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A Remark on Spectral Properties of Certain Non-selfadjoint Schrödinger Operators

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Abstract. In this paper, we study the spectral and pseudospectral properties of the differential operator $H_\varepsilon = -\partial_x^2 + x^{2m} + i\varepsilon^{-1} f(x)$ on $L^2(\mathbf{R})$, where $\varepsilon > 0$ is a small parameter, $m \in \mathbf{N}$ and f is a real-valued Morse function which satisfies $|\partial_x^l(f(x) - |x|^{-k})| \le C|x|^{-k-l-1}$ for $l = 0, 1, 2, 3$ and large $|x|$. We show that $\Psi(\varepsilon) = (\sup_{\lambda\in\mathbf{R}} \|(H_\varepsilon - i\lambda)^{-1}\|)^{-1}$ and $\Sigma(\varepsilon) = \inf \Re(\sigma(H_\varepsilon))$ satisfy $C^{-1}\varepsilon^{-\nu(m)} \le \Psi(\varepsilon) \le C\varepsilon^{-\nu(m)}$ and $\Sigma(\varepsilon) \ge C^{-1}\varepsilon^{-\nu(m)}$, $\nu(m) = \min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}$. This extends the result of I. Gallagher, T. Gallay and F. Nier [3] (2009) for the case $m = 1$ to general $m \in \mathbf{N}$.

1. Introduction

We consider Schrödinger operator with a complex potential $\tilde{H}_\varepsilon = -\partial_x^2 + x^2 + i\varepsilon^{-1} f(x)$ in $L^2(\mathbf{R})$ where $\varepsilon > 0$ is a small parameter and $f(x)$ is a real-valued function. In [ICM 2], C. Villani asked the following question: What is the condition on $f(x)$ for $\tilde{\Sigma}(\varepsilon) = \inf \Re(\sigma(\tilde{H}_\varepsilon)) \to +\infty$ as $\varepsilon \to 0$ and how the rate of divergence ? In [5], J. H. Schenker has proved that $\tilde{\Sigma}(\varepsilon) \to +\infty$ as $\varepsilon \to 0$ if $L_t \overset{\text{def}}{=} \{x \in \mathbf{R};\ f(x) = t\}$ is essentially nowhere dense for each $t \in \mathbf{R}$. Now, we say that a set S is essentially nowhere dense if $S = S' \cup N$ where S' is nowhere dense and N has Lebesgue measure zero. In [3], I. Gallagher, T. Gallay and F. Nier have studied the rate of growth of $\tilde{\Sigma}(\varepsilon)$ and the spectral quantity $\tilde{\Psi}(\varepsilon) = \left(\sup_{\lambda\in\mathbf{R}} \|(\tilde{H}_\varepsilon - i\lambda)^{-1}\|\right)^{-1}$ under the condition that $f(x)$ is a real-valued Morse function.

In this paper, we study the same problem for $H_\varepsilon = -\partial_x^2 + x^{2m} + i\varepsilon^{-1} f(x)$ where $m \ge 1$ is an integer. We consider the operator H_ε with domain $\mathcal{D} = \{u \in H^2(\mathbf{R});\ x^{2m}u \in L^2(\mathbf{R})\}$. Let $H_\infty = -\partial_x^2 + x^{2m}$ be a self-adjoint operator with domain $\mathcal{D}$. Then H_ε is H_∞-bounded and skew-symmetric. Since H_ε has a compact resolvent, the spectrum $\sigma(H_\varepsilon)$ consists of a countable number discrete eigenvalues $\{\lambda_n(\varepsilon)\}_{n\in\mathbf{N}}$ with $\Re(\lambda_n(\varepsilon)) \to +\infty$ as $n \to \infty$. The numerical range $\Theta(H_\varepsilon) = \{\langle H_\varepsilon u, u\rangle_{L^2};\ u \in \mathcal{D}, \|u\|_{L^2} = 1\}$ is obviously contained in the rectangle $\mathcal{R}_\varepsilon = \{\lambda \in \mathbf{C};\ \Re(\lambda) \ge a_0, \varepsilon\Im(\lambda) \in \overline{f(\mathbf{R})}\}$ where $a_0 = \inf \sigma(H_\infty)$. Hence, we have

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$\lambda_n(\varepsilon) \in \Theta(H_\varepsilon) \subset \mathcal{R}_\varepsilon$ for all $n \in \mathbf{N}$ and all $\varepsilon > 0$. It follows that the imaginary axis $i\mathbf{R}$ is contained in the resolvent set of H_ε. We define

$$\Sigma(\varepsilon) = \inf \Re(\sigma(H_\varepsilon)) = \min_{n\in\mathbf{N}} \Re(\lambda_n(\varepsilon)), \quad \text{and} \quad \Psi(\varepsilon) = \left(\sup_{\lambda\in\mathbf{R}} \|(H_\varepsilon - i\lambda)^{-1}\|\right)^{-1}.$$

As is proved by [3], we have $\Sigma(\varepsilon) \geq \Psi(\varepsilon) \geq a_0$. To state our main theorem, we set the following hypothesis.

HYPOTHESIS 1. Assume that $f \in C^3(\mathbf{R}, \mathbf{R})$ has the following properties:

(i) All critical points of f are nondegenerate, i.e., $f'(x) = 0$ implies $f''(x) \neq 0$,

(ii) There exist positive constants C and k such that, for all $x \in \mathbf{R}$ with $|x| \geq 1$,

$$\left|\partial_x^l \left(f(x) - \frac{1}{|x|^k}\right)\right| \leq \frac{C}{|x|^{k+l+1}}, \quad \text{for } l = 0, 1, 2, 3.$$

The main theorem of this paper is the following.

THEOREM 1. *Suppose that f satisfies Hypothesis* 1. *Then there exists* $C > 0$ *such that, for all* $0 < \varepsilon \ll 1$,

$$\frac{1}{C\varepsilon^{\nu(m)}} \leq \Psi(\varepsilon) \leq \frac{C}{\varepsilon^{\nu(m)}} \quad \text{and} \quad \Sigma(\varepsilon) \geq \frac{1}{C\varepsilon^{\nu(m)}} \quad \text{where} \quad \nu(m) = \min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}.$$

A few remarks are in order.

REMARK 1. For the case $m = 1$, Theorem 1 was proven by I. Gallagher, T. Gallay and F. Nier [3]. Our result shows that $\nu(m) > \nu(n)$ if $m > n$.

REMARK 2. Since $\Theta(H_\varepsilon) \subset \mathcal{R}_\varepsilon$, $H_\varepsilon - a_0$ is maximal accretive and H_ε is the infinitesimal generator of C_0-semigroup e^{-tH_ε}. We set that $C(\mu) = \frac{1}{\pi}\left\{\frac{\mu}{\tan\alpha} N(\mu) + \frac{2\pi}{\sin\alpha}\right\}$ and $N(\mu) = \sup_{\lambda\in\mathbf{R}} \|(H_\varepsilon - \mu - i\lambda)^{-1}\|$ where the angle α satisfies $\tan(2\alpha) = a_0\varepsilon\|f\|_\infty^{-1}$. As is proved by [3], for any $0 < \mu < \Sigma(\varepsilon)$, we have $\|e^{-tH_\varepsilon}\| \leq C(\mu)e^{-\mu t}$ for all $t \geq 0$.

In spite of $\Sigma(\varepsilon) \geq \Psi(\varepsilon)$, $\Sigma(\varepsilon)$ can be much bigger than $\Psi(\varepsilon)$ in some particular cases. The following is also a generalization of the Theorem 1.9 of [3].

THEOREM 2. *Fix* $k > 0$ *and set* $f(x) = (1 + x^2)^{-k/2}$. *Then there exists a constant* $C > 0$ *such that for all* $0 < \varepsilon \ll 1$,

$$\Sigma(\varepsilon) \geq \frac{C}{\varepsilon^{\nu'(m)}}, \quad \text{where} \quad \nu'(m) = \min\left\{\frac{1}{2}, \frac{2m}{k+2m}\right\}.$$

The rest of the paper is devoted to the proof of Theorem 1 and Theorem 2. Theorem 1 is proved in section 2 and Theorem 2 in section 3. Before going into the next, we remark that

(i) $\Psi(\varepsilon) > a_0$ if $f \in L^\infty(\mathbf{R})$ is not a constant,

(ii) $\Psi(\varepsilon) \to \infty$ as $\varepsilon \to 0$ if $f \in L^\infty(\mathbf{R}) \cap C^0(\mathbf{R})$ and for any $t \in \mathbf{R}$, L_t has empty interiors.

This can be proven similarly to Proposition 1.4 and Lemma 2.1 of [3]. Throughout this paper, we denote by C various constants whose exact values are not important. Thus they may differ from one place to the other.

2. Resolvent Estimates

In this section, we prove Theorem 1 by using the localization techniques and semiclassical subelliptic estimates. The proof patterns after that of Proposition 4.1 of [3], and we shall point out only what modifications are necessary for the generalization. We estimate

$$\kappa(\varepsilon, \lambda) = \|(H_\varepsilon - i\lambda)^{-1}\| \quad for \ \lambda \in \mathbf{R} \quad and \quad 0 < \varepsilon \ll 1 \,.$$

Under Hypothesis 1, f has only a finite number of critical points, and we denote the set of critical values of f by

$$\mathrm{cv}(f) = \{f(x);\, x \in \mathbf{R},\, f'(x) = 0\}\,.$$

PROPOSITION 1. *If f satisfies Hypothesis 1. Then for any $\lambda \in \mathbf{R}$ and $0 < \varepsilon \ll 1$, the quantity $\kappa(\varepsilon, \lambda)$ satisfies the following estimates*:

(i) *If* $\mathrm{dist}(\varepsilon\lambda, f(\mathbf{R})) \geq \delta > 0$, *then* $\kappa(\varepsilon, \lambda) \leq \varepsilon/\delta$.

(ii) *If* $\mathrm{dist}(\varepsilon\lambda, \mathrm{cv}(f) \cup \{0\}) \geq \delta > 0$, *then* $\kappa(\varepsilon, \lambda) \leq C_\delta \varepsilon^{2/3}$.

(iii) *If* $\lambda = \lambda(\varepsilon)$ *is such that* $\lim_{\varepsilon\to 0} \varepsilon\lambda(\varepsilon) = \alpha \in \mathrm{cv}(f)\backslash\{0\}$, *then*

$$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon^{-1/2}\kappa(\varepsilon, \lambda(\varepsilon)) \leq C\,.$$

(iv) *For* $\lambda = 0$, *the quantity* $\kappa(\varepsilon, 0)$ *satisfies*

$$\kappa(\varepsilon, 0) \leq \begin{cases} C\varepsilon^{\frac{2m}{k+2m}}, & if \quad 0 \notin f(\mathbf{R}), \\ C\varepsilon^{\min\left\{\frac{2m}{k+2m}, \frac{2}{3}\right\}}, & if \quad 0 \in f(\mathbf{R})\backslash \mathrm{cv}(f), \\ C\varepsilon^{\min\left\{\frac{2m}{k+2m}, \frac{1}{2}\right\}}, & if \quad 0 \in \mathrm{cv}(f). \end{cases}$$

(v) *There exists* $C > 1$ *such that, for all* $\lambda \in \mathbf{R}$ *and* $0 < \varepsilon \ll 1$,

$$\kappa(\varepsilon, \lambda) \leq C\varepsilon^{\nu(m)}. \quad where \ \ \nu(m) = \min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}.$$

For the proof of Proposition 1, we use the following localization scheme. The proof of the following two lemmas may be found in [3].

LEMMA 1. *Let $Q = -\Delta + V$ in $\mathbf{R}^d$, where V is a complex valued measurable function. Let $\{\chi_j^2\}_{j\in J}$, where $\chi_j \in C_0^\infty(\mathbf{R}^d, \mathbf{R})$ be such that*

$$\sum_{j\in J} \chi_j(x)^2 = 1\,, \quad for\ all \ \ x \in \mathbf{R}^d\,, \quad and$$

$$m_1^2 \stackrel{\text{def}}{=} \sup_{x\in\mathbf{R}^d} \sum_{j\in J} |\nabla \chi_j(x)|^2 < +\infty\,, \quad m_2^2 \stackrel{\text{def}}{=} \sup_{x\in\mathbf{R}^d} \sum_{j\in J} (\Delta \chi_j(x))^2 < +\infty\,.$$

Then the following estimates hold for any $u \in C_0^\infty(\mathbf{R}^d)$

$$2\|Qu\|^2 + 3m_2^2\|u\|^2 + 8m_1^2\|\nabla u\|^2 \geq \sum_{j\in J} \|Q\chi_j u\|^2\,.$$

In particular, if $\Re V(x) \geq 0$,

$$2\|Qu\|^2 + 3m_2^2\|u\|^2 + 8m_1^2\Re\langle Qu, u\rangle \geq \sum_{j\in J} \|Q\chi_j u\|^2\,,$$

$$\langle Qu, u\rangle_{L^2} + m_1^2\|u\|^2 \geq \sum_{j\in J} \langle Q\chi_j u, \chi_j u\rangle_{L^2}\,. \tag{1}$$

Using a dyadic partition of unity, we apply Lemma 1 to the one-dimensional operator $Q = H_\varepsilon - i\lambda$.

LEMMA 2. *For* $j \in \mathbf{N}$, $\varepsilon > 0$, *and* $\lambda \in \mathbf{R}$, *we define unitary operators* U_j, $j \in \mathbf{N}$ *by* $(U_j u)(x) = 2^{j/2}u(2^j x)$ *and transform* Q *by* U_j

$$P_{j,\varepsilon,\lambda} = U_j Q U_j^* = -2^{-2j}\partial_x^2 + 2^{2mj}x^{2m} + \frac{i}{\varepsilon}f(2^j x) - i\lambda\,,$$

and let

$$C_j(\varepsilon, \lambda) = \inf\{\|P_{j,\varepsilon,\lambda}u\| : u \in C_0^\infty(\mathbf{R}), \operatorname{supp} u \subset K_j, \|u\| = 1\}\,,$$

where $K_0 = [-1, 1]$ *and* $K_j = [-1, -3/8] \cup [1, 3/8]$ *for any* $j > 0$. *Then* $\kappa(\varepsilon, \lambda) = \|(H_\varepsilon - i\lambda)^{-1}\|$ *satisfies*

$$\left(\inf_{j\in\mathbf{N}} C_j(\varepsilon, \lambda)\right)^{-1} \leq \kappa(\varepsilon, \lambda) \leq C\left(\inf_{j\in\mathbf{N}} C_j(\varepsilon, \lambda)\right)^{-1}\,, \tag{2}$$

for some constant $C \geq 1$ *independent of* ε, λ.

It is clear that $C_j(\varepsilon, \lambda) \geq a_0$ for all $j \in \mathbf{N}$, $\varepsilon > 0$, $\lambda \in \mathbf{R}$, because

$$a_0\|u\|^2 \leq \Re\langle P_{j,\varepsilon,\lambda}u, u\rangle \leq \|P_{j,\varepsilon,\lambda}u\|\|u\|,\ \ for\ all\ u \in C_0^\infty(\mathbf{R})\,.$$

We now begin the proof of Proposition 1.

2.1. Proof of Proposition 1. (i) If $\operatorname{dist}(\varepsilon\lambda, f(\mathbf{R})) \geq \delta$, then

$$|\Im\langle(H_\varepsilon - i\lambda)u, u\rangle| = \left|\left\langle\left(\frac{f}{\varepsilon} - \lambda\right)u, u\right\rangle\right| \geq (\delta/\varepsilon)\|u\|^2 \quad \text{for all } u \in \mathcal{D}\,,$$

and we get $\kappa(\varepsilon,\lambda) \le \varepsilon/\delta$. Before we prove (ii), for f satisfying Hypothesis 1, set that

$$C_f \stackrel{\text{def}}{=} \sup_{j\in\mathbf{N}} \sup_{x\in K_j} 2^{kj}|f(2^j x)| < +\infty\,,$$

where $k > 0$ is the parameter that governs the asymptotic behavior of $f(x)$ as $|x| \to \infty$.

(ii) Suppose that $\operatorname{dist}(\varepsilon\lambda, \operatorname{cv}(f) \cup \{0\}) \ge \delta$. We also assume that $\varepsilon|\lambda| \le \|f\|_{L^\infty} + \delta$, because otherwise we can use the estimate (i). For any $u \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp} u \subset K_j$ and $u \not\equiv 0$, we have the lower bound

$$\frac{\|P_{j,\varepsilon,\lambda}\|}{\|u\|} \ge \frac{|\Im\langle P_{j,\varepsilon,\lambda}u, u\rangle|}{\|u\|^2} = \frac{\left|\left\langle\left[2^{kj} f\left(2^j\cdot\right) - 2^{kj}\varepsilon\lambda\right]u, u\right\rangle\right|}{\varepsilon 2^{kj}\|u\|^2} \ge \frac{1}{\varepsilon}\left(\varepsilon|\lambda| - \frac{C_f}{2^{kj}}\right).$$

Since $\varepsilon|\lambda| \ge \delta$, taking large enough $J \in \mathbf{N}$ such that $2^{kJ} \ge 2C_f/\delta$, we find that $C_j(\varepsilon,\lambda) \ge \delta/(2\varepsilon)$ for all $j \ge J$.

Thus, we only consider $0 \le j \le J$ and the problem is reduced to finding a lower bound on $\|(H_\varepsilon - i\lambda)u\|$ when $u \in C_0^\infty(\{x \in \mathbf{R};\, |x| < R_\delta\})$, for some $R_\delta > 0$. On a bounded domain, we can neglect the bounded term x^{2m} in H_ε and only consider the operator $\tilde{Q} = -\partial_x^2 + \frac{i}{\varepsilon}(f(x) - \varepsilon\lambda)$. Thus the method of [3] for the case $m = 1$ applies here to obtain $\kappa(\varepsilon,\lambda) \le C\varepsilon^{2/3}$.

(iii) The assumption $\lim_{\varepsilon\to 0}\varepsilon\lambda(\varepsilon) = \alpha \in \operatorname{cv}(f)\backslash\{0\}$ implies that $\varepsilon|\lambda| \ge \delta$ for some fixed $\delta > 0$ if $\varepsilon > 0$ is small enough. Thus, we can reduce the analysis to a bounded domain as in (ii) and again the analysis of [3] for the case $m = 1$ yields the statement (iii).

(iv) For any $j \ge 1$ and $u \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp} u \subset K_j = \{\frac{3}{8} \le |x| \le 1\}$, we have

$$\|u\|\|P_{j,\varepsilon,0}u\| \ge |\Re\langle P_{j,\varepsilon,0}u, u\rangle| \ge 2^{2mj}\int_{K_j}|x|^{2m}|u(x)|^2dx \ge 3^{2m}2^{2(j-3)m}\|u\|^2\,,$$

$$\|u\|\|P_{j,\varepsilon,0}u\| \ge |\Im\langle P_{j,\varepsilon,0}u, u\rangle| \ge \frac{1}{\varepsilon 2^{kj}}\int_{K_j}2^{kj}|f(2^j x)||u(x)|^2dx \ge \frac{m_j}{\varepsilon 2^{kj}}\|u\|^2\,,$$

where $m_j(x) = \inf\{2^{kj}|f(2^j x)|;\, \frac{3}{8} \le |x| \le 1\}$. From Hypothesis 1, we find that $\lim_{j\to\infty} m_j = 1$, so taking large enough $J \in \mathbf{N}$, we find that

$$C_j(\varepsilon, 0) \ge C\left(2^{mj} + \frac{1}{\varepsilon 2^{kj}}\right) \ge C\varepsilon^{-\frac{2m}{k+2m}}\,, \quad \text{for all } j \ge J\,.$$

Since $0 \le j \le J$ corresponds to a bounded spatial domain, we can treated as in (ii) and (iii). Hence, we find that

$$\|H_\varepsilon u\| \ge C\varepsilon^{-\sigma}\|u\|\,, \quad \text{where } \sigma = \begin{cases} 1\,, & \text{if } 0 \notin f(\mathbf{R})\,,\\ \frac{2}{3}\,, & \text{if } 0 \in f(\mathbf{R})\backslash\operatorname{cv}(f)\,,\\ \frac{1}{2}\,, & \text{if } 0 \in \operatorname{cv}(f)\,. \end{cases}$$

Consequently, we get $\kappa(\varepsilon, 0) \leq C\varepsilon^{\min\{\frac{2m}{k+2m}, \sigma\}}$.

(v) By Lemma 2, we need only prove that

$$C_j(\varepsilon, \lambda) \geq C\varepsilon^{-\min\left\{\frac{2m}{k+3m+1}, \frac{1}{2}\right\}}, \quad \text{for all } j \in \mathbf{N}, \quad 0 < \varepsilon \ll 1 \quad \text{and} \quad \lambda \in \mathbf{R}. \tag{3}$$

As in (ii), (iii), we have $C_j(\varepsilon, \lambda) \geq C_J \varepsilon^{-1/2}$ for $0 \leq j \leq J$. Hence, we consider the case $j > J$. We take $\widetilde{u} \in C_0^\infty(\mathbf{R})$ such that $\operatorname{supp} \widetilde{u} \subset K_j$, $\|\widetilde{u}\| = 1$ and $\|P_{j,\varepsilon,\lambda}\widetilde{u}\| \leq 2C_j(\varepsilon, \lambda)$. As in (iv), we easily find that

$$\|P_{j,\varepsilon,\lambda}\widetilde{u}\| \geq C2^{2mj}, \quad \text{and} \quad \|P_{j,\varepsilon,\lambda}\widetilde{u}\| \geq \frac{\inf_{x \in K_j} |g_j(x)|}{\varepsilon 2^{kj}}, \tag{4}$$

where

$$g_j(x) = 2^{kj} f(2^j x) - 2^{kj}\varepsilon\lambda.$$

If $2^j \geq \varepsilon^{-\frac{1}{k+3m+1}}$, the first inequality of (4) implies (3). If $2^j < \varepsilon^{-\frac{1}{k+3m+1}}$, we integrate by parts and obtain the following relation:

$$\|P_{j,\varepsilon,\lambda}\widetilde{u}\|^2 + C2^{2(m-1)j}\|x^{m-1}\widetilde{u}\|^2 = \|Q_{j,\varepsilon,\lambda}\widetilde{u}\|^2 + 2^{2(m-1)j+1}\|x^m \partial_x \widetilde{u}\|^2 + 2^{4mj}\|x^{2m}\widetilde{u}\|^2,$$

where $Q_{j,\varepsilon,\lambda} = P_{j,\varepsilon,\lambda} - 2^{2mj}x^{2m}$. Thus, we have $\|P_{j,\varepsilon,\lambda}\widetilde{u}\| \geq \|Q_{j,\varepsilon,\lambda}\widetilde{u}\| - C2^{(m-1)j}$. Combining this estimate with (4), we obtain

$$2C_j(\varepsilon, \lambda) \geq \|P_{j,\varepsilon,\lambda}\widetilde{u}\| \geq \frac{C}{3}\left(2^{2mj} + \frac{\inf_{x \in K_j} |g_j(x)|}{\varepsilon 2^{kj}} + \|Q_{j,\varepsilon,\lambda}\widetilde{u}\| - 2^{(m-1)j}\right). \tag{5}$$

As is proved by [3], we have

$$\|Q_{j,\varepsilon,\lambda}u\| \geq \frac{Ch^{2/3}}{\varepsilon 2^{kj}}\|u\|, \quad \text{for all } u \in C_0^\infty(\mathbf{R}) \quad \text{with } \operatorname{supp} u \subset K_j.$$

Returning to (5), we find that

$$C_j(\varepsilon, \lambda) \geq C\left(2^{2mj} + \frac{h^{2/3}}{\varepsilon 2^{kj}} - 2^{(m-1)j}\right) \geq C\varepsilon^{\frac{-2m}{k+3m+1}},$$

which proves (3).

2.2. Proof of Theorem 1. According to (v) in Proposition 1, it is clear that $\Psi(\varepsilon) = \left(\sup_{\lambda \in \mathbf{R}} \kappa(\varepsilon, \lambda)\right)^{-1} \geq C^{-1}\varepsilon^{-\nu(m)}$. Since $\Sigma(\varepsilon) \geq \Psi(\varepsilon)$, we find that $\Sigma(\varepsilon) \geq C^{-1}\varepsilon^{-\nu(m)}$. Hence, we need only prove $\Psi(\varepsilon) \leq C\varepsilon^{-\nu(m)}$. First, we consider the case $k > m - 1$. Fix $0 < \varepsilon \ll 1$, $3/8 < x_0 < 1$. We define $j \in \mathbf{N}$, $\lambda \in \mathbf{R}$ and $h > 0$ as follows:

$$2^j \geq \varepsilon^{-\frac{1}{k+3m+1}} > 2^{j-1}, \quad h^2 = \varepsilon 2^{(k-2)j}, \quad \varepsilon\lambda = f(2^j x_0).$$

Next, we take $v \in C_0^\infty(\mathbf{R})$ such that $\|v\| = 1$ and $\operatorname{supp} v \subset [-1, 1]$. We define

$$u_h(x) = \frac{1}{h^{1/3}} v\left(\frac{x - x_0}{h^{2/3}}\right), \quad x \in \mathbf{R}. \tag{6}$$

It is clear that $u_h \in C_0^\infty(\mathbf{R})$, $\|u_h\| = 1$ and $\operatorname{supp} u_h \subset K_j$ for sufficiently small $h > 0$. Recalling that

$$P_{j,\varepsilon,\lambda} = \frac{1}{\varepsilon 2^{kj}}\big(-h^2\partial_x^2 + h^{2/3}x^{2m} + i g_j(x)\big), \quad \text{where } g_j(x) = 2^{kj} f(2^j x) - 2^{kj}\varepsilon\lambda,$$

we find that there exists $C > 0$ independent of j, ε, λ such that

$$\|P_{j,\varepsilon,\lambda} u_h\| \le C\frac{h^{2/3}}{\varepsilon 2^{kj}} = C\varepsilon^{-\frac{2m}{k+3m+1}}. \tag{7}$$

This implies that $C_j(\varepsilon, \lambda) \le C\varepsilon^{-\frac{2m}{k+3m+1}}$, hence $\kappa(\varepsilon, \lambda) \ge C\varepsilon^{\frac{2m}{k+3m+1}}$ by (2) and $\Psi(\varepsilon) \le C\varepsilon^{-\frac{2m}{k+3m+1}}$. It is straightforward to verify (7). First, using (6), we find $\|h^2\partial_x^2 u_h\| = h^{2/3}\|v''\|$. Next, since $x^{2m} \le x_0^{2m} + 2m|x - x_0|$ for all $x \in K_j$, we have $\|x^{2m}u_h\| \le C$. Finally, since $g_j(x_0) = 0$ by our choice of λ, we have for all $x \in K_j$,

$$|g_j(x)| \le |x - x_0| \sup_{\frac{3}{8} \le |x| \le 1} |g_j'(x)| \le C|x - x_0|,$$

where C does not depend on j by Hypothesis 1. Therefore, $\|g_j u_h\| \le Ch^{2/3}$ and the proof of (7) is complete.

Secondary, we consider the case $k \le m - 1$. Let x_0 be a critical point of f. We assume without loss of generality that $x_0 = 0$. We set

$$\lambda = \frac{f(0)}{\varepsilon}, \quad g(x) = f(x) - \varepsilon\lambda.$$

Next, we take $v \in C_0^\infty(\mathbf{R})$ such that $\|v\| = 1$ and $\operatorname{supp} v \subset [-1, 1]$. We define

$$u_\varepsilon(x) = \frac{1}{\varepsilon^{1/8}} v\left(\frac{x}{\varepsilon^{1/4}}\right).$$

Using Taylor's expansion of g around $x_0 = 0$, we find that

$$\begin{aligned}
\|(H_\varepsilon - i\lambda)u_\varepsilon\| &\le \|u_\varepsilon''\| + \|x^{2m}u_\varepsilon\| + \varepsilon^{-1}\|gu_\varepsilon\| \\
&= C\varepsilon^{-1/2} + C + C\|x^2 u_\varepsilon\| + \mathcal{O}\left(\int_{\operatorname{supp} u_\varepsilon} x^6 |u_\varepsilon(x)|^2 dx\right)^{1/2} \\
&\le C\varepsilon^{-1/2}.
\end{aligned}$$

Hence, $C^{-1}\varepsilon^{1/2} \le \sup_{\lambda \in \mathbf{R}} \|(H_\varepsilon - i\lambda)^{-1}\|$ and we obtain $\Psi(\varepsilon) \le C\varepsilon^{-1/2}$.

3. Spectral Lower Bounds - Proof of Theorem 2

I. Gallagher, T. Gallay and F. Nier [3] have proved Theorem 2 for the case $m = 1$, by using a complex deformation method and the same localization techniques as in the proof of Proposition 1. They also use accurate numerical computations to show that the lower bound in Theorem 2 is optimal when $m = 1$, in the sense that the exponent $\nu'(m)$ cannot be improved. Our proof for the general case follows that of Theorem 1.9 of [3]. We only give an outline the proof of Theorem 2.

To prove Theorem 2, we use a complex deformation method using the dilation group $(U_\theta\phi)(x) = e^{\theta/2}\phi(e^\theta x)$, which are unitary operators when $\theta \in \mathbf{R}$. If f is given by $f(x) = (1+x^2)^{-k/2}$, the multiplication operator $(i/\varepsilon)f(x)$ is a dilation analytic perturbation of $H_\infty = -\partial_x^2 + x^{2m}$. According to the dilation analytic theory ([4]), when we define the operator $H_\varepsilon(\theta)$ by

$$H_\varepsilon(\theta) = U_\theta H_\varepsilon U_\theta^{-1} = -e^{-2\theta}\partial_x^2 + e^{2m\theta}x^{2m} + \frac{i}{\varepsilon}\frac{1}{(1+e^{2\theta}x^2)^{k/2}},$$

for $S = \{\theta \in \mathbf{C}; |\Im(\theta)| \le \pi/4m\}$, the spectrum of $H_\varepsilon(\theta)$ does not depend on $\theta \in S$. We choose $\theta = it_k$ where $t_k = \frac{\pi}{4m(k+2)}$. Applying localization formula (1) in Lemma 1 to the operator $H_\varepsilon(it_k)$, we obtain that

$$\sigma(H_\varepsilon) \cap \left\{ z \in \mathbf{C};\, c_1\Re(z) \le |\Im(z)| \le \frac{c_2}{\varepsilon} \right\} = \emptyset, \quad \text{for some } c_1, c_2 > 0.$$

As is proved by [3], combining this relation with the resolvent estimate of Proposition 1, we deduce that there exists $C > 0$ such that H_ε has no spectrum in the region $\{\Re(z) \le C\varepsilon^{-\nu'(m)}\}$ for sufficiently small ε. Therefore, we find that $\Sigma(\varepsilon) \ge C\varepsilon^{-\nu'(m)}$ and this concludes the proof of Theorem 2.

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