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Interface Regularity of the Solutions for the Rotation Free and the Divergence Free Systems in Euclidian Space

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Abstract. In the present paper we study the interface regularity of the solutions to the differential systems of divergence free and rotation free defined by differential forms in the $N(\geq 3)$-dimensional Euclidean space. Our results are natural extensions of the results of [3] and [5] for $N = 3$.

1. Introduction

1.1. Motivation. Let $\Omega \subset \mathbf{R}^N$ $(N \geq 3)$ be a bounded domain with $C^{1,1}$-Lipschitz boundary. Let $\mathcal{M}$ be a hypersurface in $\mathbf{R}^N$. We assume that $\mathcal{M}$ divides Ω into two domains $\Omega_\pm$. Let $\Gamma = \Gamma_\pm = \Omega \cap \mathcal{M}$, and let ν be the outer unit normal vector on Γ_-. If $\mathcal{M}$ is of $C^{k,1}$, ν has a $C^{k,1} \cap W^{k+1,\infty}$-extension to Ω, which is expressed with the same symbol ν. Let $B(x) = {}^t(B^1(x), B^2(x), \dots, B^N(x))$ and $J(x) = {}^t(J^1(x), J^2(x), \dots, J^{N(N-1)/2}(x))$ be $\mathbf{R}^{N(N-1)/2}$-valued functions, and let g be an $\mathbf{R}$-valued function. We put, for $x \in \Gamma$ (interface)

$$B_\pm(x) := \lim_{\Omega_\pm \ni \xi \to x} B(\xi)\,, \quad [B]_-^+ = B_+ - B_- \text{ on } \Gamma\,.$$

The motivation of this study arises from the results on the interface vanishing of the solution to the following equations (1) and (2) for $N = 3$

$$(1) \quad \begin{cases} \operatorname{rot} B = J\,, \\ \operatorname{div} B = 0\,, \end{cases} \text{ in } \Omega_\pm\,, \quad (2) \quad \begin{cases} \operatorname{rot} B = 0\,, \\ \operatorname{div} B = g\,, \end{cases} \text{ in } \Omega_\pm$$

by Kobayashi, Suzuki and Watanabe [5] for (1), Kanou, Sato and Watanabe [3] for (2):

THEOREM 1.1 ([5]). *Let $\mathcal{M} \subset \mathbf{R}^3$ be a $C^{1,1}$-surface and rot $J \in L^2(\Omega_\pm)$. If $B \in H^1(\Omega)^3$ is a solution to* (1), *then $\nu \cdot B \in H^2_{loc}(\Omega)$.*

THEOREM 1.2 ([3]). *Let $\mathcal{M} \subset \mathbf{R}^3$ be a $C^{1,1}$-surface*, *and $g \in H^1(\Omega_\pm)$. If $B \in H^1(\Omega)^3$ is a solution of* (2), *then $\nu \times B \in H^2_{loc}(\Omega)^3$.*

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We describe the historical background. In [1], Geselowitz studied the problem for Magnetroencepharography (MEG), which is the medical mathematics. We explain MEG, concretely. Ω_+ is a "head", Ω_- is the outside of the head and Γ is the surface of the head. Let B be the magnetic field and let J be the electric current. The problem is: can we know the electric current J by measurement of the magnetic field B in Ω_- (the outside)? In [7], T. Suzuki, K. Watanabe and M. Shimogawara examined the property of the solutions to (1) by using the Newton potential. They also studied the inverse problem under the assumption that the electric current J is a dipole.

In [4], T. Kobayashi, T. Suzuki and K. Watanabe obtained the same result as Theorem 1.1 by assuming that $\mathcal{M}$ is a C^2-surface. In [5], they improved the result and obtained Theorem 1.1 above. To prove Theorem 1.1, they used the Green and the Gauss formulas in stead of the Newton potential. In [3], M. Kanou, T. Sato and K. Watanabe obtained Theorem 1.2 above.

We remark that $B \in H^2_{loc}(\Omega)$ is not necessarily true even if B and J (resp. B and g) satisfy the assumption of Theorem 1.1 (resp. 1.2). We give a concrete example. Let $\mathcal{M} = \{x = {}^t(x_1, x_2, x_3) | x_3 = 0\}$, $\Omega = \{|x| < 1\}$, $\nu = {}^t(0, 0, 1)$. $B = {}^t(|x_3|, x_1, x_2)$ (resp. $B = {}^t(0, 0, |x_3|)$), and

$$J = \begin{cases} {}^t(1, 1, 1)\,, & (x_3 > 0) \\ {}^t(1, -1, 1)\,, & (x_3 < 0)\,. \end{cases} \left(\text{resp. } g = \begin{cases} 1\,, & (x_3 > 0) \\ -1\,, & (x_3 < 0)\,. \end{cases} \right)$$

Then we can easily check that B and J (resp. B and g) satisfy (1) (resp. (2)). In fact,

$$\nabla \times B = {}^t(\partial_2 x_2, \partial_3 |x_3|, \partial_1 x_1) = J\,, \quad \nabla \cdot B = 0\,.$$
$$(\text{resp. } \nabla \cdot B = \partial_3 B^3 = \partial_3 |x_3| = g\,, \quad \nabla \times B = {}^t(0, 0, 0).)$$

But $\nu \times B = {}^t(-x_1, |x_3|, 0) \notin H^2_{loc}(\Omega)^3$ (resp. $\nu \cdot B = |x_3| \notin H^2_{loc}(\Omega)$), which means $B \notin H^2_{loc}(\Omega)$.

In the present paper, we study an extension to the above theorems for general N by using the *differential forms*.

This paper is organized as follows: In §1.2, we give some notations. In §2, we give the main theorems. In §3, we give proofs of the theorems.

1.2. Preliminaries. Let $D \subset \mathbf{R}^N$ be a domain with smooth boundary. We consider 1 or 2-forms on D, and we write as

$$B = \sum_{i=1}^{N} B^i dx_i \text{ (1-form)}\,, \quad J = \sum_{1 \le i < j \le N} J^{ij} dx_i \wedge dx_j \text{ (2-form)}\,.$$

For 1-forms $A = \sum_{i=1}^{N} B^i dx_i$, $B = \sum_{i=1}^{N} B^i dx_i$, and 2-forms $J = \sum_{1\le i<j\le N} J^{ij}\, dx_i \wedge dx_j$, $K = \sum_{1\le i<j\le N} K^{ij} dx_i \wedge dx_j$, the inner product is respectively defined by

$$(A, B) := \sum_{i=1}^{N} A^i B^i\,, \quad (J, K) := \sum_{1\le i<j\le N} J^{ij} K^{ij}\,.$$

Furthermore, the exterior product is given by

$$A \wedge B = \sum_{1\le i<j\le N} (A^j B^i - A^i B^j) dx_i \wedge dx_j\,.$$

We define L^2-inner product $\langle\cdot,\cdot\rangle$ by

$$\langle A, B\rangle := \int_D (A, B)\,, \quad \langle J, K\rangle := \int_D (J, K)\,.$$

If $\langle A, A\rangle < \infty$, $\langle J, J\rangle < \infty$, then we write,

$$A \in L^2(D; \mathbf{R}^N)\,, \quad J \in L^2(D; \mathbf{R}^{N(N-1)/2})\,.$$

When no confusion can arise, we simply write $A \in L^2(D)$, $J \in L^2(D)$. Concisely we write $\partial B^i/\partial x_j$ as B^i_j. We define the differential operators for forms by

$$d_0 f := \sum_{i=1}^{N} f_i dx_i\,, \quad d_1 B := \sum_{1\le i<j\le N} (B^j_i - B^i_j) dx_i \wedge dx_j\,,$$

$$\delta_0 B := -\sum_{i=1}^{N} B^i_i\,, \quad \delta_1 J := -\sum_{i=1}^{N}\left(\sum_{l=1}^{N} J^{li}_l\right) dx_i\,.$$

Let $H^m(D)$ be the Sobolev space of rank m. We define function spaces as follows:

$H^m(D; \mathbf{R}^K) := \{A = (A^1, \dots, A^K) \in L^2(D; \mathbf{R}^K);\ 1 \le \forall j \le K,\ A^j \in H^m(D)\}$,

$H(d_0; D) := \{f \in L^2(D); d_0 f \in L^2(D; \mathbf{R}^N)\}$,

$H(\delta_0; D) := \{B \in L^2(D; \mathbf{R}^N); \delta_0 B \in L^2(D)\}$,

$H(d_1; D) := \{B \in L^2(D; \mathbf{R}^N) : d_1 B \in L^2(D; \mathbf{R}^{N(N-1)/2})\}$,

$H(\delta_1; D) := \{J \in L^2(D; \mathbf{R}^{N(N-1)/2}) : \delta_1 J \in L^2(D; \mathbf{R}^N)\}$.

2. Main Theorems

We denote the outer unit normal vector on Ω_- by ν, and assume that ν has an extension to Ω. Here we will not consider the regularity of extension in detail. Identifying ν with

1-form, we regard as

$$\nu = \sum_{i=1}^{N} \nu^i dx_i \,.$$

We consider the following equations:

$$(M) \quad \begin{cases} d_1 B = J \\ \delta_0 B = 0 \,, \end{cases} \quad \text{in } \Omega_\pm \,, \quad J \in H(\delta_1, \Omega_\pm) \,,$$

$$(R) \quad \begin{cases} d_1 B = 0 \\ \delta_0 B = g \,, \end{cases} \quad \text{in } \Omega_\pm, \quad g \in H(d_0, \Omega_\pm) = H^1(\Omega_\pm) \,.$$

Here these equations mean as follows: In general, the equation $F(u) = v$ in $\Omega_\pm$ means

$$F(u)|_{\Omega_+} = v|_{\Omega_+} \quad \text{in } \Omega_+ \,, \quad \text{and} \quad F(u)|_{\Omega_-} = v|_{\Omega_-} \text{ in } \Omega_- \,.$$

$J \in H(\delta_1, \Omega_\pm)$ means

$$J|_{\Omega_+} \in H(\delta_1; \Omega_+) \quad \text{and} \quad J|_{\Omega_-} \in H(\delta_1; \Omega_-) \,.$$

Also, $g \in H(d_0; \Omega_\pm)$ means

$$g|_{\Omega_+} \in H(d_0; \Omega_+) \quad \text{and} \quad g|_{\Omega_-} \in H(d_0; \Omega_-) \,.$$

And we define B^ν, B^τ by

$$B^\nu := \nu(\nu, B) \,, \quad B^\tau := B - B^\nu \,.$$

THEOREM 2.1. *Let B, J satisfy (M). If $B \in H^1(\Omega; \mathbf{R}^N)$ and $[B]_-^+ = 0$ on Γ, then we have $(\nu, B) \in H^2_{loc}(\Omega)$.*

THEOREM 2.2. *Let B, g satisfy (R). If $B \in H^1(\Omega; \mathbf{R}^N)$ and $[B]_-^+ = 0$ on Γ, then we have $B^\tau \in H^2_{loc}(\Omega)$.*

3. Proofs of Theorems

The following lemmas are needed to obtain Theorem 2.1 and 2.2.

LEMMA 3.1 (Gauss, Stokes formula). *Let $D \subset \mathbf{R}^N$ be a domain with smooth boundary. For any $\varphi \in C^\infty(D)$, $C \in C^\infty(D; \mathbf{R}^{N(N-1)/2})$, we have*

$$\langle \delta_0 B, \varphi \rangle = \langle B, d_0 \varphi \rangle - \int_{\partial D} (B, \nu) \varphi dS \,, \tag{3.1}$$

$$\langle d_1 B, C \rangle = \langle B, \delta_1 C \rangle + \int_{\partial D} (\nu \wedge B, C) dS \,, \tag{3.2}$$

where ν denotes the exterior unit normal and dS the surface element.

PROOF (Ref. [6]). Integrating both sides of the formula $(\delta_0 B)\varphi = \delta_0(B\varphi) + (d_0\varphi, B)$ over D yields (3.1).

Next we prove (3.2). Clearly we have

$$(B, \delta_1 C) = -\sum_{l=1}^{N} B^l \sum_{j=1}^{N} C_j^{jl} .$$

Noticing that $C^{ii} = 0$, $C^{ij} = -C^{ji}$, we obtain

$$\begin{aligned}
(d_1 B, C) &= \sum_{i<j} (B_i^j - B_j^i) C^{ij} \\
&= \sum_{i<j} \{(B^j C^{ij})_i - (B^i C^{ij})_j - (B^j C_i^{ij} - B^i C_j^{ij})\} \\
&= \sum_{i<j} \{(B^j C^{ij})_i - (B^i C^{ij})_j - (B^j C_i^{ij} + B^i C_j^{ji})\} \\
&= \sum_{i<j} \{(B^j C^{ij})_i - (B^i C^{ij})_j\} + (B, \delta_1 C) .
\end{aligned}$$

Integrating both sides of the above leads to (3.2). □

LEMMA 3.2. *If $p \in H^1(\Omega)$ and $[p]_-^+ = 0$ on Γ, then we have $[\nu \wedge d_0 p]_-^+ = 0$ on Γ as $H^{-1/2}(\Gamma)$.*

PROOF. Let $p_n \in C^\infty(\Omega)$ be an approximate sequence of p in $H^1(\Omega)$. For any $C \in C_0^\infty(\Omega; \mathbf{R}^{N(N-1)/2})$, we have

$$\begin{aligned}
0 &= \langle d_1 d_0 p_n, C\rangle \\
&= \int_{\Omega_+} (d_1 d_0 p_n, C) + \int_{\Omega_-} (d_1 d_0 p_n, C) \\
&= \langle d_0 p_n, \delta_1 C\rangle + \int_\Gamma [(\nu \wedge d_0 p_n, C)]_-^+ dS \\
&= \langle p_n, \delta_0 \delta_1 C\rangle + \int_\Gamma [p_n(\nu, \delta_1 C)]_-^+ dS + \int_\Gamma [(\nu \wedge d_0 p_n, C)]_-^+ dS \\
&= \int_\Gamma [(\nu \wedge d_0 p_n, C)]_-^+ dS .
\end{aligned}$$

By letting $n \to \infty$, we obtain $[\nu \wedge d_0 p]_-^+ = 0$ on Γ. □

PROOF OF THEOREM 2.1. Notice that $B \in H^2(\Omega_\pm; \mathbf{R}^N)$, since

$$-\Delta B = (\delta_1 d_1 + d_0 \delta_0) B = \delta_1 d_1 B = \delta_1 J \ \in L^2(\Omega_\pm) .$$

By the elliptic regularity theorem (ref. [2]), it is sufficient to establish the following relation: for any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega (\Delta(\nu, B))\varphi = \int_\Omega (\nu, B)\Delta\varphi . \tag{3.3}$$

Noticing that Laplacian $-\Delta = \delta_0 d_0$ for functions (0-forms) and using (3.1), we have

$$\langle(\nu, B), \delta_0 d_0\varphi\rangle = \langle d_0(\nu, B), d_0\varphi\rangle + \int_\Gamma [(\nu, B)(\nu, d_0\varphi)]_-^+ dS .$$

Since the second term of the right hand side is 0, we have

$$\langle(\nu, B), \delta_0 d_0\varphi\rangle = \langle \delta_0 d_0(\nu, B), \varphi\rangle + \int_\Gamma [(d_0(\nu, B), \nu)\varphi]_-^+ dS$$

from (3.1). Hence it suffices to prove

$$[(d_0(\nu, B), \nu)]_-^+ = [(\nu, d_0(\nu, B))]_-^+ = 0 \quad \text{on } \Gamma . \tag{3.4}$$

DEFINITION. We define the differential operators to normal direction (ν, d_0) by

$$(\nu, d_0)f := \sum_{j=1}^N \nu^j f_j , \quad (\nu, d_0)B := \sum_{i=1}^N \left(\sum_{j=1}^N \nu^j B^i_j\right) dx_i .$$

Furthermore, we put

$$d_{0\nu} f := \nu(\nu, d_0)f , \quad d_{0\tau} f := d_0 f - d_{0\nu} f , \tag{3.5}$$

$$\delta_{0\nu} B := -(\nu, (\nu, d_0)B) , \quad \delta_{0\tau} B := \delta_0 B - \delta_{0\nu} B . \tag{3.6}$$

Then (3.4) is rewritten as

$$[(\nu, d_0)(\nu, B)]_-^+ = 0 .$$

LEMMA 3.3. *We can decompose $\delta_0 B$ as follows*:

$$\delta_0 B = \delta_{0\tau} B^\tau - (\nu, d_0)(\nu, B) + (\nu, B)\delta_0(\nu) + ((\nu, d_0)\nu, B^\tau) . \tag{3.7}$$

PROOF. From (3.5) and (3.6) we have

$$\delta_0 B = \delta_{0\tau} B^\tau + \delta_{0\nu} B^\tau + \delta_{0\tau} B^\nu + \delta_{0\nu} B^\nu .$$

To prove the lemma, we prepare next two equalities for 1-form ω and 0-form f,

$$\delta_0(f\omega) = -(d_0 f, \omega) + f\delta_0\omega , \tag{3.8}$$

$$(\nu, (\nu, d_0)\nu) = 0 . \tag{3.9}$$

We obtain (3.8) by the definitions of δ_0 and d_0. (3.9) follows from

$$2(\nu, (\nu, d_0)\nu) = 2\sum_{i=1}^N \nu^i \sum_{j=1}^N \nu^j \nu^i_j = \sum_{j=1}^N \nu^j \sum_{i=1}^N \{(\nu^i)^2\}_j = \sum_{j=1}^N \left(\sum_{i=1}^N (\nu^i)^2\right)_j \nu^j$$

$$=0\,.$$

We can then compute as follows:

$$\begin{aligned}\delta_{0\nu}B^{\tau} &= -(\nu,(\nu,d_0)B^{\tau}) = -(\nu,d_0)(\nu,B^{\tau}) + ((\nu,d_0)\nu,B^{\tau})\\ &= ((\nu,d_0)\nu,B^{\tau})\,, \qquad (3.10)\\ \delta_{0\tau}B^{\nu} &= \delta_0 B^{\nu} + (\nu,(\nu,d_0)B^{\nu}) = \delta_0(\nu(\nu,B)) + (\nu,(\nu,d_0)\nu(\nu,B))\\ &= -(d_0(\nu,B),\nu) + (\nu,B)\delta_0(\nu) + (\nu,d_0)(\nu,\nu(\nu,B)) - ((\nu,d_0)\nu,\nu(\nu,B))\\ &= -(\nu,d_0)(\nu,B) + (\nu,B)\delta_0(\nu) + (\nu,d_0)(\nu,B)\\ &\quad - ((\nu,d_0)\nu,\nu)(\nu,B) \qquad (3.11)\\ &= (\nu,B)\delta_0(\nu),\\ \delta_{0\nu}B^{\nu} &= -(\nu,(\nu,d_0)\{\nu(\nu,B)\}) = -(\nu,d_0)(\nu,\nu(\nu,B)) + ((\nu,d_0)\nu,\nu(\nu,B))\\ &= -(\nu,d_0)(\nu,B).\end{aligned}$$

In order to obtain (3.11), we used (3.8). Hence we obtain (3.7). □

We continue to prove (3.4). Since $\delta_0 B = 0$ on $\Omega_\pm$ and $[B]_-^+ = 0$, we have from (3.7)

$$\begin{aligned}0 = [\delta_0 B]_-^+ &= [\delta_{0\tau}B^{\tau} - (\nu,d_0)(\nu,B) + (\nu,B)\delta_0(\nu) + \delta_{0\nu}B^{\nu}]_-^+\\ &= [\delta_{0\tau}B^{\tau}]_-^+ - [(\nu,d_0)(\nu,B)]_-^+\end{aligned}$$

on Γ.

To show (3.4), it suffices to prove $[\delta_{0\tau}B^{\tau}]_-^+ = 0$. We put

$$\delta_0^{(j)} f := -f_j\,,\quad \delta_{0\nu}^{(j)} f := -\nu^j(\nu,d_0)f\,,\quad \delta_{0\tau}^{(j)} f := \delta_0^{(j)} f - \delta_{0\nu}^{(j)} f\,.$$

The j-th element of B^{τ} is denoted by $B^{\tau j}$. It follows that

$$\delta_{0\tau}B^{\tau} = \sum_{j=1}^{N}\delta_{0\tau}^{(j)}B^{\tau j} = -\sum_{j=1}^{N}(B_j^{\tau j} - \nu^j(\nu,d_0)B^{\tau j})\,. \qquad (3.12)$$

We begin to compute $\delta_{0\tau}^{(j)}B^{\tau j}$ directly. Since $\sum_{k=1}^{N}(\nu^k)^2 = 1$,

$$\begin{aligned}-\big[\delta_{0\tau}^{(j)}B^{\tau j}\big]_-^+ &= \big[B_j^{\tau j} - \nu^j(\nu,d_0)B^{\tau j}\big]_-^+\\ &= \Big[\sum_{k=1}^{N}(\nu^k)^2 B_j^{\tau j} - \sum_{k=1}^{N}\nu^j\nu^k B_k^{\tau j}\Big]_-^+\\ &= \Big[\sum_{k=1}^{N}\nu^k(\nu^k B_j^{\tau j} - \nu^j B_k^{\tau j})\Big]_-^+\,.\end{aligned}$$

Since $[\nu \wedge d_0 B^{\tau j}]_-^+ = 0$ from Lemma 3.2 ($B^\tau \in H^1(\Omega; \mathbf{R}^N)$, $[B^\tau]_-^+ = 0$), we obtain the desired result. □

PROOF OF THEOREM 2.2. Notice that $B \in H^2(\Omega_\pm; \mathbf{R}^N)$, since

$$-\Delta B = (\delta_1 d_1 + d_0 \delta_0) B = d_0 \delta_0 B = d_0 g \in L^2(\Omega_\pm; \mathbf{R}^N)$$

from equation (R). For any $C \in C_0^\infty(\Omega; \mathbf{R}^N)$, we have

$$\begin{aligned}\langle d_0 \delta_0 B^\tau, C\rangle &= \langle \delta_0 B^\tau, \delta_0 C\rangle - \int_\Gamma [\delta_0 B^\tau (\nu, C)]_-^+ dS \\ &= \langle B^\tau, d_0 \delta_0 C\rangle - \int_\Gamma [(\nu, B^\tau)\delta_0 C]_-^+ dS - \int_\Gamma [\delta_0 B^\tau (\nu, C)]_-^+ dS \\ &= \langle B^\tau, d_0 \delta_0 C\rangle - \int_\Gamma [\delta_0 B^\tau]_-^+ (\nu, C) dS\,.\end{aligned}$$

Using (3.10), we compute $[\delta_0 B^\tau]_-^+$.

$$\begin{aligned}[\delta_{0\tau} B^\tau]_-^+ &= [\delta_0 B^\tau]_-^+ - [\delta_{0\nu} B^\tau]_-^+ = [\delta_0 B^\tau]_-^+ = [\delta_0 (B - \nu(\nu, B))]_-^+ \\ &= -\Big[\sum_{i=1}^N B_i^i - \sum_{i=1}^N (\nu^i (\nu, B))_i\Big]_-^+ \\ &= -\Big[\sum_{i=1}^N B_i^i - \sum_{i=1}^N \nu^i (\nu, B)_i - \sum_{i=1}^N \nu_i^i (\nu, B)\Big]_-^+ \\ &= -\Big[\sum_{i=1}^N B_i^i - \sum_{i=1}^N \nu^i \Big(\sum_{l=1}^N \nu^l B^l\Big)_i\Big]_-^+ \\ &= -\Big[\sum_{i=1}^N B_i^i - \sum_{i=1}^N \nu^i \sum_{l=1}^N \nu^l B_i^l - \sum_{i=1}^N \sum_{l=1}^N \nu^i \nu_i^l B^l\Big]_-^+ \\ &= -\Big[\sum_{i=1}^N B_i^i - \sum_{i=1}^N \nu^i \sum_{l=1}^N \nu^l B_i^l\Big]_-^+ \\ &= -\Big[\sum_{i=1}^N \sum_{l=1}^N (\nu^l)^2 B_i^i - \sum_{i=1}^N \nu^i \sum_{l=1}^N \nu^l B_i^l\Big]_-^+ \\ &= -\Big[\sum_{i=1}^N \sum_{l=1}^N \nu^l \big\{\nu^l B_i^i - \nu^i B_i^l\big\}\Big]_-^+ \\ &= -\Big[\sum_{i=1}^N \sum_{l=1}^N \nu^l \big\{\nu^l B_i^i - \nu^i B_l^i\big\}\Big]_-^+ &(3.13) \\ &= 0\,. &(3.14)\end{aligned}$$

In fact, (3.13) follows from $B_i^j - B_j^i = 0$ (since $d_1 B = 0$), and replacing p by B^i in Lemma 3.2 leads to (3.14). Then it follows that

$$\langle d_0\delta_0 B^\tau, C\rangle = \langle B^\tau, d_0\delta_0 C\rangle\,.$$

Next we compute $\langle B, d_1\delta_1 C\rangle$. Notice that $[B]_-^+ = 0$ and from (3.2), we have

$$\begin{aligned}\langle B^\tau, \delta_1 d_1 C\rangle &= \langle d_1 B^\tau, d_1 C\rangle + \int_\Gamma [(B^\tau \wedge \nu, d_1 C)]_-^+ dS\\ &= \langle \delta_1 d_1 B^\tau, C\rangle + \int_\Gamma [(d_1 B^\tau, C \wedge \nu)]_-^+ dS\,.\end{aligned}$$

Note that $d_1 B = 0$. By replacing p by (ν, B) in Lemma 3.2, it follows that

$$\begin{aligned}[d_1 B^\tau]_-^+ &= [d_1(B - \nu(\nu, B)) = -[d_1(\nu(\nu, B))]_-^+ = [\nu \wedge d_0(\nu, B) - (\nu, B)d_1\nu]_-^+\\ &= 0\,.\end{aligned}$$

Then

$$\langle B^\tau, \delta_1 d_1 C\rangle = \langle \delta_1 d_1 B^\tau, C\rangle\,,$$

which implies that

$$\langle -\Delta B^\tau, C\rangle = \langle B^\tau, -\Delta C\rangle\,.$$

From the elliptic regularity theorem ([2]), we obtain $B^\tau \in H^2_{loc}(\Omega; \mathbf{R}^N)$. □

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