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Surfaces with Constant Chebyshev Angle II

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Abstract. In this paper we classify a class of surfaces with negative Gaussian curvature parametrized by a generalized Chebyshev net with constant Chebyshev angle in the Euclidean 3-space. As an application we obtain for each constant Chebyshev angle a seven-parameter family of complete surfaces.

1. Introduction

Bianchi in [1]–[2] studies a class of surfaces with negative Gaussian curvature obtained by generalizing Bäcklund transformation for surfaces with constant negative Gaussian curvature. Fujioka in [6] introduces the notion of generalized Chebyshev nets which is a natural generalization of Chebyshev nets for surfaces with constant negative Gaussian curvature and shows that a Bianchi surface with constant Chebyshev angle parametrized by a generalized Chebyshev net is a piece of a right helicoid; in this case the Chebyshev angle is $\pi/2$.

Riveros and Corro in [7] obtained a characterization for a class of surfaces with a generalized Chebyshev net and constant Chebyshev angle different from $\pi/2$. The characterization is obtained by showing that the coefficients of the first and second fundamental form of these surfaces depend on a meromorphic function which satisfies a certain differential equation. The characterization is based on the results obtained in [3], [4] and [5].

In this work, we classify a class of surfaces with negative Gaussian curvature parametrized by a generalized Chebyshev net with constant Chebyshev angle. We also study the completeness of such surfaces.

2. Prelimiminaries

In the following we consider only surfaces with negative Gaussian curvature in the Euclidean 3-space $\mathbf{R}^3$. Since such a surface has two directions, called the asymptotic directions,

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in which the normal curvature vanishes, we can parametrize the surface locally by asymptotic line coordinates (x, y):

$$\chi : \Omega \subset \mathbf{R}^2 \to \mathbf{R}^3 .$$

If the Gaussian curvature is $-\frac{1}{\rho^2}$ for a positive function ρ on Ω then the fundamental forms become as follows:

$$I = A^2 dx^2 + 2AB\cos\varphi dxdy + B^2 dy^2 , \quad II = \frac{2AB\sin\varphi}{\rho} dxdy ,$$

where $A = |\chi_x|$, $B = |\chi_y|$ and φ is the angle between the asymptotic lines, called the *Chebyshev angle*. Changing the coordinates if necessary, we may assume that $0 < \varphi < \pi$.

DEFINITION 1. A parametrization of a surface is called a *generalized Chebyshev net* if $A = B$.

REMARK 1. In this paper the inner product is defined by $\langle\ ,\ \rangle : \mathbf{C} \times \mathbf{C} \to \mathbf{R}$. In the computation we use the following properties: If $f, g : \mathbf{C} \to \mathbf{C}$ are holomorphic functions of $z = x + iy \in \mathbf{C}$ then

$$\langle f, g\rangle_x = \langle f', g\rangle + \langle f, g'\rangle , \quad \langle f, g\rangle_y = \langle if', g\rangle + \langle f, ig'\rangle .$$

The following result, obtained in [7], characterizes the surfaces in $\mathbf{R}^3$ with a generalized Chebyshev net, negative Gaussian curvature and constant Chebyshev angle $\varphi \neq \pi/2$.

THEOREM 1. *Let $M \subset \mathbf{R}^3$ be a connected orientable Riemann surface and φ a constant different from $\pi/2$. There exists an immersion $X : M \to \mathbf{R}^3$ with a generalized Chebyshev net, negative Gaussian curvature $K = -\frac{1}{e^{2u}}$ and Chebyshev angle φ if and only if there exists a global meromorphic function $h : M \to \mathbf{C}$ such that $h'(z) \neq 0$ at all regular points and it admits only simple poles, satisfying the following*

$$\begin{aligned} &2e^c(1 + \varepsilon\cos\varphi)\langle h, (1 + \varepsilon i)h'\rangle\langle h', h'\rangle \\ &\qquad - [1 + e^c(1 + \varepsilon\cos\varphi) \mid h \mid^2]\langle h', (1 + \varepsilon i)h''\rangle = 0 . \end{aligned} \tag{1}$$

Moreover, locally the fundamental forms of X are given by

$$I = e^{(1-\varepsilon\cos\varphi)u+c}[dx^2 + 2\cos\varphi dxdy + dy^2] , \tag{2}$$

$$II = 2e^{-\varepsilon\cos\varphi u+c}\sin\varphi dxdy , \tag{3}$$

where

$$u(x, y) = \log\left(\frac{1 + e^c(1 + \varepsilon\cos\varphi) \mid h(z) \mid^2}{2 \mid h'(z) \mid}\right)^{\frac{2}{1+\varepsilon\cos\varphi}} , \tag{4}$$

$c \in \mathbf{R}$, $z = x + iy \in \mathbf{C}$, $\varepsilon = \pm 1$.

LEMMA 1. *If $f, g, h : \mathbf{C} \to \mathbf{C}$ are holomorphic functions of $z = x + iy \in \mathbf{C}$ such that*

$$\langle 1, f \rangle + |h|^2 \langle 1, g \rangle = 0 . \tag{5}$$

Then

$$f = -\bar{z}_1 h + ic_2 , \quad g = ic_1 + \frac{z_1}{h} , \tag{6}$$

where c_i are real constants and $z_1 \in \mathbf{C}$.

PROOF. The equation (5) can be written as

$$P = \langle 1, f \rangle + \langle h, hg \rangle = 0 . \tag{7}$$

From this equation, it follows that $P_{xx} + P_{yy} = 0$, i.e.

$$\langle h', (hg)' \rangle = 0 ,$$

therefore $(hg)' = ic_1 h'$, $c_1 \in \mathbf{R}$ and by integration we get

$$hg = ic_1 h + z_1 , \quad z_1 \in \mathbf{C} . \tag{8}$$

Substituting (8) into (7), we obtain

$$\langle 1, f \rangle + \langle h, z_1 \rangle = 0 . \tag{9}$$

Differentiating with respect to x and y respectively, we get

$$\langle 1, f' \rangle + \langle h', z_1 \rangle = 0 , \quad \langle 1, if' \rangle + \langle ih', z_1 \rangle = 0 .$$

From these equations we have

$$\overline{f'} + z_1 \overline{h'} = 0$$

and consequently

$$\frac{f'}{h'} = -\overline{z_1} .$$

Integrating, we get

$$f = -\bar{z}_1 h + z_2 , \quad z_2 \in \mathbf{C} ,$$

substituting this expression into (9), we obtain $\langle 1, z_2 \rangle = 0$, it follows from this that $z_2 = ic_2$, $c_2 \in \mathbf{R}$, consequently we obtain the expression of f given in (6). The expression of g is obtained from equation (8). □

3. Main Results

The following Theorem classifies a class of surfaces in $\mathbf{R}^3$ with a generalized Chebyshev net, negative Gaussian curvature and constant Chebyshev angle $\varphi \neq \pi/2$.

THEOREM 2. *Let* $M \subset \mathbf{R}^3$ *be a connected orientable Riemann surface and* φ *a constant different from* $\pi/2$. $X : M \to \mathbf{R}^3$ *is an immersion with a generalized Chebyshev net, negative Gaussian curvature* $K = -\frac{1}{e^{2u}}$ *and Chebyshev angle* φ *if and only if the global meromorphic function* $h : M \to \mathbf{C}$ *of the Theorem* 1 *is given by*:

1)

$$h(z) = \frac{i}{\bar{z}_1}\left[S\coth\left(\frac{\varepsilon+i}{4}Sz - z_0\right) + c_2\right], \quad \textit{in this case } M = \mathbf{C} - \{\alpha_k\}_{k\in\mathbf{Z}}, \tag{10}$$

$\alpha_k = \frac{2(\varepsilon-i)}{S}\mathrm{Re}(z_0) + \frac{2(1+\varepsilon i)}{S}[\mathrm{Im}(z_0) + k\pi]$, $\quad S = \sqrt{c_2^2 + \frac{|z_1|^2}{e^c(1+\varepsilon\cos\varphi)}}$, $\quad c, c_2 \in \mathbf{R}$, $z_0, z_1 \in \mathbf{C}$, $\quad \varepsilon = \pm 1$,

2)

$$h(z) = e^{(\varepsilon+i)\frac{c_2}{2}z + z_2}, \quad \textit{in this case } M = \mathbf{C}, \ c_2 \neq 0, \ z_2 \in \mathbf{C}, \ \varepsilon = \pm 1. \tag{11}$$

PROOF. From Theorem 1, the proof of this Theorem reduces to determining all of the solutions of the differential equation (1).

The equation (1) can be written as

$$\left\langle 1, \frac{bh''}{h'}\right\rangle + |h|^2\left\langle 1, \frac{abh''}{h'} - \frac{2abh'}{h}\right\rangle = 0, \tag{12}$$

where $a = e^c(1 + \varepsilon cos\varphi)$, $b = 1 + \varepsilon i$, $\varepsilon = \pm 1$.

From Lemma 1, we obtain

$$\frac{bh''}{h'} = -\bar{z}_1 h + ic_2, \tag{13}$$

$$\frac{abh''}{h'} - 2ab\frac{h'}{h} = ic_1 + \frac{z_1}{h}. \tag{14}$$

Substituting (13) into (14) we get

$$a(-\bar{z}_1 h + ic_2) - 2ab\frac{h'}{h} = ic_1 + \frac{z_1}{h}$$

and consequently

$$bh' = -\frac{\bar{z}_1}{2}h^2 + \frac{i}{2}\left(c_2 - \frac{c_1}{a}\right)h - \frac{z_1}{2a}. \tag{15}$$

Differentiating (15) and dividing by h', we have that

$$\frac{bh''}{h'} = -\bar{z}_1 h + \frac{i}{2}\left(c_2 - \frac{c_1}{a}\right). \tag{16}$$

From (13) and (16), it follows that $c_1 = -ac_2$, substituting this expression into (15), we get

$$h' = -\frac{\bar{z}_1}{2b}h^2 + \frac{ic_2}{b}h - \frac{z_1}{2ab}. \tag{17}$$

Now we will determine the solutions of the equation (17).

Supposing that $z_1 \neq 0$, $c_2 \in \mathbf{R}$ and substituting the expressions $w_1 = -\frac{\bar{z}_1}{2b}$, $w_2 = \frac{ic_2}{b}$, $w_3 = -\frac{z_1}{2ab}$ into (17), we obtain

$$h' = w_1\left[\left(h + \frac{w_2}{2w_1}\right)^2 - \frac{w_2^2 - 4w_1w_3}{4w_1^2}\right]. \tag{18}$$

On the other hand

$$\frac{w_2^2 - 4w_1w_3}{4w_1^2} = -\frac{1}{\bar{z}_1^2}\left(c_2^2 + \frac{|z_1|^2}{a}\right). \tag{19}$$

Putting $S^2 = c_2^2 + \frac{|z_1|^2}{a}$, from (18) and (19), we get

$$w_1 = \frac{h'}{\left[\left(h + \frac{w_2}{2w_1}\right)^2 + \frac{1}{\bar{z}_1^2}S^2\right]}. \tag{20}$$

We can show that

$$\frac{h'}{\left[\left(h + \frac{w_2}{2w_1}\right)^2 + \frac{1}{\bar{z}_1^2}S^2\right]} = \frac{h'}{\frac{i}{\bar{z}_1}S}\left[\frac{\frac{1}{2}}{h + \frac{w_2}{2w_1} - \frac{i}{\bar{z}_1}S} - \frac{\frac{1}{2}}{h + \frac{w_2}{2w_1} + \frac{i}{\bar{z}_1}S}\right]. \tag{21}$$

Using (21) in (20), we get

$$\frac{i}{\bar{z}_1}Sw_1dz = \frac{\frac{1}{2}dh}{h + \frac{w_2}{2w_1} - \frac{i}{\bar{z}_1}S} - \frac{\frac{1}{2}dh}{h + \frac{w_2}{2w_1} + \frac{i}{\bar{z}_1}S}. \tag{22}$$

Integrating

$$\frac{i}{\bar{z}_1}Sw_1z + z_0 = \frac{1}{2}\log\left(h + \frac{w_2}{2w_1} - \frac{i}{\bar{z}_1}S\right) - \frac{1}{2}\log\left(h + \frac{w_2}{2w_1} + \frac{i}{\bar{z}_1}S\right).$$

Hence,

$$e^{\frac{2i}{\bar{z}_1}Sw_1z + 2z_0} = \frac{h + \frac{w_2}{2w_1} - \frac{i}{\bar{z}_1}S}{h + \frac{w_2}{2w_1} + \frac{i}{\bar{z}_1}S}.$$

Solving for h, we get

$$h(z) = \frac{i}{\bar{z}_1} S\left(\frac{1 + e^{\frac{2i}{\bar{z}_1} S w_1 z + 2z_0}}{1 - e^{\frac{2i}{\bar{z}_1} S w_1 z + 2z_0}}\right) - \frac{w_2}{2w_1}. \tag{23}$$

Substituting the expressions $w_1 = -\frac{\bar{z}_1}{2b}$, $w_2 = \frac{ic_2}{b}$, $b = 1 + \varepsilon i$ into (23), we have that the solutions of equation (17) are given by

$$h(z) = \frac{i}{\bar{z}_1}\left[S\left(\frac{1 + e^{-\frac{\varepsilon+i}{2} Sz + 2z_0}}{1 - e^{-\frac{\varepsilon+i}{2} Sz + 2z_0}}\right) + c_2\right],$$

which is equivalent to equation (10). We observe that $\sinh\left(\frac{\varepsilon+i}{4} Sz - z_0\right) = 0$ if and only if $z = \alpha_k$.

For the case $z_1 = 0$, $c_2 \neq 0$, the equation (17) reduces to $h' = \frac{ic_2}{b} h$.

Integrating, we obtain

$$\log h = \frac{ic_2}{b} z + z_2, \quad z_2 \in \mathbf{C}.$$

Hence, $h(z) = e^{(\varepsilon+i)\frac{c_2}{2} z + z_2}$, is a solution to the equation (17) given in (11). Which concludes the proof of Theorem 2. □

COROLLARY 1. *There exists a seven-parameter family of surfaces with a generalized Chebyshev net and constant Chebyshev angle* $\varphi \neq \pi/2$ *whose first and second fundamental forms are given by*

$$I = e^c\left(\frac{\sqrt{2}\left|\sinh\left(\frac{\varepsilon+i}{4} Sz - z_0\right)\right|^2}{|\bar{z}_1| S^2}\left\{|\bar{z}_1|^2 + e^c(1 + \varepsilon\cos\varphi)\right.\right. \tag{24}$$

$$\left.\left.\times\left|S\coth\left(\frac{\varepsilon+i}{4} Sz - z_0\right) + c_2\right|^2\right\}\right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}} [dx^2 + 2\cos\varphi dx dy + dy^2],$$

$$II = 2e^c\left(\frac{\sqrt{2}\left|\sinh\left(\frac{\varepsilon+i}{4} Sz - z_0\right)\right|^2}{|\bar{z}_1| S^2}\left\{|\bar{z}_1|^2 + e^c(1 + \varepsilon\cos\varphi)\right.\right. \tag{25}$$

$$\left.\left.\times\left|S\coth\left(\frac{\varepsilon+i}{4} Sz - z_0\right) + c_2\right|^2\right\}\right)^{\frac{-2\varepsilon\cos\varphi}{1+\varepsilon\cos\varphi}} \sin\varphi dx dy.$$

Moreover, if $c_2 = 0$ *and* $c \geq -\log(1 + \varepsilon\cos\varphi)$ *then the surfaces defined by* (24) *and* (25) *are complete.*

PROOF. From Theorem 2, it follows that for the meromorphic function given by (10) there exists a seven-parameter family $X : \mathbf{C} - \{z_k\}_{k\in\mathbf{Z}} \to \mathbf{R}^3$ of surfaces with a generalized

Chebyshev net and constant Chebyshev angle. Moreover, we have that

$$h'(z) = \frac{(1-\varepsilon i)S^2}{4\bar{z}_1 \sinh^2(\frac{\varepsilon+i}{4}Sz - z_0)}. \tag{26}$$

Substituting (10) and (26) into (4), we get

$$\begin{aligned} u(x,y) &= \log\left(\frac{\sqrt{2}\left|\sinh\left(\frac{\varepsilon+i}{4}Sz-z_0\right)\right|^2}{|\bar{z}_1|S^2} \left\{ |\bar{z}_1|^2 + e^c(1+\varepsilon\cos\varphi) \right.\right. \\ &\qquad \left.\left. \times \left| S\coth\left(\frac{\varepsilon+i}{4}Sz - z_0\right) + c_2 \right|^2 \right\} \right)^{\frac{2}{1+\varepsilon\cos\varphi}}. \end{aligned} \tag{27}$$

Substituting (27) in (2) and (3), we obtain the equations (24) and (25). On the other hand, if $c_2 = 0$ and $c \geq -\log(1+\varepsilon\cos\varphi)$, we can show that

$$\left(\frac{\sqrt{2}\,|\sinh R|^2}{|\bar{z}_1|S^2} \left\{ |\bar{z}_1|^2 + e^c(1+\varepsilon\cos\varphi)\,|S\coth R|^2 \right\} \right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}} \geq \left(\frac{\sqrt{2}}{|z_1|}\right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}} \tag{28}$$

where, $R = \frac{\varepsilon+i}{4}Sz - z_0$. In fact, from the condition $c \geq -\log(1+\varepsilon\cos\varphi)$, we have

$$\begin{aligned} &\frac{\sqrt{2}\,|\sinh R|^2}{|\bar{z}_1|S^2}\left\{ |\bar{z}_1|^2 + e^c(1+\varepsilon\cos\varphi)\,|S\coth R|^2 \right\} \geq \frac{\sqrt{2}\,|\sinh R|^2}{|\bar{z}_1|S^2}\left\{ |\bar{z}_1|^2 + S^2\,|\coth R|^2 \right\} \\ &\quad = \frac{\sqrt{2}}{|\bar{z}_1|S^2}\left(|\bar{z}_1|^2\,|\sinh R|^2 + \frac{|z_1|^2}{e^c(1+\varepsilon\cos\varphi)}\,|\cosh R|^2 \right) \\ &\quad = \frac{\sqrt{2}|\bar{z}_1|}{S^2 e^c(1+\varepsilon\cos\varphi)}\left(e^c(1+\varepsilon\cos\varphi)\,|\sinh R|^2 + |\cosh R|^2 \right) \geq \frac{\sqrt{2}}{|z_1|}. \end{aligned}$$

Therefore, the equation (28), is an consequence of this inequality. From (28), it follows that

$$ds^2 \geq C d\bar{s}^2 \tag{29}$$

where,

$$\begin{aligned} ds^2 &= e^c\left(\frac{\sqrt{2}\left|\sinh\left(\frac{\varepsilon+i}{4}Sz - z_0\right)\right|^2}{|\bar{z}_1|S^2} \left\{ |\bar{z}_1|^2 + e^c(1+\varepsilon\cos\varphi) \right.\right. \\ &\qquad \left.\left. \times \left| S\coth\left(\frac{\varepsilon+i}{4}Sz - z_0\right) \right|^2 \right\} \right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}} \left[dx^2 + 2\cos\varphi\, dx dy + dy^2 \right], \\ d\bar{s}^2 &= dx^2 + 2\cos\varphi\, dx dy + dy^2 \end{aligned}$$

and $\quad C = e^c\left(\frac{\sqrt{2}}{|z_1|}\right)^{\frac{2(1-\varepsilon\cos\varphi)}{1+\varepsilon\cos\varphi}}$.

Since the metric $Cd\bar{s}^2$ is complete, it follows that the metric ds^2 also is complete. On the other hand, when $z \to \alpha_k$, we have that $R \to k\pi i$ and the metric also satisfies the equation (29) and therefore is complete. This concludes the proof of Corollary 1. □

REMARK 2. Considering the solution $h(z) = e^{(\varepsilon+i)\frac{c_2}{2}z+z_2}$, given in Theorem 2, we obtain the family of the surfaces obtained in Corollary 2 in [7].

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