into $H_0 \oplus H'$, where H_0 is the G-fixed part and H' its complement, W' can be identified with

$$W' = \sum_{k \neq j} \mathbf{C}_{k-j}^{b_j} \oplus R' \oplus H' .$$

Let us choose orientations of H and H_0 (hence H' too), and fix an arbitrary identification $H' = \mathbf{C}_1^{h_1} \oplus \cdots \oplus \mathbf{C}_{p-1}^{h_{p-1}}$ so that $H = H_0 \oplus H'$ and $H_0 \oplus \mathbf{C}_1^{h_1} \oplus \cdots \oplus \mathbf{C}_{p-1}^{h_{p-1}}$ have same orientation. (Here, we used the assumption that p is odd.) Rewrite W' as $W' = \mathbf{C}_1^{b'_1} \oplus \cdots \oplus \mathbf{C}_{p-1}^{b'_{p-1}} \oplus R'$, where $b'_k = b_{k-j} + h_k$. Let L_0 be the linear map which is the composition of the following maps:

$$L_0 \colon V' \xrightarrow{Ds_x} T_{s(x)}E \xrightarrow{p_v} V_{s(x)}E \xrightarrow{p_w} W' ,$$

where p_v and p_w are the orthogonal projections.

We will *cancel out* common parts in V' and W' by a perturbation by a G-linear map. We give a local model of this as follows. Let $e_k = \min\{a'_k, b'_k\}$. We can take an orientation-preserving G-linear map $l \colon V' \to W'$ so that $\mathrm{Im}(L_0 + l) \cong \sum_k \mathbf{C}_k^{e_k} \oplus R'$. Let $W_e = \mathrm{Im}(L_0 + l)$ and its complement in W' be W_r, and $V_r = \mathrm{Ker}(L_0 + l)$ and its complement in V' be V_e. Then,

$$V' = V_e \oplus V_r ,$$

$$W' = W_e \oplus W_r ,$$

$$V_e \cong W_e \cong \sum_k \mathbf{C}_k^{e_k} \oplus R' .$$

Next, we give a local model of perturbation in the direction of V_r. Let

$$I = \{k \mid m_k = a'_k - e_k > 0\} \text{ and } I' = \{k \mid n_k = b'_k - e_k > 0\} .$$

Then

$$V_r = \sum_{k\in I} \mathbf{C}_k^{m_k}, \quad W_r = \sum_{k\in I'} \mathbf{C}_k^{n_k}.$$

Note that $I \cap I' = \emptyset$ and $\dim V_r = \dim W_r$. We will perturb s around x by a (nonlinear) G-equivariant map $\psi\colon V_r \to W_r$. The next example will illustrate how to take ψ.

EXAMPLE 3.1. Suppose $G = \mathbf{Z}_5$, $V_r = \mathbf{C}_1 \oplus \mathbf{C}_4$ and $W_r = \mathbf{C}_2 \oplus \mathbf{C}_3$. Then take $\psi\colon \mathbf{C}_1 \oplus \mathbf{C}_4 \to \mathbf{C}_2 \oplus \mathbf{C}_3$ which is given by $\psi(z, w) = (z^2, w^2)$. If we perturb s around x by ψ, then the multiplicity of x is equal modulo 5 to $2 \times 2 = 4$. As another choice, we can take ψ given by $\psi(z, w) = (w^3, z^3)$. In this case, the multiplicity of x is also equal modulo 5 to $3 \times 3 \equiv 4$. The multiplicity 4 can be calculated by $2 \cdot 3/1 \cdot 4 \equiv 4$ in the finite field $\mathbf{F}_5$.

The general case is given as follows. Let $(z_1, \dots, z_r)$ be the coordinate of V_r where $z_k \in \mathbf{C}_{i_k}$, and $(w_1, \dots, w_r)$ be that of W_r where $w_k \in \mathbf{C}_{i'_k}$. Then $\psi\colon V_r \to W_r$ is given by

$$\psi(z_1, \dots, z_r) = (z_1^{i'_1/i_1}, \dots, z_r^{i'_r/i_r}),$$

where i'_k/i_k is calculated in $\mathbf{F}_p$, and identified with an integer which represents it.

The multiplicity $\mathfrak{m}_j$ of x is given by

$$\mathfrak{m}_j = \frac{\prod_{k=1}^r i'_k}{\prod_{k=1}^r i_k}. \tag{3.2}$$

By using an appropriate G-invariant cut-off function, perturb the section s around x by $l + \psi$. For every point in $s^{-1}(0) \cap B^G$, such a perturbation should be carried out. We also need to perturb s G-equivariantly on the free part $B \setminus B^G$. This is easy.

Now, we complete the proof of Theorem 1.1. By the perturbation so far, each of zeros of s on B^G has its multiplicity $\mathfrak{m}_j$. On the other hand, $G = \mathbf{Z}_p$ acts freely on $s^{-1}(0) \cap (B \setminus B^G)$. Hence, the relation (1.2) holds.

REMARK 3.3. In the proof above, the assumption $d(c) = 0$ is not essential. In the case when $d(c) > 0$, we can use the technique of *cutting down the moduli space* as in §3(iii) in [8]. On the other hand, the assumption $d(c, G_\alpha) \le 0$ seems essential to our proof. It would be an interesting problem to consider the case when $d(c, G_\alpha) > 0$.

REMARK 3.4. Another possibility of generalization is to consider p-fold branched coverings. As mentioned in §1, the case of 2-fold branched covering is studied by [11, 9, 2] One could try to prove similar results for higher orders.

4. Examples

In this section, we give several examples.

4.1. Example 1. Let X be the $K3$ surface of the Fermat type in $\mathbf{C}\mathrm{P}^3$ defined by the equation $z_0^4+z_1^4+z_2^4+z_3^4=0$. Let $G=\mathbf{Z}_3$ act on X by permutation of components. Let c be the Spin^c-structure determined by the spin structure, and consider the lift G_0 of the G-action to c whose induced action on the determinant line bundle is just the diagonal action $X\times\mathbf{C}_0$. Then, the G_0-index of the Dirac operator is written as $\mathrm{ind}_{G_0} D = 2\mathbf{C}_0$. (See [5].) The finite dimensional approximation of the monopole map has the form,

$$f\colon \mathbf{C}_0^{x+2}\oplus\mathbf{C}_1^{y}\oplus\mathbf{C}_2^{z}\to\mathbf{C}_0^{x}\oplus\mathbf{C}_1^{y}\oplus\mathbf{C}_2^{z}\oplus\mathbf{R}^3\,,$$

where $\mathbf{R}$ is the real 1-dimensional trivial representation. It follows that $d(c,G_0)=0$, $d(c,G_1)=d(c,G_2)<0$, and, by Theorem 1.1,

$$\mathrm{SW}(X,c)\equiv\mathrm{SW}(X,c,G_0)\mod 3\,.$$

In fact, $\mathrm{SW}(X,c)=\mathrm{SW}(X,c,G_0)=1$, because there exists the unique G-invariant solution with constant spinor by the perturbation by a G-invariant holomorphic 2-form. We remark that the action in Proposition 4.11 of [6] gives a similar example in the case of $G=\mathbf{Z}_5$.

4.2. Example 2. D.-Q. Zhang introduced a holomorphic $G=\mathbf{Z}_3$-action on a $K3$ surface X with $b_+^G=1$ ([14], Example 5.3, due to S. Tsunoda). Let c be the spin, and consider the lift G_0 as in §4.1. In this case, the finite dimensional approximation is of the form,

$$f\colon \mathbf{C}_0^{x}\oplus\mathbf{C}_1^{y+1}\oplus\mathbf{C}_2^{z+1}\to\mathbf{C}_0^{x}\oplus\mathbf{C}_1^{y}\oplus\mathbf{C}_2^{z}\oplus\mathbf{R}\oplus\mathbf{C}_1\,.$$

Then, $d(c,G_0)<0$, $d(c,G_1)=d(c,G_2)=0$. Note that $b_+^G=1$ in this case, and therefore $\mathrm{SW}(X,c,G_\alpha)$ depend on chambers. Nevertheless, the formula (1.2) in Theorem 1.1 holds for any chamber as

$$\mathrm{SW}(X,c)\equiv\mathrm{SW}(X,c,G_1)+2\,\mathrm{SW}(X,c,G_2)\mod 3\,.$$

In fact, the following occurs:

PROPOSITION 4.1. *In a chamber $\mathcal{C}_+$,* $\mathrm{SW}(X,c,G_1)=1$ *and* $\mathrm{SW}(X,c,G_2)=0$*. In another chamber $\mathcal{C}_-$,* $\mathrm{SW}(X,c,G_1)=0$ *and* $\mathrm{SW}(X,c,G_2)=-1$.

REMARK 4.2. In the chamber $\mathcal{C}_+$, the formula (1.2) holds as $1\equiv 1+2\cdot 0$. On the other hand, in $\mathcal{C}_-$, the formula (1.2) holds as $1\equiv 0+2\cdot(-1)$.

To prove Proposition 4.1, we note the next.

LEMMA 4.3. *X admits a Kähler form ω preserved by the G-action.*

PROOF. Let us recall the construction of the log Enriques surface $\bar{S}=X/G$ ([14], Example 5.3). Let x, y, z be the homogeneous coordinates of $\mathbf{C}\mathrm{P}^2$. Consider three cuspidal cubic curves in $\mathbf{C}\mathrm{P}^2$:

$$C_1\colon x^3=y^2z\,,\quad C_2\colon y^3=z^2x\,,\quad C_3\colon z^3=x^2y\,.$$

Let ξ be a primitive 7th root of the unity. Then $C_1 \cap C_2 \cap C_3 = \{(\xi^i : \xi : 1) \mid 0 \le i \le 6\}$. Let $\tau : S \to \mathbf{C}P^2$ be the blowing up of cusps $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, and 7 points in $C_1 \cap C_2 \cap C_3$. Then S contains three disjoint nonsingular (-3)-curves from C_1, C_2 and C_3. Collapsing these (-3)-curves, we obtain the surface $\bar{S}$ whose covering is a $K3$. These surfaces fit into the following diagram

$$\begin{array}{ccccc} X\#3\overline{\mathbf{C}P}^2 & \xrightarrow{\pi} & S & \xrightarrow{\tau} & \mathbf{C}P^2 \\ {\scriptstyle\sigma}\big\downarrow & & {\scriptstyle c}\big\downarrow & & \\ X & \xrightarrow{\bar{\pi}} & \bar{S}, & & \end{array}$$

where σ and τ are blowing up, c is the collapsing map, π is a G-fold covering branched along the (-3)-spheres, and $\bar{\pi}$ is a G-cover. Note that $X\#3\overline{\mathbf{C}P}^2$ has a G-invariant Kähler form obtained by pulling back a Kähler form on $\mathbf{C}P^2$ via τ and π. By blowing down, we have a Kähler form ω on X which is preserved by the G-action. $\square$

PROOF OF PROPOSITION 4.1. The positive spinor bundle S^+ of c can be written as $S^+ = I \oplus K_X^{-1}$, where I is a trivial bundle and K_X is the canonical line bundle of X (which is also trivial). Therefore, a spinor ϕ has two components $\phi = (\alpha, \beta)$. Since the G-action on K_X^{-1} is given by $X \times \mathbf{C}_2$ and we fix the lift G_0 so that $\det S^+ = I \otimes K_X^{-1} = X \times \mathbf{C}_0$, the G_0-action on I is given by $I = X \times \mathbf{C}_1$. By Taubes' perturbation [4](*cf.* [13]) adding $-ir\omega$, we have a unique solution such that $\alpha =$ const. and $\beta = 0$. This solution is G_1-invariant. On the other hand, if we use the perturbation adding $+ir\omega$, then the roles of α and β are exchanged. Therefore, we have a unique solution with $\alpha = 0$ and $\beta =$ const. which is G_2-invariant. These two belong to different chambers. By considering the orientations readily, the proof is completed. $\square$

REMARK 4.4. In this case, the formula $\mathrm{SW}(X, c) = \mathrm{SW}(X, c, G_1) - \mathrm{SW}(X, c, G_2)$ holds. In fact, the perturbation adding $+ir\omega$ corresponds to a linear but orientation-reversing perturbation by $\psi : \mathbf{C}_1 \to \mathbf{C}_2$ given by $\psi(z) = \bar{z}$.

Several actions of higher order G in [14] give similar examples.

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