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Remarks on Formal Solution and Genuine Solutions for Some Nonlinear Partial Differential Equations

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Abstract. Ōuchi ([2], [3]) found a formal solution $\tilde{u}(t,x) = \sum_{k\geq 0} u_k(x)t^k$ with

$$|u_k(x)| \leq AB^k \Gamma\left(\frac{k}{\gamma_*}+1\right) \quad 0 < \gamma_* \leq \infty$$

for some class of nonlinear partial differential equations. For these equations he showed that there exists a genuine solution $u_S(t,x)$ on a sector S with asymptotic expansion $u_S(t,x) \sim \tilde{u}(t,x)$ as $t \to 0$ in the sector S. These equations have polynomial type nonlinear terms.

In this paper we study a similar class of equations with the following nonlinear terms

$$\sum_{|q|\geq 1} t^{\sigma_q} c_q(t,x) \prod_{j+|\alpha|\leq m} \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x)\right\}^{q_{j,\alpha}}.$$

It is main purpose to get a solvability of the equation in a category $u_S(t,x) \sim 0$ as $t \to 0$ in a sector S. We give a proof by the method that is a little different from that in [3]. Further we give a remark that the similar class of equations has a genuine solution $u_S(t,x)$ with $u_S(t,x) \sim \tilde{u}(t,x)$ as $t \to 0$ in the sector S.

1. Introduction

Let $\mathbf{C}$ be the complex plane or the set of all complex numbers, t be the variable in $\mathbf{C}_t$, and $x = (x_1,\ldots,x_n)$ be the variable in $\mathbf{C}_x^n = \mathbf{C}_{x_1} \times \cdots \times \mathbf{C}_{x_n}$. We use the notations: $\mathbf{N} = \{0,1,2,\ldots\}$, $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbf{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$. Let $|x| = \max_{1\leq i\leq n}\{|x_i|\}$, $D_R = \{x \in \mathbf{C}_x^n;\ |x| < R\}$ and $S_\theta(T) = \{t \in \mathbf{C}_t;\ 0 < |t| < T \text{ and } |\arg t| < \theta\}$. $\mathcal{O}(D_R)$ is the set of all holomorphic functions on D_R . $\mathcal{O}(D_R)[[t]]$ is the set of all formal power series $\sum_{i=0}^\infty f_i(x)t^i$ with the coefficients $f_i(x)$ are in $\mathcal{O}(D_R)$ for all $i = 0,1,\ldots$. $A(S_\theta(T) \times D_R)$ is the set of all functions $f(t,x) \in \mathcal{O}(D_R)[[t]]$ that are holomorphic on $S_\theta(T) \times D_R$. $S_{\theta'}(T') \Subset S_\theta(T)$ means $\theta' < \theta$ and $T' < T$, and for $f(t,x) \in A(S_\theta(T) \times D_R)$ $f(0,x)$ means $\lim_{t\to 0, t\in S_\theta(T)} f(t,x)$.

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Let $Z = \{Z_{j,\alpha}\}_{j+|\alpha|\le m}$ with $Z_{j,\alpha} \in \mathbf{C}$ and $q = \{q_{j,\alpha}\}_{j+|\alpha|\le m}$ with $q_{j,\alpha} \in \mathbf{N}$ then put $Z^q = \prod_{j+|\alpha|\le m} Z_{j,\alpha}{}^{q_{j,\alpha}}$ and $|q| = \sum_{j+|\alpha|\le m} q_{j,\alpha}$. Let $0 < R < 1$. We define a series $L(Z)$ by

$$L(Z) = \sum_{|q|\ge 1} t^{\sigma_q} c_q(t,x) Z^q \tag{1.1}$$

where coefficients $c_q(t,x)$ are in $A(S_\theta(T) \times D_R)$ with $c_q(0,x) \not\equiv 0$ and $\sigma_q \in \mathbf{N}$ for all q. In this paper we assume the following condition:

ASSUMPTION 1. *The series $L(Z)$ converges in a neighborhood of $Z = 0$.*

Let us consider the following nonlinear partial differential equation:

$$L(u(t,x)) = f(t,x) \in \mathcal{O}(S_\theta(T) \times D_R) \tag{1.2}$$

where

$$L(u(t,x)) = \sum_{|q|\ge 1} t^{\sigma_q} c_q(t,x) \prod_{j+|\alpha|\le m} \left\{ \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x) \right\}^{q_{j,\alpha}}. \tag{1.3}$$

In [3] Ōuchi studied a similar class of the equation (1.2) in the case that the series (1.1) is a polynomial in Z with the degree M. Let us introduce some results for (1.2) by [3].

Set $l_q := \max\{j + |\alpha|;\ q_{j,\alpha} \ne 0\}$ and

$$\Pi(a,b) := \{(x,y) \in \mathbf{R}^2;\ x \le a \text{ and } y \ge b\}.$$

Then we define the Newton polygon $NP(L)$ of the nonlinear operator (1.3) by

$$NP(L) = CH\left\{ \bigcup_{|q|\ge 1} \Pi(l_q, \sigma_q);\ c_q(t,x) \not\equiv 0 \right\}$$

where $CH\{\cdot\}$ is the convex hull of a set $\{\cdot\}$.

The boundary of the Newton polygon $NP(L)$ consists of a vertical half line $\Sigma_{L,0}$, a horizontal half line Σ_{L,p^*} and segments $\Sigma_{L,i}$ $(1 \le i \le p^* - 1)$. Let $\gamma_{L,i}$ be the slope of $\Sigma_{L,i}$ for $i = 0, \ldots, p^*$. Then we have $0 = \gamma_{L,p^*} < \gamma_{L,p^*-1} < \cdots < \gamma_{L,0} = \infty$. Further the Newton polygon $NP(L)$ have p^*-point vertices, we denote them by (l_i, σ_i) with $l_{p^*-1} < \cdots < l_0 = m$.

Let us denote the linear part of the nonlinear operator (1.3) by $\mathcal{L}(u)$, that is,

$$\mathcal{L}(u) = \sum_{|q|=1} t^{\sigma_q} c_q(t,x) \prod_{j+|\alpha|\le m} \left\{ \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x) \right\}^{q_{j,\alpha}}.$$

For the linear part $\mathcal{L}(u)$ we define the Newton polygon $NP(\mathcal{L})$ by the same rule as $NP(L)$. For $NP(\mathcal{L})$, we define $\Sigma_{\mathcal{L},i}$ and $\gamma_{\mathcal{L},i}$ for $i = 0, \ldots, p^*_{\mathcal{L}}$ and $(l_{\mathcal{L},i}, \sigma_{\mathcal{L},i})$ for $i = 0, \ldots, p^*_{\mathcal{L}} - 1$ by the same rule as those of $NP(L)$.

For nonlinear operators $L(t^\nu u(t,x))$, we define the formers and denote them by $NP(L;\nu)$, $NP(\mathcal{L};\nu)$, $\Sigma_{L,i}(\nu)$, $\Sigma_{\mathcal{L},i}(\nu)$ and so on. Then $\gamma_{\mathcal{L},i} = \gamma_{\mathcal{L},i}(\nu)$ holds for $i = 0,\dots,p^*_{\mathcal{L}}$.

Let us define operators $\mathcal{L}_i$ with respect to $\Sigma_{\mathcal{L},i}$ for $i = 1,\dots,p^*_{\mathcal{L}}-1$. Set $I_i = \{q;\ \sigma_{\mathcal{L},i-1}-\sigma_q = \gamma_{\mathcal{L},i}(l_{\mathcal{L},i-1}-l_q)$ and $|q|=1\}$. Then we define

$$\begin{aligned}\mathcal{L}_i u(t,x) &= \sum_{q\in I_i} t^{\sigma_q} c_q(t,x) \prod_{j+|\alpha|\le m} \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x)\right\}^{q_{j,\alpha}} \\ &= \sum_{(j,\alpha)\in J_i} t^{\sigma_{j,\alpha}} c_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x)\end{aligned}$$

where $J_i = \{(j,\alpha)\in \mathbf{N}\times\mathbf{N}^n;\ j+|\alpha|\le m$ and $\sigma_{\mathcal{L},i-1}-\sigma_{j,\alpha} = \gamma_{\mathcal{L},i}(l_{\mathcal{L},i-1}-j-|\alpha|)\}$. Let m_i be the differential order with respect to x of $\mathcal{L}_i$.

The equation (1.2) is studied in Ōuchi [3] under the following three conditions.

CONDITION 1. *The series* (1.1) *is a polynomial in Z with the degree M.*

CONDITION 2. *The equation* (1.2) *has a linear part with the order m.*

CONDITION 3. *The operators $\mathcal{L}_i$ hold.*
(1) *If* $j+|\alpha| < l_{\mathcal{L},i-1}$ *then* $|\alpha| < m_i$ *and* (2) $\sum_{j+|\alpha|=l_{\mathcal{L},i-1},|\alpha|=m_i} c_{j,\alpha}(0,0)\hat{\xi}^\alpha \neq 0$ *where* $\hat{\xi} = (1,0,\dots,0)$.

Under Condition 3 the operators $\mathcal{L}_i$ is rewritten by

$$\begin{aligned}\mathcal{L}_i u(t,x) = t^{\sigma_{\mathcal{L},i-1}} &\sum_{\substack{j+|\alpha|=l_{\mathcal{L},i-1}\\ |\alpha|=m_i}} c_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x_1}\right)^\alpha u(t,x) \\ &+ t^{\sigma_{\mathcal{L},i-1}} \sum_{\substack{j+|\alpha|\le l_{\mathcal{L},i-1}\\ |\alpha|<m_i}} t^{-\gamma_{\mathcal{L},i}(l_{\mathcal{L},i-1}-j-|\alpha|)} c_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x)\end{aligned} \tag{1.4}$$

and $c_{j_{\mathcal{L},i-1},m_ie_1}(0,0)\neq 0$ holds where $j_{\mathcal{L},i-1} = l_{\mathcal{L},i-1}-m_i$ and $m_ie_1 = (m_i,0,\dots,0)$.

LEMMA 1.1. *If the equation* (1.2) *satisfies Condition* 2, *then there exists a sufficiently large $\nu_0 > 0$ such that for $\nu \ge \nu_0$ $NP(L;\nu) = NP(\mathcal{L};\nu)$ holds.*

We can show Lemma 1.1 as in Proposition 1.7 in [3].

Let us define the function class that is treated in this paper. Set $S = S_\theta(T)$ and $S' = S_{\theta'}(T')$

DEFINITION 1.2. Let $\gamma > 0$. $Asy^0_{\{\gamma\}}(S\times D_R)$ is the set of all functions $f(t,x)\in \mathcal{O}(S\times D_R)$ such that for any $S' \Subset S$

$$|f(t,x)| \le C\exp(-c|t|^{-\gamma})$$

where c depends on S'.

Set $S_i = S_{\theta_i}(T_i)$ with $0 < \theta_i < \pi/(2\gamma_i)$. Then the following results on the function class $Asy^0_{\{\gamma\}}(S \times D_R)$ were obtained by [3]:

THEOREM 1.3. *Let* $f(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},i}\}}(S_i \times D_R)$. *Suppose that Condition* 1, 2 *and* 3 *on* $\mathcal{L}_i$ *hold. Then we have*;

(1) *If* $2 \le i \le p^*_{\mathcal{L}} - 1$, *then there exists a function* $u_{S_{i-1}}(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},i-1}\}}(S_{i-1} \times D_r)$ *for* $0 < r < R$ *such that*

$$L(u_{S_{i-1}}(t,x)) - f(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},i-1}\}}(S_{i-1} \times D_r).$$

(2) *If* $i = 1$, *then we get a solution* $u_{S^*}(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},1}\}}(S_1 \times D_r)$ *of* (1.2) *for* $0 < r < R$.

By Theorem 1.3 we have the following corollary:

COROLLARY 1.4. *Let* $f(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},p^*_{\mathcal{L}}-1}\}}(S_i \times D_R)$. *Suppose that Condition* 1, 2 *and* 3 *on* $\mathcal{L}_i$ *for* $i = 1, \ldots, p^*_{\mathcal{L}} - 1$ *hold. Then we get a solution* $u_{S^*}(t,x) \in Asy_{\{\gamma_{\mathcal{L},p^*_{\mathcal{L}}-1}\}}(S_1 \times D_r)$ *of* (1.2) *for* $0 < r < R$.

In this paper we have the same results without Condition 1.

THEOREM 1.5. *Let* $f(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},i}\}}(S_i \times D_R)$. *Suppose that Condition* 2 *and* 3 *on* $\mathcal{L}_i$ *hold. Then we have*;

(1) *If* $2 \le i \le p^*_{\mathcal{L}} - 1$, *then there exists a function* $u_{S_{i-1}}(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},i-1}\}}(S_{i-1} \times D_r)$ *for* $0 < r < R$ *such that*

$$L(u_{S_{i-1}}(t,x)) - f(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},i-1}\}}(S_{i-1} \times D_r).$$

(2) *If* $i = 1$, *then we get a solution* $u_{S^*}(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},1}\}}(S_1 \times D_r)$ *of the equation* (1.2) *for* $0 < r < R$.

By Theorem 1.5 we have the following corollary:

COROLLARY 1.6. *Let* $f(t,x) \in Asy^0_{\{\gamma_{\mathcal{L},p^*_{\mathcal{L}}-1}\}}(S_i \times D_R)$. *Suppose that Condition* 2 *and* 3 *on* $\mathcal{L}_i$ *for* $i = 1, \ldots, p^*_{\mathcal{L}} - 1$ *hold. Then we get a solution* $u_{S^*}(t,x) \in Asy_{\{\gamma_{\mathcal{L},p^*_{\mathcal{L}}-1}\}}(S_1 \times D_r)$ *of* (1.2) *for* $0 < r < R$.

REMARK 1.7. The relations between formal solutions and genuine solutions of an equation

$$L(u(t,x)) = f(t,x) \tag{1.5}$$

where $c_q(t,x)$ and $f(t,x)$ are in $\mathcal{O}(\{|t| < T\}) \times D_R)$ with $c_q(0,x) \not\equiv 0$ and $\sigma_q \in \mathbf{N}$ were studied in [3]. Corollary 1.4 is used to show that genuine solutions exist. Let explain the main point of the proof.

Assume that the equation (1.5) has formal power series solutions $\widetilde{u}(t,x) = \sum_{k=0}^{\infty} u_k(x)t^k$ with

$$|u_k(x)| \leq AB^k \Gamma\left(\frac{k}{\gamma_*}+1\right).$$

There exist functions $u_*(t,x) \in \mathcal{O}(S_\theta(T') \times D_{R'})$ with $0 < \theta < \pi/(2\gamma_*)$, $0 < T' < T$ and $0 < R' < R$ such that

$$(1.6) \qquad |u_*(t,x) - \sum_{k=0}^{K-1} u_k(x)t^k| \leq A_0 B_0^K |t|^K \Gamma\left(\frac{K}{\gamma_*}+1\right) \quad \text{for } t \in S' \Subset S_\theta(T').$$

Set $L^{u_*}(u) := L(u_* + u) - L(u_*)$. The linear part of $L^{u_*}(u)$ is denoted by $\mathcal{L}^{u_*}$. Suppose $\gamma_* = \gamma_{\mathcal{L}^{u_*},p^*-1}$ and $L(u_*) - f(t,x) \in Asy^0_{\{\gamma_*\}}(S' \times D_{R'})$. Further we assume that $L^{u_*}(u)$ satisfies Condition 1, 2 and 3. Then by Corollary 1.4, the equation (1.5) has genuine solutions with the estimate (1.6).

By Corollary 1.6, we can get the same results without Condition 1 by the same way as in [3].

2. Preparation of Theorem 1.5

In this section we give one theorem and some lemmas to show Theorem 1.5, and we give a proof of the theorem of this section.

2.1. Preparatory lemmas. Set

$$\mathcal{L}^* = \sum_{\substack{j+|\alpha|=l^* \\ |\alpha|=m^*}} a_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha + \sum_{\substack{j+|\alpha|\leq l^* \\ |\alpha|<m^*}} t^{-\gamma(l^*-j-|\alpha|)} a_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha$$

where the coefficients $a_{j,\alpha}(t,x)$ belong to $A(S_\theta(T) \times D_R)$ with $a_{j,\alpha}(0,x) \not\equiv 0$, $a_{j^*,m^*e_1}(0,0) \equiv 1$, $j^* = l^* - m^*$ and $m^*e_1 = (m^*, 0, \ldots, 0) \in \mathbf{N}^n$. Let us treat the following series:

$$\sum_{|q|\geq 1} t^{\sigma_q} a_q(t,x) Z^q$$

where the coefficients and $a_q(t,x)$ belong to $A(S_\theta(T) \times D_R)$ with and $a_q(0,x) \not\equiv 0$. Further numbers σ_q are integers and satisfy the follows:

$$(2.1) \qquad \sigma_q = \begin{cases} -\gamma(l^* - l_q) + J_q^1 & (J_q^1 > 0) \quad \text{for } l_q \leq l^* \\ \gamma^*(l_q - l^*) + J_q^2 & (J_q^2 \geq 0) \quad \text{for } l_q > l^* \end{cases}$$

where $0 \le \gamma < \gamma^* \le \infty$. If $\{q;\ l_q > l^*\} = \emptyset$ then we define $\gamma^* = \infty$.

Assume that the series $\sum_{|q|\ge 1} t^{\sigma_q} a_q(t,x) Z^q$ converges in a neighborhood of $Z = 0$. For a function $g(t,x) \in Asy^0_{\{\gamma\}}(S_\theta(T) \times D_R)$ we consider the following equation:

$$\mathcal{L}^* u = \sum_{|q|\ge 1} t^{\sigma_q} a_q(t,x) \prod_{j+|\alpha|\le m} \left\{ \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u \right\}^{q_{j,\alpha}} + g(t,x)\,. \tag{2.2}$$

Set

$$L(u) := \mathcal{L}^* u - \sum_{|q|\ge 1} t^{\sigma_q} a_q(t,x) \prod_{j+|\alpha|\le m} \left\{ \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u \right\}^{q_{j,\alpha}}.$$

Let us define the following functional class $X_{p,q,c,\gamma}$ where $p \in \mathbf{N}, q, c, \gamma \ge 0$ and $\zeta > 0$. The definition of $X_{p,q,c,\gamma}(S_\theta(T) \times D_\rho)$ is a little different from that in [3].

Let $\rho > 0$, and let $\tau > 0$ be a sufficiently small fixed number. For $\varphi(x) = \sum_{\beta\in\mathbf{N}^n} a_\beta x^\beta$ we define the norm $\|\varphi\|_\rho$ by

$$\|\varphi\|_\rho = \sum_{\beta\in\mathbf{N}^n} |a_\beta| \frac{\beta!}{|\beta|!} \tau^{\beta_1} \rho^{|\beta|}\,. \tag{2.3}$$

For a fixed number $a > 0$ we set

$$\Theta^{(k)} = \frac{ak!}{(k+1)^{m+2}} \quad \text{and} \quad \Theta^{(k)}_{R-\rho} = \frac{1}{(R-\rho)^k} \Theta^{(k)}$$

for $k = 0, 1, \ldots$.

DEFINITION 2.1. $X_{p,q,c,\gamma}(S_\theta(T) \times D_\rho)$ is the set of all functions $\varphi(t,x) \in \mathcal{O}(S_\theta(T) \times D_\rho)$ with the following bounds; There exists a positive constant Φ such that for all $s \in \mathbf{N}$

$$\left\| \left(t\frac{\partial}{\partial t}\right)^s \varphi(t,\cdot) \right\|_\rho \le \Phi \zeta^s |t|^q \exp(-c|t|^{-\gamma}) \Theta^{(s+p)}_{R-\rho} \quad \text{for } t \in S_\theta(T). \tag{2.4}$$

The norm of $\varphi(t,x)$ is defined by the infimum of Φ in (2.4) and is denoted by $\|\varphi\|_{p,q,c,\gamma}$.

We can define for a function $u(t,x) \in X_{p,q,c,\gamma}(S_\theta(T) \times D_\rho)$

$$\left(t\frac{\partial}{\partial t}\right)^{-1} u(t,x) := \int_0^t \tau^{-1} u(\tau,x)\,d\tau \quad \text{and} \quad \left(\frac{\partial}{\partial x_1}\right)^{-1} u(t,x) := \int_0^{x_1} u(t,\chi,x')\,d\chi$$

where $x' = (x_2, \ldots, x_n)$.

We fix a positive constant δ so that $0 < \delta < \min\{J^1_q;\ q \text{ with } l_q \le l^*\}$ and $\gamma^*/\delta \in \mathbf{N}$. We define p_k by $p_k = [\delta k/\gamma^*] + j^* k$ where $j^* = l^* - m^*$. If $\{q;\ l_q > l^*\} = \emptyset$ then $p_k = j^* k$ by $\gamma^* = \infty$ where $[a]$ denote the integral part of a. Set $|k(q)| = \sum_{j+|\alpha|\le m} \sum_{i=1}^{q_{j,\alpha}} k(j,\alpha,i)$.

REMARK 2.2. We remark that there exists $\min\{J_q^1;\ q \text{ with } l_q \le l^*\}$. Points $(l_q, \gamma(l_q - l^*))$ are on the segment $\{(x, y) \in \mathbf{R}^2;\ y = \gamma(x - l^*) \text{ and } 0 \le x \le l^*\}$. By $(l_q, \sigma_q) \in \mathbf{N} \times \mathbf{Z}$ and $\sigma_q > \gamma(l_q - l^*)$ for $0 \le l_q \le l^*$, the set $\{J_q^1 = \sigma_q - \gamma(l^* - l_q);\ q \text{ with } l_q \le l^*\}$ is lower bound and lower close. Hence there exists $\min\{J_q^1;\ q \text{ with } l_q \le l^*\}$.

Let us construct a formal solution $u(t, x) = \sum_{k \ge 1} u_k(t, x)$ of (2.2) with

$$\mathcal{L}^* u_1(t, x) = g(t, x)$$

$$\mathcal{L}^* u_k(t, x)$$

$$(2.5)\quad = \sum_{\substack{1 \le |q| \le k \\ l_q \le l^*}} t^{\sigma_q} a_q(t, x) \sum_{|k(q)|+1=k} \prod_{j+|\alpha| \le m} \prod_{i=1}^{q_{j,\alpha}} \left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u_{k(j,\alpha,i)}(t, x)$$

$$+ \sum_{\substack{1 \le |q| \le k \\ l_q > l^*}} t^{\sigma_q} a_q(t, x) \sum_{|k(q)| + \frac{\gamma^*}{\delta}(l_q - l^*) = k} \prod_{j+|\alpha| \le m} \prod_{i=1}^{q_{j,\alpha}} \left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u_{k(j,\alpha,i)}(t, x) .$$

Then we have the following theorem for $u_k(t, x)$ in the relation (2.5):

THEOREM 2.3. *Let $S = S_\theta(T)$. For the function $g(t, x) \in X_{p_1+m^*, \delta, c_0, \gamma}(S \times D_\rho)$ $(0 < \forall\rho < R)$ suppose that there exists a positive constant G such that*

$$(2.6)\qquad \left\| \left(t \frac{\partial}{\partial t}\right)^s g(t, \cdot) \right\|_\rho \le G \zeta^s |t|^\delta \exp(-c_0 |t|^{-\gamma}) \Theta^{(s+p_1+m^*)} \quad \text{for } t \in S$$

for all $s \in \mathbf{N}$. Then for $k \ge 1$ the functions $u_k(t, x)$ belong to $X_{p_k, \delta k, c_1, \gamma}(S \times D_\rho)$ and satisfy that there exist positive constants U_k such that

$$(2.7)\qquad \left\| \left(t \frac{\partial}{\partial t}\right)^s u_k(t, \cdot) \right\|_\rho \le U_k \zeta^s |t|^{\delta k} \exp(-c_1 |t|^{-\gamma}) \frac{1}{(j^* k)!} \Theta_{R-\rho}^{(s+p_k)} \quad \text{for } t \in S$$

for a sufficiently small $T > 0$, $0 < \forall\rho < R$ and $0 < c_1 < c_0$. Further a series $\sum_{k \ge 1} U_k t^k$ converges in a neighborhood of $t = 0$.

Let us give some lemmas on the functional class $X_{p,q,c,\gamma}$.

LEMMA 2.4. *Assume*

$$(2.8)\qquad \|u\|_\rho \le \Theta_{R-\rho}^{(k)} \quad \text{for } 0 < \rho < R .$$

(1) *Let $k > 0$. Then we have*

$$(2.9)\qquad \left\| \frac{\partial}{\partial x_1} u \right\|_\rho \le \frac{M_0 e}{\tau} \Theta_{R-\rho}^{(k+1)} \quad \text{for } 0 < \rho < R$$

and have

$$\left\| \frac{\partial}{\partial x_i} u \right\|_\rho \le M_0 e \Theta^{(k+1)}_{R-\rho} \quad \textit{for} \ \ 0 < \rho < R \tag{2.10}$$

for $i = 2, \dots, n$ *where* $M_0 = 2^{m+2}$.
(2) *Let* $k > 1$. *Then we have*

$$\left\| \left(\frac{\partial}{\partial x_1} \right)^{-1} u \right\|_\rho \le 2\tau \Theta^{(k-1)}_{R-\rho} \quad \textit{for} \ \ 0 < \rho < R\,. \tag{2.11}$$

PROOF. We use for any $k \ge 0$

$$\frac{(k+2)^{m+2}}{(k+1)^{m+2}} \le 2^{m+2}\,.$$

We can show Lemma 2.4 as in [1] (Chapter10, Lemma 10.4.1). We omit the details. Q.E.D.

LEMMA 2.5. (1) *For* $k = 1, 2, \dots,$ *the following inequality holds*:

$$\frac{\{j^*k\}!}{k^{j^*}\{j^*(k-1)\}!} \le j^{*j^*}\,.$$

(2) *There exists a positive constant* $M_1 > 1$ *such that*

$$\Theta^{(l)} \le \frac{M_1}{l+1} \Theta^{(l+1)}\,.$$

We omit a proof.

LEMMA 2.6. *Let* $0 \le l' \le l \le m$. *Then for any* $k \in \mathbf{N}$ *there exists a positive constant* $a > 0$ *such that*

$$\sum_{k_1+k_2=k} \frac{1}{k_1!} \Theta^{(k_1+l)} \frac{1}{k_2!} \Theta^{(k_2+l')} \le \frac{1}{k!} \Theta^{(k+l)}\,. \tag{2.12}$$

Lemma 2.6 is the case $t = 0$ in Lemma 2.1 in [3].

Form now we fix a number $a > 0$ so that the estimate (2.12) holds.

LEMMA 2.7. *Let* $0 \le l' \le l \le m$ *and* p, $p' > 0$. *Then the following inequality holds:*

$$\sum_{i=0}^{s} \frac{s!}{(s-i)!i!} \Theta^{(s-i+p+l)} \Theta^{(i+p'+l')} \le \frac{p!p'!}{(p+p')!} \Theta^{(s+p+p'+l)}\,.$$

Lemma 2.7 is the case $t = 0$ in Proposition 2.3 in [3].

PROPOSITION 2.8. *For* $0 \le l' \le l \le m$ *and* p, $p' > 0$ *let* $u(t,x) \in X_{p+l,q,c,\gamma}(S \times D_\rho)$ *and* $v(t,x) \in X_{p'+l',q',c',\gamma}(S \times D_\rho)$ *and we assume that there exist positive constants* U

and V *such that for* $t \in S$

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s u \right\|_\rho \le U\zeta^s |t|^q \exp(-c|t|^{-\gamma}) \frac{1}{p!} \Theta_{R-\rho}^{(s+p+l)}$$

and

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s v \right\|_\rho \le V\zeta^s |t|^{q'} \exp(-c'|t|^{-\gamma}) \frac{1}{p'!} \Theta_{R-\rho}^{(s+p'+l')} .$$

Then we have $(uv)(t,x) \in X_{p+p'+l,q+q',c+c',\gamma}(S \times D_\rho)$ *and for* $t \in S$

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s (uv) \right\|_\rho \le \frac{1}{(R-\rho)^{l'}} UV\zeta^s |t|^{q+q'} \exp(-(c+c')|t|^{-\gamma}) \frac{1}{(p+p')!} \Theta_{R-\rho}^{(s+p+p'+l)} .$$

PROOF. By

$$\left(t\frac{\partial}{\partial t} \right)^s (uv) = \sum_{i=0}^{s} \frac{s!}{(s-i)!i!} \left(t\frac{\partial}{\partial t} \right)^{s-i} u \left(t\frac{\partial}{\partial t} \right)^i v$$

and Lemma 2.7 we obtain the desired result. Q.E.D.

By Proposition 2.8 we have:

PROPOSITION 2.9. *Let an* I *be a finite subset of* $\mathbf{N}$ *and* $|I|$ *be the cardinal of* I. *For functions* $u_i(t,x) \in X_{p_i+l_i,q_i,c,\gamma}(S \times D_\rho)$ *for all* $i \in I$ *and* $0 \le l_i \le m$ *we assume that there exist positive constants* U_i *such that*

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s u_i \right\|_\rho \le U_i\zeta^s |t|^{q_i} \exp(-c|t|^{-\gamma}) \frac{1}{p_i!} \Theta_{R-\rho}^{(s+p_i+l_i)} .$$

Then we have

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s \left(\prod_{i\in I} u_i \right) \right\|_\rho \le \frac{1}{(R-\rho)^{l_I(|I|-1)}} \left(\prod_{i\in I} U_i \right) \zeta^s |t|^{\sum_{i\in I} q_i} \exp(-c|t|^{-\gamma})$$
$$\times \frac{1}{\sum_{i\in I} p_i!} \Theta_{R-\rho}^{(s+\sum_{i\in I} p_i + l_I)}$$

where $l_I = \max\{l_i;\ i \in I\}$.

PROPOSITION 2.10. *Let* $p \ge 0$ *and* $q > 0$. *For a function* $u(t,x) \in X_{p,q,c,\gamma}(S \times D_\rho)$ *we assume that there exists a positive constant* U *such that for* $t \in S$

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s u \right\|_\rho \le U\zeta^s |t|^q \exp(-c|t|^{-\gamma}) \frac{1}{p!} \Theta_{R-\rho}^{(s+p)} .$$

Then we have the following estimates:

There exists a positive constant C such that for $t \in S$

$$(1)\qquad \left\|\left(t\frac{\partial}{\partial t}\right)^s\left\{\left(t\frac{\partial}{\partial t}\right)^{-1}u\right\}\right\|_\rho \le \frac{1}{q}U\zeta^s|t|^q\exp(-c|t|^{-\gamma})\frac{1}{p!}\Theta_{R-\rho}^{(s+p)},$$

$$(2)\qquad \left\|\left(t\frac{\partial}{\partial t}\right)^s\left\{t^{-\gamma}\left(t\frac{\partial}{\partial t}\right)^{-1}u\right\}\right\|_\rho \le \frac{C}{c\gamma}U\zeta^s|t|^q\exp(-c|t|^{-\gamma})\frac{1}{p!}\Theta_{R-\rho}^{(s+p)},$$

$$(3)\qquad \left\|\left(t\frac{\partial}{\partial t}\right)^s(t^{-\gamma}u)\right\|_\rho \le \frac{C}{c\gamma}U\zeta^{s+1}|t|^q\exp(-c|t|^{-\gamma})\frac{1}{p!}\Theta_{R-\rho}^{(s+p+1)}.$$

PROOF. We can show this proposition by the same way as the proof of Proposition (2.10) in [3]. Then we give a proof for only (1). We have
$(t\partial/\partial t)^s(t\partial/\partial)^{-1}u = (t\partial/\partial)^{-1}(t\partial/\partial t)^s u$ and

$$\int_0^{|t|}\tau^q\exp(-c\tau^{-\gamma})\frac{d\tau}{\tau} \le \frac{1}{q}|t|^q\exp(-c|t|^{-\gamma}).$$

Hence we can obtain (1). Q.E.D.

To prove Theorem 2.3 we consider the following equation:

$$(2.13)\qquad \mathcal{L}^{**}w(t,x) = W(t,x) \in X_{p+m^*,\delta k,c,\gamma}(S\times D_R)$$

where

$$(2.14)\qquad \mathcal{L}^{**} := \mathcal{L}^*\left(t\frac{\partial}{\partial t}\right)^{-j^*}\left(\frac{\partial}{\partial x_1}\right)^{-m^*}.$$

Let $A_{j,\alpha} = \|a_{j,\alpha}\|_{0,0,0,\gamma}$.

PROPOSITION 2.11. *Let $p \ge 0$ and $k \ge 1$. For the equation* (2.13) *we assume that there exists a positive constant $\mathcal{W}$ such that*

$$(2.15)\qquad \left\|\left(t\frac{\partial}{\partial t}\right)^s W\right\|_\rho \le \mathcal{W}\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{(j^*(k-1))!}\Theta_{R-\rho}^{(s+p+m^*)}$$

for $0 < \rho < R$ and $t \in S$. Then we get the solution $w(t,x) \in X_{p+m^*,\delta k,c,\gamma}(S\times D_\rho)$ of (2.13) *that satisfies*

$$(2.16)\qquad \left\|\left(t\frac{\partial}{\partial t}\right)^s w\right\|_\rho \le \frac{1}{1-C(\zeta,\tau)}\mathcal{W}\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{(j^*(k-1))!}\Theta_{R-\rho}^{(s+p+m^*)}$$

for $t \in S$ *and* $0 < \rho < R$ *where*

$$C(\zeta,\tau) := \sum_{\substack{j+|\alpha|=l^* \\ |\alpha|=m^*, \alpha_1<m^*}} A_{j,\alpha}(M_0 e)^{|\alpha'|}(2\tau)^{m^*-\alpha_1} + \sum_{\substack{j+|\alpha|\le l^* \\ |\alpha|<m^*}} A_{j,\alpha}\left(\frac{C}{c\gamma}\right)^{l^*-j-|\alpha|}\zeta^{m^*-|\alpha|}(M_0 e)^{|\alpha'|}(2\tau)^{m^*-\alpha_1}$$

and $|\alpha'| = \alpha_2 + \cdots + \alpha_n$.

PROOF. We construct a formal solution $w(t,x) = \sum_{i=0}^{\infty} w_i(t,x)$ of (2.13) with

$$w_0(t,x) = W(t,x)$$

$$w_i(t,x) = -\sum_{\substack{j+|\alpha|=l^* \\ |\alpha|=m^*, \alpha_1<m^*}} a_{j,\alpha}(t,x)\left(\frac{\partial}{\partial x}\right)^{\alpha-m^*e_1} w_{i-1}(t,x) - \sum_{\substack{j+|\alpha|\le l^* \\ |\alpha|<m^*}} t^{-\gamma(l^*-j-|\alpha|)}a_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^{j-(l^*-m^*)}\left(\frac{\partial}{\partial x}\right)^{\alpha-m^*e_1} w_{i-1}(t,x)$$

for $i \ge 1$. Then for $i \ge 0$ we get

$$\left\| \left(t\frac{\partial}{\partial t}\right)^s w_i \right\|_\rho \le \{C(\zeta,\tau)\}^i \mathcal{W}\zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma}) \frac{1}{(j^*(k-1))!}\Theta_{R-\rho}^{(s+p+m^*)} \tag{2.17}$$

for $t \in S$ and $0 < \rho < R$.

Let us show the estimate (2.17). It is trivial that the estimate (2.17) holds for $i = 0$ by $w_0(t,x) = W(t,x)$.

For $i \ge 1$ we show the estimate (2.17) on induction. We assume that the estimate (2.17) holds for $i' = 0, 1, \ldots, i-1$.

For $\alpha_1 < m^*$ by Lemma 2.4 we have

$$\begin{aligned} &\left\| \left(t\frac{\partial}{\partial t}\right)^s \left\{ \left(\frac{\partial}{\partial x}\right)^{\alpha-m^*e_1} w_{i-1} \right\} \right\|_\rho \\ &\quad \le \{C(\zeta,\tau)\}^{i-1}(2\tau)^{m^*-\alpha_1}(M_0 e)^{|\alpha'|}\mathcal{W}\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma}) \\ &\quad\quad \times \frac{1}{(j^*(k-1))!}\Theta_{R-\rho}^{(s+p+|\alpha|)}. \end{aligned} \tag{2.18}$$

Therefore by (2.18) and Proposition 2.8 we get

$$\begin{aligned}&\left\|\left(t\frac{\partial}{\partial t}\right)^s\left\{\sum_{\substack{j+|\alpha|=l^*\\|\alpha|=m^*,\alpha_1<m^*}} a_{j,\alpha}\left(\frac{\partial}{\partial x}\right)^{\alpha-m^*e_1} w_{i-1}\right\}\right\|_\rho\\ \text{(2.19)}\qquad &\le \sum_{\substack{j+|\alpha|=l^*\\|\alpha|=m^*,\alpha_1<m^*}} A_{j,\alpha}\{C(\zeta,\tau)\}^{i-1}(2\tau)^{m^*-\alpha_1}(M_0e)^{|\alpha'|}\mathcal{W}\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\\ &\quad\times\frac{1}{(j^*(k-1))!}\Theta_{R-\rho}^{(s+p+m^*)}\end{aligned}$$

for $t \in S$ and $0 < \rho < R$.

For $|\alpha| < m^*$ and $j + |\alpha| \le l^*$, by the estimate (2.18) we get

$$\begin{aligned}&\left\|\left(t\frac{\partial}{\partial t}\right)^s\left\{\left(t\frac{\partial}{\partial t}\right)^{m^*-|\alpha|}\left(\frac{\partial}{\partial x}\right)^{\alpha-m^*e_1} w_{i-1}\right\}\right\|_\rho\\ &\quad\le \{C(\zeta,\tau)\}^{i-1}(2\tau)^{m^*-\alpha_1}(M_0e)^{|\alpha'|}\zeta^{m^*-|\alpha|}\mathcal{W}\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\\ &\qquad\times\frac{1}{(j^*(k-1))!}\Theta_{R-\rho}^{(s+p+m^*)}\end{aligned}$$

and by Proposition 2.8 and 2.10-(2) we get

$$\begin{aligned}&\left\|\left(t\frac{\partial}{\partial t}\right)^s\left\{\sum_{\substack{j+|\alpha|\le l^*\\|\alpha|<m^*}} t^{-\gamma(l^*-j-|\alpha|)} a_{j,\alpha}\left(t\frac{\partial}{\partial t}\right)^{j-(l^*-m^*)}\left(\frac{\partial}{\partial x}\right)^{\alpha-m^*e_1} w_{i-1}\right\}\right\|_\rho\\ \text{(2.20)}\qquad &\le \sum_{\substack{j+|\alpha|\le l^*\\|\alpha|<m^*}} A_{j,\alpha}\{C(\zeta,\tau)\}^{i-1}\left(\frac{C}{c\gamma}\right)^{l^*-j-|\alpha|}(2\tau)^{m^*-\alpha_1}(M_0e)^{|\alpha'|}\zeta^{m^*-|\alpha|}\mathcal{W}\zeta^s|t|^{\delta k}\\ &\quad\times\exp(-c|t|^{-\gamma})\frac{1}{(j^*(k-1))!}\Theta_{R-\rho}^{(s+p+m^*)}\end{aligned}$$

for $t \in S$ and $0 < \rho < R$. By the estimates (2.19) and (2.20) we obtain the estimate (2.17) for $i \ge 0$. By the definition of $C(\zeta,\tau)$ we have $C(\zeta,\tau) < 1$ for a sufficiently small $\tau > 0$. Hence the solution $w(t,x) = \sum_{i\ge 0} w_i(t,x)$ converges and holds the estimate (2.16).

Q.E.D.

Let us consider the following equation:

$$\text{(2.21)}\qquad \mathcal{L}^*u(t,x) = W(t,x)\,.$$

By Proposition 2.11 following proposition holds for (2.21);

PROPOSITION 2.12. *Let $p \geq 0$ and $k \geq 1$. For the function $W(t,x) \in X_{p+m^*,\delta k,c,\gamma}(S \times D_\rho)$ assume that there exists a positive constant $\mathcal{W}$ such that*

$$\left\| \left(t\frac{\partial}{\partial t}\right)^s W \right\|_\rho \leq \mathcal{W}\zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma}) \frac{1}{(j^*(k-1))!} \Theta^{(s+p+m^*)}_{R-\rho} \quad \textit{for } t \in S\,.$$

Then we get the solution $u(t,x) \in X_{p,\delta k,c,\gamma}(S \times D_\rho)$ of (2.21) *that satisfies*

$$\left\| \left(t\frac{\partial}{\partial t}\right)^s u \right\|_\rho \leq \left(\frac{j^*}{\delta}\right)^{j^*} (2\tau)^{m^*} \frac{1}{1 - C(\zeta,\tau)} \mathcal{W}\zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma})$$
$$\times \frac{1}{(j^*k)!} \Theta^{(s+p)}_{R-\rho} \quad \textit{for } t \in S.$$

PROOF. For the solution $w(t,x)$ of the equation (2.13), we have

$$u(t,x) = \left(t\frac{\partial}{\partial t}\right)^{-j^*} \left(\frac{\partial}{\partial x_1}\right)^{-m^*} w(t,x)\,. \tag{2.22}$$

By Proposition 2.11 and Lemma 2.4-(2), we have

$$\left\| \left(t\frac{\partial}{\partial t}\right)^s \left\{ \left(\frac{\partial}{\partial x_1}\right)^{-m^*} w \right\} \right\|_\rho \leq (2\tau)^{m^*} \frac{1}{1 - C(\zeta,\tau)} \mathcal{W}\zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma})$$
$$\times \frac{1}{(j^*(k-1))!} \Theta^{(s+p)}_{R-\rho} \quad \text{for } t \in S\,.$$

By Proposition 2.10-(1) and Lemma 2.5-(1) we obtain the desired result. Q.E.D.

2.2. Proof of Theorem 2.3. Let us give a proof of Theorem 2.3. We will show the estimate (2.7) by the same way as the proof of Proposition 3.6 in [3], and show that the series $\sum_{k\geq 0} U_k t^k$ converges in a neighborhood of $t = 0$ by majorant functions and Implicit's function theorem as in [1].

Set $A_q := \|a_q\|_{0,0,0,\gamma}$. Then we can assume that a series

$$\sum_{|q|\geq 1} \frac{A_q}{(R-\rho)^{m(|q|-1)}} Z^q$$

converges in a neighborhood of $Z = 0$.

Set

$$W_{1,k}(u_{k'};\ k'<k) := \sum_{\substack{1\le|q|\le k\\ l_q\le l^*}} t^{\sigma_q} a_q(t,x)$$

$$\times \sum_{|k(q)|+1=k} \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u_{k(j,\alpha,i)}(t,x)$$

(2.23) $$W_{2,k}(u_{k'};\ k'<k) := \sum_{\substack{1\le|q|\le k\\ l_q> l^*}} t^{\sigma_q} a_q(t,x)$$

$$\times \sum_{|k(q)|+\frac{\gamma^*}{\delta}(l_q-l^*)=k} \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u_{k(j,\alpha,i)}(t,x)$$

$$W_k(u_{k'};\ k'<k) := W_{1,k}(u_{k'};\ k'<k) + W_{2,k}(u_{k'};\ k'<k)\,.$$

We show that the estimate (2.7) holds for $k\ge 1$. We give the assumption on the function $g(t,x)$ again:

$$\left\|\left(t\frac{\partial}{\partial t}\right)^s g\right\|_\rho \le G\zeta^s|t|^\delta \exp(-c_0|t|^{-\gamma})\Theta_{R-\rho}^{(s+p_1+m^*)} \quad \text{for } t\in S=S_\theta(T)\,.$$

Let us show the estimate (2.7) on $k=1$. We solve an equation

$$\mathcal{L}^* u_1(t,x) = g(t,x)\,.$$

We get a solution $u_1(t,x)$ of the above equation by

$$\mathcal{L}^{**} w_1(t,x) = g(t,x) \quad \text{and} \quad u_1(t,x) = \left(t\frac{\partial}{\partial t}\right)^{-j^*}\left(\frac{\partial}{\partial x_1}\right)^{-m^*} w_1(t,x)\,.$$

By Proposition 2.11 we get

$$\left\|\left(t\frac{\partial}{\partial t}\right)^s w_1\right\|_\rho \le \frac{1}{1-C(\zeta,\tau)} G\zeta^s|t|^\delta \exp(-c|t|^{-\gamma})\frac{1}{j^*!}\Theta_{R-\rho}^{(s+p_1+m^*)} \quad \text{for } t\in S$$

and by Proposition 2.12

(2.24) $$\left\|\left(t\frac{\partial}{\partial t}\right)^s u_1\right\|_\rho \le \left(\frac{j^*}{\delta}\right)^{j^*}(2\tau)^{m^*}\frac{1}{1-C(\zeta,\tau)} G\zeta^s|t|^\delta \exp(-c|t|^{-\gamma})\frac{1}{j^*!}\Theta_{R-\rho}^{(s+p_1)} \quad \text{for } t\in S.$$

If we take a sufficiently small $\tau>0$ so that

$$\left(\frac{j^*}{\delta}\right)^{j^*}(2\tau)^{m^*}\frac{1}{1-C(\zeta,\tau)} \le 1,$$

then by the estimate (2.24) we get

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s u_1 \right\|_\rho \leq G\zeta^s |t|^\delta \exp(-c|t|^{-\gamma}) \frac{1}{j^*!} \Theta_{R-\rho}^{(s+p_1)} \quad \text{for } t \in S.$$

By setting $U_1 = G$, the estimate (2.7) holds for $k = 1$.

For $k \geq 2$ let us show the estimate (2.7) on induction. Let us assume that the estimate (2.7) holds for $k' = 1, 2, \ldots, k-1$. By Lemma 2.4-(1) for $k(j, \alpha, i) < k$ we get

$$\begin{aligned} &\left\| \left(t\frac{\partial}{\partial t} \right)^s \left\{ \left(t\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{k(j,\alpha,i)} \right\} \right\|_\rho \\ &\quad \leq \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k(j,\alpha,i)} \zeta^s |t|^{\delta k(j,\alpha,i)} \exp(-c|t|^{-\gamma}) \frac{1}{(j^* k(j,\alpha,i))!} \Theta_{R-\rho}^{(s+p_{k(j,\alpha,i)}+j+|\alpha|)} . \end{aligned}$$

Here we use an inequality $\prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} p_{k(j,\alpha,i)}!/(j^* k(j,\alpha,i))! \leq |p(q)|!/(j^*|k(q)|)!$ where $|p(q)| = \sum_{j+|\alpha|\leq m} \sum_{i=1}^{q_{j,\alpha}} p_{k(j,\alpha,i)}$. Then by Proposition 2.8 and 2.9 we obtain

$$\begin{aligned} &\left\| \left(t\frac{\partial}{\partial t} \right)^s \left\{ a_q \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \left(t\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{k(j,\alpha,i)} \right\} \right\|_\rho \\ &\quad \leq \frac{A_q}{(R-\rho)^{l_q(|q|-1)}} \left\{ \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k(j,\alpha,i)} \right\} \zeta^s |t|^{\delta|k(q)|} \exp(-c|t|^{-\gamma}) \\ &\quad \times \frac{1}{(j^*|k(q)|)!} \Theta_{R-\rho}^{(s+|p(q)|+l_q)} . \end{aligned} \tag{2.25}$$

Let us give an estimate for $W_{1,k}$. In the case $1 \leq |q| \leq k$ and $|k(q)| + 1 = k$, it follows from $\sigma_q = -\gamma(l^* - l_q) + J_q^1$ that by Proposition 2.10-(3) we get

$$\begin{aligned} \left\| \left(t\frac{\partial}{\partial t} \right)^s W_{1,k} \right\|_\rho &\leq \sum_{\substack{1\leq|q|\leq k \\ l_q \leq l^*}} \left(\frac{C}{c\gamma} \right)^{l^*-l_q} \zeta^{l^*-l_q} \frac{A_q}{(R-\rho)^{l_q(|q|-1)}} \\ &\quad \times \sum_{|k(q)|+1=k} \left\{ \prod_{j+|\alpha|\leq m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k(j,\alpha,i)} \right\} \\ &\quad \times \zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma}) \frac{1}{(j^*(k-1))!} \Theta_{R-\rho}^{(s+|p(q)|+l^*)} . \end{aligned}$$

Further we have

$$\begin{aligned} 0 &\leq p_k + m^* - (|p(q)| + l^*) \\ &= \left[\frac{\delta}{\gamma^*} (|k(q)| + 1) \right] - \sum_{j+|\alpha|\leq m} \sum_{i=1}^{q_{j,\alpha}} \left[\frac{\delta}{\gamma^*} k(j,\alpha,i) \right] := I_{k(q)} . \end{aligned} \tag{2.26}$$

For all $i = 1, \dots, q_{j,\alpha}$ and (j, α) with $j + |\alpha| \le m$ if $\delta k(j, \alpha, i)/\gamma^* = n(j, \alpha, i) - \varepsilon(j, \alpha, i)$ with $n(j, \alpha, i) \in \mathbf{N}$ and $0 \le \varepsilon(j, \alpha, i) < 1$ then we have the maximum of $I_{k(q)}$. Then we get $[\delta k(j, \alpha, i)/\gamma^*] = n(j, \alpha, i) - \eta(j, \alpha, i)$ with $\eta(j, \alpha, i) = 0$ or 1,

$$\sum_{j+|\alpha|\le m} \sum_{i=1}^{q_{j,\alpha}} \left[\frac{\delta}{\gamma^*} k(j, \alpha, i)\right] = \sum_{j+|\alpha|\le m} \sum_{i=1}^{q_{j,\alpha}} n(j, \alpha, i) - \sum_{j+|\alpha|\le m} \sum_{i=1}^{q_{j,\alpha}} \eta(j, \alpha, i) \tag{2.27}$$

and

$$\begin{aligned}\left[\frac{\delta}{\gamma^*}(|k(q)| + 1)\right] &= \left[\sum_{j+|\alpha|\le m} \sum_{i=1}^{q_{j,\alpha}} n(j, \alpha, i) - \sum_{j+|\alpha|\le m} \sum_{i=1}^{q_{j,\alpha}} \varepsilon(j, \alpha, i) + \frac{\delta}{\gamma^*}\right] \\ &\le \sum_{j+|\alpha|\le m} \sum_{i=1}^{q_{j,\alpha}} n(j, \alpha, i) + 1\end{aligned} \tag{2.28}$$

by $\gamma^*/\delta \in \mathbf{N}$. Then by the inequalities (2.26), (2.27) and (2.28)

$$p_k + m^* - (|p(q)| + l^*) \le \sum_{j+|\alpha|\le m} \sum_{i=1}^{q_{j,\alpha}} \eta(j, \alpha, i) + 1 \le |q| + 1 \le 2|q| \tag{2.29}$$

holds for $|q| \ge 1$. By the inequality (2.29) and Lemma 2.5-(2) we obtain

$$\begin{aligned}\left\| \left(t\frac{\partial}{\partial t}\right)^s W_{1,k} \right\|_\rho &\le \sum_{\substack{1\le|q|\le k \\ l_q \le l^*}} \left(\frac{C\zeta}{c\gamma}\right)^{l^*-l_q} \\ &\quad \times \frac{M_1^{2|q|} A_q}{(R-\rho)^{l_q(|q|-1)}} \sum_{|k(q)|+1=k} \left\{ \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k(j,\alpha,i)} \right\} \\ &\quad \times \zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma}) \frac{1}{(j^*(k-1))!} \Theta_{R-\rho}^{(s+p_k+m^*)}.\end{aligned} \tag{2.30}$$

Let us give an estimate for $W_{2,k}(t, x)$. In the case $1 \le |q| \le k$ and $|k(q)| + \frac{\gamma^*}{\delta}(l_q - l^*) = k$, we have $p_k + m^* - (|p(q)| + l_q) \ge j^*(k - |k(q)| - 1) \ge 0$. It follows from $\sigma_q = \gamma^*(l_q - l^*) + J_q^2$ that by the estimate (2.25) we have

$$\begin{aligned}\left\| \left(t\frac{\partial}{\partial t}\right)^s W_{2,k} \right\|_\rho &\le \sum_{\substack{1\le|q|\le k \\ l_q > l^*}} \frac{A_q}{(R-\rho)^{l_q(|q|-1)}} \\ &\quad \times \sum_{|k(q)|+\gamma^*(l_q-l^*)/\delta=k} \left\{ \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k(j,\alpha,i)} \right\} \\ &\quad \times \zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma}) \frac{(j^*(k-1))!}{(j^*|k(q)|)!} \frac{1}{(j^*(k-1))!} \Theta_{R-\rho}^{(s+|p(q)|+l_q)}.\end{aligned}$$

By Lemma 2.5-(2) we get

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s W_{2,k} \right\|_\rho \le \sum_{\substack{1\le |q| \le k \\ l_q > l^*}} \frac{A_q}{(R-\rho)^{l_q(|q|-1)}}$$
$$\times \sum_{|k(q)|+\gamma^*(l_q-l^*)/\delta = k} \left\{ \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k(j,\alpha,i)} \right\}$$
$$\times \zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma}) \frac{(j^*(k-1))!}{(j^*|k(q)|)!}$$
$$\times \frac{M_1^{p_k+m^*-(|p(q)|+l_q)}}{(p_k+m^*)\cdots(|p(q)|+l_q+1)} \frac{1}{(j^*(k-1))!} \Theta_{R-\rho}^{(s+p_k+m^*)}$$

for $t \in S$. By $(p_k+m^*)-(|p(q)|+l_q) \ge j^*(k-|k(q)|-1)$,

$$\frac{(j^*(k-1))!}{(j^*|k(q)|)!} \frac{1}{(p_k+m^*)\cdots(|p(q)|+l_q+1)} \le 1$$

holds. By the same way as in (2.29) we have $p_k+m^*-(|p(q)|+l_q) \le p_k+l^*-(|p(q)|+l_q) \le |q|+\frac{\gamma^*}{\delta} j^*(l_q-l^*)$ and $|q|+\frac{\gamma^*}{\delta} j^*(l_q-l^*) \le (1+\frac{\gamma^*}{\delta} j^*(m-l^*))|q| =: \kappa|q|$. Then we obtain

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s W_{2,k} \right\|_\rho \le \sum_{\substack{1\le |q| \le k \\ l_q > l^*}} \frac{M_1^{\kappa|q|} A_q}{(R-\rho)^{l_q(|q|-1)}}$$
$$\times \sum_{|k(q)|+\gamma^*(l_q-l^*)/\delta = k} \left\{ \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k(j,\alpha,i)} \right\} \tag{2.31}$$
$$\times \zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma}) \frac{1}{(j^*(k-1))!} \Theta_{R-\rho}^{(s+p_k+m^*)}$$

for $t \in S$. By the estimates (2.30) and (2.31) for $0 < \zeta < 1$ the following estimate holds:

$$\left\| \left(t\frac{\partial}{\partial t} \right)^s W_k \right\|_\rho \le \sum_{\substack{1\le |q| \le k \\ l_q \le l^*}} \left(\frac{C\zeta}{c\gamma} \right)^{l^*-l_q} \frac{M_1^{2|q|} A_q}{(R-\rho)^{l_q(|q|-1)}}$$
$$\times \sum_{|k(q)|+1 = k} \left\{ \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k(j,\alpha,i)} \right\}$$
$$\times \zeta^s |t|^{\delta k} \exp(-c|t|^{-\gamma}) \frac{1}{(j^*(k-1))!} \Theta_{R-\rho}^{(s+p_k+m^*)}$$

$$+\sum_{\substack{1\le|q|\le k\\ l_q>l^*}}\frac{M_1^{\kappa|q|}A_q}{(R-\rho)^{l_q(|q|-1)}}$$

$$\times\sum_{|k(q)|+\gamma^*(l_q-l^*)/\delta=k}\left\{\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k(j,\alpha,i)}\right\}$$

$$\times\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{(j^*(k-1))!}\Theta_{R-\rho}^{(s+p_k+m^*)}\,.$$

Set

$$U_k=M\sum_{\substack{1\le|q|\le k\\ l_q\le l^*}}\left(\frac{C\zeta}{c\gamma}\right)^{l^*-l_q}$$

$$\times\frac{M_1^{2|q|}A_q}{(R-\rho)^{l_q(|q|-1)}}\sum_{|k(q)|+1=k}\left\{\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k(j,\alpha,i)}\right\}$$

(2.32)

$$+M\sum_{\substack{1\le|q|\le k\\ l_q>l^*}}\frac{M_1^{\kappa|q|}A_q}{(R-\rho)^{l_q(|q|-1)}}$$

$$\times\sum_{|k(q)|+\gamma^*(l_q-l^*)/\delta=k}\left\{\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k(j,\alpha,i)}\right\}$$

where

$$M:=\left(\frac{j^*}{\delta}\right)^{j^*}(2\tau)^{m^*}\frac{1}{1-C(\zeta,\tau)}\quad\text{(in Proposition 2.12)}\,.$$

By Proposition 2.12, we get

$$\left\|\left(t\frac{\partial}{\partial t}\right)^s u_k\right\|_\rho\le U_k\xi^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{(j^*k)!}\Theta_{R-\rho}^{(s+p_k)}\quad\text{for }\ t\in S\,.$$

Hence the estimate (2.7) holds for $k\ge 1$.

Let us show that $\sum_{k\ge1}U_kt^k$ is a convergent power series in a neighborhood of the origin $t=0$. Coefficients U_k ($k\ge 1$) are given by $U_1=G$ and the relation (2.32) for $k\ge 2$. Let us consider the following equation:

$$
\begin{aligned}
Y = Gt + Mt \sum_{\substack{|q|\geq 1 \\ l_q \leq l^*}} \left(\frac{C\zeta}{c\gamma}\right)^{l^*-l_q} \frac{M_1^{2|q|}A_q}{(R-\rho)^{m(|q|-1)}} \prod_{j+|\alpha|\leq m} \left(\frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} Y\right)^{q_{j,\alpha}} \\
+ M \sum_{\substack{|q|\geq 1 \\ l_q > l^*}} t^{(l_q-l^*)\gamma^*/\delta} \frac{M_1^{\kappa|q|}A_q}{(R-\rho)^{m(|q|-1)}} \prod_{j+|\alpha|\leq m} \left(\frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} Y\right)^{q_{j,\alpha}} .
\end{aligned} \tag{2.33}
$$

We can show that the equation (2.33) has a holomorphic solution $Y(t) = \sum_{k\geq 1} Y_k t^k$ in a neighborhood of $t = 0$ with $U_k \leq Y_k$ for $k \geq 1$ by Implicit's function theorem at $(t, Y) = (0, 0)$. Hence $\sum_{k\geq 1} U_k t^k$ converges in a neighborhood of the origin $t = 0$. Q.E.D.

3. Proof of Theorem 1.5

In this section we prove Theorem 1.5 by Theorem 2.3.

Set $l^* = l_{\mathcal{L},i-1}$, $m^* = m_i$, $j^* = j_{\mathcal{L},i-1} = l_{\mathcal{L},i-1} - m_i$, $\gamma = \gamma_{\mathcal{L},i}$, $\gamma^* = \gamma_{\mathcal{L},i-1}$ and $c^*(t,x) = t^{\sigma_{\mathcal{L},i-1}} c_{j_{\mathcal{L},i-1}, m_i e_1}(t,x)$ where $c_{j_{\mathcal{L},i-1}, m_i e_1}(0,0) \neq 0$.

We consider an equation

$$L(u(t,x))/c^*(t,x) = f(t,x)/c^*(t,x) \tag{3.1}$$

for the equation (1.2).

REMARK 3.1. For $L(t^\nu u(t,x))$ with $t^\nu u(t,x) \in t^\nu Asy^0_{\{\gamma\}}(S_\theta(T) \times D_R)$, $NP(L;\nu) = NP(\mathcal{L};\nu)$ holds for a sufficiently large $\nu \in \mathbf{N}$ by Lemma 1.1. Hence we can assume $NP(L) = NP(\mathcal{L})$ for the equation (1.2).

We set

$$\mathcal{L}^* = \{c^*(t,x)\}^{-1}\mathcal{L}_i , \tag{3.2}$$

$$L^*(u(t,x)) = (L(u(t,x)) - \mathcal{L}_i u(t,x))/c^*(t,x) \tag{3.3}$$

and

$$g(t,x) = f(t,x)/c^*(t,x) =: t^\delta h(t,x) \tag{3.4}$$

where $L^*(u)$ is in (2.2) and $h(t,x) = f(t,x)/\{t^\delta c^*(t,x)\}$. By Remark 3.1 $L^*(u(t,x))$ is the following form:

$$L^*(u(t,x)) = \sum_{|q|\geq 1} t^{\sigma_q - \sigma_{\mathcal{L},i-1}} a_q(t,x) \prod_{j+|\alpha|\leq m} \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x)\right\}^{q_{j,\alpha}} \tag{3.5}$$

where $a_q(t,x) = c_q(t,x)/c_{j_{\mathcal{L},i-1},m_i e_1}(t,x)$, and numbers $\sigma_q - \sigma_{\mathcal{L},i-1} \in \mathbf{Z}$ satisfy

$$\sigma_q - \sigma_{\mathcal{L},i-1} = \begin{cases} -\gamma(l^* - l_q) + J_q^1 & (J_q^1 > 0) \quad \text{for } l_q \le l^* \\ \gamma^*(l_q - l^*) + J_q^2 & (J_q^2 \ge 0) \quad \text{for } l_q > l^* . \end{cases} \tag{3.6}$$

If $i = 1$ then we define $\gamma^* = \infty$. Further if we take $0 < c_0 < c$ then we have

$$h(t,x) \in X_{p_1+m^*,0,c_0,\gamma}(S \times D_R) . \tag{3.7}$$

Therefore it is sufficient to show Theorem 1.5 for the equation (3.1).

Let us show Theorem 1.5. By Theorem 2.3 we have the following estimate: There exist positive constants A and B such that for $k \ge 1$ and $t \in S_\theta(T)$

$$\begin{aligned} &\|u_k(t,\cdot)\|_\rho \le AB^k |t|^{\delta k} \exp(-c|t|^{-\gamma}) \quad \text{if} \quad \{q;\ l_q > l^*\} = \emptyset \\ &\|u_k(t,\cdot)\|_\rho \le AB^k |t|^{\delta k} \exp(-c|t|^{-\gamma}) \Gamma\left(\frac{\delta k}{\gamma^*} + 1\right) \quad \text{if} \quad \{q;\ l_q > l^*\} \ne \emptyset . \end{aligned} \tag{3.8}$$

By the estimate (3.8), if $\{q;\ l_q > l^*\} = \emptyset$ then the formal solution $\sum_{k\ge1} u_k(t,x)$ becomes a genuine solution of the equation (3.1). We get Theorem 1.5-(2).

From now we will show Theorem 1.5-(1) in the case $\{q;\ l_q > l^*\} \ne \emptyset$. It is our purpose to show the following two propositions;

Let $S = S_\theta(T)$ and $S_0 = S_{\theta_0}(T_0)$ with $0 < \theta_0 < \pi/(2\gamma^*)$ and $S_0 \Subset S$.

PROPOSITION 3.2. *Let $u_k(t,x)$ be constructed in the relation* 2.5 *for $k \ge 1$. Then there exists a function $u_{S_0}(t,x) \in Asy^0_{\{\gamma*\}}(S_0 \times D_{R_0})$ such that*

$$\|u_{S_0} - \sum_{k=0}^{N} u_k\|_\rho \le AB^{N+1}\Gamma\left(\frac{(N+1)\delta}{\gamma*} + 1\right)|t|^{(N+1)\delta} \exp(-c|t|^{-\gamma}) \quad \text{for } t \in S_0 .$$

For the function $u_{S_0}(t,x)$ set

$$\begin{aligned} g_{S_0}(t,x) := \mathcal{L}^* u_{S_0}(t,x) - \sum_{|q|\ge1} t^{\sigma_q - \sigma_{\mathcal{L},i-1}} a_q(t,x) \prod_{j+|\alpha|\le m} \left\{ \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u_{S_0}(t,x) \right\}^{q_{j,\alpha}} \\ - g(t,x). \end{aligned}$$

PROPOSITION 3.3. *We have $g_{S_0}(t,x) \in Asy^0_{\{\gamma*\}}(S_0 \times D_{R_1})$ for $0 < R_1 < R_0$.*

We get Theorem 1.5-(1) by Proposition 3.3.

Let us give proofs of Proposition 3.2 and 3.3. We can show these proposition by the same way as in [3] for the norm (2.3).

We define the follows for the functions $u_k(t,x)$ in Theorem 2.3:

$$\begin{cases} \widehat{u}_k(t,x,\xi) = \dfrac{u_k(t,x)}{t^{\delta k+\gamma^*}} \dfrac{\xi^{\delta k/\gamma^*}}{\Gamma(\delta k/\gamma^*+1)} \\ \widetilde{u}_N(t,x,\xi) = \displaystyle\sum_{k=N+1}^{\infty} \widehat{u}_k(t,x,\xi) \\ \widetilde{u}(t,x,\xi) = \displaystyle\sum_{k=0}^{\infty} \widehat{u}_k(t,x,\xi)\,. \end{cases}$$

By Theorem 2.3 there exists a positive constant $\hat{\xi}_0$ such that $\widetilde{u}_N(t,x,\xi)$ and $\widetilde{u}(t,x,\xi)$ converge in $S \times D_\rho \times \{|\xi| \le \hat{\xi}_0\}$.

LEMMA 3.4. *There exist positive constants $\hat{\hat{\xi}}$ with $0 < \hat{\hat{\xi}} < \hat{\xi}_0$, A_i and B_i $(i = 0, 1)$ such that for $S \times \{|\xi| \le \hat{\hat{\xi}}\}$*

$$\|\widetilde{u}_N(t,\cdot,\xi)\|_\rho \le A_0 {B_0}^{N+1} |t|^{-\gamma^*} \exp(-c|t|^{-\gamma}) |\xi|^{(N+1)\delta/\gamma^*} \tag{3.9}$$

and for $S \times \{|\xi| > \hat{\hat{\xi}}\}$

$$\sum_{k=0}^{N} \|\widehat{u}_k(t,\cdot,\xi)\|_\rho \le A_1 {B_1}^{N+1} |t|^{-\gamma^*} \exp(-c|t|^{-\gamma}) |\xi|^{(N+1)\delta/\gamma^*}. \tag{3.10}$$

We can show Lemma 3.4 as in Lemma 4.1 in [3]. We omit the details.

PROOF OF PROPOSITION 3.2. Set

$$u_{S_0}(t,x) = \int_0^{\hat{\hat{\xi}}} \exp(-\xi t^{-\gamma *}) \widetilde{u}(t,x,\xi) d\xi\,.$$

Then we have

$$\begin{aligned} u_{S_0}(t,x) - \sum_{k=0}^{N} u_k(t,x) &= \int_0^{\hat{\hat{\xi}}} \exp(-\xi t^{-\gamma *}) \widetilde{u}_N(t,x,\xi) d\xi \\ &\quad - \int_{\hat{\hat{\xi}}}^{\infty} \exp(-\xi t^{-\gamma *}) \sum_{k=0}^{N} \widehat{u}_k(t,x,\xi) d\xi \\ &= I_{1,N} + I_{2,N}\,. \end{aligned}$$

By Lemma 3.4 for $t \in S_0$ the following estimates hold:

$$\|I_{1,N}\|_\rho \le A_0 {B_0}^{N+1} \exp(-c|t|^{-\gamma}) |t|^{(N+1)\delta} \Gamma\left(\frac{(N+1)\delta}{\gamma *} + 1\right)$$

and

$$\|I_{2,N}\|_\rho \le A_1 {B_1}^{N+1} \exp(-c|t|^{-\gamma}) |t|^{(N+1)\delta} \Gamma\left(\frac{(N+1)\delta}{\gamma *} + 1\right).$$

Hence we obtain Proposition 3.2. Q.E.D.

Let us show Proposition 3.3. Set

$$v_N(t,x) = \sum_{k=1}^{N} u_k(t,x), \quad w_N(t,x) = u_{S_0}(t,x) - v_N(t,x)$$

and

$$W(u) := \sum_{|q|\geq 1} t^{\sigma_q} c_q(t,x) \prod_{j+|\alpha|\leq m} \left\{ \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u \right\}^{q_{j,\alpha}}$$

for the equation (3.1). Then we have

$$g_{S_0}(t,x) = \mathcal{L}^*(v_N + w_N) - W(v_N + w_N) - g(t,x)\,, \tag{3.11}$$

$\mathcal{L}^* v_N = \sum_{k=1}^{N} \mathcal{L}^* u_k(t,x)$ and

$$\begin{cases} \mathcal{L}^* u_1(t,x) = g(t,x) \\ \mathcal{L}^* u_k(t,x) = W_k(u_{k'} :\ k' < k) \quad \text{for } k \geq 2\,. \end{cases} \tag{3.12}$$

By the relation (3.12) we have

$$\mathcal{L}^* v_N - W(v_N) - g(t,x) = \sum_{k=2}^{N} W_k(u_{k'} :\ k' < k) - W(v_N) \tag{3.13}$$

and by relations (3.11) and (3.13)

$$\begin{aligned} g_{S_0}(t,x) &= \left\{ \sum_{k=2}^{N} W_k(u_{k'} :\ k' < k) - W(v_N) \right\} \\ &\quad + \left\{ \mathcal{L}^* w_N(t,x) - W(v_N + w_N) + W(v_N) \right\} \\ &= J_{1,N} + J_{2,N}\,. \end{aligned}$$

Set

$$v_{N,k}(t,x) := \begin{cases} u_k(t,x) & \text{for } 1 \leq k \leq N \\ 0 & \text{for } k \geq N+1 \end{cases}$$

Then we have $v_N(t,x) = \sum_{k=1}^{\infty} v_{N,k}(t,x)$ and

$$J_{1,N} = \sum_{k\geq N+1} \{W_{1,k}(v_{N,k'};\ k' < k) + W_{2,k}(v_{N,k'};\ k' < k)\}\,.$$

LEMMA 3.5. *For* $0 < \rho_0 < \rho$ *there exist positive constants* c_1 *and* c_2 *such that if* $c_1/(N+1) \leq |t|^{\gamma*} \leq c_1/N$ *then*

$$\|J_{1,N}\|_{\rho_0},\ \|J_{2,N}\|_{\rho_0} \leq A \exp(-c_2|t|^{-\gamma*})\,.$$

We can show Lemma 3.5 as in Lemma 4.4 in [3]. We omit the details.

PROOF OF PROPOSITION 3.3. By Lemma 3.5 we obtain

$$\|g_{S_0}\|_{\rho_0} \le A \exp(-c_2|t|^{-\gamma *})$$

for $c_1/(N+1) \le |t|^{\gamma *} \le c_1/N$ where c_2 and A are independent of N. Therefore we can show $g_{S_0}(t,x) \in Asy^0_{\{\gamma *\}}(S_0 \times D_{\rho_0})$ and we get Proposition 3.3. Q.E.D.

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