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# Intersection of Stable and Unstable Manifolds for Invariant Morse Function

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**Abstract.** We study the structure of the smooth manifold which is defined as the intersection of a stable manifold and an unstable manifold for an invariant Morse-Smale function.

## 1. Introduction

The aim of this paper is to investigate invariant Morse functions on compact smooth manifolds with action of compact Lie groups.

Let $M$ be a compact $n$-dimensional Riemannian manifold, $\langle \cdot, \cdot \rangle$ its Riemannian metric, and $\Phi$ a Morse function on $M$. We denote by $-\nabla\Phi$ the negative gradient vector field of $\Phi$ with respect to the metric $\langle \cdot, \cdot \rangle$, and let $\gamma_p(t)$ be the corresponding negative gradient flow which passes through a point $p$ of $M$ at $t = 0$. For a critical point $p$ of $\Phi$, the **unstable manifold** and the **stable manifold** of $p$ are defined by

$$W^u(p) = \left\{ x \in M \,\middle|\, \lim_{t \to -\infty} \gamma_x(t) = p \right\},$$

$$W^s(p) = \left\{ x \in M \,\middle|\, \lim_{t \to \infty} \gamma_x(t) = p \right\}$$

respectively. Since $\Phi$ is a Morse function, $W^u(p)$ and $W^s(p)$ are smoothly embedded open disks of dimensions $n - \lambda(p), \lambda(p)$ respectively, where $\lambda(p)$ denotes the Morse index of $p$ (see [2, Theorem 4.2]). We say that a Morse function $\Phi$ is **Morse-Smale** if $W^u(p)$ and $W^s(q)$ intersect transversally for all critical points $p, q$. If the Morse function $\Phi$ is Morse-Smale, then $\widetilde{\mathcal{M}}(p, q) := W^u(p) \cap W^s(q)$ is also a submanifold of $M$ which has dimension $\lambda(p) - \lambda(q)$.

$\widetilde{\mathcal{M}}(p, q)$ has a natural $\mathbb{R}$-action which is defined by $t \cdot x := \gamma_x(t)$ where $t \in \mathbb{R}, x \in \widetilde{\mathcal{M}}(p, q)$. The quotient space of $\widetilde{\mathcal{M}}(p, q)$ by the $\mathbb{R}$-action is denoted by $\mathcal{M}(p, q)$. Witten's

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Morse theory [5] asserts that the homology group of $M$ with integral coefficient is recovered from the structure of $\mathcal{M}(p,q)$'s such that $\lambda(p)-\lambda(q)=1$. However, there is a Morse function which has no critical points $p, q$ such that $\lambda(p)-\lambda(q)=1$. For example, for a certain Morse function on the partial flag manifold, every unstable manifold is given by the Bruhat cell $BwP/P$. In particular, every Morse index is even (see [1]).

This phenomenon leads us to the study of the structure of $\widetilde{\mathcal{M}}(p,q)$ for $p, q \in \mathrm{Cr}(\Phi)$, $\lambda(p)-\lambda(q)=2$.

In this paper, we investigate the structure of $\widetilde{\mathcal{M}}(p,q)$ for $p, q \in \mathrm{Cr}(\Phi)$ such that $\lambda(p)-\lambda(q)=2$ under the assumption that $M$ admits an action of a compact connected Lie group $G$ and $\Phi$ is $G$-invariant.

Our main theorem is the following.

THEOREM 1. *Let $\Phi$ be a $G$-invariant Bott-Morse function on $M$. Let $p, q$ be $G$-fixed points. Assume the following conditions*:

(1) $M^G \subset \mathrm{Cr}(\Phi)$.

(2) $\lambda(p)-\lambda(q)=2$.

(3) *$W^u(p)$ and $W^s(q)$ intersect transversally.*

*Then every connected component of $\widetilde{\mathcal{M}}(p,q)$ is diffeomorphic to $S^1\times\mathbb{R}$.*

We also show that the action of $G$ on $\widetilde{\mathcal{M}}(p,q)$ is given by the rotation of sphere (see Proposition 3.4 below). By these results geometric structure of $\widetilde{\mathcal{M}}(p,q)$ in our setting is similar to the one treated in the GKM theory [3].

This paper is organized as follows. In Section 2, we study the critical point set of an invariant Morse function and apply it to an invariant Morse function on a homogenious space. In Section 3, we prove Theorem 1.

## 2. Critical points

Let $G$ be a compact Lie group and $M$ be a compact $G$-manifold. Denote by $M^G$ the fixed point set of the action of $G$ on $M$. We say a smooth function $\Phi : M \longrightarrow \mathbb{R}$ is **$G$-invariant** if it satisfies $\Phi(g\cdot p)=\Phi(p)$ for all $g\in G$, $p\in M$. For a smooth function $\Phi$ on $M$, we denote by $\mathrm{Cr}(\Phi)$ the critical point set of $\Phi$.

PROPOSITION 1. *Let $G$ be a compact connected Lie group, $M$ be a compact smooth $G$-manifold, and $\Phi : M \longrightarrow \mathbb{R}$ be a $G$-invariant Morse function on $M$. Assume that there exist only finitely many $G$-fixed points on $M$. Then we have* $\mathrm{Cr}(\Phi)=M^G$.

Since $G$ and $M$ are both compact, there exists a $G$-invariant metric $\langle\cdot,\cdot\rangle$ on $M$. Consider the negative gradient flow equation

$$\gamma(0)=p,\quad \frac{d}{dt}\gamma(t)=-(\nabla\Phi)_{\gamma(t)}\,.$$

Here, we denote by $\nabla\Phi$ the gradient vector field for $\Phi$ with respect to the $G$-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on $M$. Let $\gamma_p(t)$ be the unique solution of this equation. By the uniqueness of the solution we see easily the following.

LEMMA 1. *We have* $\gamma_{g\cdot p}(t) = g\cdot\gamma_p(t)$ *for all* $g \in G$, $p \in M$.

PROOF OF PROPOSITION 2.1. Take $p \in \mathrm{Cr}(\Phi)$. By Lemma 1, we have

$$\lim_{t\to-\infty}\gamma_{g\cdot p}(t) = \lim_{t\to-\infty} g\cdot\gamma_p(t) = g\cdot p\,.$$

This means $g\cdot p$ is also a critical point for $\Phi$, so we have $G\cdot p \subset \mathrm{Cr}(\Phi)$. However, since $M$ is compact, $\mathrm{Cr}(\Phi)$ is a finite set. Thus by the connectedness of $G$, we have $G\cdot p = \{p\}$. This shows $p \in M^G$.

Take $p \in M^G$. By Lemma 1 we have

$$g\cdot\gamma_p(t) = \gamma_{g\cdot p}(t) = \gamma_p(t)$$

for all $g \in G$. This means $\{\gamma_p(t) | t \in \mathbb{R}\} \subset M^G$. Since $M^G$ is a finite set, this implies $\{\gamma_p(t) | t \in \mathbb{R}\} = \{p\}$. Thus we have $p \in \mathrm{Cr}(\Phi)$. □

COROLLARY 1. *Let* $p_0$ *be a point of* $M$ *and* $H$ *be its stabilizer. Assume the following three conditions*:

(1) $H$ *is connected.*

(2) $W_H := N_G(H)/H$ *is a finite group.*

(3) *The fixed point set of the* $H$*-action on* $M$ *is contained in the* $G$*-orbit of* $p_0$.

*Then*, *we have*

$$\mathrm{Cr}(\Phi) = W_H\cdot p_0$$

*for any* $H$*-invariant Morse function* $\Phi : M \longrightarrow \mathbb{R}$.

PROOF. First, we prove $M^H = W_H\cdot p_0$. The inclusion $M^H \supset W_H\cdot p_0$ is clear. Take $p \in M^H$. Then by the condition (3), it is contained in the $G$-orbit of $p_0$. So we can write $p = g\cdot p_0$ where $g$ is an element of $G$. Since $p \in M^H$, we have $h\cdot(g\cdot p_0) = g\cdot p_0$ for all $h \in H$. So we have $g^{-1}Hg \subset H$. Since $g^{-1}Hg$ and $H$ are connected Lie subgroups with the same Lie algebra, the inclusion implies $g^{-1}Hg = H$. Thus we have $p = g\cdot p_0 \in W_H\cdot p_0$, as desired.

In particular, by the condition (2), $M^H = W_H\cdot p_0$ is a finite set. Thus by Proposition 1, we have $\mathrm{Cr}(\Phi) = W_H\cdot p_0$. □

As an application to homogeneous spaces, we have the following corollaries:

COROLLARY 2. *Let* $G$ *be a compact Lie group and* $H$ *be its connected closed subgroup. If* $N_G(H)/H$ *is a finite group*, *we have*

$$\mathrm{Cr}(\Phi) = N_G(H)/H$$

*for any* $H$*-invariant Morse function* $\Phi : G/H \longrightarrow \mathbb{R}$.

COROLLARY 3. *Let $G$ be a compact Lie group and $T$ be a maximal torus. Then, the critical point set of any $T$-invariant Morse function on the flag manifold $G/T$ is given by its Weyl group.*

## 3. Intersections

Let $G$ be a compact connected Lie group and $M$ be a compact smooth $G$-manifold. The following is our main result in this paper.

THEOREM 2. *Let $\Phi$ be a $G$-invariant Bott-Morse function on $M$. Let $p, q$ be $G$-fixed points. Assume the following conditions*:

(1) $M^G \subset \mathrm{Cr}(\Phi)$.
(2) $\lambda(p) - \lambda(q) = 2$.
(3) *$W^u(p)$ and $W^s(q)$ intersect transversally.*

*Then every connected component of $\widetilde{\mathcal{M}}(p, q)$ is diffeomorphic to $S^1 \times \mathbb{R}$.*

PROOF. Let $C$ be a connected component of $\widetilde{\mathcal{M}}(p, q)$. By Lemma 1 and the connectedness of $G$, $C$ is a $G$-invariant subset of $\widetilde{\mathcal{M}}(p, q)$. We note that $C$ is non-compact. To see this, assume that $C$ is compact. Take $c' \in C$. Since the negative gradient flow $\gamma_{c'}(\mathbb{R})$ is connected, it must be contained in $C$. Therefore the assumption implies that $p = \lim_{t \to -\infty} \gamma_c(t) \in C$. This is a contradiction, because $p \notin \widetilde{\mathcal{M}}(p, q)$. So $C$ is non-compact. Since $\mathrm{Cr}(\Phi) \cap C = \emptyset$, the assumption (1) implies that $M^G \cap C = \emptyset$. Let us show the following.

**(3.1)** $\dim G \cdot c = 1$ .

Assume that $\dim G \cdot c = 2$. Then $G \cdot c$ is a codimension 0 submanifold of $C$. Therefore $G \cdot c$ is an open subset of $C$. On the other hand, by the compactness of $G$, $G \cdot c$ is a closed subset of $C$. So we have $C = G \cdot c$ since $C$ is connected. This is a contradiction, because $C$ is non-compact. Assume that $\dim G \cdot c = 0$. Then by the connectedness of $G$, we have $G \cdot c = \{c\}$. This is also a contradiction, because $c \notin M^G$. Hence we have $\dim G \cdot c = 1$. The proof of (3.1) is complete.

Define an action of $G \times \mathbb{R}$ on $C$ by $(g, t) \cdot c = g \cdot \gamma_c(t)$. In fact, this gives an action on $C$, because

$$\begin{aligned}(gg', t + t') \cdot c &= gg' \cdot \gamma_c(t + t') \\ &= g \cdot \gamma_{g' \cdot c}(t + t') \\ &= g \cdot \gamma_{\gamma_{g' \cdot c}(t')}(t) \\ &= (g, t) \cdot \gamma_{g' \cdot c}(t') \\ &= (g, t) \cdot ((g', t') \cdot c)\end{aligned}$$

for all $(g, t), (g', t') \in G \times \mathbb{R}$. We next show the following.

**(3.2)** $(G \times \mathbb{R})_c = G_c \times \{0\}$.

Here, $(G \times \mathbb{R})_c$ (resp. $G_c$) is the stabilizer of $c$ for the action of $G \times \mathbb{R}$ (resp. $G$) on $C$. It is enough to show that $(G \times \mathbb{R})_c \subset G_c \times \{0\}$. Let $(g, t)$ be an element of $(G \times \mathbb{R})_c$. It is sufficient to show $t = 0$. Assume that $t > 0$. Since $(g^n, nt) \in (G \times \mathbb{R})_c$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} g^n \cdot c = \lim_{n\to\infty} \gamma_c(-nt) = p$. This implies that $p \in C$ since $G \cdot c$ is a closed subset of $C$. This is a contradiction. If we assume that $t < 0$, a similar argument implies the same contradiction. The proof of (3.2) is complete.

Let us consider the natural embedding $G \times \mathbb{R}/(G \times \mathbb{R})_c \longrightarrow \widetilde{\mathcal{M}}(p, q)$. By (3.1) and (3.2), we have $\dim G \times \mathbb{R}/(G \times \mathbb{R})_c = \dim \widetilde{\mathcal{M}}(p, q) = 2$. Thus $G \cdot \gamma_c(\mathbb{R})$ is open in $\widetilde{\mathcal{M}}(p, q)$. In particular, every orbit of the action of $G \times \mathbb{R}$ on $C$ is open. Since $C$ is connected, this implies that $C = G \cdot \gamma_c(\mathbb{R})$. Therefore we obtain the following isomorphisms:

$$C \cong G \times \mathbb{R}/G_c \times \{0\} \cong G/G_c \times \mathbb{R} \cong G \cdot c \times \mathbb{R}.$$

By (3.1), $G \cdot c$ is a compact connected 1-dimensional manifold. Thus $G \cdot c$ is diffeomorphic to $S^1$. Hence $C$ is diffeomorphic to $S^1 \times \mathbb{R}$.

The proof is complete. □

COROLLARY 4. *Let $\Phi$ be a $G$-invariant Morse-Smale function on $M$. Let $p, q$ be critical points of $\Phi$ such that $\lambda(p) - \lambda(q) = 2$. If $M^G$ is a finite set, every connected component of $\widetilde{\mathcal{M}}(p, q)$ is diffeomorphic to $S^1 \times \mathbb{R}$.*

PROOF. By Proposition 1, we have $M^G = \mathrm{Cr}(\Phi)$. So this corollary follows from Theorem 2. □

REMARK 1. If the function $\Phi$ is not invariant under the group action, Theorem 2 and Corollary 4 do not hold. For example, let us consider the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the standard metric. We define a smooth function $\Phi : T^2 \longrightarrow \mathbb{R}$ by $\Phi(x, y) = \cos(2\pi x) + \cos(2\pi y)$. Then the function $\Phi$ gives a counter example.

In the rest of this section, we study the stabilizer $G_c$. Let $G$ be a compact connected Lie group which acts smoothly on $S^1$. We denote by $\mathfrak{g}$ the Lie algebra of $G$. Consider the following commutative diagram:

$$\begin{array}{ccc} G & \to & \mathrm{Diff}(S^1) \\ \uparrow & & \uparrow \\ \mathfrak{g} & \to & \Gamma(TS^1)\,. \end{array}$$

Here, vertical arrows are exponential maps and horizontal arrows are induced by the action of $G$ on $S^1$. Since $G$ is a compact connected Lie group, the exponential map $\mathfrak{g} \longrightarrow G$

is surjective. Thus the image of $G \longrightarrow \mathrm{Diff}(S^1)$ is completely determined by the image of $\mathfrak{g} \longrightarrow \Gamma(TS^1)$. We need the following result of Plante [5, Theorem 1.2].

LEMMA 2. *Let $G$ be a Lie group and $\mathfrak{g}$ be its Lie algebra. Assume that $G$ acts smoothly and transitively on $S^1$. Then the image of $\mathfrak{g} \longrightarrow \Gamma(TS^1)$ is conjugate via a diffeomorphism to one of the following subalgebras of $\Gamma(TS^1)$*

(1) $\left\langle \frac{\partial}{\partial x} \right\rangle$,

(2) $\left\langle (1+\cos x)\frac{\partial}{\partial x}, (\sin x)\frac{\partial}{\partial x}, (1-\cos x)\frac{\partial}{\partial x} \right\rangle$.

Note that we have the isomorphism

$$\left\langle (1+\cos x)\frac{\partial}{\partial x}, (\sin x)\frac{\partial}{\partial x}, (1-\cos x)\frac{\partial}{\partial x} \right\rangle \cong \mathfrak{sl}_2(\mathbb{R})$$

of Lie algebras.

PROPOSITION 2. *In the setting of Theorem 2, let $C$ be a connected component of $\widetilde{\mathcal{M}}(p,q)$. Then there is a surjective group homomorphism $\alpha : G \longrightarrow S^1$ and a diffeomorphism $C \cong S^1 \times \mathbb{R}$ such that the action of $G \times \mathbb{R}$ on $C \cong S^1 \times \mathbb{R}$ is given by*

$$(g,t)\cdot(x,s) = (\alpha(g)x, t+s)$$

*for all $(g,t) \in G \times \mathbb{R}$, $(x,s) \in S^1 \times \mathbb{R}$.*

PROOF. Take $c \in C$. We consider the action of $G$ on $G \cdot c$ and identify $G \cdot c$ with $S^1$. Let $\alpha_0 : G \longrightarrow \mathrm{Diff}(S^1)$ be the representation of the action of $G$ on $S^1$, $\alpha_0' : \mathfrak{g} \longrightarrow \Gamma(TS^1)$ the corresponding Lie algebra homomorphism.

Since $\mathfrak{g}$ is the Lie algebra of the compact Lie group $G$, it does not admit $\mathfrak{sl}_2$ as a quotient Lie algebra . Hence by Lemma 2 we can take $\varphi \in \mathrm{Diff}(S^1)$ such that

$$\varphi_*(\alpha_0'(\mathfrak{g})) = \left\langle \frac{\partial}{\partial x} \right\rangle.$$

This shows that $\varphi(\alpha_0(G))\varphi^{-1}$ consists of rotations of $S^1$. Now we define a group homomorphism $\alpha : G \longrightarrow S^1$ by $\alpha(g) := \varphi \circ \alpha_0(g) \circ \varphi^{-1}$. This map satisfies the required properties. □

COROLLARY 5. *In the setting of Theorem 2, let $C$ be a connected component of $\widetilde{\mathcal{M}}(p,q)$. Then the stabilizer of $c \in C$ is independent of choice of $c$ and is a codimension 1 closed normal Lie subgroup of $G$.*

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