TOKYO J. MATH.
VOL. 36, NO. 1, 2013

On the Fourth Moment of the Epstein Zeta Functions and the Related Divisor Problem

Keiju SONO

University of Tokyo

(Communicated by M. Tsuzuki)

Abstract. In this paper, we study the fourth moment of the Epstein zeta function $\zeta(s; Q)$ associated to a $n \times n$ positive definite symmetric matrix Q $(n \geq 4)$ on the line $\mathrm{Re}(s) = \frac{n-1}{2}$. We prove that the integral $\int_0^T |\zeta(\frac{n-1}{2} + it; Q)|^4 dt$ is evaluated by $O(T(\log T)^4)$ if Q satisfies some conditions. As an application, we consider the divisor problem with respect to the coefficients of the Dirichlet series of Epstein zeta functions. Certain estimates for the error term of the sum of the Dirichlet coefficients are obtained by combining our results and Fomenko's estimates for $\zeta(\frac{n-1}{2} + it; Q)$.

1. Introduction

The moments of the Riemann zeta function and other L-functions have been studied for about one hundred years, from the age of Hardy and Littlewood. In 1918, Hardy and Littlewood ([2]) proved that

$$\int_1^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 dt \sim T \log T\,, \tag{1.1}$$

and in 1926, Ingham ([8]) showed that

$$\int_1^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 dt \sim \frac{1}{2\pi^2} T(\log T)^4\,. \tag{1.2}$$

It is conjectured that

$$\int_1^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \sim C_k T(\log T)^{k^2} \tag{1.3}$$

holds for $k \geq 0$ with some positive constant number C_k, but this has not been proved except for the cases $k = 0, 1, 2$.

In this paper, we deal with the Epstein zeta function $\zeta(s; Q)$, where Q is a $n \times n$ positive definite symmetric matrix $(n \geq 4)$ which gives an integer-valued quadratic form. The mean

Received March 26, 2012

square (second moment) of zeta functions of this kind is studied by several mathematicians. For example, Müller proved that the Epstein zeta function $\zeta(s; Q)$ associated to a primitive positive definite quadratic form in n variables with integral coefficients satisfies

$$\int_0^T \left|\zeta\left(\frac{n-1}{2}+it; Q\right)\right|^2 dt = D_Q T\log T + O(T) \quad (T \to \infty)$$

when $n \geq 3$, and

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it; Q\right)\right|^2 dt = A_Q T(\log T)^2 + O(T\log T) \quad (T \to \infty)$$

when $n = 2$, where D_Q and A_Q are computable constants (see [15]). But as far as the author knows, little is known about the fourth moment of the Epstein zeta functions. We consider the fourth moment of $\zeta(s; Q)$ on the line $\mathrm{Re}(s) = \frac{n-1}{2}$. We prove that the integral $\int_0^T |\zeta(\frac{n-1}{2}+it; Q)|^4 dt$ is evaluated by $O(T(\log T)^4)$ when $T \to \infty$ (Theorem 2.5, Theorem 2.6). Further, we apply our results to the divisor problem for the coefficients of the Dirichlet series of $\zeta(s; Q)$, and obtain certain estimates for the error terms of them.

Let us introduce the basic idea of this paper. For a $n \times n$ positive definite symmetric matrix Q, the quadratic form associated to Q is defined by $Q[\mathbf{x}] = {}^t\mathbf{x}Q\mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^n$. We assume that $Q[\mathbf{x}] \in \mathbf{N}$ for any $\mathbf{x} \in \mathbf{Z}^n\backslash\{\mathbf{0}\}$. For $k \in \mathbf{Z}_{\geq 0}$, we define $r_Q(k)$ by the number of $\mathbf{x} \in \mathbf{Z}^n$ which satisfies $Q[\mathbf{x}] = k$. Then the Epstein zeta function $\zeta(s; Q)$ is expressed by

$$\zeta(s; Q) = \sum_{k=1}^{\infty} \frac{r_Q(k)}{k^s} \tag{1.4}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. It is well-known that the corresponding theta series

$$\theta(z; Q) = \sum_{k=0}^{\infty} r_Q(k) e^{2\pi ikz}$$

becomes a modular form of weight $\frac{n}{2}$, which can be written as the sum of an Eisenstein series and a cusp form. Therefore, $\zeta(s; Q)$ decomposes into the sum of the L-function associated to the Eisenstein series and the L-function associated to the cusp form. Hence to establish the upper bound for the moments of $\zeta(s; Q)$, it suffices to evaluate the moments of these two L-functions. We can easily prove that the fourth moment of the L-function associated to the cusp form on the line $\mathrm{Re}(s) = \frac{n-1}{2}$ is evaluated by $O(T)$ (Lemma 2.1), and our main problem is to evaluate the moment of L-function associated to the Eisenstein series. For this purpose, we use the classical results by Hecke ([7]), Malyshev ([13]), and Siegel ([17]). By using their theorems, we prove that the L-function associated to the Eisenstein series is expressed by some series consisting of Dirichlet L-functions and thus we can use the theory of the moments of Dirichlet L-functions.

As the easiest example, we take $Q = I_4$, the 4×4 unit matrix. Then the Epstein zeta function $\zeta(s; I_4)$ is expressed by

$$\zeta(s; I_4) = 8(1 - 2^{2-2s})\zeta(s)\zeta(s-1). \tag{1.5}$$

Since the factor $(1 - 2^{2-2s})\zeta(s)$ is bounded on the line $\mathrm{Re}(s) = \frac{3}{2}$, the fourth moment $\int_0^T |\zeta(\frac{3}{2} + it; I_4)|^4 dt$ is evaluated by $O(T(\log T)^4)$, by using Ingham's asymptotic formula (1.2). The general case is more complicated, but the underlying idea is similar.

In section 3, we apply our results to the divisor problem. For a positive integer l (in this paper, we assume that $l \geq 4$), we write

$$\zeta(s; Q)^l = \sum_{k=1}^{\infty} \frac{r_Q^{(l)}(k)}{k^s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right).$$

Evaluating the magnitude of the sum $\sum_{k \leq x} r_Q^{(l)}(k)$ $(x \to \infty)$ is called the divisor problem. It is well-known that the following asymptotic formula holds (see [11], [16]):

$$\sum_{k \leq x} r_Q^{(l)}(k) = M_l^{(n)}(x) + \Delta_l^{(n)}(x) \quad (x \to \infty),$$

where $M_l^{(n)}(x)$ is the main term, expressed by $x^{\frac{n}{2}} P_l(\log x)$ with some polynomial P_l of degree $l-1$, and $\Delta_l^{(n)}(x)$ is the error term which becomes $o(x^{\frac{n}{2}})$. In [16], Sankaranarayanan showed that the error term $\Delta_l^{(n)}(x)$ is evaluated by $O(x^{\frac{n}{2} - \frac{1}{l} + \varepsilon})$ for $n \geq 3, l \geq 2$ by using the order estimate for $\zeta(\frac{n-1}{2} + it; Q)$ and the fact that the mean-square $\int_0^T |\zeta(\frac{n-1}{2} + it; Q)|^2 dt$ is evaluated by $O(T^{1+\varepsilon})$ $(\forall \varepsilon > 0)$ (this fact is proved in [10]). Note that the order estimate for $\zeta(\frac{n-1}{2} + it; Q)$ above relies on Stirling's formula for the gamma function and the Phragmén-Lindelöf principle, which is weaker than Fomenko's ones. Further, Lü ([11], [12]) obtained sharper estimates for $\Delta_l^{(n)}(x)$ for special kinds of Q, whose the "Eisenstein part"of the associated Epstein zeta functions are expressed by using the Riemann zeta function, like (1.5). His method relies on the sophisticated theories for the Riemann zeta function, for example, the order estimate for $\zeta(s)$ on the critical line, and the estimate for the twelfth moment of $\zeta(s)$ by Heath-Brown ([6]). We investigate the divisor problem for general Q ($n \times n$ positive definite symmetric matrix ($n \geq 4$) which gives an integer valued quadratic form). Instead of the theories for the Riemann zeta function above, we use Fomenko's estimates for the order of $\zeta(\frac{n-1}{2} + it; Q)$ ([1]), and our theorems for the fourth moment of $\zeta(s; Q)$. Certain estimates for $\Delta_l^{(n)}(x)$ are obtained (Theorem 3.2).

2. Fourth moment of the Epstein zeta functions

2.1. Notation and some basic results. Let n be a positive integer and Q be a $n \times n$ positive definite symmetric matrix. The Epstein zeta function associated to Q is defined by

$$\zeta(s; Q) = \sum_{\mathbf{x} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} Q[\mathbf{x}]^{-s} \quad \left(\mathrm{Re}(s) > \frac{n}{2}\right),$$

where $Q[\mathbf{x}] := {}^t\mathbf{x} Q \mathbf{x}$. This function has the meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(s; Q) = (\det Q)^{-\frac{1}{2}}\pi^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2} - s\right)\zeta\left(\frac{n}{2} - s; Q^{-1}\right). \tag{2.1}$$

$\zeta(s; Q)$ is holomorphic everywhere except for a simple pole at $s = \frac{n}{2}$ with residue $\pi^{\frac{n}{2}}/(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})$. Throughout this paper, we assume that $Q[\mathbf{x}] \in \mathbf{N}$ for any $\mathbf{x} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}$. Let $r_Q(k)$ be the number of $\mathbf{x} \in \mathbf{Z}^n$ which satisfies $Q[\mathbf{x}] = k$. Then $\zeta(s; Q)$ has the following Dirichlet series expansion for $\mathrm{Re}(s) > \frac{n}{2}$:

$$\zeta(s; Q) = \sum_{k=1}^{\infty} \frac{r_Q(k)}{k^s}.$$

Hereafter, we assume that $n \geq 4$. We consider the theta series corresponding to $\zeta(s; Q)$ defined by

$$\theta(z; Q) = \sum_{k=0}^{\infty} r_Q(k) e^{2\pi i k z}.$$

It is well-known that $\theta(z; Q)$ is written as the sum of an Eisenstein series and a cusp form:

$$\theta(z; Q) = E(z) + S(z), \tag{2.2}$$

where

$$E(z) = \sum_{k=0}^{\infty} e(k) e^{2\pi i k z}$$

is the Eisenstein series and

$$S(z) = \sum_{k=1}^{\infty} s(k) e^{2\pi i k z}$$

is the cusp form. Moreover, it is known that the coefficient $s(k)$ of $S(z)$ is evaluated by

$$s(k) \ll k^{\frac{n}{4} - \frac{1}{2} + \varepsilon} \tag{2.3}$$

if n is even, and

$$s(k) \ll k^{\frac{n}{4}-\frac{1}{4}+\varepsilon} \tag{2.4}$$

if n is odd, where ε is always an arbitrary positive number in this paper.

2.2. The fourth moment of $\zeta(s; Q)$ on the line $\mathrm{Re}(s) = \frac{n-1}{2}$. We want to establish some upper bound for the magnitude of the integral

$$\int_0^T \left| \zeta\left(\frac{n-1}{2} + it; Q \right) \right|^4 dt$$

when $T \to \infty$. Since $\zeta(s; Q)$ is expressed by

$$\zeta(s; Q) = \hat{E}(s) + \hat{S}(s)\,, \tag{2.5}$$

where $\hat{E}(s)$ and $\hat{S}(s)$ are defined by

$$\hat{E}(s) = \sum_{k=1}^{\infty} \frac{e(k)}{k^s}\,, \quad \hat{S}(s) = \sum_{k=1}^{\infty} \frac{s(k)}{k^s}$$

for $\mathrm{Re}(s) > \frac{n}{2}$ and analytically continued to the whole s-plane, it is enough to evaluate the two integrals

$$\int_0^T \left| \hat{E}\left(\frac{n-1}{2} + it \right) \right|^4 dt\,, \quad \int_0^T \left| \hat{S}\left(\frac{n-1}{2} + it \right) \right|^4 dt\,.$$

Firstly, the following lemma gives an upper bound for the integral $\int_0^T |\hat{S}(\frac{n-1}{2} + it)|^4 dt$.

LEMMA 2.1. *When $T \to \infty$, we have*

$$\int_0^T \left| \hat{S}\left(\frac{n-1}{2} + it \right) \right|^4 dt = O(T)\,. \tag{2.6}$$

PROOF. If $n \geq 6$, by the estimates (2.3), (2.4), the series $\sum_{k=1}^{\infty} \frac{s(k)}{k^s}$ converges absolutely on the line $\mathrm{Re}(s) = \frac{n-1}{2}$. Therefore, the estimate (2.6) trivially holds. In case of $n = 4$ or 5, the line $\mathrm{Re}(s) = \frac{n-1}{2}$ is so to say the "absolute convergence line", that is, it is assured that the Dirichlet series for $\hat{S}(s)$ converges absolutely for $\mathrm{Re}(s) > \frac{n-1}{2}$ by the estimates (2.3), (2.4), but not on the line $\mathrm{Re}(s) = \frac{n-1}{2}$. To prove (2.6), we use a classical method. Assume that

$$s(k) = O(k^{\alpha+\varepsilon})$$

holds. Then the Dirichlet series

$$\hat{S}(s) = \sum_{k=1}^{\infty} \frac{s(k)}{k^s} \tag{2.7}$$

converges absolutely for $\mathrm{Re}(s) > \alpha + 1$. If $n = 4$, $\alpha = \frac{1}{2}$ and if $n = 5$, $\alpha = 1$. Firstly, as an easy consequence of the functional equation of the Hecke L-functions of cusp forms, the estimate $\hat{S}(\sigma + it) \ll |t|^{\mu(\sigma)}$ ($|t| \to \infty$) holds for some $\mu(\sigma) < \infty$. Put

$$\hat{S}(s)^2 = \sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^s} \quad (\mathrm{Re}(s) > \alpha + 1)\,.$$

By using the Mellin inversion formula

$$e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s} ds \quad (c > 0, x > 0)\,,$$

for $\delta > 0$,

$$\sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^s} e^{-\delta k} = \frac{1}{2\pi i} \int_{\alpha+2-i\infty}^{\alpha+2+i\infty} \Gamma(z-s) \hat{S}(z)^2 \delta^{s-z} dz \tag{2.8}$$

holds when $\mathrm{Re}(s) < \alpha + 2$. We take a real number β which satisfies $\alpha < \beta < \alpha + 1$, and assume that s satisfies

$$\max\left\{\alpha + \frac{1}{2}, \beta\right\} < \mathrm{Re}(s) < \beta + 1\ .$$

We move the path of integral to the line $\mathrm{Re}(s) = \beta$. The pole of integrant between the lines $\mathrm{Re}(s) = \alpha + 2$, $\mathrm{Re}(s) = \beta$ is $z = s$, and the residue is $\hat{S}(s)^2$. Therefore, by Cauchy's residue theorem, we have

$$\begin{aligned} \hat{S}(s)^2 = &\sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^s} e^{-\delta k} \\ &- \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(z-s) \hat{S}(z)^2 \delta^{s-z} dz\,. \end{aligned} \tag{2.9}$$

We put

$$S_1 = \sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^s} e^{-\delta k}\,,$$

$$S_2 = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(z-s) \hat{S}(z)^2 \delta^{s-z} dz\,.$$

Firstly,

$$\int_0^T |S_1|^2 dt = \int_0^T \Big(\sum_{k=1}^{\infty} \frac{s^{(2)}(k)}{k^{\sigma+it}} e^{-\delta k} \Big) \Big(\sum_{l=1}^{\infty} \frac{s^{(2)}(l)}{l^{\sigma-it}} e^{-\delta l} \Big) dt$$

$$= T \sum_{k=1}^{\infty} \frac{s^{(2)}(k)^2}{k^{2\sigma}} e^{-2\delta k} \tag{2.10}$$

$$+ O\Big(\sum_{l<k}\sum \frac{s^{(2)}(k) s^{(2)}(l) e^{-(k+l)\delta}}{(kl)^{\sigma} \log(\frac{k}{l})} \Big).$$

Since $s^{(2)}(k)$ is evaluated by

$$s^{(2)}(k) = O(k^{\alpha+\varepsilon}),$$

the second term of the right hand side of (2.10) is evaluated by $O(\delta^{2\sigma-2\alpha-2-\varepsilon})$ when $\delta \to +0$ (see [18], p117). Therefore,

$$\int_0^T |S_1|^2 dt = T \sum_{k=1}^{\infty} \frac{s^{(2)}(k)^2}{k^{2\sigma}} e^{-2\delta k} + O(\delta^{2\sigma-2\alpha-2-\varepsilon}). \tag{2.11}$$

Note that the series $\sum_{k=1}^{\infty} \frac{s^{(2)}(k)^2}{k^{2\sigma}}$ converges for $\sigma > \alpha + \frac{1}{2}$. Next, by using the Cauchy-Schwarz inequality, we have

$$S_2 \ll \delta^{\sigma-\beta} \int_{\beta-i\infty}^{\beta+i\infty} |\Gamma(z-s)||\hat{S}(z)|^2 dz$$

$$\ll \delta^{\sigma-\beta} \left(\int_{-\infty}^{\infty} |\Gamma(\beta+iv-s)| dv \int_{-\infty}^{\infty} |\Gamma(\beta+iv-s)||\hat{S}(\beta+iv)|^4 dv \right)^{\frac{1}{2}}$$

$$\ll \delta^{\sigma-\beta} \left(\int_{-\infty}^{\infty} |\Gamma(\beta+iv-s)||\hat{S}(\beta+iv)|^4 dv \right)^{\frac{1}{2}}.$$

Therefore,

$$\int_0^T |S_2|^2 dt \ll \delta^{2\sigma-2\beta} \int_0^T \int_{-\infty}^{\infty} |\Gamma(\beta+iv-s)||\hat{S}(\beta+iv)|^4 dv dt. \tag{2.12}$$

The contribution of the integral in $|v| \geq 2T$ is evaluated by

$$\ll \delta^{2\sigma-2\beta} \int_0^T \int_{|v|\geq 2T} e^{-C_1|v-t|} |v|^{C_2} dv dt$$

$$\ll \delta^{2\sigma-2\beta} \int_0^T e^{-C_3 T} dt \ll \delta^{2\sigma-2\beta}.$$

Here, C_1, C_2, C_3 are some positive constants. The remaining part is evaluated by

$$\begin{aligned}&\ll \delta^{2\sigma-2\beta}\int_{-2T}^{2T}|\hat{S}(\beta+iv)|^4\int_0^T|\Gamma(\beta+iv-s)|dtdv\\&\ll \delta^{2\sigma-2\beta}\int_{-2T}^{2T}|\hat{S}(\beta+iv)|^4 dv\\&\ll \delta^{2\sigma-2\beta}T^{4\mu(\beta)+1}.\end{aligned}$$

Therefore,

$$\int_0^T |S_2|^2 dt \ll \delta^{2\sigma-2\beta}T^{4\mu(\beta)+1}. \tag{2.13}$$

By combining (2.11) and (2.13), we have

$$\begin{aligned}\int_0^T|\hat{S}(s)|^4 dt &\ll \int_0^T|S_1|^2dt+\int_0^T|S_2|^2dt\\&\ll O(T)+O(\delta^{2\sigma-2\alpha-2-\varepsilon})+O(\delta^{2\sigma-2\beta}T^{4\mu(\beta)+1}).\end{aligned} \tag{2.14}$$

We put

$$\delta = T^{-\frac{4\mu(\beta)+1}{2\alpha-2\beta+2}}.$$

Then (2.14) becomes

$$\int_0^T|\hat{S}(s)|^4dt \ll O(T)+O(T^{\frac{(\alpha+1-\sigma)(4\mu(\beta)+1)}{\alpha-\beta+1}+\varepsilon}). \tag{2.15}$$

The second term of the right hand side of (2.15) becomes $o(T)$ if $\sigma=\mathrm{Re}(s)$ satisfies

$$\sigma > \alpha+1-\frac{\alpha-\beta+1}{4\mu(\beta)+1}.$$

In particular, the left hand side of (2.15) becomes $O(T)$ for $\sigma=\alpha+1$. □

Our main problem is to evaluate the integral $\int_0^T|\hat{E}(\frac{n-1}{2}+it)|^4dt$. For this purpose, we use the relations between the L-function associated to the Eisenstein series and the Dirichlet L-functions. Before stating our main theorems, we prepare three lemmas. The first lemma is the estimate for the fourth moment of Dirichlet L-functions (for example, see [5]):

LEMMA 2.2. *When $T\to\infty$, we have*

$$\sum_{\chi\,(\mathrm{mod}\,q)}\int_0^T\left|L\left(\frac{1}{2}+it,\chi\right)\right|^4dt \ll qT(\log qT)^4. \tag{2.16}$$

Here, $\sum_{\chi\,(\mathrm{mod}\,q)}$ denotes the sum over all Dirichlet characters modulo q.

The second lemma is the estimates for the order of Dirichlet L-functions on the critical line by Heath-Brown ([3], [4]):

LEMMA 2.3. *Let $L(s,\chi)$ be a Dirichlet L-function associated to a Dirichlet character modulo q. Then, when $t \to \infty$, the following estimates hold*:

$$L\left(\frac{1}{2}+it,\chi\right) \ll q^{\frac{1}{2}} t^{\frac{1}{6}} \log(qt)\,, \tag{2.17}$$

$$L\left(\frac{1}{2}+it,\chi\right) \ll (qt)^{\frac{3}{16}+\varepsilon}\,. \tag{2.18}$$

The third lemma is a simple inequality, used to evaluate the fourth moment of $\hat{E}(s)$ in case of n is odd and $n \geq 7$.

LEMMA 2.4. *For $x_1,\dots,x_m \geq 0$, we have*

$$x_1^{\frac{1}{4}} + \cdots + x_m^{\frac{1}{4}} \leq m^{\frac{3}{4}} (x_1+\cdots+x_m)^{\frac{1}{4}}\,. \tag{2.19}$$

PROOF. The inequality (2.19) is equivalent to

$$\frac{x_1^{\frac{1}{4}} + \cdots + x_m^{\frac{1}{4}}}{m} \leq \left(\frac{x_1+\cdots+x_m}{m}\right)^{\frac{1}{4}}$$

which directly follows from the convexity of the function $f(x) = x^{\frac{1}{4}}$. □

THEOREM 2.5. *Assume that n is even and $n \geq 4$, or n is odd and $n \geq 7$. Then, when $T \to \infty$, the following estimate holds*:

$$\int_0^T \left|\zeta\left(\frac{n-1}{2}+it;Q\right)\right|^4 dt = O(T(\log T)^4)\,. \tag{2.20}$$

PROOF. Firstly, we assume that n is even and $n \geq 4$. Then, the Eisenstein series $E(z)$ is a modular form of weight $\frac{n}{2}$ and level N, where N is a positive integer such that NA^{-1} becomes an integral matrix for $A = 2Q$ (see [9]). According to Hecke ([7], Theorem 44), the series $\hat{E}(s)$ is expressed by some linear combination of the form

$$(t_1t_2)^{-s} L(s,\chi_1) L\left(s-\frac{n}{2}+1,\chi_2\right),$$

where t_1, t_2 are positive divisors of level N and χ_1, χ_2 are Dirichlet characters modulo $\frac{N}{t_1}, \frac{N}{t_2}$, respectively. We write

$$\hat{E}(s) = \sum_{l=1}^{L} c_l (t_{1,l}t_{2,l})^{-s} L(s,\chi_{1,l}) L\left(s-\frac{n}{2}+1,\chi_{2,l}\right).$$

Then

$$\begin{aligned}\int_0^T \left|\hat{E}\left(\frac{n-1}{2}+it\right)\right|^4 dt &\ll \sum_{l=1}^{L}\int_0^T \left|L\left(\frac{n-1}{2}+it, \chi_{1,l}\right)L\left(\frac{1}{2}+it, \chi_{2,l}\right)\right|^4 dt \\ &\ll \sum_{l=1}^{L}\int_0^T \left|L\left(\frac{1}{2}+it, \chi_{2,l}\right)\right|^4 dt\end{aligned} \tag{2.21}$$

since $L(\frac{n-1}{2}+it, \chi_{1,l})$ is bounded with respect to t. Moreover, by the estimate (2.16), each integral $\int_0^T |L(\frac{1}{2}+it, \chi_{2,l})|^4 dt$ is evaluated by $O(T(\log T)^4)$. Therefore, the statement of theorem is proved in this case.

Next, we assume that n is odd and $n \geq 7$. The following argument is a straightforward adaption of the technique of Fomenko in his treatment of pointwise bounds for $\zeta(\frac{n-1}{2}+it; Q)$ (see [1]). In this case, the Fourier coefficient of the Eisenstein series $E(z)$ has the following expression (see [13]):

$$e(k) = \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})}k^{\frac{n}{2}-1}H(Q; k),$$

where

$$H(Q; k) = \sum_{q=1}^{\infty}\left\{\sideset{}{'}\sum_{h(\mathrm{mod}\, q)} q^{-n}S(hQ; q)e^{-2\pi i\frac{kh}{q}}\right\}$$

is a singular series, $\sum'$ means the sum over a reduced residue system, and

$$S(Q; q) = \sum_{x_1,\dots,x_n=0}^{q-1} e^{\frac{2\pi i Q(x_1,\dots,x_n)}{q}}$$

is a Gaussian sum. Therefore, the associated Dirichlet series is given by

$$\hat{E}(s) = \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})}\sum_{k=1}^{\infty}\frac{1}{k^{s-\frac{n}{2}+1}}\sum_{q=1}\sideset{}{'}\sum_{h(\mathrm{mod}\, q)} q^{-n}S(hQ; q)e^{-2\pi i\frac{kh}{q}}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. Let $(k, q) = d$, $k = k_1 d$, $q = q_1 d$, $(k_1, q_1) = 1$ and $k_1 = k_2 q_1 + l$, $(q_1, l) = 1$. Then the right hand side becomes

$$\begin{aligned}&\frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}}\Gamma(\frac{n}{2})}\sum_{d=1}^{\infty}\frac{1}{d^{s-\frac{n}{2}+1}}\sum_{q_1=1}\sideset{}{'}\sum_{h(\mathrm{mod}\, q_1 d)}(q_1 d)^{-n}S(hQ; q_1 d) \\ &\quad\cdot \sideset{}{'}\sum_{l(\mathrm{mod}\, q_1)} e^{-\frac{2\pi i h l}{q_1 d}}\sum_{k_1\equiv l(\mathrm{mod}\, q_1)}\frac{1}{k_1^{s-\frac{n}{2}+1}}.\end{aligned}$$

By using the well-known identity

$$\sum_{\chi (\mathrm{mod}\, q_1)} \overline{\chi}(l)\chi(k) = \begin{cases} \phi(q_1) & (k \equiv l \quad (\mathrm{mod}\, q_1)) \\ 0 & (\text{otherwise}), \end{cases}$$

we have

$$\begin{aligned} \sum_{k_1 \equiv l (\mathrm{mod}\, q_1)} \frac{1}{k_1^{s-\frac{n}{2}+1}} &= \frac{1}{\phi(q_1)} \sum_{\chi (\mathrm{mod}\, q_1)} \overline{\chi}(l) \sum_{k_1=1}^{\infty} \frac{\chi(k_1)}{k_1^{s-\frac{n}{2}+1}} \\ &= \frac{1}{\phi(q_1)} \sum_{\chi (\mathrm{mod}\, q_1)} \overline{\chi}(l) L\left(s - \frac{n}{2} + 1, \chi\right) \end{aligned}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. Therefore,

$$\begin{aligned} \hat{E}(s) =& \frac{\pi^{\frac{n}{2}}}{(\det Q)^{\frac{1}{2}} \Gamma(\frac{n}{2})} \sum_{d=1}^{\infty} \frac{1}{d^{s-\frac{n}{2}+1}} \sum_{q_1=1} \sideset{}{'}\sum_{h (\mathrm{mod}\, q_1 d)} \frac{S(hQ; q_1 d)}{(q_1 d)^n} \\ & \cdot \sideset{}{'}\sum_{l (\mathrm{mod}\, q_1)} e^{-\frac{2\pi i h l}{q_1}} \frac{1}{\phi(q_1)} \sum_{\chi (\mathrm{mod}\, q_1)} \overline{\chi}(l) L\left(s - \frac{n}{2} + 1, \chi\right) \end{aligned} \tag{2.22}$$

holds for $\mathrm{Re}(s) > \frac{n}{2}$. It is known that the following estimate holds (see [13]):

$$S(hQ; q) \ll q^{\frac{n}{2}},$$

where the constant occuring in $\ll$ depends only on Q, not on h. Therefore, the absolute value of the right hand side of (2.22) is estimated by

$$\begin{aligned} &\ll \sum_{d=1}^{\infty} \frac{1}{d^{\sigma-\frac{n}{2}+1}} \sum_{q_1=1}^{\infty} \phi(q_1 d) \frac{(q_1 d)^{\frac{n}{2}}}{(q_1 d)^n} \cdot \phi(q_1) \frac{1}{\phi(q_1)} \sum_{\chi (\mathrm{mod}\, q_1)} \left| L\left(s - \frac{n}{2} + 1, \chi\right) \right| \\ &\ll \sum_{d=1}^{\infty} \frac{1}{d^{\sigma}} \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} \sum_{\chi (\mathrm{mod}\, q_1)} \left| L\left(s - \frac{n}{2} + 1, \chi\right) \right| . \end{aligned} \tag{2.23}$$

By the estimate (2.18), the right hand side of (2.23) converges on the line $\mathrm{Re}(s) = \frac{n-1}{2}$, hence $\hat{E}(s)$ is continued analytically to some domain containing the line $\mathrm{Re}(s) = \frac{n-1}{2}$ by (2.22) and the estimate

$$\left| \hat{E}\left(\frac{n-1}{2} + it\right) \right| \ll \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} \sum_{\chi (\mathrm{mod}\, q_1)} \left| L\left(\frac{1}{2} + it, \chi\right) \right| \tag{2.24}$$

holds. By applying Minkowski's inequality to (2.24), we have

$$\begin{aligned}\left(\int_0^T \left|\hat{E}\left(\frac{n-1}{2}+it\right)\right|^4 dt\right)^{\frac{1}{4}} \\ \ll \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} \sum_{\chi \,(\mathrm{mod}\, q_1)} \left(\int_0^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\right)^{\frac{1}{4}}.\end{aligned} \tag{2.25}$$

By applying the inequality (2.19) to the sum in $\chi\,(\mathrm{mod}\, q_1)$ and using the estimate (2.16), the right hand side of (2.25) is evaluated by

$$\begin{aligned}&\le \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} \phi(q_1)^{\frac{3}{4}} \left(\sum_{\chi\,(\mathrm{mod}\, q_1)} \int_0^T \left|L\left(\frac{1}{2}+it,\chi\right)\right|^4 dt\right)^{\frac{1}{4}} \\ &\ll \sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-1}} q_1^{\frac{3}{4}} (q_1 T(\log q_1 T))^{\frac{1}{4}} \\ &\ll \left(\sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-2-\varepsilon}}\right) T^{\frac{1}{4}} \log T\,.\end{aligned}$$

The series $\sum_{q_1=1}^{\infty} \frac{1}{q_1^{\frac{n}{2}-2-\varepsilon}}$ converges when $n > 6$. Therefore, the estimate

$$\left(\int_0^T \left|\hat{E}\left(\frac{n-1}{2}+it\right)\right|^4 dt\right)^{\frac{1}{4}} \ll T^{\frac{1}{4}} \log T$$

holds when $n \ge 7$, hence the statement of theorem is proved. □

Next, we consider the case of $n = 5$. In this case, we cannot use the method we used in the proof of Theorem 2.5, since the right hand side of (2.23) may not converge on the line $\mathrm{Re}(s) = 2$ if $n = 5$. We use another formula proved by Siegel ([17]).

THEOREM 2.6. *Let Q be a 5×5 positive definite symmetric integral matrix which satisfies* $\det Q = 1$. *Then, when $T \to \infty$, we have*

$$\int_0^T |\zeta(2+it;Q)|^4 dt = O(T(\log T)^4)\,. \tag{2.26}$$

PROOF. Assume that Q satisfies the conditions of theorem. In this case, Siegel showed that $\hat{E}(s)$ has the following expression (see [17], Theorem 12):

$$\hat{E}(s) = 2\pi^s \frac{\Gamma(\frac{5}{2}-s)}{\Gamma(\frac{5}{2})} \left\{\psi(s) + \psi\left(\frac{5}{2}-s\right)\right\} \tag{2.27}$$

for $1 < \mathrm{Re}(s) < \frac{3}{2}$, where

$$\begin{aligned}\psi(s) =& 2^{s-\frac{5}{2}}\left\{\cos\frac{\pi}{4}(2s-5)\sum_{a,b\ b\equiv 1\,(\mathrm{mod}\,4)}\chi_b(a)a^{s-\frac{5}{2}}b^{-s}\right.\\ &\left.+\cos\frac{\pi}{4}(2s+5)\sum_{a,b\ b\equiv 3\,(\mathrm{mod}\,4)}\chi_b(a)a^{s-\frac{5}{2}}b^{-s}\right\}\end{aligned} \tag{2.28}$$

and $\chi_b(a) = (\frac{a}{b})$ denoting the Legendre-Jacobi symbol. For fixed b, we have

$$\sum_a \chi_b(a)a^{s-\frac{5}{2}} = L\left(\frac{5}{2}-s, \chi_b\right)$$

for $\mathrm{Re}(s) < \frac{3}{2}$. Therefore,

$$\sum_{a,b\ b\equiv j\,(\mathrm{mod}\,4)}\chi_b(a)a^{s-\frac{5}{2}}b^{-s} = \sum_{b\equiv j\,(\mathrm{mod}\,4)} b^{-s}L\left(\frac{5}{2}-s, \chi_b\right) \tag{2.29}$$

$(j = 1, 3)$ holds for $\mathrm{Re}(s) < \frac{3}{2}$. By using the estimate (2.17), the series of the right hand side of (2.29) converges absolutely on $\mathrm{Re}(s) = 2$, so the left hand side of (2.29) can be continued analytically to some domain containing the line $\mathrm{Re}(s) = 2$ by (2.29). Therefore, $\psi(s)$ can be continued analytically to some domain containing the line $\mathrm{Re}(s) = 2$ by

$$\begin{aligned}\psi(s) =& 2^{s-\frac{5}{2}}\left\{\cos\frac{\pi}{4}(2s-5)\sum_{b\equiv 1\,(\mathrm{mod}\,4)}b^{-s}L\left(\frac{5}{2}-s, \chi_b\right)\right.\\ &\left.+\cos\frac{\pi}{4}(2s+5)\sum_{b\equiv 3\,(\mathrm{mod}\,4)}b^{-s}L\left(\frac{5}{2}-s, \chi_b\right)\right\}.\end{aligned} \tag{2.30}$$

On the other hand, for fixed a,

$$\begin{aligned}&\sum_{b,\ b\equiv j\,(\mathrm{mod}\,4)}\chi_b(a)b^{-s}\\ &\qquad = \frac{1}{\phi(4)}\sum_{\chi\,(\mathrm{mod}\,4)}\overline{\chi}(j)\sum_{b=1}^{\infty}\chi(b)\chi_b(a)b^{-s}\\ &\qquad = \frac{1}{\phi(4)}\sum_{\chi\,(\mathrm{mod}\,4)}\overline{\chi}(j)L(s, \tilde{\chi}_{a,\chi})\end{aligned}$$

$(j = 1, 3)$ holds for $\mathrm{Re}(s) > 1$, where

$$\tilde{\chi}_{a,\chi}(b) = \chi(b)\chi_b(a) = \chi(b)\left(\frac{a}{b}\right). \tag{2.31}$$

Note that $\tilde{\chi}_{a,\chi}$ is a Dirichlet character modulo $4a$. Therefore, we have proved that the identity

$$\begin{aligned}\psi(s) = \frac{2^{s-\frac{5}{2}}}{\phi(4)}\Bigg\{&\cos\frac{\pi}{4}(2s-5)\sum_{a=1}^{\infty} a^{s-\frac{5}{2}} \sum_{\chi\,(\mathrm{mod}\,4)} \overline{\chi}(1)L(s,\tilde{\chi}_{a,\chi}) \\ &+\cos\frac{\pi}{4}(2s+5)\sum_{a=1}^{\infty} a^{s-\frac{5}{2}} \sum_{\chi\,(\mathrm{mod}\,4)} \overline{\chi}(3)L(s,\tilde{\chi}_{a,\chi})\Bigg\}\end{aligned} \tag{2.32}$$

holds for $1 < \mathrm{Re}(s) < \frac{3}{2}$, where $\tilde{\chi}_{a,\chi}$ is a Dirichlet character modulo $4a$. By using the estimate (2.17) again, the right hand side of (2.32) converges absolutely at $s = \frac{1}{2} + it$, so $\psi(s)$ can be continued analytically to some domain containing the line $\mathrm{Re}(s) = \frac{1}{2}$ by (2.32). Therefore, by combining these results, the L-function $\hat{E}(s)$ has the following expression on the line $\mathrm{Re}(s) = 2$:

$$\begin{aligned}&\hat{E}(2+it) \\ &= 2^{\frac{1}{2}+it}\pi^{2+it}\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{5}{2})}\Bigg\{\cos\frac{\pi}{4}(-1+2it)\sum_{b\equiv 1\,(\mathrm{mod}\,4)} b^{-2-it}L\Big(\frac{1}{2}-it,\chi_b\Big) \\ &\qquad + \cos\frac{\pi}{4}(9+2it)\sum_{b\equiv 3\,(\mathrm{mod}\,4)} b^{-2-it}L\Big(\frac{1}{2}-it,\chi_b\Big)\Bigg\} \\ &+ 2^{-2-it}\pi^{2+it}\frac{\Gamma(\frac{1}{2}-it)}{\Gamma(\frac{5}{2})}\Bigg\{\cos\frac{\pi}{4}(-4-2it)\sum_{a=1}^{\infty} a^{-2-it}\sum_{\chi\,(\mathrm{mod}\,4)}\overline{\chi}(1)L\Big(\frac{1}{2}-it,\tilde{\chi}_{a,\chi}\Big) \\ &\qquad + \cos\frac{\pi}{4}(6-2it)\sum_{a=1}^{\infty} a^{-2-it}\sum_{\chi\,(\mathrm{mod}\,4)}\overline{\chi}(3)L\Big(\frac{1}{2}-it,\tilde{\chi}_{a,\chi}\Big)\Bigg\}.\end{aligned} \tag{2.33}$$

Note that $\Gamma(\frac{1}{2}-it)\cos\frac{\pi}{4}(\cdot \pm 2it)$ (4 terms) are bounded when $t \to \infty$. By applying Minkowski's inequality, we have

$$\begin{aligned}&\left(\int_0^T |\hat{E}(2+it)|^4 dt\right)^{\frac{1}{4}} \\ &\ll \sum_{b\equiv 1\,(\mathrm{mod}\,4)} b^{-2}\left(\int_0^T \left|L\Big(\frac{1}{2}-it,\chi_b\Big)\right|^4 dt\right)^{\frac{1}{4}} \\ &\quad + \sum_{b\equiv 3\,(\mathrm{mod}\,4)} b^{-2}\left(\int_0^T \left|L\Big(\frac{1}{2}-it,\chi_b\Big)\right|^4 dt\right)^{\frac{1}{4}}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{a=1}^{\infty} a^{-2}\biggl(\int_0^T \Bigl|L\Bigl(\frac{1}{2}-it,\tilde{\chi}_{a,\chi}\Bigr)\Bigr|^4 dt\biggr)^{\frac{1}{4}} \\
&\ll \sum_{b\equiv 1,3 (\mathrm{mod}\, 4)} b^{-2}(bT(\log bT)^4)^{\frac{1}{4}} + \sum_{a=1}^{\infty} a^{-2}(aT(\log aT)^4)^{\frac{1}{4}} \\
&\ll T^{\frac{1}{4}}\log T\,.
\end{aligned}
$$

Therefore, since

$$\int_0^T |\hat{E}(2+it)|^4 dt \ll T(\log T)^4\,,$$

we obtain the estimate (2.26). □

REMARK 2.7. Assume that Q satisfies the conditions of Theorem 2.6. By applying the estimate (2.17) to the identity (2.33), the estimate

$$|\hat{E}(2+it)| \ll |t|^{\frac{1}{6}} \quad (|t|\to\infty)$$

holds. Since the Dirichlet series associated to the cusp form satisfies

$$|\hat{S}(2+it)| = O(|t|^{\varepsilon}) \quad (|t|\to\infty)\,,$$

the Epstein zeta function associated to Q satisfies

$$|\zeta(2+it;Q)| \ll |t|^{\frac{1}{6}} \quad (|t|\to\infty)\,. \tag{2.34}$$

3. Application to the divisor problem

In this section, we evaluate the sum of the coefficients of Dirichlet series of $\zeta(s;Q)^l$, where $l \geq 4$ is a positive integer. For $l \in \mathbf{N}$, write

$$\zeta(s;Q)^l = \sum_{k=1}^{\infty} \frac{r_Q^{(l)}(k)}{k^s}$$

for $\mathrm{Re}(s) > \frac{n}{2}$. It is known that the following asymptotic formula holds (see [11], [16]):

$$\sum_{k\leq x} r_Q^{(l)}(k) = M_l^{(n)}(x) + \Delta_l^{(n)}(x) \quad (x\to\infty)\,,$$

where $M_l^{(n)}(x)$ is called the main term, expressed by $x^{\frac{n}{2}}P_l(\log x)$ with a polynomial P_l of degree $l-1$, and $\Delta_l^{(n)}(x)$ is called the error term which becomes $o(x^{\frac{n}{2}})$ in general. Our main problem is to evaluate the error term $\Delta_l^{(n)}(x)$ as small as possible. The following lemma is due to Fomenko ([1]), which plays a fundamental role in the proof of our theorem:

LEMMA 3.1. *Let Q be a $n \times n$ positive definite matrix which defines an integer valued quadratic form. Then the estimate*

$$\left|\zeta\left(\frac{n-1}{2}+it;\,Q\right)\right| \ll |t|^{\alpha_n+\varepsilon} \quad (|t| \to \infty) \tag{3.1}$$

holds, where

$$\alpha_n = \begin{cases} \dfrac{9}{56} & (n \geq 4,\ \text{even}), \\ \dfrac{1}{6} & (n \geq 9,\ \text{odd}), \\ \dfrac{3}{16} & (n = 7). \end{cases} \tag{3.2}$$

By combining this lemma and our theorems in Section 2, we obtain the following estimates for the error term $\Delta_l^{(n)}(x)$:

THEOREM 3.2. *Let Q be a $n \times n$ positive definite symmetric matrix which satisfies $Q[\mathbf{x}] \in \mathbf{N}$ for any $\mathbf{x} \in \mathbf{Z}^n \backslash \{\mathbf{0}\}$. If $n = 5$, we assume that Q is integral and $\det Q = 1$. Then, for $l \geq 4$, the following asymptotic formula holds when $x \to \infty$:*

$$\sum_{k \leq x} r_Q^{(l)}(k) = M_l^{(n)}(x) + \begin{cases} O\left(x^{\frac{n}{2}-\frac{28}{9l+20}+\varepsilon}\right) & (n \geq 4,\ \text{even}), \\ O\left(x^{\frac{n}{2}-\frac{3}{l+2}+\varepsilon}\right) & (n \geq 9,\ \text{odd or } n = 5), \\ O\left(x^{\frac{n}{2}-\frac{8}{3l+4}+\varepsilon}\right) & (n = 7). \end{cases} \tag{3.3}$$

Here, $M_l^{(n)}(x)$ is expressed by $x^{\frac{n}{2}} P_l(\log x)$ with a polynomial P_l of degree $l-1$.

PROOF. We start from Perron's formula (see [16])

$$\sum_{k \leq x} r_Q^{(l)}(k) = \frac{1}{2\pi i}\int_{\frac{n}{2}+\varepsilon-iT}^{\frac{n}{2}+\varepsilon+iT} \zeta(s;\,Q)^l \frac{x^s}{s} ds + O\left(\frac{x^{\frac{n}{2}+\varepsilon}}{T}\right) + O(x^\varepsilon).$$

We move the path of integral to the parallel segment with $\mathrm{Re}(s) = \frac{n-1}{2}$. Then,

$$\begin{aligned} \sum_{k \leq x} r_Q^{(l)}(k) = \frac{1}{2\pi i}\left\{\int_{\frac{n-1}{2}-iT}^{\frac{n-1}{2}+iT} + \int_{\frac{n-1}{2}+iT}^{\frac{n}{2}+\varepsilon+iT} + \int_{\frac{n}{2}+\varepsilon-iT}^{\frac{n-1}{2}-iT}\right\}\zeta(s;\,Q)^l \frac{x^s}{s} ds \\ + \mathrm{Res}\left[\zeta(s;\,Q)^l \frac{x^s}{s}, s = \frac{n}{2}\right] + O\left(\frac{x^{\frac{n}{2}+\varepsilon}}{T}\right) + O(x^\varepsilon). \end{aligned} \tag{3.4}$$

We put

$$I_1 := \frac{1}{2\pi i}\int_{\frac{n-1}{2}-iT}^{\frac{n-1}{2}+iT} \zeta(s;\,Q)^l \frac{x^s}{s} ds,$$

$$I_2 := \frac{1}{2\pi i}\int_{\frac{n-1}{2}+iT}^{\frac{n}{2}+\varepsilon+iT} \zeta(s;\,Q)^l \frac{x^s}{s} ds\,,$$

$$I_3 := \frac{1}{2\pi i}\int_{\frac{n}{2}+\varepsilon-iT}^{\frac{n-1}{2}-iT} \zeta(s;\,Q)^l \frac{x^s}{s} ds\,,$$

$$M_l^{(n)}(x) := \mathrm{Res}\left[\zeta(s;\,Q)^l \frac{x^s}{s},\, s = \frac{n}{2}\right].$$

Firstly, since $\zeta(s;\,Q)^l \frac{x^s}{s}$ has a pole of order l at $s = \frac{n}{2}$, $M_l^{(n)}(x)$ is expressed by

$$M_l^{(n)}(x) = x^{\frac{n}{2}} P_l(\log x) \tag{3.5}$$

with some polynomial P_l of degree $l-1$. Next, we evaluate three integrals I_i $(i = 1, 2, 3)$. Assume that the estimate

$$\left|\zeta\left(\frac{n-1}{2} + it;\,Q\right)\right| \ll |t|^{\alpha_n+\varepsilon} \quad (|t| \to \infty)$$

holds. Then, by using Theorem 2.5 or Theorem 2.6, $|I_1|$ is evaluated by

$$\begin{aligned} |I_1| &\ll x^{\frac{n-1}{2}} \int_{-T}^{T} \left|\zeta\left(\frac{n-1}{2} + it;\,Q\right)\right|^{l-4} \left|\zeta\left(\frac{n-1}{2} + it;\,Q\right)\right|^4 \frac{dt}{t} \\ &\ll x^{\frac{n-1}{2}} T^{(l-4)\alpha_n+\varepsilon} \int_{-T}^{T} \left|\zeta\left(\frac{n-1}{2} + it;\,Q\right)\right|^4 \frac{dt}{t} \\ &\ll x^{\frac{n-1}{2}} T^{(l-4)\alpha_n+\varepsilon}\,. \end{aligned} \tag{3.6}$$

By Phragmén-Lindelöf principle,

$$|\zeta(\sigma \pm iT;\,Q)| \ll T^{-2\alpha_n(\sigma-\frac{n}{2})+\varepsilon}$$

holds for $\frac{n-1}{2} \le \sigma \le \frac{n}{2}$. Therefore, $|I_2| + |I_3|$ is evaluated by

$$\begin{aligned} |I_2| + |I_3| &\ll \int_{\frac{n-1}{2}}^{\frac{n}{2}+\varepsilon} T^{-2l\alpha_n(\sigma-\frac{n}{2})+\varepsilon}\, \frac{x^\sigma}{T} d\sigma \\ &= \int_{\frac{n-1}{2}}^{\frac{n}{2}+\varepsilon} \left(\frac{x}{T^{2l\alpha_n}}\right)^\sigma T^{nl\alpha_n-1+\varepsilon} d\sigma \\ &\ll x^{\frac{n-1}{2}} T^{l\alpha_n-1+\varepsilon} + x^{\frac{n}{2}+\varepsilon} T^{-1+\varepsilon}\,. \end{aligned} \tag{3.7}$$

By inserting (3.5), (3.6), (3.7) into (3.4), we have

$$\sum_{k\le x} r_Q^{(l)}(k) = M_l^{(n)}(x) + O(x^{\frac{n-1}{2}} T^{(l-4)\alpha_n+\varepsilon}) + O(x^{\frac{n-1}{2}} T^{l\alpha_n-1+\varepsilon}) + O(x^{\frac{n}{2}+\varepsilon} T^{-1+\varepsilon}) + O(x^{\varepsilon}) . \tag{3.8}$$

We put

$$T = x^{\frac{1}{2(l-4)\alpha_n+2}} .$$

Then (3.8) becomes

$$\sum_{k\le x} r_Q^{(l)}(k) = M_l^{(n)}(x) + O\Big(x^{\frac{n}{2}-\frac{1}{2(l-4)\alpha_n+2}+\varepsilon}\Big) + O\Big(x^{\frac{n}{2}-\frac{1-2\alpha_n}{(l-4)\alpha_n+1}+\varepsilon}\Big) . \tag{3.9}$$

Finally, by inserting the values given in Lemma 3.1 or (2.34) into α_n, we obtain the result. □

References

[1] O. M. FOMENKO, Order of the Epstein zeta-function in the critical strip, J. of Math. Sci. **110** (2002), No. 6, 3150–3163.

[2] G. H. HARDY and J. E. LITTLEWOOD, Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, Acta Math. **41** (1918), 119–196.

[3] D. R. HEATH-BROWN, Hybrid bounds for Dirichlet L-functions I, Invent. Math. **47** (1978), 149–170.

[4] D. R. HEATH-BROWN, Hybrid bounds for Dirichlet L-functions II, Quart. J. Math. **31** (1980), 157–167.

[5] D. R. HEATH-BROWN, Fractional moments of Dirichlet L-functions, Acta Arith. **145** (2010), No. 4, 397–409.

[6] D. R. HEATH-BROWN, The twelfth power moment of the Riemann zeta-function, Quart. J. Math. Oxford Ser. (2) **29** (1978), 443–462.

[7] E. HECKE, Über Modulfunktionen und Dirichletschen Reihen mit Eulerscher Productentwicklung I, II, Math. Ann. **114** (1937), 1–28, 316–351.

[8] A. E. INGHAM, Mean-value theorems in the theory of the Riemann zeta-function, Proc. London Math. Soc. **27** (1926), 273–300.

[9] H. IWANIEC, *Topics in Classical Automorphic Forms*, Amer. Math. Soc. Graduate Studies in Mathematics **17**.

[10] S. KANEMITSU, A. SANKARANARAYANAN and Y. TANIGAWA, A Mean Value Theorem for Dirichlet Series and a General Divisor Problem, Monatsh. Math. **136** (2002), 17–34.

[11] G. LÜ, On a divisor problem related to the Epstein zeta-function, Bull. London Math. Soc. **42** (2010), 267–274.

[12] G. LÜ, J. WU and W. ZHAI, On a divisor problem related to the Epstein zeta-function II, J. of Number Theory **131** (2011), 1734–1742.

[13] A. V. MALYSHEV, Representation of integers by positive quadratic forms (in Russian), Trudy Mat. Inst. Akad. Nauk SSSR, **65** (1962).

[14] H. L. MONTGOMERY, *Topics in Multiplicative Number Theory*, Lecture Notes in Math. **227**, Springer, Berlin, 1971.

[15] W. MÜLLER, The mean square of Dirichlet series associated with automorphic forms, Mh. Math. **113** (1992), 121–159.

[16] A. SANKARANARAYANAN, On a divisor problem related to the Epstein zeta-function, Arch. Math. **65** (1995), 303–309.

[17] C. L. SIEGEL, Contribution to the theory of the Dirichlet L-series and the Epstein zeta-functions, Ann. of Math. **44** (1943), 143–172.

[18] E. C. TITCHMARSH, *The theory of the Riemann zeta-function*, Oxford, Calendon Press, 1951.

Present Address:
GRADUATE SCHOOL OF MATHEMATICAL SCIENCES,
UNIVERSITY OF TOKYO,
KOMABA, MEGURO, TOKYO, JAPAN.
e-mail: souno@ms.u-tokyo.ac.jp