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Fourth Moment of the Riemann Zeta-function with a Shift along the Real Line

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Abstract. For the Riemann zeta-function we consider the fourth moment whose square part admits a shift along the real line. Asymptotic formulas are obtained for the shift parameter around 0 and for the one in unbounded regions.

1. Introduction

The asymptotic formula for the fourth moment of the Riemann zeta-function

$$\mathcal{I}_4(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = \frac{1}{2\pi^2}T(\log T)^4 + O(T(\log T)^3)$$

was proved by Ingham [4]. Ramachandra [12], [13] gave a simplified proof of this formula, in which the identity of Montgomery and Vaughan [9] (see also [6, §5.3]) plays an essential role. A precise representation with lower order terms of the form

$$\mathcal{I}_4(T) = \sum_{j=0}^{4} c_j T(\log T)^j + O(T^{7/8+\varepsilon})$$

was presented by Heath-Brown [2]. For a weighted fourth moment Motohashi established an explicit formula based on spectral theory of automorphic forms, which brought about the improvement $\ll T^{2/3}(\log T)^8$ on the error term above (cf. [10]).

The second moment with shift parameters

$$\int_1^T \zeta\left(\frac{1}{2}+a+it\right)\zeta\left(\frac{1}{2}-b-it\right)dt$$

was also treated in [4], and Bettin [1] gave an asymptotic formula for unbounded shifts near the critical line. A shifted fourth moment was discussed in [10], and an asymptotic formula

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for a twisted fourth moment presented by Hughes and Young [3] contains small shift parameters $\ll (\log T)^{-1}$. Concerning shifted moments of another type Kösters [8] pointed out an interesting occurrence of the sine kernel for random matrix ensembles. For a shifted fourth moment of the form

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\zeta\left(\frac{1}{2}+a+it\right)\right|^2 dt$$

with $ai = \lambda \in \mathbf{R}$ we [15] studied the asymptotic behaviour in the shift range $|\lambda| \ll \exp((\log T)^{3/8-\varepsilon})$, in which the sine kernel appears if $|\lambda| \ll (\log T)^{-\beta}$ with $0 < \beta \leq 1/2$.

In this paper we consider this type of fourth moment for $a \in \mathbf{R}$. Asymptotic formulas are obtained for a around 0 and for a in ranges unbounded as $T \to \infty$. In deriving them we apply the method of Ramachandra to $\zeta(s)\zeta(s+a)$. In addition to the identity of Montgomery and Vaughan we need a summation formula related to $\sigma_{-a}(n) = \sum_{d\,|\,n} d^{-a}$, which is given in Section 3. Our results are proved in Section 6 by using the facts described in Sections 4 and 5. Throughout this paper ε denotes an arbitrary small positive number permitted to vary in each appearance.

2. Main results

For $a \in \mathbf{R}$ and $T \gg 1$, let

$$\Psi(a,T) := \frac{1}{T}\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\zeta\left(\frac{1}{2}+a+it\right)\right|^2 dt\,.$$

Our results are stated as follows:

THEOREM 2.1. *As* $|a| + (\log T)^{-1} \to 0$,

$$\Psi(a,T) = 3\pi^{-2}a^{-4}\phi(a\log T)\big(1 + O(|a| + (\log T)^{-1})\big)$$

with $\phi(y) := -2 + y + 4e^{-y} - (2+y)e^{-2y}$, *the implied constant being absolute.*

THEOREM 2.2. *Suppose that* $\theta(a) := |2\pi a|^{2|a|} + |a|^{-1} = o(\log T)$. *If* $a < 0$,

$$\Psi(a,T) = \frac{(2\pi)^{2a}\zeta(1-2a)\zeta(1-a)^2}{(1-2a)\zeta(2-2a)}T^{-2a}\log T\big(1 + O(\theta(a)(\log T)^{-1})\big)\,;$$

and if $a > 0$,

$$\Psi(a,T) = \frac{\zeta(1+2a)\zeta(1+a)^2}{\zeta(2+2a)}\log T\big(1 + O(\theta(a)(\log T)^{-1})\big)\,.$$

Here the implied constants are absolute.

REMARK 2.1. Since $y^{-4}\phi(y) = 1/6 + O(y)$ around $y = 0$, Theorem 2.1 gives $\Psi(0, T) = (2\pi^2)^{-1}(\log T)^4\big(1 + O((\log T)^{-1})\big)$ as $(\log T)^{-1} \to 0$, which coincides with the result of Ingham [4].

REMARK 2.2. Let $0 < \beta \le 1$. For $|a| \ll (\log T)^{-\beta}$ (respectively, $(\log T)^{-1+\beta} \ll |a| \ll 1$) the error term in Theorem 2.1 (respectively, Theorem 2.2) is $\ll (\log T)^{-\beta}$. The formulas in Theorem 2.2 are significant for a in regions unbounded as $T \to \infty$. Indeed, if $(\log T)^{-\varepsilon} \ll |a| \ll (\log\log T)^{1-\varepsilon}$, then each error term is $\ll (\log T)^{-1+\varepsilon}$.

3. Sums related to $\sigma_{-a}(n)$

Recall the formula

$$Z(a, s) := \frac{\zeta(s)\zeta(s+a)^2\zeta(s+2a)}{\zeta(2s+2a)} = \sum_{n=1}^{\infty} \frac{\sigma_{-a}(n)^2}{n^s}$$

for $\mathrm{Re}(s) > \max\{1, 1-2a\}$ (cf. [6, p. 37] or [16, (1.3.3)]). The function $(x^s/s)Z(a, s)$ with $x > 0$ possesses poles at least at $s = 1, 1-a, 1-2a$ if $|a| < 1/2$, at $s = 1$ if $a \ge 1/2$, and at $s = 1-2a$ if $a \le -1/2$. If $a \ne 0$, the residues are, respectively,

$$\begin{aligned}
R_1 &:= \frac{\zeta(1+2a)\zeta(1+a)^2}{\zeta(2+2a)} x\,,\\
R_{1-a} &:= \frac{\zeta(1-a)\zeta(1+a)}{\zeta(2)(1-a)} x^{1-a}\\
&\quad \times \left(\log x - \frac{1}{1-a} + \frac{\zeta'}{\zeta}(1-a) + \frac{\zeta'}{\zeta}(1+a) - 2\frac{\zeta'}{\zeta}(2) + 2\gamma_{\mathrm{e}} \right),\\
R_{1-2a} &:= \frac{\zeta(1-2a)\zeta(1-a)^2}{(1-2a)\zeta(2-2a)} x^{1-2a}\,,
\end{aligned}$$

where γ_{e} is the Euler constant. These three quantities are also meromorphic in $a \in \mathbf{C}$, and are holomorphic around $a = \pm 1/2$. Furthermore the sum of them is holomorphic around $a = 0$. Indeed, for $0 < |a| < 1/16$

$$R_1 + R_{1-a} + R_{1-2a} = \frac{1}{2\pi i} \int_{\Delta} \frac{x^s}{s} Z(a, s)\,ds$$

with the positively oriented circle $\Delta : |s-1| = 1/4$, where the integrand $(x^s/s)Z(a, s)$ restricted to Δ is holomorphic around $a = 0$. The holomorphy of the sum around $a = 0$ may also be checked by a direct computation of the Laurent series expansion (cf. Section 4.1). For $x \ge 2$, restricting to $a \in \mathbf{R}$ again, we set

$$\Lambda(a, x) := \begin{cases} x^{-1} R_{1-2a} & \text{if } a \le -1/2\,, \\ x^{-1}(R_1 + R_{1-a} + R_{1-2a}) & \text{if } -1/2 < a < 1/2\,, \\ x^{-1} R_1 & \text{if } a \ge 1/2\,, \end{cases} \tag{3.1}$$

which is continuous in $a \in \mathbf{R} \setminus \{\pm 1/2\}$.

PROPOSITION 3.1. *For $a \in \mathbf{R}$, $\sum_{n \le x} \sigma_{-a}(n)^2 = x\Lambda(a, x) + E_a(x)$, the error term $E_a(x)$ satisfying $\ll x^{1/2-2a+\varepsilon}$ if $a \le 0$, and $\ll x^{1/2+\varepsilon}$ if $a \ge 0$. Here the implied constant depends on ε only.*

For each a, better error estimates are known [5], [11], [14]. In particular, $E_a(x) \ll x^{1/2-2a/3}(\log x)^5$ for $0 < a \le 3/8$ (cf. [11]), while $E_0(x) \ll x^{1/2}(\log x)^5 \log\log x$ (cf. [14]). In the proof below we put emphasis on the uniformity in a.

PROOF. Let δ be a given small positive number. First we consider the case $0 \le a \le 1/4 - \delta$. Let ω be a number such that $\omega \ge x^{1/2}$. By Perron's formula,

$$\sum_{n \le x} \sigma_{-a}(n)^2 = \frac{1}{2\pi i} \int_{1+\delta-i\omega}^{1+\delta+i\omega} \frac{x^s}{s} Z(a, s) ds + O(x^{1/2+\delta}) \tag{3.2}$$

with $s = \sigma + it$. The poles of the integrand inside the rectangle with vertices $1 + \delta \pm i\omega$, $1/2 \pm i\omega$ are $s = 1, 1 - a, 1 - 2a$, and hence

$$I_{1+\delta-i\omega}^{1+\delta+i\omega} - I_{1/2+i\omega}^{1+\delta+i\omega} - I_{1/2-i\omega}^{1/2+i\omega} + I_{1/2-i\omega}^{1+\delta-i\omega} = R_1 + R_{1-a} + R_{1-2a} = x\Lambda(a, x)\,,$$

provided that $a \ne 0$, where

$$I_{z_-}^{z_+} := \frac{1}{2\pi i} \int_{z_-}^{z_+} \frac{x^s}{s} Z(a, s) ds\,.$$

This equality is also valid for $a = 0$, because both sides are holomorphic around $a = 0$. Hence

$$\sum_{n \le x} \sigma_{-a}(n)^2 = x\Lambda(a, x) + I_0 - I_- + I_+ + O(x^{1/2+\delta})$$

with $I_+ = -\overline{I_-} := I_{1/2+i\omega}^{1+\delta+i\omega}$ and $I_0 := I_{1/2-i\omega}^{1/2+i\omega}$. To evaluate I_+ set

$$g(t) := \int_{1/2}^{1+\delta} \Big| \zeta(\sigma + it)\zeta(\sigma + a + it)^2 \zeta(\sigma + 2a + it) \Big| d\sigma\,.$$

Since

$$\int_{x^{1/2}}^{2x^{1/2}} g(t)dt \ll \int_{1/2}^{1+\delta} \bigg(\int_{x^{1/2}}^{2x^{1/2}} |\zeta(\sigma + it)|^4 dt \bigg(\int_{x^{1/2}}^{2x^{1/2}} |\zeta(\sigma + a + it)|^4 dt \bigg)^2 \times \int_{x^{1/2}}^{2x^{1/2}} |\zeta(\sigma + 2a + it)|^4 dt \bigg)^{1/4} d\sigma \ll x^{1/2+\delta}\,,$$

we may choose $\omega \in [x^{1/2}, 2x^{1/2}]$ so that $g(\omega) \ll x^{\delta}$. Then, observing $\zeta(2s)^{-1} \ll \log t$ for

$1/2 \le \sigma$, we have $I_+ \ll x^{1+\delta}\omega^{-1} \cdot \log x \cdot g(\omega) \ll x^{1/2+3\delta}$. Furthermore

$$I_0 \ll x^{1/2} \log \omega \left(\int_1^\omega \left| \zeta(1/2+it)\zeta(1/2+a+it)^2 \zeta(1/2+2a+it) \right| \frac{dt}{t} + \delta^{-1} \right)$$
$$\ll x^{1/2} \log \omega \big(J(0)^{1/4} J(a)^{1/2} J(2a)^{1/4} + \delta^{-1} \big) ,$$

where

$$J(v) := \int_1^\omega |\zeta(1/2+v+it)|^4 \frac{dt}{t} .$$

Integrating by parts, we have, uniformly in $v \ge 0$,

$$J(v) \ll \frac{1}{\omega} \int_1^\omega |\zeta(1/2+v+it)|^4 dt + \int_1^\omega \int_1^t |\zeta(1/2+v+i\rho)|^4 d\rho \, \frac{dt}{t^2} \ll \omega^\delta .$$

This gives $I_0 \ll x^{1/2+2\delta}$, the implied constant depending on δ only. Thus we obtain $\sum_{n \le x} \sigma_{-a}(n)^2 = x\Lambda(a,x) + O(x^{1/2+3\delta})$ for $0 \le a \le 1/4 - \delta$. Next suppose that $1/4 - \delta \le a \le 1/2 - 4\delta$. The poles are arrayed as $1 - 2a < 1/2 + 3\delta < 1 - a < 1$, and hence $x\Lambda(a,x) = R_1 + R_{1-a} + O(x^{1/2+3\delta})$. In (3.2), shift of the path of integration to $\sigma = 1/2 + 3\delta$ takes the right-hand side to

$$x\Lambda(a,x) + I_{1/2+3\delta-i\omega}^{1/2+3\delta+i\omega} + I_{1/2+3\delta+i\omega}^{1+\delta+i\omega} + \overline{I_{1/2+3\delta+i\omega}^{1+\delta+i\omega}} + O(x^{1/2+3\delta}) .$$

Then $I_{1/2+3\delta+i\omega}^{1+\delta+i\omega} \ll x^{1/2+3\delta}$ and $I_{1/2+3\delta-i\omega}^{1/2+3\delta+i\omega} \ll x^{1/2+5\delta}$, which implies the formula with the error term $\ll x^{1/2+5\delta}$. If $a \ge 1/2 - 4\delta$, then $x\Lambda(a,x) = R_1 + O(x^{1/2+5\delta})$, since $1 - a < 1/2 + 5\delta$. Shifting the path of integration to $\sigma = 1/2 + 5\delta$, and noting the bound of $|\zeta(1/2+a+it)|$ uniform in $a \ge 1$, we similarly obtain the error term $\ll x^{1/2+7\delta}$ with the implied constant independent of a. Therefore $E_a(x) \ll x^{1/2+\varepsilon}$ for $a \ge 0$. The formula for $a \le 0$ is an immediate consequence of the fact

$$\sum_{n \le x} \sigma_{-a}(n)^2 = \sum_{n \le x} n^{-2a} \sigma_a(n)^2 = x^{-2a} \sum_{n \le x} \sigma_a(n)^2 + 2a \int_1^x \xi^{-2a-1} \Bigg(\sum_{n \le \xi} \sigma_a(n)^2 \Bigg) d\xi .$$

Thus we obtain the proposition. □

We need the following estimate as well (for the summation formula see [7]).

PROPOSITION 3.2. *For* $a \in \mathbf{R}$, $\sum_{n \le x} \sigma_{-a}(n) \ll (1+|a|)|x(1-x^{-a})/a|$, *the implied constant being absolute.*

PROOF. Note that $\sum_{n \le x} \sigma_{-a}(n) = \sum_{n \le x} \sum_{d|n} d^{-a} = \sum_{m \le x} \sum_{d \le x/m} d^{-a}$, where the last sum satisfies, for $|a| \le 1/2$,

$$\ll x^{1-a} \sum_{m \le x} m^{a-1} \ll x^{1-a} \left(1 + \int_1^x \xi^{a-1} d\xi \right) = x^{1-a} + \frac{x}{a}(1 - x^{-a}) .$$

Using $|ax^{-a}/(1-x^{-a})| = |Xe^{-X}/(1-e^{-X})|(\log x)^{-1} \ll (1+|X|)(\log x)^{-1} \ll 1+|a|$ with $X = a\log x$, we obtain the desired estimate for $|a| \le 1/2$. Since, for $a \ge 1/2$, $\sum_{n\le x}\sigma_{-a}(n) \le \sum_{n\le x}\sigma_{-1/2}(n) \ll x$, the lemma is valid for $a \ge -1/2$. Hence, for $a \le -1/2$ as well, we have $\sum_{n\le x}\sigma_{-a}(n) = \sum_{n\le x} n^{-a}\sigma_a(n) \le x^{-a}\sum_{n\le x}\sigma_a(n) \ll (1+|a|)|x(1-x^{-a})/a|$. This completes the proof. □

4. Lemmas

Set, for $x \ge 1$ and for $a \in \mathbf{R}$,

$$\Phi(a,x) := \int_1^x \xi^{-1}\Lambda(a,\xi)d\xi\,.$$

Let N be a positive integer such that $x^{1/4} \le 2^{-N}x \le 2x^{1/4}$. Define related sums by

$$\Phi_N(a,x) := \sum_{n=0}^{N} 2^{-n}\Phi(a,2^{-n}x)\,, \quad |\Lambda|_N(a,x) := \sum_{n=0}^{N} 2^{-n}|\Lambda(a,2^{-n}x)|\,,$$

$$\widetilde{\Phi}_N(a,x) := \sum_{n=0}^{N} 2^{(2a-1)n}x^{-2a}\Phi(-a,2^{-n}x)\,.$$

Then we have the following, in which the implied constants are absolute.

LEMMA 4.1. *As* $|a| + (\log x)^{-1} \to 0$,

$$\Phi_N(a,x) = 6\pi^{-2}a^{-4}\psi(a,x)\big(1+O(|a|+(\log x)^{-1})\big) \gg \log x\,,$$
$$\widetilde{\Phi}_N(a,x) = 6\pi^{-2}a^{-4}x^{-2a}\psi(-a,x)\big(1+O(|a|+(\log x)^{-1})\big)\,,$$

where $\psi(a,x) := -5/2 + a\log x + 2(1+a\log x)x^{-a} + x^{-2a}/2$.

In the lemma above $\hat{\psi}(a,x) := a^{-4}\psi(a,x)$ satisfies $\hat{\psi}(0,x) = (\log x)^4/12$ (cf. (4.5) in the proof of Lemma 4.1).

LEMMA 4.2. *Let* $a < 0$. *As* $|a\log x|^{-1} \to 0$,

$$\widetilde{\Phi}_N(a,x) = \frac{\Lambda_0(-a)}{1-2^{2a-1}}x^{-2a}\log x\big(1+O(|a\log x|^{-1})\big)\,,$$

where $\Lambda_0(a) := \zeta(1+2a)\zeta(1+a)^2\zeta(2+2a)^{-1}$.

LEMMA 4.3. *As* $|a| + (\log x)^{-1} \to 0$, $|\Lambda|_N(a,x) \ll \Phi_N(a,x)(|a| + (\log x)^{-1})$. *Furthermore*, *if* $a < 0$, *then* $|\Lambda|_N(a,x) \ll \widetilde{\Phi}_N(a,x)(\log x)^{-1}$ *as* $|a\log x|^{-1} \to 0$.

Let τ be a sufficiently large positive number. Set

$$\Lambda^*(a,x) := (a^2+1)\Lambda(a,x)\,.$$

LEMMA 4.4. (1) *For* $a \in \mathbf{R}$,

$$\sum_{n \le \tau} \frac{\sigma_{-a}(n)^2}{n} = \Phi(a, \tau) + O\big(\Lambda^*(a, \tau)\big),$$

the implied constant being absolute.

(2) *For* $a \ge 0$, *if* $\mu - 1 \ge c_0 > 0$ (*respectively*, $\mu - 1 \le -c_0 < 0$),

$$\sum_{n > \tau} \frac{\sigma_{-a}(n)^2}{n^\mu} \left(\textit{respectively}, \ \sum_{n \le \tau} \frac{\sigma_{-a}(n)^2}{n^\mu} \right) \ll \tau^{1-\mu} \Lambda^*(a, \tau),$$

the implied constant depending on c_0 *only.*

(3) *For* $a < 0$, *if* $\mu + 2a - 1 \ge c_0 > 0$ (*respectively*, $\mu - 1 \le -c_0 < 0$), *we have the same estimate as in* (2) *with the implied constant depending on* c_0 *only.*

LEMMA 4.5. *For* $a \in \mathbf{R}$ *and for* $\sigma - 1 - \max\{0, a\} \ge c_0 > 0$ (*respectively*, $0 < \sigma \le 1 - c_0 < 1$),

$$\sum_{n > \tau} \frac{\sigma_a(n)}{n^\sigma} \left(\textit{respectively}, \ \sum_{n \le \tau} \frac{\sigma_a(n)}{n^\sigma} \right) \ll (1 + a^2) \tau^{1-\sigma} \left| \frac{1 - \tau^a}{a} \right|,$$

the implied constant depending on c_0 *only.*

These lemmas are verified in the remaining part of this section.

4.1. $\Lambda(a, x)$ **and** $\Phi(a, x)$ **around** $a = 0$**.** The function $\Lambda(a, x)$ is holomorphic for $|a| < 1/2$. The Laurent series expansion of $\zeta(1 + a)$ gives the expression

$$\Lambda(a, x) = \pi^{-2} l_3(a, x) + c_2^1 l_2^1(a, x) + c_2^2 l_2^2(a, x) + c_1^1 l_1^1(a, x) + c_1^2 l_1^2(a, x) + g_0(a) x^{-a} \log x + g_1(a) x^{-a} + g_2(a) x^{-2a} + g_3(a). \tag{4.1}$$

Here c_i^j are certain real numbers; $g_k(a)$ are holomorphic for $|a| < 1/2$; and

$$l_3(a, x) := 3a^{-3}(1 - 2ax^{-a} \log x - x^{-2a}),$$

$$l_2^1(a, x) := a^{-2} x^{-a}(1 - a \log x - x^{-a}), \quad l_2^2(a, x) := a^{-2}(1 - x^{-a})^2,$$

$$l_1^1(a, x) := a^{-1} x^{-a}(1 - x^{-a}), \quad l_1^2(a, x) := a^{-1}(1 - x^{-2a}).$$

These functions are holomorphic around $a = 0$. Furthermore, for $x \ge 1$, by definition

$$\Phi(a, x) = 3\pi^{-2} a^{-4} \psi(a, x) + c_2^1 l_2^{1*}(a, x) + c_2^2 l_2^{2*}(a, x) + c_1^1 l_1^{1*}(a, x) + c_1^2 l_1^{2*}(a, x) + g_0(a) \eta_0(a, x) + g_1(a) \eta_1(a, x) + g_2(a) \eta_1(2a, x) + g_3(a) \log x.$$

Here

$$\psi(a, x) = -5/2 + a \log x + 2(1 + a \log x) x^{-a} + x^{-2a}/2,$$

$$l_2^{1*}(a, x) := a^{-3}(-1/2 + ax^{-a} \log x + x^{-2a}/2),$$

$$l_2^{2*}(a,x) := a^{-3}(-3/2 + a\log x + 2x^{-a} - x^{-2a}/2)\,,$$

$$l_1^{1*}(a,x) := a^{-2}(1-x^{-a})^2/2\,, \quad l_1^{2*}(a,x) := a^{-2}(-1/2 + a\log x + x^{-2a}/2)\,,$$

$$\eta_0(a,x) := a^{-2}(1 - ax^{-a}\log x - x^{-a})\,, \quad \eta_1(a,x) := a^{-1}(1-x^{-a})\,.$$

4.2. Proof of Lemma 4.1. Note that, for $a < 1/4$,

$$\sum_{n\geq 0}(2^{-n})^{1-a}\log(2^{-n}x) = \frac{\log x}{1-2^{a-1}} - \frac{2^{a-1}\log 2}{(1-2^{a-1})^2}\,, \tag{4.2}$$

$$\sum_{n\geq 0} 2^{-n}\left((2^{-n}x)^{-a} - 1\right) = \frac{x^{-a} - 2 + 2^a}{1-2^{a-1}}\,. \tag{4.3}$$

For $|a| < 1/4$, $x \geq 2$, we have $\sum_{n\geq 0} 2^{-n}\psi(a, 2^{-n}x) = h(a, a\log x)$, where

$$h(a,y) := 2(y-a') + 2e^{-y}\left(\frac{y}{1-e^{a'}/2} - \frac{a'e^{a'}/2}{(1-e^{a'}/2)^2}\right) + \frac{2(e^{-y} - 2 + e^{a'})}{1-e^{a'}/2} + \frac{e^{-2y} - 2 + e^{2a'}}{2(1-e^{2a'}/2)}$$

with $a' := a\log 2$. Furthermore $\psi(a,x) = h_0(a\log x)$ with

$$h_0(y) := -5/2 + y + 2(1+y)e^{-y} + e^{-2y}/2\,.$$

Computation of the series expansion leads us to

$$h(a,y) - 2h_0(y) \ll |a|(|y| + |a|)^3\,, \tag{4.4}$$

provided that $|y| \ll 1$, $|a| < 1/4$. The function $h_0(y)$ satisfies

$$h_0(y) = (y^4/12)(1 + O(y)) \tag{4.5}$$

for $|y| \ll 1$; and

$$h_0(y) \gg |y| + e^{-2y} \tag{4.6}$$

for $|y| \gg 1$. If $|y| \ll 1$ and if $|a| < 1/4$, then, by (4.4) and (4.5), $h(a,y)/h_0(y) - 2 \ll |a/y| + |a/y|^4$. For $y \in \mathbf{R}$, $h(a,y) - 2h_0(y) \ll |a|(1 + e^{-2y})$, since $h(0,y) \equiv 2h_0(y)$. Hence, if $|y| \gg 1$ and if $|a| < 1/4$, then, by (4.6), $h(a,y)/h_0(y) - 2 \ll |a|$. Thus we have $h(a,y) = 2h_0(y)\left(1 + O(|a| + |a/y| + |a/y|^4)\right)$ for $|a| < 1/4$, which implies

$$\sum_{n\geq 0} 2^{-n}\psi(a, 2^{-n}x) = h(a, a\log x) = 2h_0(a\log x)\big(1 + O(|a| + (\log x)^{-1})\big)$$

as $|a| + (\log x)^{-1} \to 0$. Note that $h_0(y) > h_0(y') \ge 0$ if $y > y' \ge 0$ or if $y < y' \le 0$. Since $2^{-N}x \asymp x^{1/4}$, it follows that

$$\sum_{n=0}^{N} 2^{-n}\psi(a, 2^{-n}x) = h(a, a\log x) - 2^{-N-1}h\big(a, a\log(2^{-N-1}x)\big) \tag{4.7}$$
$$= 2h_0(a\log x)\big(1 + O(|a| + (\log x)^{-1})\big) \gg \min\{(a\log x)^4, |a|\log x\}$$

with $h_0(a\log x) = \psi(a, x)$. To show the formula for $\Phi_N(a, x)$ it remains to evaluate the sums $\sum_{n=0}^{N} 2^{-n} l_i^{j*}(a, 2^{-n}x)$ and $\sum_{n=0}^{N} 2^{-n}\eta_i(a, 2^{-n}x)$. Write $l_2^{1*}(a, x) = a^{-3}\varphi_2^{1*}(a\log x)$, where $\varphi_2^{1*}(y) := -1/2 + ye^{-y} + e^{-2y}/2$ has the properties:

(a) $|\varphi_2^{1*}(y)| > |\varphi_2^{1*}(y')|$ if $y > y' \ge 0$ or if $y < y' \le 0$;

(b) $|\varphi_2^{1*}(y)| \asymp |y|^3$ for $|y| \ll 1$;

(c) $|\varphi_2^{1*}(y)| \ll |y| + e^{-2y}$ for $|y| \gg 1$.

Then, by (4.5) and (4.6), $|\varphi_2^{1*}(y)| \ll h_0(y)(1 + |y|^{-1})$ uniformly in $y \in \mathbf{R}$. Since $2^{-n}x > 1$ for $n \le N$, we have

$$\sum_{n=0}^{N} 2^{-n} l_2^{1*}(a, 2^{-n}x) \ll \sum_{n=0}^{N} 2^{-n}|a|^{-3}\big|\varphi_2^{1*}\big(a\log(2^{-n}x)\big)\big|$$
$$\ll |a|^{-3}|\varphi_2^{1*}(a\log x)| \ll a^{-4}h_0(a\log x)\big(|a| + (\log x)^{-1}\big).$$

The other sums for $l_i^{j*}(a, x)$, $\eta_i(a, x)$ and $\log x$ admit the same estimates. From (4.7) combined with these estimates the first expression of Lemma 4.1 immediately follows.

To derive the formula for $\widetilde{\Phi}_N(a, x)$ we calculate, for $x \ge 2$, $|a| < 1/4$,

$$\Sigma := \sum_{n\ge 0} 2^{(2a-1)n}x^{-2a}\psi(-a, 2^{-n}x) = \sum_{n\ge 0} 2^{-n}\tilde{\psi}(a, 2^{-n}x),$$
$$\tilde{\psi}(a, x) := x^{-2a}\psi(-a, x) = 1/2 + 2(1 - a\log x)x^{-a} - (5/2 + a\log x)x^{-2a}.$$

Note that $\tilde{\psi}(a, x) = \tilde{h}_0(a\log x)$ with $\tilde{h}_0(y) := e^{-2y}h_0(-y)$. By (4.2) and (4.3) we have $\Sigma = \tilde{h}(a, a\log x)$ with

$$\tilde{h}(a, y) := -e^{-2y}\left(\frac{y}{1 - e^{2a'}/2} - \frac{a'e^{2a'}/2}{(1 - e^{2a'}/2)^2}\right) - \frac{(5/2)e^{-2y}}{1 - e^{2a'}/2}$$
$$-2e^{-y}\left(\frac{y}{1 - e^{a'}/2} - \frac{a'e^{a'}/2}{(1 - e^{a'}/2)^2}\right) + \frac{2e^{-y}}{1 - e^{a'}/2} + 1$$

satisfying $\tilde{h}(a, y) - 2\tilde{h}_0(y) \ll |a|(|a| + |y|)^3$ for $|y| \ll 1$, $|a| < 1/4$. From this we get $\sum_{n=0}^{N} 2^{-n}\tilde{\psi}(a, 2^{-n}x) = 2\tilde{h}_0(a\log x)(1 + O(|a| + (\log x)^{-1}))$. Write $\tilde{l}_2^{1*}(a, x) := x^{-2a}l_2^{1*}(-a, x) = a^{-3}\tilde{\varphi}_2^{1*}(a\log x)$ with $\tilde{\varphi}_2^{1*}(y) := -1/2 + ye^{-y} + e^{-2y}/2$ admitting the same property as of $\varphi_2^{1*}(y)$. Then it follows that $\sum_{n=0}^{N} 2^{-n}\tilde{l}_2^{1*}(a, 2^{-n}x) \ll$

$a^{-4}\tilde{h}_0(a\log x)(|a| + (\log x)^{-1})$. The same estimates are verified for the sums related to the remaining $l_i^{j*}(-a, x)$. Furthermore, $\tilde{\eta}_0(a, x) := x^{-2a}\eta_0(-a, x) = a^{-2}\rho_0(a\log x)$ and $\tilde{\eta}_1(a, x) := x^{-2a}\eta_1(-a, x) = a^{-1}\rho_1(a\log x)$, where $\rho_0(y) := e^{-2y} + ye^{-y} - e^{-y}$ and $\rho_1(y) := e^{-y} - e^{-2y}$, respectively. The functions $\rho_i(y)$ satisfy

(a) $|\rho_0(y)| \asymp y^2$ and $|\rho_1(y)| \asymp |y|$ for $|y| \ll 1$;

(b) $|\rho_i(y)| \ll e^{-2y} + |y|e^{-y}$ for $|y| \gg 1$;

(c) $|\rho_i(y)| > |\rho_i(y')|$ if $y < y' \le 0$;

(d) there exists a positive number $c_0^{(i)}$ such that $|\rho_i(y)| > |\rho_i(y')|$ if $0 \le y' < y \le c_0^{(i)}$ and that $0 < \rho_i(y) \ll 1$ if $y \ge c_0^{(i)}$.

Hence, if $a\log x \le 0$ or if $a\log x \ll 1$,

$$\sum_{n=0}^{N} 2^{-n}\tilde{\eta}_0(a, 2^{-n}x) \ll a^{-2}|\rho_0(a\log x)| \ll a^{-4}\tilde{h}_0(a\log x)\big(a^2 + (\log x)^{-2}\big),$$

and otherwise, $\ll a^{-2} \ll a^{-4}\tilde{h}_0(a\log x)\cdot a^2$. Combining the facts above, we arrive at the second formula of Lemma 4.1.

4.3. Proof of Lemma 4.2. By (4.2), for $a < 0$,

$$\sum_{n=0}^{N} 2^{(2a-1)n}\log(2^{-n}x) = \frac{\log x}{1 - 2^{2a-1}}\big(1 + O(|a\log x|^{-1})\big), \tag{4.8}$$

since $2^{-N}x \asymp x^{1/4}$. Recalling (3.1) for $|a| < 1/2$, we write

$$\Lambda(a, x) = \Lambda_0(a) + \Lambda_1(a)x^{-a} + \Lambda_2(a)x^{-a}\log x + \frac{\Lambda_0(-a)}{1-2a}x^{-2a} \tag{4.9}$$

with $\Lambda_2(a) := \zeta(1-a)\zeta(1+a)(1-a)^{-1}/\zeta(2)$,

$$\Lambda_1(a) := -\frac{\zeta(1-a)\zeta(1+a)}{\zeta(2)(1-a)^2}\left(1 - (1-a)\left(\frac{\zeta'}{\zeta}(1-a) + \frac{\zeta'}{\zeta}(1+a) - 2\frac{\zeta'}{\zeta}(2) + 2\gamma_{\mathrm{e}}\right)\right)$$

and $\Lambda_0(a)$ as in Lemma 4.2. Suppose that $-1/2 < a < 0$. By definition

$$\Phi(-a, x) = \Lambda_0(-a)\log x + \Lambda_1(-a)\eta_1(-a, x) + \Lambda_2(-a)\eta_0(-a, x) + \frac{\Lambda_0(a)}{1+2a}\eta_1(-2a, x).$$

Observe that $\eta_0(-a, x) = a^{-2}\varphi_0^*(-a\log x)$, where $\varphi_0^*(y) := 1 - ye^{-y} - e^{-y}$ is monotone increasing and bounded for $y \ge 0$. Then

$$\left|\frac{\Lambda_2(-a)}{\Lambda_0(-a)\log x}\sum_{n=0}^{N} 2^{(2a-1)n}\eta_0(-a, 2^{-n}x)\right| \ll |a\log x|^{-1}\varphi_0^*(-a\log x) \ll |a\log x|^{-1}$$

as $|a \log x|^{-1} \to 0$. Similarly

$$\left| \frac{1}{\Lambda_0(-a) \log x} \sum_{n=0}^{N} 2^{(2a-1)n} \left(\Lambda_1(-a)\eta_1(-a, 2^{-n}x) + \frac{\Lambda_0(a)}{1+2a}\eta_1(-2a, 2^{-n}x) \right) \right|$$

is also $\ll |a \log x|^{-1}$. Combining these estimates with (4.8) we obtain the asymptotic formula for $-1/2 < a < 0$. In the case $a \le -1/2$, the desired formula immediately follows from $\Phi(-a, x) = \Lambda_0(-a) \log x$. Thus the lemma is verified.

4.4. Proof of Lemma 4.3. We need the following:

LEMMA 4.6. *As* $|a| + (\log x)^{-1} \to 0$, $\Lambda(a, x) \asymp l_3(a, x)$ (> 0).

PROOF. Recall expression (4.1) with $l_3(a, x) = 3a^{-3}\varphi_3(a \log x)$ and $l_2^1(a, x) = a^{-2}\varphi_2^1(a \log x)$, where $\varphi_3(y) = 1 - 2ye^{-y} - e^{-2y}$ and $\varphi_2^1(y) := e^{-y}(1 - y - e^{-y})$. These functions satisfy $|\varphi_3(y)| \asymp |y|^3$ and $|\varphi_2^1(y)| \asymp y^2$ for $|y| \ll 1$. If $|a \log x| \ll 1$, then $l_2^1(a, x)/l_3(a, x) \ll |a\varphi_2^1(a \log x)/\varphi_3(a \log x)| \asymp (\log x)^{-1}$, and if $|a \log x| \gg 1$, then $l_2^1(a, x)/l_3(a, x) \ll |a|$. Hence we have $l_2^1(a, x)/l_3(a, x) \ll |a| + (\log x)^{-1}$. Similarly $l_3(a, x)^{-1} \left(|l_2^2(a, x)| + |l_1^1(a, x)| + |l_1^2(a, x)| + x^{-a} \log x + x^{-2a} + 1\right) \ll |a| + (\log x)^{-1}$. These estimates imply the lemma. □

LEMMA 4.7. *Let* $0 < \beta \le 1$. *Then*, $\Lambda(a, x) \gg (1 + x^{-2a})(\log x)^{3\beta}$ *if* $|a| \ll (\log x)^{-\beta}$. *Furthermore*, *as* $|a \log x|^{-1} \to 0$, $\Lambda(a, x) \asymp \Lambda_0(a)$ *if* $a > 0$ (*respectively*, $\Lambda(a, x) \asymp (1 - 2a)^{-1}\Lambda_0(-a)x^{-2a}$ *if* $a < 0$).

PROOF. If $0 \le a \ll (\log x)^{-\beta}$, then, by Lemma 4.6,

$$\Lambda(a, x) \asymp l_3(a, x) = 3|a^{-3}\varphi_3(a \log x)| \gg \min\{(\log x)^3, a^{-3}\} \gg (\log x)^{3\beta},$$

since $|\varphi_3(y)| = |1 - 2ye^{-y} - e^{-2y}| \gg \min\{y^3, 1\}$ for $y \ge 0$. Using $|\varphi_3(y)| \gg e^{-2y} \min\{|y|^3, 1\}$ for $y \le 0$, we have $\Lambda(a, x) \gg x^{-2a}(\log x)^{3\beta}$ if $0 \le -a \ll (\log x)^{-\beta}$. If $0 < a < 1/2$ (respectively, $-1/2 < a < 0$) and if $(a \log x)^{-1}$ is sufficiently small, we obtain from (4.9) that $\Lambda(a, x) \asymp \Lambda_0(a)$ (respectively, $\asymp (1 - 2a)^{-1}\Lambda_0(-a)x^{-2a}$), since $(1 + |a| \log x)x^{-|a|} \ll \exp(-|a| \log x/2)$. If $a \ge 1/2$ (respectively, $a \le -1/2$) the same estimate immediately follows. □

To prove Lemma 4.3 we first suppose that $|a| + (\log x)^{-1}$ is sufficiently small. Since $2^{-N}x \asymp x^{1/4}$, $(\log(2^{-n}x))^{-1} \asymp (\log x)^{-1}$ for $n \le N$. Using Lemma 4.6 and observing that $\varphi_3(y)$ is monotone increasing for $y \in \mathbf{R}$, we have

$$|\Lambda|_N(a, x) = \sum_{n=0}^{N} 2^{-n}|\Lambda(a, 2^{-n}x)| \ll \sum_{n=0}^{N} 2^{-n}l_3(a, 2^{-n}x) \ll l_3(a, x).$$

By Lemma 4.1, $\Phi_N(a,x) \asymp a^{-4}\psi(a,x) = a^{-4}h_0(a\log x)$. Estimates (4.5) and (4.6) yield $\varphi_3(y)/h_0(y) \ll 1+|y|^{-1}$ uniformly in $y \in \mathbf{R}$. Hence

$$|\Lambda|_N(a,x)/\Phi_N(a,x) \ll |a\varphi_3(a\log x)/h_0(a\log x)| \ll |a| + (\log x)^{-1}$$

as $|a| + (\log x)^{-1} \to 0$. If $a < 0$, then, by Lemma 4.7, $\Lambda(a,x) \ll \Lambda_0(-a)x^{-2a}$ as $|a\log x|^{-1} \to 0$. Hence by Lemma 4.2, $|\Lambda|_N(a,x) \ll \Lambda_0(-a)x^{-2a} \ll \widetilde{\Phi}_N(a,x)(\log x)^{-1}$. Thus we obtain Lemma 4.3.

4.5. Proof of Lemma 4.4. Lemma 4.4 immediately follows from Lemma 4.7 and the following with $c_0 > 0$:

LEMMA 4.8. (1) *Suppose that $a \geq 0$. For $\mu - 1 \geq c_0$ (respectively, $\mu - 1 \leq -c_0$),*

$$\int_x^\infty \xi^{-\mu}\Lambda(a,\xi)d\xi \left(\textit{respectively, } \int_1^x \xi^{-\mu}\Lambda(a,\xi)d\xi\right) \ll \frac{x^{1-\mu}\Lambda(a,x)}{|\mu-1|},$$

the implied constant depending on c_0 only.

(2) *Suppose that $a < 0$. For $\mu + 2a - 1 \geq c_0$ (respectively, $\mu - 1 \leq -c_0$), the integral above is $\ll |\mu-1|^{-1}(|a|+1)x^{1-\mu}\Lambda(a,x)$, the implied constant depending on c_0 only.*

Indeed, for example, in the case $a < 0$, denoting $\Sigma(a,x) := \sum_{n\leq x}\sigma_{-a}(n)^2$ and using Proposition 3.1, we derive

$$\begin{aligned}\sum_{n\leq\tau}\frac{\sigma_{-a}(n)^2}{n} &= \tau^{-1}\Sigma(a,\tau) + \int_1^\tau \xi^{-2}\Sigma(a,\xi)d\xi \\ &= \Lambda(a,\tau) + \int_1^\tau \xi^{-1}\Lambda(a,\xi)d\xi + O(\tau^{-2a-1/2+\varepsilon}+1) \\ &= \Phi(a,\tau) + O\big((1+|a|)\Lambda(a,\tau)\big) = \Phi(a,\tau) + O\big(\Lambda^*(a,\tau)\big)\end{aligned}$$

with the absolute implied constants, and if $\mu + 2a - 1 \geq c_0$,

$$\begin{aligned}\sum_{n>\tau}\frac{\sigma_{-a}(n)^2}{n^\mu} &= \tau^{-\mu}\Sigma(a,\tau) + \mu\int_\tau^\infty \xi^{-\mu-1}\Sigma(a,\xi)d\xi \\ &\ll \tau^{1-\mu}\Lambda(a,\tau) + \mu\int_\tau^\infty \xi^{-\mu}\Lambda(a,\xi)d\xi + \frac{\mu\tau^{-2a+1/2-\mu+\varepsilon}}{|2a+\mu|+1} \\ &\ll (1+|a|)\tau^{1-\mu}(\Lambda(a,\tau) + O(\tau^{-2a})) \ll (1+|a|)^2\tau^{1-\mu}\Lambda(a,\tau) \ll \tau^{1-\mu}\Lambda^*(a,\tau)\end{aligned}$$

with the implied constants depending on c_0 only.

4.5.1. Proof of Lemma 4.8 in the case $a \geq 0$

Suppose that $\mu - 1 \geq c_0$. We have

$$\int_x^\infty \xi^{-\mu}\Lambda(a,\xi)d\xi = \frac{x^{1-\mu}\Lambda(a,x)}{\mu-1} + \frac{J(a,x)}{\mu-1}, \quad J(a,x) := \int_x^\infty \xi^{1-\mu}\Lambda_\xi(a,\xi)d\xi\,,$$

where $\Lambda_\xi(a,\xi) := (\partial/\partial\xi)\Lambda(a,\xi)$, provided that $J(a,x)$ converges. If $a \geq 1/2$, then $\Lambda_\xi(a,\xi) \equiv 0$, and hence the conclusion immediately follows. For $0 \leq a < 1/2$, by (4.9)

$$\xi\Lambda_\xi(a,\xi) = \Lambda_2(a)\xi^{-a}(1 - a\log\xi) - a\Lambda_1(a)\xi^{-a} - \frac{2a\Lambda_0(-a)}{1-2a}\xi^{-2a} \ll \Lambda_0(a)\,,$$

which is checked by using $ax^{-a}\log x \ll 1$ and $(|\Lambda_2(a)| + |\Lambda_1(a)|)/\Lambda_0(a) \ll a$. This gives $J(a,x) \ll \Lambda_0(a)x^{1-\mu}$ with the implied constant depending on c_0 only. Then, by Lemma 4.7, we obtain the conclusion, say for $(\log x)^{-5/6} \ll a < 1/2$. To complete the proof for $a \geq 0$ it is sufficient to show the estimate for $0 \leq a \ll (\log x)^{-2/3}$. To do so we recall (4.1). Under this condition, $l_3'(a,x) \ll a^{-2}x^{-a-1}\left|-1 + a\log x + x^{-a}\right| \ll x^{-a-1}(\log x)^2$, and similarly $|(l_i^j)'(a,x)| \ll x^{-a-1}(\log x)^2$. Hence

$$J(a,x) \ll \int_x^\infty \xi^{-a-\mu}(\log\xi)^2 d\xi \ll x^{1-\mu-a}(\log x)^2.$$

By Lemma 4.7 with $\beta = 2/3$, $J(a,x) \ll x^{1-\mu}(\log x)^2 \ll x^{1-\mu}\Lambda(a,x)$. Thus we obtain the lemma for $\mu - 1 \geq c_0$. The integral with $\mu - 1 \leq -c_0$ may be treated similarly.

4.5.2. Proof of Lemma 4.8 in the case $a < 0$

If $a \leq -1/2$, the conclusion is immediately obtained. For $-1/2 < a < 0$ and for $\mu + 2a - 1 \geq c_0$, write $\xi^{1-\mu}\Lambda_\xi(a,\xi) = \xi^{-\mu-2a}\cdot\xi^{1+2a}\Lambda_\xi(a,\xi) \ll \Lambda_0(-a)\xi^{-\mu-2a}$. If $(\log x)^{-5/6} \ll -a < 1/2$, we have $J(a,x) \ll x^{1-\mu}\Lambda(a,x)$. Furthermore, if $0 < -a \ll (\log x)^{-2/3}$, observing $l_3'(a,x) \ll a^{-2}x^{-2a-1}|1 + ax^a\log x - x^a| \ll x^{-2a-1}(\log x)^2$ and so on, we deduce that $J(a,x) \ll x^{1-\mu}\Lambda(a,x)$. The remaining case $\mu - 1 \leq -c_0$ is similarly treated.

4.6. Proof of Lemma 4.5. Set $U_a(x) := \sum_{n\leq x}\sigma_a(n)$. If $c_0 \leq 1 - \sigma < 1$,

$$\sum_{n\leq\tau}\frac{\sigma_a(n)}{n^\sigma} \ll \tau^{-\sigma}U_a(\tau) + \int_1^\tau \xi^{-\sigma-1}U_a(\xi)d\xi \ll (1+|a|)\left|\frac{1-\tau^a}{a}\right|\frac{\tau^{1-\sigma}}{c_0}\,,$$

since, by Proposition 3.2, $(1+|a|)^{-1}\xi^{-\sigma-1}U_a(\xi) \ll \xi^{-\sigma}|(1-\xi^a)/a| \leq \xi^{-\sigma}|(1-\tau^a)/a|$ for $1 \leq \xi \leq \tau$ and for $a \in \mathbf{R}$. If $\sigma \geq 1 + \max\{0, a\} + c_0$,

$$\sum_{n>\tau}\frac{\sigma_a(n)}{n^\sigma} \ll \tau^{-\sigma}U_a(\tau) + \sigma\int_\tau^\infty \xi^{-\sigma-1}U_a(\xi)d\xi$$

$$\ll \tau^{-\sigma}U_a(\tau) + \frac{(1+|a|)\sigma}{|a|}\int_\tau^\infty |\xi^{-\sigma} - \xi^{-\sigma+a}|d\xi \ll (1+|a|)\frac{\tau^{1-\sigma}}{c_0}\left(\left|\frac{1-\tau^a}{a}\right| + \frac{\tau^a}{c_0}\right).$$

Thus we obtain Lemma 4.5.

5. Integral related to the proofs of the main results

For $\tau \gg 1$, consider the integral

$$I(\tau/2,\tau) := \int_{\tau/2}^{\tau} \left| \zeta\left(\frac{1}{2}+it\right)\zeta\left(\frac{1}{2}+a+it\right)\right|^2 dt\,.$$

In this section we show the following, which will be used in the proofs of our main results.

PROPOSITION 5.1. *If* $-\tau^{1/4} \le a \le 1/16$, *then*

$$I(\tau/2,\tau) = \frac{\tau}{2}\big(\Phi(a,\tau) + (2\pi)^{2a}\lambda(a)\tau^{-2a}\Phi(-a,\tau)(1+O(\tau^{-1/4})) + R_0(a,\tau)\big)$$

with $R_0(a,\tau) \ll (|a|^{2|a|}+1)(\Lambda(a,\tau)+|(1-\tau^{-a})/a|^2)$ *and* $\lambda(a) := 2(1-2^{2a-1})/(1-2a)$. *Here the implied constants are absolute.*

5.1. Approximate formula. For $\zeta(s)\zeta(s+a)$ we give an approximate formula by the reflection principle. Let a satisfy $-\tau^{1/4} \le a \le 1/16$. Suppose that $\tau/2 \le t \le \tau$ and that $|\sigma - 1/2| < 1/4$. Then

$$\sum_{n=1}^{\infty} \sigma_{-a}(n) e^{-n/\tau} n^{-s} = \frac{1}{2\pi i}\int_{(2+|a|)} \zeta(s+w)\zeta(s+a+w)\Gamma(w)\tau^w dw\,, \tag{5.1}$$

where $s=\sigma+it$, $w=u+iv$, and the symbol (α) denotes the vertical line $w=\alpha+iv$, $-\infty < v < \infty$, since $\zeta(s)\zeta(s+a) = \sum_{n=1}^{\infty}\sigma_{-a}(n)n^{-s}$ for $\mathrm{Re}(s) > \max\{1, 1-a\}$. If $a \ne 0$, the poles of the integrand in the strip $-3/4 < u < 2+|a|$ are $w=0$, $1-s$, $1-s-a$, whose residues are $R_0 := \zeta(s)\zeta(s+a)$, $R_{1-s} := \zeta(1+a)\Gamma(1-s)\tau^{1-s}$ and $R_{1-s-a} := \zeta(1-a)\Gamma(1-s-a)\tau^{1-s-a}$, respectively. Note that

$$R_{1-s} + R_{1-s-a} = \Gamma(1-s)\tau^{1-s}\big(\zeta(1+a)+\zeta(1-a) - a\zeta(1-a)g(-a,s)\big)$$

with $g(z,s) = z^{-1}(\Gamma(1-s+z)\tau^z/\Gamma(1-s) - 1)$, and that

$$|\Gamma(\sigma+it)| = \sqrt{2\pi}|t|^{\sigma-1/2}e^{-(\pi/2)|t|}\big(1+O(|t|^{-1/2})\big) \tag{5.2}$$

for $|\sigma| \ll |t|^{1/4}$, $|t| \ge 1$, which may be checked by using Stirling's formula. Since $g(z,s)$ is holomorphic around $z=0$, by the maximal modulus principle together with (5.2), we have $g(-a,s) \ll \max_{|z|=1}\big|\Gamma(1-s+z)\tau^z/\Gamma(1-s)\big| \ll \tau^2$ for $|a| \le 1/16$, so that $R_{1-s} + R_{1-s-a} \ll e^{-t}$. For $-\tau^{1/4} \le a \le -1/16$, we have $R_{1-s-a} \ll |\Gamma(1-s-a)|\tau^{1-\sigma-a} \ll \tau^{1-2a}e^{-(\pi/2)t} \ll e^{-t}$, and $R_{1-s} \ll |\zeta(-a)\chi(1+a)\Gamma(1-s)|\tau^{1-\sigma} \ll \Gamma(-a)|\Gamma(1-s)|\tau \ll e^{-t}$, where $\chi(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)$. In (5.1) we shift the path of integration to $(-3/4)$ and use the functional equation for $\zeta(s)$ to find

$$\frac{1}{2\pi i}\int_{(-3/4)} \chi(s+w)\chi(s+w+a)\Gamma(w)\tau^w\left(\sum_{n=1}^{\infty}\frac{\sigma_a(n)}{n^{1-s-w}}\right)dw$$

$$= \sum_{n=1}^{\infty} \sigma_{-a}(n) e^{-n/\tau} n^{-s} - \zeta(s)\zeta(s+a) + O(e^{-\tau/2})$$

for $|\sigma - 1/2| < 1/4$ and for $-\tau^{1/4} \le a \le 1/16$. The sum in the integrand above converges. Split the sum into two parts $\sum_{n\le\tau}$ and $\sum_{n>\tau}$. For the integral related to $\sum_{n\le\tau}$, shift the path of integration to the line $(1/4)$. Between $(-3/4)$ and $(1/4)$, the integrand possesses a pole only at $w = 0$, whose residue is $\chi(s)\chi(s+a)\sum_{n\le\tau}\sigma_a(n)n^{s-1}$. Consequently

$$\zeta(s)\zeta(s+a) = S_1(a,s) + S_2(a,s) + \sum_{j=1}^{4} F_j(a,s) + O(e^{-\tau/2}) \tag{5.3}$$

for $|\sigma - 1/2| \le 1/8$, $\tau/2 \le t \le \tau$, $-\tau^{1/4} \le a \le 1/16$, where

$$S_1(a,s) := \sum_{n\le\tau} \sigma_{-a}(n) n^{-s}, \quad S_2(a,s) := \chi(s)\chi(s+a) \sum_{n\le\tau} \sigma_a(n) n^{s-1},$$

$$F_1(a,s) := \sum_{n\le\tau} \sigma_{-a}(n) n^{-s}(e^{-n/\tau} - 1), \quad F_2(a,s) := \sum_{n>\tau} \sigma_{-a}(n) n^{-s} e^{-n/\tau},$$

$$F_3(a,s) := -\frac{1}{2\pi i}\int_{(-3/4)} \chi(s+w)\chi(s+w+a)\Gamma(w)\tau^w \left(\sum_{n>\tau} \frac{\sigma_a(n)}{n^{1-s-w}}\right) dw,$$

$$F_4(a,s) := -\frac{1}{2\pi i}\int_{(1/4)} \chi(s+w)\chi(s+w+a)\Gamma(w)\tau^w \left(\sum_{n\le\tau} \frac{\sigma_a(n)}{n^{1-s-w}}\right) dw.$$

5.2. Expression of the integral. Suppose that $|\sigma - 1/2| \le 1/8$. By (5.2), for $\tau/2 \le t \le \tau$, for $a \le 1/16$ and for each fixed $u \ll 1$,

$$\begin{aligned} &\chi(s+w)\chi(s+w+a)\Gamma(w)\tau^w \\ &\quad \ll (2\pi)^{2u+a}(1+|v+t|)^{1-2\sigma-2u-a}(1+|v|)^{u-1/2}e^{-(\pi/2)|v|}\tau^u \\ &\quad \ll \tau^u t^{1-2\sigma-2u-a}\left(1+\frac{|v|}{t}\right)^{1-2\sigma-2u-a}(1+|v|)^{u-1/2}e^{-(\pi/2)|v|} \\ &\quad \ll \tau^{1-2\sigma-u-a}(1+|v|)^{1/2-2\sigma-u-a}e^{-(\pi/2)|v|}. \end{aligned} \tag{5.4}$$

Along the line $u = -3/4$, $\sum_{n>\tau}\sigma_a(n)/n^{1-\sigma-u} \ll \sum_{n>\tau}(1+n^a)d(n)/n^{7/4-\sigma} \ll \tau^{\sigma-3/4+\varepsilon}(1+\tau^a)$ if $a \le 1/16$. Hence $F_3(a,s) \ll \Gamma(|a|+2)\tau^{1-\sigma+\varepsilon}(1+\tau^{-a}) \ll \exp(2\tau^{1/4+\varepsilon})$ for $|\sigma-1/2| \le 1/8$ and for $-\tau^{1/4} \le a \le 1/16$. Furthermore $F_2(a,s) \ll (|a|+1)^{|a|}\sum_{n>\tau}\sigma_{-a}(n)n^{-\sigma}(\tau/n)^{|a|+1} \ll \tau^{2|a|}\sum_{n>\tau}(1+n^{-a})d(n)n^{-\sigma-|a|-1} \ll \tau^{2|a|} \ll \exp(2\tau^{1/4+\varepsilon})$, since $e^{-n/\tau} \ll ((|a|+1)/e)^{|a|+1}(\tau/n)^{|a|+1}$. Similarly, $S_j(a,s)$, $F_{j'}(a,s) \ll \exp(2\tau^{1/4+\varepsilon})$ ($j = 1, 2$; $j' = 1, 4$). Using formula (5.3) together with these estimates, we derive

$$I(\tau/2,\tau)=\sum_{k=1}^{6}J_k+O(\tau^{-1}) \tag{5.5}$$

for $-\tau^{1/4}\le a\le 1/16$, where

$$J_1:=\int_{\tau/2}^{\tau}|S_1(a,1/2+it)|^2dt+\int_{\tau/2}^{\tau}|S_2(a,1/2+it)|^2dt\,,$$

$$J_2:=\sum_{j=1}^{4}\int_{\tau/2}^{\tau}|F_j(a,1/2+it)|^2dt\,,$$

$$J_3:=\int_{\tau/2}^{\tau}\big(S_1(a,1/2+it)S_2(a,1/2-it)+S_1(a,1/2-it)S_2(a,1/2+it)\big)dt\,,$$

$$J_4=\overline{J_5}:=\sum_{j=1}^{4}\int_{\tau/2}^{\tau}\big(S_1(a,1/2+it)+S_2(a,1/2+it)\big)(-1)^{\iota(j)}F_j(a,1/2-it)dt\,,$$

$$J_6:=\sum_{\substack{1\le j\le 4\\ 1\le j'\le 4\\ j\ne j'}}\int_{\tau/2}^{\tau}(-1)^{\iota(j')}F_j(a,1/2+it)F_{j'}(a,1/2-it)dt$$

with $\iota(1)=\iota(2)=0$, $\iota(3)=\iota(4)=1$. Evaluation of these integrals basically depends on the identity of Montgomery and Vaughan ([9], [6, §§5.1, 5.2]):

PROPOSITION 5.2. *Let $\gamma_1,\dots,\gamma_N$ be arbitrary complex numbers. Then*

$$\int_0^{\tau}\Big|\sum_{n\le N}\gamma_n n^{it}\Big|^2dt=\tau\sum_{n\le N}|\gamma_n|^2+O\Big(\sum_{n\le N}n|\gamma_n|^2\Big)\,.$$

This remains valid for $N=\infty$ as well, provided that the series on the right-hand side converge.

Furthermore we need

LEMMA 5.3. *For $a\le 1/16$ we have $\tau^{-2a}\Lambda^*(-a,\tau)\asymp(|a|+1)\Lambda^*(a,\tau)$.*

PROOF. For $|a|\ll(\log\tau)^{-1/3}$, by Lemma 4.6,

$$\Lambda(a,\tau)\asymp l_3(a,\tau)\asymp|a^{-3}(1-2\tau^{-a}\cdot a\log\tau-\tau^{-2a})|$$
$$=\tau^{-2a}|a^{-3}(1-2\tau^{a}(-a\log\tau)-\tau^{2a})|\asymp\tau^{-2a}l_3(-a,\tau)\asymp\tau^{-2a}\Lambda(-a,\tau)\,.$$

For $(\log\tau)^{-1/2}\ll|a|$, by Lemma 4.7, we have $\Lambda(a,\tau)\asymp\Lambda_0(a)$ if $a>0$, and $\Lambda(a,\tau)\asymp(1-2a)^{-1}\Lambda_0(-a)\tau^{-2a}$ if $a<0$. From these facts and the definition of $\Lambda^*(a,\tau)$ the lemma immediately follows. □

In the remaining part of this section we give an asymptotic expression for J_1 and estimates for J_j $(2 \le j \le 6)$, from which Proposition 5.1 immediately follows.

5.3. Main terms. Since $|S_1(a, 1/2 + it)|^2 = \left|\sum_{n\le\tau} \sigma_{-a}(n) n^{-1/2-it}\right|^2$, we apply Lemma 4.4 and Proposition 5.2 to obtain

$$\int_{\tau/2}^{\tau} |S_1(a, 1/2 + it)|^2 dt = \frac{\tau}{2} \sum_{n\le\tau} \frac{\sigma_{-a}(n)^2}{n} + O\Big(\sum_{n\le\tau} \sigma_{-a}(n)^2\Big) = \frac{\tau}{2}\big(\Phi(a, \tau) + O(\Lambda^*(a, \tau))\big). \tag{5.6}$$

By (5.2), $|\chi(1/2+it)\chi(1/2+a+it)| = (2\pi/t)^a(1 + O(t^{-1/2}))$ for $-\tau^{1/4} \le a \le 1/16$ and for $\tau/2 \le t \le \tau$. Hence

$$I := \int_{\tau/2}^{\tau} |S_2(a, 1/2 + it)|^2 dt = (2\pi)^{2a}\bigg(\int_{\tau/2}^{\tau} t^{-2a} F'(t) dt + O\Big(\int_{\tau/2}^{\tau} t^{-2a-1/2} F'(t) dt\Big)\bigg)$$

with

$$F(x) := \int_{\tau/2}^{x} \bigg|\sum_{n\le\tau} \frac{\sigma_a(n)}{n^{1/2-it}}\bigg|^2 dt\,.$$

Integrating by parts and observing

$$\int_{\tau/2}^{\tau} t^{-2a-1/2} F'(t) dt \ll \tau^{-2a-1/2} \int_{\tau/2}^{\tau} F'(t) dt = \tau^{-2a-1/2} F(\tau)\,,$$

we find

$$I = (2\pi)^{2a}\bigg(\tau^{-2a} F(\tau)(1 + O(\tau^{-1/2})) + 2a \int_{\tau/2}^{\tau} t^{-2a-1} F(t) dt\bigg)\,.$$

Since, by Lemma 4.4 and Proposition 5.2, $F(x) = (x - \tau/2)\Phi(-a, \tau) + O(\tau\Lambda^*(-a, \tau))$ for $\tau/2 \le x \le \tau$, it follows that

$$2a \int_{\tau/2}^{\tau} t^{-2a-1} F(t) dt = \frac{2a + 1 - 2^{2a}}{2(1 - 2a)} \tau^{1-2a} \Phi(-a, \tau) + O(\tau^{1-2a} \Lambda^*(-a, \tau))\,,$$

and hence, by Lemma 5.3,

$$I = (2\pi)^{2a}\tau\bigg(\frac{1 - 2^{2a-1}}{1 - 2a} \tau^{-2a} \Phi(-a, \tau)(1 + O((|a| + 1)\tau^{-1/2})) + O((|a| + 1)\Lambda^*(a, \tau))\bigg).$$

Combining this with (5.6), we have, for $-\tau^{1/4} \le a \le 1/16$,

$$J_1 = \frac{\tau}{2}\big(\Phi(a, \tau) + (2\pi)^{2a}\lambda(a)\tau^{-2a}\Phi(-a, \tau)(1 + O(\tau^{-1/4})) + O((1 + a^2)\Lambda(a, \tau))\big). \tag{5.7}$$

5.4. Evaluation of J_2 and J_6. Along the line $w = -3/4 + iv$, $-\infty < v < \infty$,

$$\big|\chi(w+1/2+it)\chi(w+1/2+a+it)\Gamma(w)\tau^w\big| \ll \tau^{3/4-a}(1+|v|)^{1/4-a}e^{-\pi|v|/2}$$

(cf. (5.4)), and hence

$$\begin{aligned}|F_3(a,1/2+it)|^2 &\ll \int_{-\infty}^{\infty}\bigg|\sum_{n>\tau}\frac{\sigma_a(n)n^{iv}}{n^{5/4-it}}\bigg|^2 e^{-\pi|v|/2}dv \\ &\quad\times \int_{(-3/4)}\big|\chi(w+1/2+it)\chi(w+1/2+a+it)\Gamma(w)\tau^w e^{\pi|v|/4}\big|^2|dw| \\ &\ll \Gamma(2|a|+2)\tau^{3/2-2a}\int_{-\infty}^{\infty}\bigg|\sum_{n>\tau}\frac{\sigma_a(n)n^{iv}}{n^{5/4-it}}\bigg|^2 e^{-\pi|v|/2}dv\,.\end{aligned}$$

By Lemma 4.4,

$$\tau\sum_{n>\tau}\bigg|\frac{\sigma_a(n)n^{iv}}{n^{5/4}}\bigg|^2 + \sum_{n>\tau}n\bigg|\frac{\sigma_a(n)n^{iv}}{n^{5/4}}\bigg|^2 \ll \tau^{-1/2}\Lambda^*(-a,\tau)\,.$$

This gives

$$\begin{aligned}\int_{\tau/2}^{\tau}\frac{|F_3(a,1/2+it)|^2}{\Gamma(2|a|+2)}dt &\ll \tau^{3/2-2a}\int_{-\infty}^{\infty}\bigg(\int_{\tau/2}^{\tau}\bigg|\sum_{n>\tau}\frac{\sigma_a(n)n^{iv}}{n^{5/4-it}}\bigg|^2 dt\bigg)e^{-\pi|v|/2}dv \\ &\ll \tau^{3/2-2a}\cdot\tau^{-1/2}\Lambda^*(-a,\tau) \ll (1+|a|^3)\tau\Lambda(a,\tau)\,.\end{aligned}$$

Similarly we have

$$\begin{aligned}\int_{\tau/2}^{\tau}\frac{|F_4(a,1/2+it)|^2}{\Gamma(2|a|+2)}dt &\ll \tau^{-1/2-2a}\int_{-\infty}^{\infty}\bigg(\int_{\tau/2}^{\tau}\bigg|\sum_{n\le\tau}\frac{\sigma_a(n)n^{iv}}{n^{1/4-it}}\bigg|^2 dt\bigg)e^{-\pi|v|/2}dv \\ &\ll \tau^{-1/2-2a}\cdot\tau^{3/2}\Lambda^*(-a,\tau) \ll (1+|a|^3)\tau\Lambda(a,\tau)\,.\end{aligned}$$

Since $e^{-2n/\tau} \ll ((|a|+1)/e)^{2|a|+2}(\tau/n)^{2|a|+2} \ll (1+|a|^{2|a|-2})(\tau/n)^{2|a|+2}$ for $n \ge \tau$,

$$\begin{aligned}\int_{\tau/2}^{\tau}|F_2(a,1/2+it)|^2dt &\ll \tau\sum_{n>\tau}\sigma_{-a}(n)^2n^{-1}e^{-2n/\tau} + \sum_{n>\tau}\sigma_{-a}(n)^2e^{-2n/\tau} \\ &\ll (1+|a|^{2|a|-2})\bigg(\tau^{2|a|+3}\sum_{n>\tau}\sigma_{-a}(n)^2n^{-2|a|-3} + \tau^{2|a|+2}\sum_{n>\tau}\sigma_{-a}(n)^2n^{-2|a|-2}\bigg) \\ &\ll (1+|a|^{2|a|-2})\tau\Lambda^*(a,\tau) \ll (1+|a|^{2|a|})\tau\Lambda(a,\tau)\,.\end{aligned}$$

The integral corresponding to $F_1(a,1/2+it)$ may be treated in a similar way by using $e^{-n/\tau} - 1 \ll n/\tau$ for $n \le \tau$. Thus we obtain $J_2 \ll (1+|a|^{2|a|})\tau\Lambda(a,\tau)$. The inequalities

$$\int_{\tau/2}^{\tau}|F_j(a,1/2+it)F_{j'}(a,1/2-it)|dt$$

$$\leq \left(\int_{\tau/2}^{\tau} |F_j(a, 1/2 + it)|^2 dt \int_{\tau/2}^{\tau} |F_{j'}(a, 1/2 + it)|^2 dt \right)^{1/2},$$

for $1 \leq j \leq 4$, $1 \leq j' \leq 4$, $j \neq j'$ yield $J_6 \ll (1 + |a|^{2|a|})\tau \Lambda(a, \tau)$.

5.5. Evaluation of J_3. Let $G(\sigma)$ denote the integral

$$\int_{\tau/2}^{\tau} S_1(a, s)S_2(a, 1 - s)dt = \int_{\tau/2}^{\tau} \chi(1 - s)\chi(1 - s + a)\left(\sum_{n \leq \tau} \frac{\sigma_{-a}(n)}{n^s}\right)\left(\sum_{n \leq \tau} \frac{\sigma_a(n)}{n^s}\right)dt.$$

Then J_3 is written in the form $J_3 = G(1/2) + \overline{G(1/2)}$. To evaluate $G(1/2)$ shift the segment of integration $[1/2 + i\tau/2, 1/2 + i\tau]$ to $[3/8 + i\tau/2, 3/8 + i\tau]$. By Lemma 4.5 together with $\chi(1 - s)\chi(1 - s + a) \ll t^{2\sigma - 1 - a}$, the integrand along the horizontal segment $[3/8 + i\tau/2, 1/2 + i\tau/2]$ or $[3/8 + i\tau, 1/2 + i\tau]$ is $\ll (1 + a^4)\tau|(1 - \tau^{-a})/a|^2$, and hence $J_3 \ll |G(1/2)| \ll (1 + a^4)\tau|(1 - \tau^{-a})/a|^2 + |G(3/8)|$. Using Lemma 4.4 and Proposition 5.2, we have

$$G(3/8) \ll \tau^{-1/4-a}\left(\int_{\tau/2}^{\tau}\left|\sum_{n \leq \tau} \frac{\sigma_{-a}(n)}{n^{3/8+it}}\right|^2 dt \int_{\tau/2}^{\tau}\left|\sum_{n \leq \tau} \frac{\sigma_a(n)}{n^{3/8+it}}\right|^2 dt\right)^{1/2}$$

$$\ll \tau^{-1/4-a}(\tau^{5/4}\Lambda^*(a, \tau) \cdot \tau^{5/4}\Lambda^*(-a, \tau))^{1/2} \ll (1 + |a|^3)\tau\Lambda(a, \tau),$$

so that $J_3 \ll (1 + a^4)(\tau\Lambda(a, \tau) + \tau|(1 - \tau^{-a})/a|^2)$.

5.6. Evaluation of J_4 and J_5. Let

$$G_{pq}(\sigma) := \int_{\tau/2}^{\tau} S_p(a, s)F_q(a, 1 - s)dt.$$

Let us evaluate $G_{13}(1/2)$. By Lemma 4.5 and (5.4), for $|\sigma - 1/2| \leq 1/8$, we have $|F_3(a, 1 - s)| \ll \Gamma(|a|+2)(1+a^2)\tau^{\sigma}|(1-\tau^{-a})/a|$ and $S_1(a, s) \ll (1+a^2)\tau^{1-\sigma}|(1-\tau^{-a})/a|$. Hence $G_{13}(1/2) \ll |G_{13}(3/8)| + (1 + |a|^{|a|})\tau|(1 - \tau^{-a})/a|^2$, where

$$G_{13}(3/8) \ll \left(\int_{\tau/2}^{\tau} |S_1(a, 3/8 + it)|^2 dt \int_{\tau/2}^{\tau} |F_3(a, 5/8 - it)|^2 dt\right)^{1/2}.$$

Using

$$\int_{(-3/4)}\left|\chi(w + 5/8 - it)\chi(w + 5/8 + a - it)\Gamma(w)\tau^w e^{\pi|v|/4}\right|^2 |dw| \ll \Gamma(2|a| + 2)\tau^{1-2a},$$

we have

$$\Gamma(2|a| + 2)^{-1}\int_{\tau/2}^{\tau} |F_3(a, 5/8 - it)|^2 dt$$

$$\ll \tau^{1-2a}\int_{-\infty}^{\infty}\left(\int_{\tau/2}^{\tau}\left|\sum_{n>\tau} \frac{\sigma_a(n)n^{iv}}{n^{9/8+it}}\right|^2 dt\right)e^{-\pi|v|/2}dv \ll \tau^{3/4-2a}\Lambda^*(-a, \tau).$$

From this estimate combined with

$$\int_{\tau/2}^{\tau} |S_1(a, 3/8 + it)|^2 dt = \int_{\tau/2}^{\tau} \left| \sum_{n \le \tau} \frac{\sigma_{-a}(n)}{n^{3/8+it}} \right|^2 dt \ll \tau^{5/4} \Lambda^*(a, \tau) ,$$

it follows that $G_{13}(3/8) \ll \Gamma(2|a|+2)^{1/2}(|a|+1)\tau\Lambda^*(a, \tau)$, which implies $G_{13}(1/2) \ll (1+|a|^{|a|})(\tau\Lambda(a, \tau) + \tau|(1-\tau^{-a})/a|^2)$. In treating $G_{22}(1/2)$ we use $\chi(s)\chi(s+a) \ll t^{1-2\sigma-a}$, $\sum_{n\le\tau} \sigma_a(n)/n^{1-s} \ll (1+a^2)\tau^{\sigma}|(\tau^a - 1)/a|$ and

$$\sum_{n>\tau} \frac{\sigma_{-a}(n)e^{-n/\tau}}{n^{1-s}} \ll (1+|a|^{|a|-4}) \sum_{n>\tau} \frac{\sigma_{-a}(n)}{n^{1-\sigma}} \left(\frac{\tau}{n}\right)^{|a|+1} \ll (1+|a|^{|a|-2})\tau^{\sigma} \left| \frac{1-\tau^{-a}}{a} \right|$$

for $|\sigma - 1/2| \le 1/8$. Then $G_{22}(1/2) \ll |G_{22}(5/8)| + (1+|a|^{|a|})\tau|(1-\tau^{-a})/a|^2$, where

$$G_{22}(5/8) \ll \tau^{-\kappa-a} \left(\int_{\tau/2}^{\tau} \left| \sum_{n\le\tau} \frac{\sigma_a(n)}{n^{3/8-it}} \right|^2 dt \int_{\tau/2}^{\tau} \left| \sum_{n>\tau} \frac{\sigma_{-a}(n)e^{-n/\tau}}{n^{3/8-it}} \right|^2 dt \right)^{1/2} .$$

Using $\sum_{n>\tau} \sigma_{-a}(n)^2 n^{-3/4} e^{-2n/\tau} \ll (1+|a|^{2|a|-3}) \sum_{n>\tau} \sigma_{-a}(n)^2 n^{-7/4-2|a|} \tau^{2|a|+1}$, we have $G_{22}(5/8) \ll (1+|a|^{2|a|})\tau\Lambda(a, \tau)$, so that $G_{22}(1/2) \ll (1+|a|^{2|a|})(\tau\Lambda(a, \tau) + \tau|(1-\tau^{-a})/a|^2)$. The remaining integrals $G_{pq}(1/2)$ are similarly treated by considering $G_{11}(3/8)$, $G_{12}(3/8)$, $G_{14}(3/8)$, $G_{21}(5/8)$, $G_{23}(5/8)$ and $G_{24}(5/8)$. Thus we have $J_4 = \overline{J_5} \ll (1+|a|^{2|a|})(\tau\Lambda(a, \tau) + \tau|(1-\tau^{-a})/a|^2)$. Combining (5.7) with the estimates for J_j $(2 \le j \le 6)$ given above, we immediately obtain Proposition 5.1.

6. Proofs of the main results

For $T \gg 1$, choose the positive integer $N_0 := [(3/4)(\log 2)^{-1} \log T]$. Then

$$\bigcup_{0\le n\le N_0} [2^{-n-1}T, 2^{-n}T] = [2^{-N_0-1}T, T] , \quad T^{1/4}/2 \le 2^{-N_0-1}T \le T^{1/4} . \tag{6.1}$$

Split the integral $T\Psi(a, T)$ into two parts: $T\Psi(a, T) = V_*(a, T) + V_0(a, T)$ with

$$V_*(a, T) := \int_1^{2^{-N_0-1}T} |\zeta(1/2+it)\zeta(1/2+a+it)|^2 dt ,$$

$$V_0(a, T) := \int_{2^{-N_0-1}T}^{T} |\zeta(1/2+it)\zeta(1/2+a+it)|^2 dt .$$

The estimate $\zeta(1/2+it) \ll t^{1/6}$ together with the convexity property gives $\zeta(\sigma+it) \ll t^{1/4+\varepsilon}$ for $\sigma \ge 3/8$, $t \ge 1$. Hence, if $a \ge -1/8$,

$$V_*(a, T) \ll \left(\int_1^{T^{1/4}} |\zeta(1/2+it)|^4 dt \int_1^{T^{1/4}} |\zeta(1/2+a+it)|^4 dt \right)^{1/2} \ll T . \tag{6.2}$$

By the convexity property for an integral of $|\zeta(\sigma + it)|^4$ (cf. [6, Lemma 8.3] or [16, §7.8]), if $a \le -1/8$,

$$V_*(a, T) \ll T^{(1-2a)/4+\varepsilon} \ll T^{-2a+1/2}\,. \tag{6.3}$$

Moreover by (6.1) and Proposition 5.1 with $T^{1/4} \le \tau \le T$, for $-T^{1/4} \le a \le 1/16$,

$$\begin{aligned}
V_0(a, T) &= \sum_{n=0}^{N_0} I(2^{-n-1}T, 2^{-n}T) = \frac{T}{2}\Big(\sum_{n=0}^{N_0} 2^{-n}\Phi(a, 2^{-n}T) \\
&\quad + (2\pi)^{2a}\lambda(a)\sum_{n=0}^{N_0} 2^{(2a-1)n}T^{-2a}\Phi(-a, 2^{-n}T)\big(1 + O((2^{-n}T)^{-1/4})\big) \\
&\quad + \sum_{n=0}^{N_0} 2^{-n}R_0(a, 2^{-n}T)\Big) \\
&= \frac{T}{2}\big(\Phi_{N_0}(a, T) + (2\pi)^{2a}\lambda(a)\widetilde{\Phi}_{N_0}(a, T) + R_{N_0}(a, T)\big)
\end{aligned} \tag{6.4}$$

with

$$\begin{aligned}
R_{N_0}(a, T) &\ll T^{-2a-1/16}\sum_{n=0}^{N_0} 2^{(2a-1)n}|\Phi(-a, 2^{-n}T)| \\
&\quad + (|a|^{2|a|} + 1)\big(|\Lambda|_{N_0}(a, T) + |(1 - T^{-a})/a|^2\big)\,,
\end{aligned}$$

since $\sum_{n=0}^{N_0} 2^{-n}|(1 - (2^{-n}T)^{-a})/a|^2 \le 2|(1 - T^{-a})/a|^2$. Here the implied constants are absolute; and $\Phi_N(a, x)$, $\widetilde{\Phi}_N(a, x)$ and $|\Lambda|_N(a, x)$ are the quantities defined at the beginning of Section 4.

6.1. Proof of Theorem 2.1. By Lemma 4.6, $\Lambda(-a, \tau) \ll l_3(-a, \tau)$ for $T^{1/4} \le \tau \le T$ if $|a| + (\log T)^{-1} \to 0$, so that, for $n \le N_0$,

$$\begin{aligned}
\Phi(-a, 2^{-n}T) &\ll \int_1^{2^{-n}T} \xi^{-1}|\Lambda(-a, \xi)|d\xi \ll \int_1^{2^{-n}T} \xi^{-1}l_3(-a, \xi)d\xi \\
&\ll a^{-4}\psi(-a, 2^{-n}T) \ll a^{-4}\psi(-a, T)\,.
\end{aligned}$$

Furthermore $|(1 - T^{-a})/a|^2 \cdot (a^{-4}\psi(a, T))^{-1} \ll (\log T)^{-2} + a^2$. These estimates together with Lemmas 4.1 and 4.3 yield

$$R_{N_0}(a, T) \ll a^{-4}\big(T^{-2a}\psi(-a, T) + \psi(a, T)\big)(|a| + (\log T)^{-1})$$

with $T^{-2a}\psi(-a, T) + \psi(a, T) = \phi(a \log T)$ as $|a| + (\log T)^{-1} \to 0$. Substitute this and $(2\pi)^{2a}\lambda(a) = 1 + O(a)$ into (6.4). Using Lemma 4.1 and (6.2), we obtain Theorem 2.1.

6.2. Proof of Theorem 2.2. Suppose that $-T^{1/4} \le a < 0$ and that $|a \log T|^{-1}$ is sufficiently small. By Lemma 4.7 we have $\Lambda(a, \tau) \ll \Lambda_0(-a)\tau^{-2a}$ and $\Lambda(-a, \tau) \ll \Lambda_0(-a)$ for $T^{1/4} \le \tau \le T$. Hence $\Phi(a, 2^{-n}T) \ll \Lambda_0(-a)|a|^{-1}(2^{-n}T)^{-2a}$ and $\Phi(-a, 2^{-n}T) \ll \Lambda_0(-a) \log T$ for $n \le N_0$. Then, by Lemmas 4.2 and 4.3,

$$\Phi_{N_0}(a, T) \ll \Lambda_0(-a)|a|^{-1}T^{-2a} \ll \widetilde{\Phi}_{N_0}(a, T)|a \log T|^{-1},$$

$$R_{N_0}(a, T) \ll \Lambda_0(-a)T^{-2a-1/16} \log T + (|a|^{2|a|} + 1)\big(|\Lambda|_{N_0}(a, T) + a^{-2}T^{-2a}\big)$$

$$\ll (|a|^{2|a|} + 1)\widetilde{\Phi}_{N_0}(a, T)(\log T)^{-1}.$$

Using Lemma 4.2, (6.2), (6.3) and (6.4), we obtain the desired formula for $a < 0$. For $0 < a \le T^{1/4}$, we note the following:

$$\begin{aligned} T\Psi(a, T) &= \int_1^T |\zeta(1/2 + it)\zeta(1/2 - a - it)\chi(1/2 + a + it)|^2 dt \\ &= (2\pi)^{2a} \int_1^T (t\Psi(-a, t))' t^{-2a}\big(1 + O(t^{-1/2})\big)dt \\ &= (2\pi)^{2a}\big(H(2a, T) + O\big(H(2a + 1/2, T)\big)\big) \end{aligned}$$

with

$$H(\kappa, T) := T^{1-\kappa}\Psi(-a, T) + \kappa \int_1^T t^{-\kappa}\Psi(-a, t)dt,$$

since $(t\Psi(-a, t))' \ge 0$. Substitution of the asymptotic expression for $\Psi(-a, T)$ with $a > 0$ leads us to the second formula.

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