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# $S^1$-equivariant CMC-hypersurfaces in the Hyperbolic 3-space and the Corresponding Lagrangians

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**Abstract.** A family of $S^1$-equivariant hypersurfaces of constant mean curvature can be obtained by using the Lagrangians with suitable potentials in the hyperbolic 3-space. The conservation law is effectively applied to the construction of $S^1$-equivariant hypersurfaces of constant mean curvature in the hyperbolic 3-space.

## 1. Introduction

W-Y. Hsiang [5] investigated the rotation hypersurfaces of constant mean curvature in the spherical or hyperbolic $n$-space. In [2], Eells and Ratto have constructed rotation($S^1$-equivariant) minimal hypersurfaces in the unit 3-sphere, where they used a certain first integral which is invariant with respect to the horizontal rotation angle of generating curves on the orbit space. In [8], A certain family of $S^1$-equivariant CMC (constant mean curvature) hypersurfaces was constructed in the unit 3-sphere equipped with parametrized metric. In its construction, the Lagrangians with potentials appear and the corresponding Hamiltonians and the conservation laws are used effectively. In the construction of $S^1$-equivariant CMC hypersurfaces in the hyperbolic 3-space, it is cleared that the conserved quantity can be obtained by using the Lagrangian of the corresponding dynamical system with respect to the Hsiang-Lawson metric [2], [6] on the orbit space via the Hamilton equation [10] when we consider the horizontal angle of generating curves as “time”. We should remark that the corresponding Lagrangian has the vanishing potential when we construct the $S^1$-equivariant minimal hypersurfaces. However, in case that we construct $S^1$-equivariant non-minimal CMC hypersurfaces in the hyperbolic 3-space, the corresponding potentials are nonvanishing functions. We determine the potential function of the Lagrangian which corresponds to $S^1$-equivariant CMC-surfaces immersed in the hyperbolic 3-space (Theorem 4.3). As a result we can see that the corresponding potential function depends on the constant mean curvature $H$ itself.

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## 2. Preliminaries

We identify $\mathbf{R}^4$ with the space of quaternions $\mathbf{H} = span\{1, i, j, k\}$. The Minkowski inner product $\langle\ ,\ \rangle_M$ on $\mathbf{H}$ is defined by

$$\langle w_1, w_2\rangle_M := -a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 ,$$

where $w_m = a_m + b_m i + c_m j + d_m k,\ m = 1, 2$.

The hyperbolic space $H^3$ is defined by

$$H^3 = \{w \in \mathbf{H};\ \langle w, w\rangle_M = -1,\ \Re(w) > 0\} .$$

Then the orbit space $X$ by the $S^1$-action $r_t$ on $H^3$:

$$r_t(w) = a + bi + e^{it}(cj + dk) , \quad w = a + bi + cj + dk \in H^3 ,$$

is represented by

$$X = \{(\cosh\theta)e_{i\phi} + (\sinh\theta)j\ ;\ 0 \leq \theta < +\infty,\ -\infty < \phi < +\infty\} ,$$

where $e_{i\phi} := \cosh\phi + \sinh\phi\ i$.

Let $X \setminus \partial X$ denote by $X^\circ$. The orbital metric $h$ on $X$ is given by $h = h_1 d\theta^2 + h_2 d\phi^2$, where $h_1=1$, $h_2 = \cosh^2\theta$. Moreover, the volume function is $V=2\pi\sinh\theta$ and the Hsiang-Lawson metric $\hat{h} = \hat{h_1}d\theta^2 + \hat{h_2}d\phi^2$, where $\hat{h_1} = 4\pi^2\sinh^2\theta$, $\hat{h_2} = 4\pi^2\sinh^2\theta\cosh^2\theta$.

$\gamma : J \subset \mathbf{R} \to (X^\circ, h)$ denotes a curve parametrized by arclength $s$. $\tau(\gamma) = \nabla_{\dot\gamma}\dot\gamma$ and $\hat\tau(\gamma) = \hat\nabla_{\dot\gamma}\dot\gamma$ stand for the tension fields of $\gamma$ with respect to the metrics $h$ and $\hat h$, respectively.

The geodesic curvature $\kappa(\gamma)$ at $\gamma(s)$ is defined by $\kappa(\gamma) := h(\tau(\gamma), \eta)$ where $\eta$ denotes the unit normal vector field to $\gamma$.

## 3. $S^1$-equivariant CMC-immersion

For a curve $\gamma : J \to X^\circ$, we consider an $S^1$-equivariant map $\mu : M = \gamma^{-1}(H^3) \to H^3$ such that $\gamma \circ \pi = \sigma \circ \mu$, where $\pi : M \to J$ and $\sigma : H^3 \to X^\circ$ are Riemannian submersions. Throughout the paper, we assume that $\mu$ is an $S^1$-equivariant constant mean curvature $H$ immersion. Then we have

$$\kappa(\gamma) - \eta(\log V) = 2H , \tag{1}$$

since

$$h(\tau(\gamma), \eta) - \eta(\log V) = h(\hat\tau(\gamma), \eta) .$$

On the orbit space $(X^\circ, h)$, the velocity vector field of a curve $\gamma(s) = (\theta(s), \phi(s))$ is given by the following component functions:

$$\theta'(s) = \cos\lambda(s) , \quad \phi'(s) = \frac{\sin\lambda(s)}{\cosh\theta(s)} . \tag{2}$$

LEMMA 3.1. *The following formulas hold on* $(X^\circ, h)$

$$\eta(s) = -\sin\lambda(s)\frac{\partial}{\partial\theta} + \frac{\cos\lambda(s)}{\cosh\theta(s)}\frac{\partial}{\partial\phi}\,, \tag{3}$$

$$\tau(\gamma) = \tau(\gamma)_1\frac{\partial}{\partial\theta} + \tau(\gamma)_2\frac{\partial}{\partial\phi}\,, \tag{4}$$

*where*

$$\tau(\gamma)_1 = \theta''(s) - \sinh\theta(s)\cosh\theta(s)\phi'(s)^2$$

*and*

$$\tau(\gamma)_2 = \phi''(s) + \frac{2\sinh\theta(s)}{\cosh\theta(s)}\theta'(s)\phi'(s)\,.$$

Then, using the formula (1), we have the following differential equation (5) of generating curves on $X^\circ$ which corresponds to $S^1$-equivariant CMC-hypersurfaces immersed in $H^3$, since using Lemma 3.1 the geodesic curvature $\kappa(\gamma)$ is given by

$$\kappa(\gamma) = \lambda'(s) + \tanh\theta(s)\sin\lambda(s)\,,$$

$$\lambda'(s) + (\tanh\theta(s) + \coth\theta(s))\sin\lambda(s) - 2H = 0\,. \tag{5}$$

## 4. An application of conservation laws

We consider a generating curve $\gamma(s) = (\theta(s), \phi(s))$ on $X^\circ$ such that $\theta = \theta(\phi)$ and $\phi'(s) > 0$. Then we can consider the space $\Xi(\theta, \theta^{\#})$ of motion with $\theta^{\#} = d\theta\,/\,d\phi$ and time $\phi$. Let $\mathcal{L} = \mathcal{L}(\theta, \theta^{\#})$ be a Lagrangian on $\Xi(\theta, \theta^{\#})$. Via the Legendre transformation, we have the Hamiltonian $\mathcal{H}$ on the phase space $\Xi^*(\theta, p)$:

$$\mathcal{H} = \theta^{\#}p - \mathcal{L}\,, \quad p = \frac{\partial\mathcal{L}}{\partial\theta^{\#}}$$

The conservation laws of our system imply the following

PROPOSITION 4.1. *Let the Lagrangian* $\mathcal{L}$ *on* $\Xi(\theta, \theta^{\#})$ *be the following form*:

$$\mathcal{L} = \sqrt{\hat{h}_1(\theta^{\#})^2 + \hat{h}_2} + G(\theta)\,,$$

*where* $\hat{h}$ *is the Hsiang-Lawson metric on* $X^\circ$ *and* $G(\theta)$ *is a potential function on the configuration space. Then we have*

$$\frac{d}{d\phi}\left\{\frac{\hat{h}_2}{\sqrt{\hat{h}_1(\theta^{\#})^2 + \hat{h}_2}} + G(\theta)\right\} = 0\,, \tag{6}$$

*where the conserved quantity in* (6) *represents the Hamiltonian of our system.*

By means of the Hamilton equation (6), we shall determine the potential $G(\theta)$ which corresponds to $S^1$-equivariant CMC-hypersurfaces immersed in $H^3$ via the differential equation (5) of generating curves on the orbit space $X^\circ$.

The direct computation yields the following

LEMMA 4.2. *Assume that* $\theta$ *and* $\lambda$ *are functions of* $\phi$ *and* $d\lambda \,/\, d\phi = \lambda'(s) \,/\, \phi'(s)$. *Then we have*

$$\frac{d}{d\phi}\frac{\hat{h}_2}{\sqrt{\hat{h}_1(\theta^\#)^2+\hat{h}_2}} = \Psi\{\lambda'(s) + (\tanh\theta(s) + \coth\theta(s))\sin\lambda(s)\}\,, \tag{7}$$

*where*

$$\Psi = 2\pi\sinh\theta(s)\cosh^2\theta(s)\cot\lambda(s)\,.$$

By using the conservation law (6) and (7), we have the following

THEOREM 4.3. *On our system we have the following potential function* $G(\theta)$, *Lagrangian* $\mathcal{L}$ *and Hamiltonian* $\mathcal{H}$.

$$G(\theta) = -\pi H\cosh 2\theta\,,$$

$$\mathcal{L} = \frac{\pi\sinh 2\theta(s)}{\sin\lambda(s)} - \pi H\cosh 2\theta(s)\,,$$

*and*

$$\mathcal{H} = -2\pi\sinh\theta(s)\cosh\theta(s)\sin\lambda(s) + \pi H\cosh 2\theta(s)\,.$$

Let $\gamma(s) = (\theta(s), \phi(s))$ be a generating curve on $X^\circ$ such that $\theta = \theta(\phi)$ and $\phi'(s) > 0$. Then we set the initial conditions: $\theta_0 := \theta(0)$, $\phi(0) = 0$, $\theta'(0) = 0$ and $\lambda(0) = \frac{\pi}{2}$. Then we have the following

LEMMA 4.4.

$$\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} = 2\cosh^2\theta_0(\coth 2\theta_0 - H)\,. \tag{8}$$

PROOF. The conservation law implies that

$$\frac{\hat{h}_2}{\sqrt{\hat{h}_1(\theta^\#)^2+\hat{h}_2}} + G(\theta) = C\,,$$

where

$$C = 2\pi\sinh\theta_0\cosh\theta_0 - \pi H\cosh 2\theta_0\,,$$

then

$$\left(\frac{d\theta}{d\phi}\right)^2 = \frac{\hat{h}_2}{\hat{h}_1}\left\{\frac{\hat{h}_2}{(C - G(\theta))^2} - 1\right\}. \tag{9}$$

Since $(C - G(\theta_0))^2 = \hat{h}_2(\theta_0)$ and

$$\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} = \frac{1}{2}\left(\frac{d}{d\theta}\right)_{s=0}\left(\frac{d\theta}{d\phi}\right)^2, \tag{10}$$

using (9) we have

$$2(C - G(\theta_0))^3\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0}$$

$$= \frac{\hat{h}_2(\theta_0)}{\hat{h}_1(\theta_0)}\left\{2\pi^2\left(\frac{\partial G}{\partial\theta}\right)_{s=0}\sinh^2 2\theta_0 + (C - G(\theta_0))\left(\frac{\partial\hat{h}_2}{\partial\theta}\right)_{s=0}\right\},$$

from which, a direct computation implies the formula (8).

Lemma 4.4 implies the following.

LEMMA 4.5. *Under the initial conditions above with respect to a generating curve* $\theta = \theta(\phi(s))$ *on* $X^\circ$, *assume that* $H > 1$. *Then*

$$\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} \geq 0 \quad (resp., \leq 0)$$

*if and only if*

$$\theta_0 \leq \theta_H \quad (resp., \geq \theta_H), \tag{11}$$

*where*

$$\theta_H := \frac{1}{4}\log\left(\frac{H+1}{H-1}\right).$$

Let $H > 1$ and we choose $\theta_0$ such that $\theta_H < \theta_0 < 3\theta_H$. Under the initial conditions above with respect to a generating curve $\theta = \theta(\phi(s))$, using Lemma 4.5 we have $\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} < 0$ and there exists the value $\phi_1 = \phi(s_1)$ of $\phi$ such that $\theta = \theta(\phi(s))$ decreases strictly until $\phi_1$, where the value of $d\theta\,/\,d\phi$ equals to zero at $\phi = \phi_1$, and $\theta = \theta(\phi(s))$ takes a local minimum at $\phi = \phi_1$.

In fact, suppose that this is not valid, then we may assume that there exists $a$ such that $0 \leq a < \theta_0 < +\infty$ and $\lim_{s\to+\infty}\theta(s) = a$, $\lim_{s\to+\infty}\theta'(s) = 0$, $\lim_{s\to+\infty}\lambda(s) = \frac{\pi}{2}$.

Then, by making use of (9), we have

$$\left(\frac{d\theta}{d\phi}\right)^2 = \frac{4\cosh^2\theta(s)\sinh(\theta(s)+\theta_0)\sinh(\theta(s)-\theta_0)\Phi\Omega}{\{\sinh 2\theta_0 + 2H\sinh(\theta(s)+\theta_0)\sinh(\theta(s)-\theta_0)\}^2}\,, \tag{12}$$

where

$$\Phi = \cosh(\theta(s)-\theta_0) + H\sinh(\theta(s)-\theta_0)\,,$$

$$\Omega = \cosh(\theta(s)+\theta_0) - H\sinh(\theta(s)+\theta_0)\,.$$

Furthermore, by using the differential equation (5) of generating curves, we obtain $a = \theta_H$. Hence we have $\theta_0 < a + arctanh(1/H)$, since $\theta_H < \theta_0 < 3\theta_H$ and $2\theta_H = arctanh(1/H)$, which implies $\cosh(a-\theta_0) + H\sinh(a-\theta_0) > 0$. Moreover, we can easily prove that $\cosh(a+\theta_0) - H\sinh(a+\theta_0)$ is not equal to zero. Consequently, since $0 \le a < \theta_0$, from the formula (12) we see that $\lim_{s\to+\infty}(d\theta\,/\,d\phi)^2$ is not zero, which contradicts the assumption $\lim_{s\to+\infty}\theta'(s) = 0$.

Consequently we can continue $\theta = \theta(\phi(s))$ as the generating curve satisfying the differential equation (5) by the reflection. Thus we have

THEOREM 4.6. *In case that $H > 1$ and $\theta_H < \theta_0 < 3\theta_H$, we obtain $S^1$-equivariant CMC-H hypersurfaces immersed in $H^3$, whose generating curves have the periodicity on the orbit space $X^\circ$. If $\theta_0 = \theta_H$, then we obtain $S^1$-equivariant CMC-H hypersurfaces embedded in $H^3$. The corresponding Lagrangian and Hamiltonian are given by the formulas in Theorem* 4.3.

OBSERVATION. Let $0 < H < 1$. Then we can choose the initial value $\theta_0 = \theta(0)$ such that $\tanh 2\theta_0 = H$. This initial condition implies that $C = 0$ and using the formula (9) we have

$$\left(\frac{d\theta}{d\phi}\right)^2 = \frac{1}{H^2}\cosh^2\theta(\tanh^2 2\theta - H^2)\,.$$

Now we consider the generating curve $\theta(s) = \theta(\phi(s))$ issuing from the point $(\theta_0, \phi(0))$ on $X^\circ$. Let $s_1 > 0$ be sufficiently small. Then, noting that $\left(\frac{d\theta}{d\phi}\right)_{s=0} = 0$ and $\left(\frac{d^2\theta}{d\phi^2}\right)_{s=0} > 0$, we have

$$\phi(s) = H\int_{\theta(s_1)}^{\theta(s)} \frac{1}{\cosh\theta\sqrt{\tanh^2 2\theta - H^2}}d\theta\,,$$

where $s \ge s_1 > 0$ and assume $\phi = \phi(s) \ge 0$.

This formula describes the behavior of the generating curves in case that $0 < H < 1$.

COROLLARY 4.7. *In case that $H > 1$ and $\theta_0 = \theta_H$, we have a CMC-H embedding*

$\mu : L \times S^1 \to H^3$,

$$\mu(\phi, t) = (\cosh\theta_H)e_{i\phi} + (\sinh\theta_H)e^{it} j$$

$$= \cosh\theta_H \cosh\phi + (\cosh\theta_H \sinh\phi)i + (\sinh\theta_H \cos t) j + (\sinh\theta_H \sin t)k \,,$$

*where* $\theta_H = \frac{1}{4}\log\left(\frac{H+1}{H-1}\right)$, $L = (-\infty, +\infty)$, $0 \le t < 2\pi$.

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