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Manin Triples and Differential Operators on Quantum Groups

To Jiro Sekiguchi on his 60th birthday

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Abstract. Let G be a simple algebraic group over $\mathbf{C}$. By taking the quasi-classical limit of the ring of differential operators on the corresponding quantized algebraic group at roots of 1 we obtain a Poisson manifold $\Delta G \times K$, where ΔG is the subgroup of $G \times G$ consisting of the diagonal elements, and K is a certain subgroup of $G \times G$. We show that this Poisson structure coincides with the one introduced by Semenov-Tyan-Shansky geometrically in the framework of Manin triples.

1. Introduction

In this paper we will explicitly compute the Poisson bracket of a certain Poisson manifold arising from the ring of differential operators on a quantized algebraic group at roots of 1. This result will be a foundation in the author's recent works regarding the Beilinson-Bernstein type localization theorem for representations of quantized enveloping algebras at roots of 1 (see [16], [17]).

Let G be a simple algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. Take Borel subgroups B^+ and B^- of G such that $H = B^+ \cap B^-$ is a maximal torus of G. Set $N^\pm = [B^\pm, B^\pm]$. We define a subgroup K of $G \times G$ by

$$K = \{(tx, t^{-1}y) \mid t \in H, x \in N^+, y \in N^-\} \subset B^+ \times B^- \subset G \times G\,.$$

Let $\zeta \in \mathbf{C}^\times$ be a primitive ℓ-th root of 1, where ℓ is an odd positive integer satisfying certain conditions depending on $\mathfrak{g}$, and let U_ζ be the De Concini-Kac type quantized enveloping algebra of $\mathfrak{g}$ at ζ. It is expected that there exists a certain correspondence between representations of U_ζ and modules over the ring $D_{\mathcal{B}_\zeta}$ of differential operators on the quantized flag manifold $\mathcal{B}_\zeta$. Since $D_{\mathcal{B}_\zeta}$ is closely related to the ring D_{G_ζ} of differential operators on the quantized algebraic group G_ζ, it is an important step in establishing the expected correspondence to investigate the ring D_{G_ζ} in detail. Note that D_{G_ζ} is nothing but the Heisenberg

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double $\mathbf{C}[G_\zeta]\otimes U_\zeta$ of the Hopf algebras $\mathbf{C}[G_\zeta]$ and U_ζ, where $\mathbf{C}[G_\zeta]$ is the coordinate algebra of G_ζ. We have natural central embeddings $\mathbf{C}[G]\subset\mathbf{C}[G_\zeta]$, $\mathbf{C}[K]\subset U_\zeta$ of Hopf algebras, and hence G and K become Poisson algebraic groups. By De Concini-Procesi [4] and De Concini-Lyubashenko [3] these Poisson algebraic group structures of G and K turn out to be the ones defined geometrically from the Manin triple $(G\times G, \Delta G, K)$, where ΔG is the subgroup of $G\times G$ consisting of diagonal elements. The aim of the present paper is to give a description of the Poisson algebra structure of $\mathbf{C}[G]\otimes\mathbf{C}[K]$ induced by the central embedding

$$\mathbf{C}[G]\otimes\mathbf{C}[K]\subset\mathbf{C}[G_\zeta]\otimes U_\zeta \tag{1.1}$$

of algebras.

Let $(\mathfrak{a},\mathfrak{m},\mathfrak{l})$ be a Manin triple over $\mathbf{C}$. Assume that we are given a connected algebraic group A with Lie algebra $\mathfrak{a}$ and connected closed subgroups M and L of A with Lie algebras $\mathfrak{m}$ and $\mathfrak{l}$ respectively. Then Semenov-Tyan-Shansky [13], [14] showed that A has a natural structure of Poisson manifold. Hence by considering the pull-back with respect to the local isomorphism $M\times L\to A$ $((m,l)\mapsto ml)$ the manifold $M\times L$ also turns out to be a Poisson manifold.

THEOREM 1.1. *The Poisson structure of $G\times K$ induced from the central embedding* (1.1) *coincides with the one defined geometrically from the Manin triple* $(G\times G,\Delta G,K)$.

As explained above, the coincidence of the two Poisson brackets

$$\mathbf{C}[G\times K]\times\mathbf{C}[G\times K]\to\mathbf{C}[G\times K]$$

is already known for the parts $\mathbf{C}[G]\times\mathbf{C}[G]\to\mathbf{C}[G]$ and $\mathbf{C}[K]\times\mathbf{C}[K]\to\mathbf{C}[K]$ by [4], [3]. Hence we will be only concerned with the mixed part of the Poisson bracket between $\mathbf{C}[G]$ and $\mathbf{C}[K]$. We point out that a closely related result in the case of $\zeta=1$ for general Manin triples already appeared in [14].

In [14] it is noted that the Poisson manifold L associated to a Manin triple $(\mathfrak{a},\mathfrak{m},\mathfrak{l})$ can also be recovered as a Hamiltonian reduction with respect to the action of M on $M\times L$. In order to pass from D_{G_ζ} to $D_{\mathcal{B}_\zeta}$ we need to consider Hamiltonian reduction for more general situation. As a result we obtain the following.

PROPOSITION 1.2. *The varieties*

$$\overline{Y}=\{(N^-g,(k_1,k_2))\in(N^-\backslash G)\times K\mid gk_1k_2^{-1}g^{-1}\in HN^-\},$$
$$\overline{Y}_t=\{(B^-g,(k_1,k_2)\in(B^-\backslash G)\times K\mid gk_1k_2^{-1}g^{-1}\in tN^-\}\qquad(t\in H)$$

turn out to be Poisson manifolds with respect to the Poisson tensors induced from that of $G\times K$. Moreover, the Poisson tensors of $\overline{Y}$ and $\overline{Y}_t$ are non-degenerate. Hence they are symplectic manifolds.

In fact the Poisson manifold arising from the Poisson structure of the center of $D_{\mathcal{B}_\zeta}$ coincides with $\overline{Y}$ above (see [16]). The non-degeneracy of the Poisson tensor plays a crucial role in the argument of [16].

The contents of this paper is as follows. In Section 2 we recall the definition of the Poisson structure due to Semenov-Tyan-Shansky, and show that the technique of the Hamiltonian reduction works for certain cases. The case of the typical Manin triple $(\mathfrak{g} \oplus \mathfrak{g}, \Delta\mathfrak{g}, \mathfrak{k})$ is discussed in detail. In Section 3 we give a summary of some of the known results on quantized enveloping algebras at roots of 1 due to Lusztig [9], De Concini-Kac [2], De Concini-Lyubashenko [3], De Concini-Procesi [4], Gavarini [6]. In Section 4 we show that the Poisson structure arising from the algebra of differential operators acting on quantized coordinate algebra of G at roots of 1 coincides with the one coming from the typical Manin triple.

2. Poisson structures arising from Manin triples

2.1. Manin triples. We first recall standard facts on Poisson structures (see e.g., [5], [4]). A commutative associative algebra $\mathcal{R}$ over $\mathbf{C}$ equipped with a bilinear map $\{\ ,\ \} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is called a Poisson algebra if it satisfies

(a) $\{a, a\} = 0 \quad (a \in \mathcal{R})$,

(b) $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \quad (a, b, c \in \mathcal{R})$,

(c) $\{a, bc\} = b\{a, c\} + \{a, b\}c \quad (a, b, c \in \mathcal{R})$.

A map $F : \mathcal{R} \to \mathcal{R}'$ between Poisson algebras $\mathcal{R}$, $\mathcal{R}'$ is called a homomorphism of Poisson algebras if it is a homomorphism of associative algebras and satisfies $F(\{a_1, a_2\}) = \{F(a_1), F(a_2)\}$ for any $a_1, a_2 \in \mathcal{R}$. The tensor product $\mathcal{R} \otimes_{\mathbf{C}} \mathcal{R}'$ of two Poisson algebras $\mathcal{R}$, $\mathcal{R}'$ over $\mathbf{C}$ is equipped with a canonical Poisson algebra structure given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2\,,$$

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}$$

for $a_1, a_2 \in \mathcal{R}$, $b_2, b_2 \in \mathcal{R}'$. A commutative Hopf algebra $\mathcal{R}$ over a field $\mathbf{C}$ equipped with a bilinear map $\{\ ,\ \} : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is called a Poisson Hopf algebra if it is a Poisson algebra and the comultiplication $\mathcal{R} \to \mathcal{R} \otimes_{\mathbf{C}} \mathcal{R}$ is a homomorphism of Poisson algebras (in this case the counit $\mathcal{R} \to \mathbf{C}$ and the antipode $\mathcal{R} \to \mathcal{R}$ become automatically a homomorphism and an anti-homomorphism of Poisson algebras respectively).

For a smooth algebraic variety X over $\mathbf{C}$ let $\mathcal{O}_X$ (resp. Θ_X, Ω_X) be the sheaf of regular functions (resp. vector fields, 1-forms). We denote the tangent and the cotangent bundles of X by TX and T^*X respectively. A smooth affine algebraic variety X over $\mathbf{C}$ is called a Poisson variety if we are given a bilinear map $\{\ ,\ \} : \mathbf{C}[X] \times \mathbf{C}[X] \to \mathbf{C}[X]$ so that $\mathbf{C}[X]$ is a Poisson algebra. In this case $\{f, g\}(x)$ for $f, g \in \mathbf{C}[X]$ and $x \in X$ depends only on df_x, dg_x, and hence we have $\delta \in \Gamma(X, \bigwedge^2 \Theta_X)$ (called the Poisson tensor of the Poisson variety X) such that

$$\{f, g\}(x) = \delta_x(df_x, dg_x)\,.$$

Consequently we also have the notion of Poisson variety which is not necessarily affine.

Let S be a linear algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{s}$. For $a \in \mathfrak{s}$ we define vector fields $R_a, L_a \in \Gamma(S, \Theta_S)$ by

$$(R_a(f))(s) = \frac{d}{dt} f(\exp(-ta)s)|_{t=0} \quad (f \in \mathcal{O}_S,\ s \in S),$$
$$(L_a(f))(s) = \frac{d}{dt} f(s\exp(ta))|_{t=0} \quad (f \in \mathcal{O}_S,\ s \in S).$$

For $\xi \in \mathfrak{s}^*$ we also define 1-forms $L^*_\xi, R^*_\xi \in \Gamma(S, \Omega_S)$ by

$$\langle L^*_\xi, L_a\rangle = \langle R^*_\xi, R_a\rangle = \langle \xi, a\rangle \quad (a \in \mathfrak{s}).$$

For $s \in S$ we define $\ell_s : S \to S$ by $\ell_s(x) = sx$.

A linear algebraic group S over $\mathbf{C}$ is called a Poisson algebraic group if we are given a bilinear map $\{\ ,\ \} : \mathbf{C}[S] \times \mathbf{C}[S] \to \mathbf{C}[S]$ so that $\mathbf{C}[S]$ is a Poisson Hopf algebra. Let δ be the Poisson tensor of S as a Poisson variety, and define $\varepsilon : S \to \bigwedge^2 \mathfrak{s}$ by $(d\ell_s)(\varepsilon(s)) = \delta_s$ for $s \in S$. Here, we identify the tangent space $(TS)_1$ at the identity element $1 \in S$ with $\mathfrak{s}$ by $L_a \leftrightarrow a$ $(a \in \mathfrak{s})$. By differentiating ε at 1 we obtain a linear map $\mathfrak{s} \to \bigwedge^2 \mathfrak{s}$. It induces an alternating bilinear map $[\ ,\ ] : \mathfrak{s}^* \times \mathfrak{s}^* \to \mathfrak{s}^*$. Then this $[\ ,\ ]$ gives a Lie algebra structure of $\mathfrak{s}^*$. Moreover, the following bracket product gives a Lie algebra structure of $\mathfrak{s} \oplus \mathfrak{s}^*$:

$$[(a, \varphi), (b, \psi)] = ([a, b] + \varphi b - \psi a, a\psi - b\varphi + [\varphi, \psi]).$$

Here, $\mathfrak{s} \times \mathfrak{s}^* \ni (a, \varphi) \to a\varphi \in \mathfrak{s}^*$ and $\mathfrak{s}^* \times \mathfrak{s} \ni (\varphi, a) \to \varphi a \in \mathfrak{s}$ are the coadjoint actions of $\mathfrak{s}$ and $\mathfrak{s}^*$ on $\mathfrak{s}^*$ and $\mathfrak{s}$ respectively. In other words $(\mathfrak{s} \oplus \mathfrak{s}^*, \mathfrak{s}, \mathfrak{s}^*)$ is a Manin triple with respect to the symmetric bilinear form ρ on $\mathfrak{s} \oplus \mathfrak{s}^*$ given by $\rho((a, \varphi), (b, \psi)) = \varphi(b) + \psi(a)$. We say that $(\mathfrak{a}, \mathfrak{m}, \mathfrak{l})$ is a Manin triple with respect to a symmetric bilinear form ρ on $\mathfrak{a}$ if

(a) $\mathfrak{a}$ is a finite-dimensional Lie algebra,
(b) ρ is $\mathfrak{a}$-invariant and non-degenerate,
(c) $\mathfrak{m}$ and $\mathfrak{l}$ are subalgebras of $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{m} \oplus \mathfrak{l}$ as a vector space,
(d) $\rho(\mathfrak{m}, \mathfrak{m}) = \rho(\mathfrak{l}, \mathfrak{l}) = \{0\}$.

Conversely, for each Manin triple we can associate a Poisson algebraic group by reversing the above process as follows. Let $(\mathfrak{a}, \mathfrak{m}, \mathfrak{l})$ be a Manin triple with respect to a bilinear form ρ on $\mathfrak{a}$ and let M be a linear algebraic group with Lie algebra $\mathfrak{m}$. Denote by $\pi_{\mathfrak{m}} : \mathfrak{a} \to \mathfrak{m}$, $\pi_{\mathfrak{l}} : \mathfrak{a} \to \mathfrak{l}$ the projections with respect to the direct sum decomposition $\mathfrak{a} = \mathfrak{m} \oplus \mathfrak{l}$. We sometimes identify $\mathfrak{m}^*$ and $\mathfrak{l}^*$ with $\mathfrak{l}$ and $\mathfrak{m}$ respectively via the non-degenerate bilinear form $\rho|_{\mathfrak{m}\times\mathfrak{l}} : \mathfrak{m} \times \mathfrak{l} \to \mathbf{C}$. Hence we have also a natural identification

$$\mathfrak{a}^* = (\mathfrak{m} \oplus \mathfrak{l})^* \cong \mathfrak{m}^* \oplus \mathfrak{l}^* \cong \mathfrak{l} \oplus \mathfrak{m} = \mathfrak{a}. \tag{2.1}$$

For $m \in M$ we denote by $\mathrm{Ad}(m) : \mathfrak{a} \to \mathfrak{a}$ the adjoint action. Then we have the following (see e.g., [5], [4]).

PROPOSITION 2.1. *The algebraic group M is endowed with a structure of Poisson algebraic group whose Poisson tensor δ^M is given by*

$$\delta^M_m(L^*_\xi, L^*_\eta) = \rho(\pi_{\mathfrak{m}}(\mathrm{Ad}(m)(\xi)), \mathrm{Ad}(m)(\eta)) \qquad (\xi, \eta \in \mathfrak{l} = \mathfrak{m}^*)\,,$$
$$\delta^M_m(R^*_\xi, R^*_\eta) = -\rho(\pi_{\mathfrak{m}}(\mathrm{Ad}(m^{-1})(\xi)), \mathrm{Ad}(m^{-1})(\eta)) \quad (\xi, \eta \in \mathfrak{l} = \mathfrak{m}^*)$$

for $m \in M$.

2.2. Semenov-Tyan-Shansky Poisson structure. Let $(\mathfrak{a}, \mathfrak{m}, \mathfrak{l})$ be a Manin triple over **C** with respect to a bilinear form ρ on $\mathfrak{a}$. We assume that we are given a connected algebraic group A and its closed connected subgroups M and L with Lie algebras $\mathfrak{a}, \mathfrak{m}, \mathfrak{l}$ respectively. Define an alternating bilinear form ω on $\mathfrak{a}$ by

$$\omega(a + b, a' + b') = \rho(a, b') - \rho(b, a') \quad (a, a' \in \mathfrak{m}, b, b' \in \mathfrak{l})\,.$$

Denote the adjoint action of A on $\mathfrak{a}$ by $\mathrm{Ad} : A \to GL(\mathfrak{a})$.

PROPOSITION 2.2 (Semenov-Tyan-Shansky [13], [14]). *The smooth affine variety A is endowed with a structure of Poisson variety whose Poisson tensor $\tilde{\delta}$ is given by*

$$\tilde{\delta}_g(L^*_\xi, L^*_\eta) = \frac{1}{2}(\omega(\mathrm{Ad}(g)(\xi), \mathrm{Ad}(g)(\eta)) + \omega(\xi, \eta)) \quad (\xi, \eta \in \mathfrak{a}^*, g \in A)\,.$$

Here, we identify $\mathfrak{a}$ with $\mathfrak{a}^*$ via (2.1).

Note that we can rewrite $\tilde{\delta}$ in terms of ρ as

$$\begin{aligned}\tilde{\delta}_g(R^*_a, R^*_b) &= \rho(a, (-\pi_{\mathfrak{m}} + \mathrm{Ad}(g)\pi_{\mathfrak{l}}\,\mathrm{Ad}(g^{-1}))(b)) \\ &= \rho(a, (\pi_{\mathfrak{l}} - \mathrm{Ad}(g)\pi_{\mathfrak{m}}\,\mathrm{Ad}(g^{-1}))(b))\,, \\ \tilde{\delta}_g(L^*_a, L^*_b) &= \rho(a, (-\pi_{\mathfrak{m}} + \mathrm{Ad}(g^{-1})\pi_{\mathfrak{l}}\,\mathrm{Ad}(g))(b)) \\ &= \rho(a, (\pi_{\mathfrak{l}} - \mathrm{Ad}(g^{-1})\pi_{\mathfrak{m}}\,\mathrm{Ad}(g))(b)) \quad (g \in A, a, b \in \mathfrak{a})\,.\end{aligned}$$

Consider the map

$$\Phi : M \times L \to A \quad ((m, l) \mapsto ml)\,. \tag{2.2}$$

Since Φ is a local isomorphism, we obtain a Poisson structure of $M \times L$ whose Poisson tensor δ is the pull-back of $\tilde{\delta}$ with respect to Φ. Let us give a concrete description of δ. By Proposition 2.1 M is endowed with a structure of Poisson algebraic group. By the symmetry of the notion of a Manin triple L is also a Poisson algebraic group whose Poisson tensor δ^L is given by

$$\delta^L_l(L^*_\xi, L^*_\eta) = \rho(\pi_{\mathfrak{l}}(\mathrm{Ad}(l)(\xi)), \mathrm{Ad}(l)(\eta)) \qquad (l \in L, \xi, \eta \in \mathfrak{m} = \mathfrak{l}^*)\,,$$
$$\delta^L_l(R^*_\xi, R^*_\eta) = -\rho(\pi_{\mathfrak{l}}(\mathrm{Ad}(l^{-1})(\xi)), \mathrm{Ad}(l^{-1})(\eta)) \quad (l \in L, \xi, \eta \in \mathfrak{m} = \mathfrak{l}^*)\,.$$

By a standard computation we have the following.

PROPOSITION 2.3. *The Poisson tensor δ is given by*

$$\delta_{(m,l)} : ((T^*M)_m \oplus (T^*L)_l) \times ((T^*M)_m \oplus (T^*L)_l) \to \mathbf{C}$$

for $(m, l) \in M \times L$ *with*

$$\delta_{(m,l)}|_{(T^*M)_m \times (T^*M)_m} = \delta^M_m \,, \tag{2.3}$$

$$\delta_{(m,l)}|_{(T^*L)_l \times (T^*L)_l} = \delta^L_l \,, \tag{2.4}$$

$$\delta_{(m,l)}(L^*_a, R^*_\xi) = \rho(a, \xi) \qquad (a \in \mathfrak{l} = \mathfrak{m}^*, \xi \in \mathfrak{m} = \mathfrak{l}^*) \,. \tag{2.5}$$

As noted in [14] the Poisson tensors $\tilde{\delta}$ and δ are non-degenerate at generic points, and hence some open subsets of A and $M \times L$ turn out to be symplectic manifolds. We give below the condition on the point of A and $M \times L$ so that the Poisson tensor is non-degenerate.

LEMMA 2.4. (i) *Let* $g \in A$. *Then* $\tilde{\delta}_g$ *is non-degenerate if and only if*

$$\mathrm{Ad}(g)(\mathfrak{l}) \cap \mathfrak{m} = \mathrm{Ad}(g)(\mathfrak{m}) \cap \mathfrak{l} = \{0\} \,.$$

(ii) *Let* $(m, l) \in M \times L$. *Then we have*

$$\dim \mathrm{rad}\, \delta_{(m,l)} = \dim(\mathfrak{l} \cap \mathrm{Ad}(ml)(\mathfrak{m})) \,.$$

Especially, $\delta_{(m,l)}$ *is non-degenerate if and only if*

$$\mathrm{Ad}(m^{-1})(\mathfrak{l}) \cap \mathrm{Ad}(l)(\mathfrak{m}) = \{0\} \,.$$

PROOF. (i) Set $F = -\pi_{\mathfrak{m}} + \mathrm{Ad}(g)\pi_{\mathfrak{l}}\,\mathrm{Ad}(g^{-1}) : \mathfrak{a} \to \mathfrak{a}$ for simplicity. By definition $\tilde{\delta}_g$ is non-degenerate if and only if F is an isomorphism.

Assume that F is an isomorphism. Since F is surjective, we must have $\mathfrak{a} = \mathfrak{m} + \mathrm{Ad}(g)(\mathfrak{l})$ by the definition of F. By $\dim \mathfrak{a} = \dim \mathfrak{m} + \dim \mathfrak{l}$ we have $\mathfrak{a} = \mathfrak{m} \oplus \mathrm{Ad}(g)(\mathfrak{l})$ and $\mathfrak{m} \cap \mathrm{Ad}(g)(\mathfrak{l}) = 0$. Then

$$\mathrm{Ker}\, F = \{a \in \mathfrak{a} \mid \pi_{\mathfrak{m}}(a) = \mathrm{Ad}(g)\pi_{\mathfrak{l}}\,\mathrm{Ad}(g^{-1})(a) = 0\} = \mathfrak{l} \cap \mathrm{Ad}(g)(\mathfrak{m}) \,.$$

Hence the injectivity of F implies $\mathfrak{l} \cap \mathrm{Ad}(g)(\mathfrak{m}) = \{0\}$.

Assume $\mathrm{Ad}(g)(\mathfrak{l}) \cap \mathfrak{m} = \mathrm{Ad}(g)(\mathfrak{m}) \cap \mathfrak{l} = \{0\}$. By $\mathrm{Ad}(g)(\mathfrak{l}) \cap \mathfrak{m} = \{0\}$ we have $\mathfrak{a} = \mathfrak{m} \oplus \mathrm{Ad}(g)(\mathfrak{l})$. Then $\mathrm{Ker}\, F = \mathfrak{l} \cap \mathrm{Ad}(g)(\mathfrak{m}) = \{0\}$. Hence F is an isomorphism.

(ii) For $g = ml$ we have

$$\mathrm{Ad}(g)(\mathfrak{l}) \cap \mathfrak{m} = \mathrm{Ad}(m)(\mathrm{Ad}(l)(\mathfrak{l}) \cap \mathrm{Ad}(m^{-1})(\mathfrak{m})) = \mathrm{Ad}(m)(\mathfrak{l} \cap \mathfrak{m}) = \{0\}.$$

Hence by the proof of (i) we obtain

$$\begin{aligned} \dim \mathrm{rad}\, \delta_{(m,k)} &= \dim \mathrm{Ker}(-\pi_{\mathfrak{m}} + \mathrm{Ad}(g)\pi_{\mathfrak{l}}\,\mathrm{Ad}(g^{-1})) \\ &= \dim(\mathfrak{l} \cap \mathrm{Ad}(g)(\mathfrak{m})) \,. \end{aligned}$$

□

COROLLARY 2.5. (i) *The Poisson structure of* A *induces a symplectic structure of the open subset*

$$\tilde{U} = \{g \in A \mid \mathrm{Ad}(g)(\mathfrak{l}) \cap \mathfrak{m} = \mathrm{Ad}(g)(\mathfrak{m}) \cap \mathfrak{l} = \{0\}\}$$

of A

(ii) *The Poisson structure of $M \times L$ induces a symplectic structure of the open subset*

$$U := \{(m, l) \in M \times L \mid \mathrm{Ad}(m^{-1})(\mathfrak{l}) \cap \mathrm{Ad}(l)(\mathfrak{m}) = \{0\}\}$$

of $M \times L$.

2.3. A variant of Hamiltonian reduction. Let X be a Poisson variety with Poisson tensor δ and let S be a connected linear algebraic group acting on the algebraic variety X (we do not assume that S preserves the Poisson structure of X). Assume also that we are given an S-stable smooth subvariety Y of X on which S acts locally freely. Denote by $\mathfrak{s}$ the Lie algebra of S.

For $y \in Y$ the linear map

$$\mathfrak{s} \ni a \mapsto \partial_a \in (TY)_y\,, \quad (\partial_a f)(y) = \frac{d}{dt} f(\exp(-ta)y)|_{t=0}$$

is injective by the assumption. Hence we may regard $\mathfrak{s} \subset (TY)_y$ for $y \in Y$. This gives an embedding

$$Y \times \mathfrak{s} \subset TY \subset (TX|_Y)$$

of vector bundles on Y. Correspondingly, we have

$$T^*_Y X \subset (Y \times \mathfrak{s})^{\perp} \subset (T^*X|_Y)$$

where

$$(Y \times \mathfrak{s})^{\perp} = \{v \in (T^*X|_Y) \mid \langle v, Y \times \mathfrak{s}\rangle = 0\}\,,$$

and $T^*_Y X$ denotes the conormal bundle.

By restricting $\delta \in \Gamma(\wedge^2(TX))$ to Y we obtain $\delta|_Y \in \Gamma(\wedge^2(TX|_Y))$. For $y \in Y$ restricting the anti-symmetric bilinear form $(\delta|_Y)_y$ on $(T^*X)_y$ to $((Y \times \mathfrak{s})^{\perp})_y$ we obtain an anti-symmetric bilinear form $\hat{\delta}_y$ on $((Y \times \mathfrak{s})^{\perp})_y$. Then we have $\hat{\delta} \in \Gamma(\wedge^2((TX|Y)/(Y \times \mathfrak{s})))$. Denote the action of $g \in S$ by $r_g : X \to X$. Then for $y \in Y$ the isomorphism $(dr_g)_y : (TX)_y \to (TX)_{gy}$ induces

$$(dr_g)_y : (TY)_y \to (TY)_{gy}\,, \quad (dr_g)_y : \mathfrak{s} \ni a \mapsto \mathrm{Ad}(g)(a) \in \mathfrak{s}\,,$$

where $\mathfrak{s}$ is identified with subspaces of $(TY)_y$ and $(TY)_{gy}$. In particular, S naturally acts on $\Gamma(\wedge^2((TX|Y)/(Y \times \mathfrak{s})))$.

PROPOSITION 2.6. *Assume that $\hat{\delta}$ is S-invariant and $(T^*_Y X)_y \subset \mathrm{rad}(\hat{\delta}_y)$ for any $y \in Y$. Then the quotient space $S\backslash Y$ admits a natural structure of Poisson variety as follows. Let φ, ψ be functions on $S\backslash Y$, and let $\tilde{\varphi}, \tilde{\psi}$ be the corresponding S-invariant functions on Y. Take extensions $\hat{\varphi}, \hat{\psi}$ of $\tilde{\varphi}, \tilde{\psi}$ to X (not necessarily S-invariant). Then $\{\hat{\varphi}, \hat{\psi}\}|_Y$ is S-invariant and* *does not depend on the choice of* $\hat{\varphi}, \hat{\psi}$. *We define* $\{\varphi, \psi\}$ *to be the function corresponding to* $\{\hat{\varphi}, \hat{\psi}\}|_Y$.

Moreover, if we have $(T^*_Y X)_y = \mathrm{rad}(\hat{\delta}_y)$ *for any* $y \in Y$, *then the Poisson tensor of* $S\backslash Y$ *is non-degenerate. Hence* $S\backslash Y$ *turns out to be a symplectic variety.*

PROOF. For $F \in \mathcal{O}_X$, $\partial \in \Theta_X$, $y \in Y$ we have $\langle (dF)_y, \partial \rangle = (\partial(F))(y)$, and hence $F|_Y$ is S-invariant (resp. $F|_Y$ is a locally constant function) if and only if $dF|_Y \in (Y \times \mathfrak{s})^{\perp}$ (resp. $dF|_Y \in T^*_Y X$).

Take φ, ψ and $\tilde{\varphi}, \tilde{\psi}, \hat{\varphi}, \hat{\psi}$ as above. We first show that $\{\hat{\varphi}, \hat{\psi}\}|_Y$ does not depend on the choice of $\hat{\varphi}, \hat{\psi}$. For that it is sufficient to show that $\{\hat{\varphi}, \hat{\psi}\}|_Y = 0$ if $\tilde{\psi} = 0$. By $d\hat{\varphi}|_Y \in (Y \times \mathfrak{s})^{\perp}$, $d\hat{\psi}|_Y \in T^*_Y X$ we have

$$\{\hat{\varphi}, \hat{\psi}\}(y) = \delta_y((d\hat{\varphi})_y, (d\hat{\psi})_y) = \hat{\delta}_y((d\hat{\varphi})_y, (d\hat{\psi})_y) = 0$$

by the assumption.

Let us show that $\{\hat{\varphi}, \hat{\psi}\}|_Y$ is S-invariant. For $g \in S$, $y \in Y$ we have

$$\begin{aligned}\{\hat{\varphi}, \hat{\psi}\}(gy) &= \hat{\delta}_{gy}((d\hat{\varphi})_{gy}, (d\hat{\psi})_{gy}) = \hat{\delta}_y(d(\hat{\varphi} \circ r_g)_y, d(\hat{\psi} \circ r_g)_y) \\ &= \{\hat{\varphi} \circ r_g, \hat{\psi} \circ r_g\}(y)\end{aligned}$$

by the S-invariance of $\hat{\delta}$. Since $\tilde{\varphi}, \tilde{\psi}$ are S-invariant, we have $\hat{\varphi} \circ r_g|_Y = \tilde{\varphi}$ and $\hat{\psi} \circ r_g|_Y = \tilde{\psi}$. Hence the independence of $\{\hat{\varphi}, \hat{\psi}\}|_Y$ on the choice of $\hat{\varphi}, \hat{\psi}$ implies

$$\{\hat{\varphi} \circ r_g, \hat{\psi} \circ r_g\}(y) = \{\hat{\varphi}, \hat{\psi}\}(y)$$

for $g \in S$ and $y \in Y$.

The remaining assertions are now clear. □

Now we apply the above general result to our Poisson varieties $M \times L$ and A.

Assume that we are given a connected closed subgroup F of M. Let $\mathfrak{f}$ be the Lie algebra of F and set $\mathfrak{f}^{\perp} = \{a \in \mathfrak{a} \mid \rho(\mathfrak{f}, a) = 0\}$. The action $F \times A \ni (x, g) \mapsto xg \in A$ of F on A induces an injection

$$\mathfrak{f} \ni a \mapsto R_a \in (TA)_g \quad (g \in A)\,.$$

Define a subbundle $(A \times \mathfrak{f})^{\perp}$ of T^*A by

$$((A \times \mathfrak{f})^{\perp})_g = \{R^*_c \mid c \in \mathfrak{f}^{\perp}\} \subset (T^*A)_g\,,$$

and set $\hat{\delta} = \tilde{\delta}|_{(A\times\mathfrak{f})^{\perp}\times(A\times\mathfrak{f})^{\perp}}$.

LEMMA 2.7. *If* $\mathfrak{f}^{\perp} \cap \mathfrak{l}$ *is a Lie subalgebra of* $\mathfrak{l}$, *then* $\hat{\delta}$ *is* F*-invariant.*

PROOF. By definition $\hat{\delta}_g$ for $g \in A$ is given by

$$\hat{\delta}_g(R^*_c, R^*_{c'}) = \rho(c, (-\pi_{\mathfrak{m}} + \mathrm{Ad}(g)\pi_{\mathfrak{l}}\,\mathrm{Ad}(g^{-1}))(c')) \quad (c, c' \in \mathfrak{f}^{\perp})\,.$$

On the other hand for $x \in F$, $g \in A$ the isomorphism $(T^*A)_g \cong (T^*A)_{xg}$ induced by the action of x is given by

$$(T^*A)_g \cong (T^*A)_{xg} \quad \bigl(R_b^* \mapsto R_{\mathrm{Ad}(x)(b)}^*\bigr)\,.$$

Hence it is sufficient to show

$$\rho(\mathrm{Ad}(x)(c), \pi_{\mathfrak{m}}\,\mathrm{Ad}(x)(c')) = \rho(c, \pi_{\mathfrak{m}}(c')) \quad (x \in F,\ c, c' \in \mathfrak{f}^{\perp})\,.$$

Since F is connected, this is equivalent to its infinitesimal counterpart

$$\rho([a,c], \pi_{\mathfrak{m}}(c')) + \rho(c, \pi_{\mathfrak{m}}([a,c'])) = 0 \quad (a \in \mathfrak{f},\ c, c' \in \mathfrak{f}^{\perp})\,.$$

Note that $\mathfrak{f}^{\perp} = \mathfrak{m} \oplus (\mathfrak{f}^{\perp} \cap \mathfrak{l})$. If $c \in \mathfrak{m}$, then we have $[a,c] \in \mathfrak{m}$ and hence $\rho([a,c], \pi_{\mathfrak{m}}(c')) = \rho(c, \pi_{\mathfrak{m}}([a,c'])) = 0$. If $c' \in \mathfrak{m}$, then

$$\rho([a,c], \pi_{\mathfrak{m}}(c')) + \rho(c, \pi_{\mathfrak{m}}([a,c'])) = \rho([a,c], c') + \rho(c, [a,c']) = 0$$

by the invariance of ρ. Hence we may assume that $c, c' \in \mathfrak{f}^{\perp} \cap \mathfrak{l}$. In this case we have

$$\begin{aligned}\rho([a,c], \pi_{\mathfrak{m}}(c')) + \rho(c, \pi_{\mathfrak{m}}([a,c'])) &= \rho(c, \pi_{\mathfrak{m}}([a,c'])) = \rho(c, [a,c']) \\ = -\rho([c',c], a) &\in \rho\bigl(\mathfrak{f}^{\perp} \cap \mathfrak{l}, \mathfrak{f}\bigr) = 0\,.\end{aligned}$$

□

By Proposition 2.6 and Lemma 2.7 we have the following.

PROPOSITION 2.8. *Assume that $\mathfrak{f}^{\perp} \cap \mathfrak{l}$ is a Lie subalgebra of $\mathfrak{l}$. Let V be an F-stable smooth subvariety of A such that the action of F on V is locally free. Assume also that for $g \in V$ we have*

$$\mathrm{rad}(\hat{\delta}_g) \supset (T_V^*A)_g\,.$$

Then $F\backslash V$ has a structure of Poisson variety whose Poisson bracket is defined as follows: *Let φ, ψ be functions on $F\backslash V$, and denote by $\tilde{\varphi}, \tilde{\psi}$ the corresponding F-stable functions on V. Take extensions $\hat{\varphi}, \hat{\psi}$ of $\tilde{\varphi}, \tilde{\psi}$ respectively to A. Then $\{\hat{\varphi}, \hat{\psi}\}|_V$ is F-stable and dose not depend on the choice of $\hat{\varphi}, \hat{\psi}$. We define $\{\varphi, \psi\}$ to be the function on $F\backslash V$ corresponding to $\{\hat{\varphi}, \hat{\psi}\}|_V$.*

If, moreover,

$$\mathrm{rad}(\hat{\delta}_g) = (T_V^*A)_g$$

holds for any $g \in V$, then the Poisson tensor of $F\backslash V$ is non-degenerate (*hence $F\backslash V$ turns out to be a symplectic variety*).

2.4. A special case. Let G be a connected simple algebraic group over $\mathbf{C}$, and let H be its maximal torus. We take Borel subgroups B^+, B^- of G such that $H = B^+ \cap B^-$, and set $N^{\pm} = [B^{\pm}, B^{\pm}]$. Denote the Lie algebras of G, H, $B^{\pm}$, $N^{\pm}$ by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}^{\pm}$, $\mathfrak{n}^{\pm}$. Define subalgebras $\Delta\mathfrak{g}$ and $\mathfrak{k}$ of $\mathfrak{g} \oplus \mathfrak{g}$ by

$$\Delta\mathfrak{g} = \{(a, a) \mid a \in \mathfrak{g}\},$$
$$\mathfrak{k} = \{(h + x, -h + y) \mid h \in \mathfrak{h}, x \in \mathfrak{n}^+, y \in \mathfrak{n}^-\},$$

and denote by ΔG, K the connected closed subgroups of $G \times G$ with Lie algebras $\Delta\mathfrak{g}$, $\mathfrak{k}$ respectively. In particular, $\Delta G = \{(g, g) \mid g \in G\}$. We fix an invariant non-degenerate symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}$, and define a bilinear form $\rho : (\mathfrak{g} \oplus \mathfrak{g}) \times (\mathfrak{g} \oplus \mathfrak{g}) \to \mathbf{C}$ by

$$\rho((a, b), (a', b')) = \kappa(a, a') - \kappa(b, b').$$

Then $(\mathfrak{g} \oplus \mathfrak{g}, \Delta\mathfrak{g}, \mathfrak{k})$ is a Manin triple with respect to the bilinear form ρ.

By Proposition 2.2 (resp. Proposition 2.3) we have a Poisson structure of $G \times G$ (resp. $\Delta G \times K$) with Poisson tensor $\tilde{\delta}$ (resp. δ). Moreover, the Poisson structure of $\Delta G \times K$ is the pull-back of that of $G \times G$ with respect to

$$\Phi : \Delta G \times K \to G \times G \quad (((g, g), (k_1, k_2)) \mapsto (gk_1, gk_2)).$$

LEMMA 2.9.

$$\operatorname{Im} \Phi = \{(g_1, g_2) \in G \times G \mid g_1^{-1} g_2 \in N^+ H N^-\}.$$

PROOF. We have

$$(gk_1)^{-1}(gk_2) = k_1^{-1} k_2 \in N^+ H N^-.$$

Assume $g_1^{-1} g_2 \in N^+ H N^-$. Then for $(k_1, k_2) \in K$ with $k_1^{-1} k_2 = g_1^{-1} g_2$ we have

$$(g_1, g_2) = (g_1 k_1^{-1}, g_2 k_2^{-1})(k_1, k_2) \in \operatorname{Im} \Phi.$$

□

PROPOSITION 2.10. *$\delta_{((g,g),(k_1,k_2))}$ is non-degenerate if and only if we have $gk_1 k_2^{-1} g^{-1} \in N^+ H N^-$.*

PROOF. Note that

$$\dim \operatorname{rad}(\delta_{((g,g),(k_1,k_2))}) = \dim(\mathfrak{k} \cap \operatorname{Ad}(gk_1, gk_2)(\Delta\mathfrak{g})) \tag{2.6}$$

by Lemma 2.4. In general for $(g_1, g_2) \in G \times G$ set $d(g_1, g_2) := \dim(\mathfrak{k} \cap \operatorname{Ad}(g_1, g_2)(\Delta\mathfrak{g}))$. For $(k_1, k_2) \in K$ and $(g, g) \in \Delta G$ we have

$$d((k_1, k_2)(g_1, g_2)(g, g)) = d(g_1, g_2),$$

and hence $d(g_1, g_2)$ is regarded as a function on $K \backslash (G \times G) / \Delta G$. Denote by $W = N_G(H)/H$ the Weyl group of G. A standard fact on simple algebraic groups tells us that for any $(g_1, g_2) \in$

$G \times G$ there exists some $w \in W$ and $t \in H$ such that $K(g_1, g_2)\Delta G \ni (t\dot{w}, 1)$, where $\dot{w}$ is a representative of w. By

$$d(t\dot{w}, 1) = \dim(\mathfrak{k} \cap \mathrm{Ad}(t\dot{w}, 1)(\Delta\mathfrak{g})) = \dim(\mathrm{Ad}((t\dot{w}, 1)^{-1})(\mathfrak{k}) \cap \Delta\mathfrak{g}),$$

$$\mathrm{Ad}((t\dot{w}, 1)^{-1})(\mathfrak{k}) = \{(w^{-1}h + \dot{w}^{-1}x, -h + y) \mid h \in \mathfrak{h}, x \in \mathfrak{n}^+, y \in \mathfrak{n}^-\}$$

we see easily that $d(t\dot{w}, 1) = 0$ if and only if $w = 1$. The assertion follows from this easily. □

COROLLARY 2.11. *The Poisson structure of $\Delta G \times K$ induces a symplectic structure of the open subset*

$$U := \{((g, g), (k_1, k_2)) \in \Delta G \times K \mid gk_1k_2^{-1}g^{-1} \in N^+HN^-\}.$$

Set

$$Y = \{((g, g), (k_1, k_2)) \in \Delta G \times K \mid gk_1k_2^{-1}g^{-1} \in B^-\} \subset U \subset \Delta G \times K,$$
$$\tilde{Y} = \Phi(Y) \subset G \times G.$$

Then we have

$$\tilde{Y} = \{(g_1, g_2) \in G \times G \mid g_1g_2^{-1} \in B^-,\ g_1^{-1}g_2 \in N^+HN^-\}. \tag{2.7}$$

Moreover, setting

$$\tilde{Z} = \{(g, b) \in G \times B^- \mid g^{-1}b^{-1}g \in N^+HN^-\}$$

we have

$$\tilde{Y} \cong \tilde{Z} \quad ((g_1, g_2) \leftrightarrow (g_1, g_1g_2^{-1}), \quad (g, b^{-1}g) \leftrightarrow (g, b)). \tag{2.8}$$

Since N^+HN^- is an open subset of G, $\tilde{Z}$ is open in $G \times B^-$. In particular, $\tilde{Z}$ is a smooth variety. Hence $\tilde{Y}$ is also smooth. Define an action of N^- on $G \times G$ by

$$x(g_1, g_2) = (xg_1, xg_2) \quad (x \in N^-, (g_1, g_2) \in G \times G).$$

Then $\tilde{Y}$ is N^--invariant. Moreover, (2.8) preserves the action of N^-, where the action of N^- on $\tilde{Z}$ is given by

$$x(g, b) = (xg, xbx^{-1}) \quad (x \in N^-, (g, b) \in \tilde{Z}).$$

For $C \subset G$ such that $C \ni c \mapsto N^-c \in N^-\backslash G$ is an open embedding we have

$$\begin{aligned}\{(g, b) \in \tilde{Z} \mid g \in N^-C\} &= \{(yc, yby^{-1}) \mid y \in N^-, c \in C, b \in B^-, c^{-1}b^{-1}c \in N^+HN^-\} \\ &\cong N^- \times \{(c, b) \in C \times B^- \mid c^{-1}b^{-1}c \in N^+HN^-\},\end{aligned}$$

and hence the action of N^- on $\tilde{Z}$ is locally free. Hence we have the following.

LEMMA 2.12. *$\tilde{Y}$ is a smooth variety, and the action of N^- on $\tilde{Y}$ is locally free.*

Set $\Delta\mathfrak{n}^- = \{(a, a) \mid a \in \mathfrak{n}^-\}$. We have obviously the following.

LEMMA 2.13. *We have*

$$(\Delta\mathfrak{n}^-)^\perp \cap \mathfrak{k} = \{(h, -h + y) \mid y \in \mathfrak{n}^-\}.$$

In particular, $(\Delta\mathfrak{n}^-)^\perp \cap \mathfrak{k}$ is a Lie subalgebra of $\mathfrak{k}$.

For $(g_1, g_2) \in \tilde{Y}$ we have

$$T(G \times G)_{(g_1,g_2)} = \{R_{(a_1,a_2)} \mid (a_1, a_2) \in \mathfrak{g} \oplus \mathfrak{g}\},$$
$$T^*(G \times G)_{(g_1,g_2)} = \{R^*_{(u_1,u_2)} \mid (u_1, u_2) \in \mathfrak{g} \oplus \mathfrak{g}\},$$
$$\langle R_{(a_1,a_2)}, R^*_{(u_1,u_2)}\rangle = \kappa(a_1, u_1) - \kappa(a_2, u_2).$$

By (2.8) we have also

$$(T\tilde{Y})_{(g_1,g_2)} = \{R_{(a,\mathrm{Ad}(g_2 g_1^{-1})(a))} \mid a \in \mathfrak{g}\} \oplus \{R_{(0,b)} \mid b \in \mathfrak{b}^-\}$$

for $(g_1, g_2) \in \tilde{Y}$. By Lemma 2.12 the natural map $\mathfrak{n}^- \to (T\tilde{Y})_{(g_1,g_2)}$ is injective and is given by

$$\mathfrak{n}^- \ni c \mapsto R_{(c,c)} \in (T\tilde{Y})_{(g_1,g_2)}.$$

Hence under the identification $\mathfrak{n}^- \subset (T\tilde{Y})_{(g_1,g_2)} \subset T(G \times G)_{(g_1,g_2)}$ we have

$$(\mathfrak{n}^-)^\perp = \{R^*_{(u_1,u_2)} \mid u_1 - u_2 \in \mathfrak{b}^-\} = \{R^*_{(u,u+v)} \mid u \in \mathfrak{g}, v \in \mathfrak{b}^-\},$$
$$((T\tilde{Y})_{(g_1,g_2)})^\perp = \{R^*_{(\mathrm{Ad}(g_2 g_1^{-1})(y),y)} \mid y \in \mathfrak{n}^-\}.$$

LEMMA 2.14. *For $(g_1, g_2) \in \tilde{Y}$ we have*

$$\mathrm{rad}(\tilde{\delta}_{(g_1,g_2)}|_{(\mathfrak{n}^-)^\perp \times (\mathfrak{n}^-)^\perp}) = ((T\tilde{Y})_{(g_1,g_2)})^\perp.$$

PROOF. For $u \in \mathfrak{g}$, $v \in \mathfrak{b}^-$ we have $R^*_{(u,u+v)} \in \mathrm{rad}(\tilde{\delta}_{(g_1,g_2)}|_{(\mathfrak{n}^-)^\perp \times (\mathfrak{n}^-)^\perp})$ if and only if $\tilde{\delta}_{(g_1,g_2)}(R^*_{(a,a+b)}, R^*_{(u,u+v)}) = 0$ for any $a \in \mathfrak{g}$, $b \in \mathfrak{b}^-$. Setting

$$(-\pi_{\Delta\mathfrak{g}} + \mathrm{Ad}(g_1, g_2)\pi_{\mathfrak{k}}\,\mathrm{Ad}(g_1^{-1}, g_2^{-1}))(u, u + v) = (x, y)$$

we have

$$\tilde{\delta}_{(g_1,g_2)}(R^*_{(a,a+b)}, R^*_{(u,u+v)}) = \kappa(a, x) - \kappa(a + b, y) = \kappa(a, x - y) - \kappa(b, y).$$

Hence $R^*_{(u,u+v)} \in \mathrm{rad}(\tilde{\delta}_{(g_1,g_2)}|_{(\mathfrak{n}^-)^\perp \times (\mathfrak{n}^-)^\perp})$ if and only if $x = y \in \mathfrak{n}^-$. By $(g_1, g_2) \in \Phi(\Delta G \times K)$ we have $\mathfrak{g} \oplus \mathfrak{g} = \Delta\mathfrak{g} \oplus \mathrm{Ad}(g_1, g_2)(\mathfrak{k})$. Therefore,

$$R^*_{(u,u+v)} \in \mathrm{rad}\big(\tilde{\delta}_{(g_1,g_2)}|_{(\mathfrak{n}^-)^\perp \times (\mathfrak{n}^-)^\perp}\big)$$

$$\iff \pi_{\Delta\mathfrak{g}}(u, u+v) = (y, y)\ (\exists y \in \mathfrak{n}^-)\,,\quad \pi_{\mathfrak{k}}\,\mathrm{Ad}(g_1^{-1}, g_2^{-1})(u, u+v) = 0$$

$$\iff u \in \mathfrak{n}^-\,,\quad \mathrm{Ad}(g_1^{-1}, g_2^{-1})(u, u+v) \in \Delta\mathfrak{g}$$

$$\iff u \in \mathfrak{n}^-\,,\quad v = \mathrm{Ad}(g_2 g_1^{-1})(u) - u\,.$$

It follows that

$$\mathrm{rad}\big(\tilde{\delta}_{(g_1,g_2)}|_{(\mathfrak{n}^-)^\perp\times(\mathfrak{n}^-)^\perp}\big) = \big\{R^*_{(u,\mathrm{Ad}(g_2g_1^{-1})(u))}\,\big|\,u \in \mathfrak{n}^-\big\} = \big((T\tilde{Y})_{(g_1,g_2)}\big)^\perp\,.$$

□

By Proposition 2.8 and the above argument we obtain the following.

PROPOSITION 2.15. *We have a natural Poisson structure of $N^-\backslash\tilde{Y}$ whose Poisson tensor is non-degenerate and defined as follows (hence $N^-\backslash\tilde{Y}$ turns out to be a symplectic variety) : Let φ, ψ be functions on $N^-\backslash\tilde{Y}$, and let $\tilde{\varphi}, \tilde{\psi}$ be the corresponding N^--invariant functions on $\tilde{Y}$. Take extensions $\hat{\varphi}, \hat{\psi}$ of $\tilde{\varphi}, \tilde{\psi}$ to $G\times G$. Then $\{\hat{\varphi}, \hat{\psi}\}|_{\tilde{Y}}$ is N^--invariant and does not depend on the choice of $\hat{\varphi}, \hat{\psi}$. We define $\{\varphi, \psi\}$ to be the function on $N^-\backslash\tilde{Y}$ corresponding to $\{\hat{\varphi}, \hat{\psi}\}|_{\tilde{Y}}$.*

By considering the pull-back to Y via Φ we also obtain the following.

PROPOSITION 2.16. *Consider the action of N^- on Y given by*

$$x((g, g), (k_1, k_2)) = ((xg, xg), (k_1, k_2)) \quad (x \in N^-, ((g, g), (k_1, k_2)) \in Y)\,.$$

Then we have a natural Poisson structure of $N^-\backslash Y$ whose Poisson tensor is non-degenerate and defined as follows (hence $N^-\backslash Y$ turns out to be a symplectic variety): Let φ, ψ be functions on $N^-\backslash Y$, and let $\tilde{\varphi}, \tilde{\psi}$ be the corresponding N^--invariant functions on Y. Take extensions $\hat{\varphi}, \hat{\psi}$ of $\tilde{\varphi}, \tilde{\psi}$ to $\Delta G\times K$. Then $\{\hat{\varphi}, \hat{\psi}\}|_Y$ is N^--invariant and does not depend on the choice of $\hat{\varphi}, \hat{\psi}$. We define $\{\varphi, \psi\}$ to be the function on $N^-\backslash Y$ corresponding to $\{\hat{\varphi}, \hat{\psi}\}|_Y$.

Note that

$$N^-\backslash Y \cong \{(N^-g, (k_1, k_2)) \in (N^-\backslash G)\times K \mid gk_1k_2^{-1}g^{-1} \in B^-\}. \tag{2.9}$$

Fix $t \in H$ and set

$$Y_t = \{((g, g), (k_1, k_2)) \in \Delta G\times K \mid gk_1k_2^{-1}g^{-1} \in tN^-\} \subset U \subset \Delta G\times K\,.$$

Then by a similar argument we have the following.

PROPOSITION 2.17. *Consider the action of B^- on Y_t given by*

$$x((g, g), (k_1, k_2)) = ((xg, xg), (k_1, k_2))\ (x \in B^-, ((g, g), (k_1, k_2)) \in Y_t)\,.$$

Then we have a natural Poisson structure of $B^-\backslash Y_t$ whose Poisson tensor is non-degenerate and defined as follows (hence $B^-\backslash Y_t$ turns out to be a symplectic variety) : Let φ, ψ be *functions on $B^-\backslash Y_t$, and let $\tilde{\varphi}, \tilde{\psi}$ be the corresponding B^--invariant functions on Y_t. Take extensions $\hat{\varphi}, \hat{\psi}$ of $\tilde{\varphi}, \tilde{\psi}$ to $\Delta G \times K$. Then $\{\hat{\varphi}, \hat{\psi}\}|_{Y_t}$ is B^--invariant and does not depend on the choice of $\hat{\varphi}, \hat{\psi}$. We define $\{\varphi, \psi\}$ to be the function on $B^-\backslash Y_t$ corresponding to $\{\hat{\varphi}, \hat{\psi}\}|_{Y_t}$.*

Note that we have

$$B^-\backslash Y_t \cong \{(B^-g, (k_1, k_2)) \in (B^-\backslash G) \times K \mid gk_1k_2^{-1}g^{-1} \in tN^-\}. \tag{2.10}$$

3. Quantized enveloping algebras

3.1. Lie algebras. In the rest of this paper we will use the notation of Section 2.4. In particular, $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbf{C}$, and G is a connected algebraic group with Lie algebra $\mathfrak{g}$. We further assume that G is simply-connected and the symmetric bilinear form

$$(\ ,\ ) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbf{C} \tag{3.1}$$

induced by κ satisfies $(\beta, \beta)/2 = 1$ for short roots β. We denote by $\Delta \subset \mathfrak{h}^*$, $Q \subset \mathfrak{h}^*$, $\Lambda \subset \mathfrak{h}^*$ and $W \subset GL(\mathfrak{h}^*)$ the set of roots, the root lattice $\sum_{\alpha\in\Delta} \mathbf{Z}\alpha$, the weight lattice and the Weyl group respectively. By our normalization of (3.1) we have

$$(\Lambda, Q) \subset \mathbf{Z}, \quad (\Lambda, \Lambda) \subset \frac{1}{|\Lambda/Q|}\mathbf{Z}.$$

For $\beta \in \Delta$ we set

$$\mathfrak{g}_\beta = \{x \in \mathfrak{g} \mid [h, x] = \beta(h)x \ (h \in \mathfrak{h})\}.$$

We choose a system of positive roots $\Delta^+ \subset \mathfrak{h}^*$ so that $\mathfrak{n}^\pm = \bigoplus_{\beta\in\Delta^+} \mathfrak{g}_{\pm\beta}$. Let $\{\alpha_i\}_{i\in I}$, $\{s_i\}_{i\in I} \subset W$ be the corresponding sets of simple roots and simple reflections respectively. Set

$$Q^+ = \sum_{\alpha\in\Delta^+} \mathbf{Z}_{\geqq 0}\alpha = \bigoplus_{i\in I} \mathbf{Z}_{\geqq 0}\alpha_i \subset \mathfrak{h}^*.$$

We denote the longest element of W by w_0. For each $i \in I$ we take $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$, $h_i \in \mathfrak{h}$ such that $[e_i, f_i] = h_i$ and $\alpha_i(h_i) = 2$.

Define subalgebras $\mathfrak{k}^0$, $\mathfrak{k}^+$, $\mathfrak{k}^-$ of $\mathfrak{k}$ by

$$\mathfrak{k}^0 = \{(h, -h) \mid h \in \mathfrak{h}\}, \quad \mathfrak{k}^+ = \{(x, 0) \mid x \in \mathfrak{n}^+\}, \quad \mathfrak{k}^- = \{(0, y) \mid y \in \mathfrak{n}^-\}.$$

Then we have $\mathfrak{k} = \mathfrak{k}^+ \oplus \mathfrak{k}^0 \oplus \mathfrak{k}^-$. For $i \in I$ set

$$x_i = (e_i, 0) \in \mathfrak{k}^+, \quad y_i = (0, f_i) \in \mathfrak{k}^-, \quad t_i = (h_i, -h_i) \in \mathfrak{k}^0.$$

We denote by K^0, $K^\pm$ the connected closed subgroups of K with Lie algebras $\mathfrak{k}^0$, $\mathfrak{k}^\pm$ respectively.

3.2. Quantized enveloping algebra of $\mathfrak{g}$. For $n \in \mathbf{Z}$ and $m \in \mathbf{Z}_{\geqq 0}$ we set

$$[n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbf{Z}[t, t^{-1}], \quad [m]_t! = [m]_t[m-1]_t \cdots [2]_t[1]_t \in \mathbf{Z}[t, t^{-1}],$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_t = [n]_t[n-1]_t \cdots [n-m+1]_t/[m]_t! \in \mathbf{Z}[t, t^{-1}].$$

The quantized enveloping algebra $U = U_q(\mathfrak{g})$ of $\mathfrak{g}$ is an associative algebra over $\mathbf{F} = \mathbf{C}(q^{1/|\Lambda/Q|})$ with identity element 1 generated by the elements K_λ $(\lambda \in \Lambda)$, E_i, F_i $(i \in I)$ satisfying the following defining relations:

$$K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda+\mu} \qquad (\lambda, \mu \in \Lambda), \tag{3.2}$$

$$K_\lambda E_i K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_i \qquad (\lambda \in \Lambda, i \in I), \tag{3.3}$$

$$K_\lambda F_i K_\lambda^{-1} = q^{-(\lambda, \alpha_i)} F_i \qquad (\lambda \in \Lambda, i \in I), \tag{3.4}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \qquad (i, j \in I), \tag{3.5}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n E_i^{(1-a_{ij}-n)} E_j E_i^{(n)} = 0 \qquad (i, j \in I, i \neq j), \tag{3.6}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n F_i^{(1-a_{ij}-n)} F_j F_i^{(n)} = 0 \qquad (i, j \in I, i \neq j), \tag{3.7}$$

where $q_i = q^{(\alpha_i, \alpha_i)/2}$, $K_i = K_{\alpha_i}$, $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ for $i, j \in I$, and

$$E_i^{(n)} = E_i^n/[n]_{q_i}!, \quad F_i^{(n)} = F_i^n/[n]_{q_i}!$$

for $i \in I$ and $n \in \mathbf{Z}_{\geqq 0}$. Algebra homomorphisms $\Delta : U \to U \otimes U, \varepsilon : U \to \mathbf{F}$ and an algebra anti-automorphism $S : U \to U$ are defined by:

$$\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \tag{3.8}$$

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\varepsilon(K_\lambda) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \tag{3.9}$$

$$S(K_\lambda) = K_\lambda^{-1}, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \tag{3.10}$$

and U is endowed with a Hopf algebra structure with the comultiplication Δ, the counit ε and the antipode S.

We define subalgebras $U^0, U^{\geqq 0}, U^{\leqq 0}, U^+, U^-$ of U by

$$U^0 = \langle K_\lambda \mid \lambda \in \Lambda \rangle, \tag{3.11}$$

$$U^{\geqq 0} = \langle K_\lambda, E_i \mid \lambda \in \Lambda, i \in I \rangle, \tag{3.12}$$

(3.13) $$U^{\leqq 0} = \langle K_\lambda, F_i \mid \lambda \in \Lambda, i \in I \rangle\,,$$

(3.14) $$U^{+} = \langle E_i \mid i \in I \rangle\,,$$

(3.15) $$U^{-} = \langle F_i \mid i \in I \rangle\,.$$

The following result is standard.

PROPOSITION 3.1. (i) *$\{K_\lambda \mid \lambda \in \Lambda\}$ is an $\mathbf{F}$-basis of U^0.*
(ii) *The linear maps*

$$U^- \otimes U^0 \otimes U^+ \to U \leftarrow U^+ \otimes U^0 \otimes U^-\,,$$

$$U^+ \otimes U^0 \to U^{\geqq 0} \leftarrow U^0 \otimes U^+\,, \quad U^- \otimes U^0 \to U^{\leqq 0} \leftarrow U^0 \otimes U^-$$

induced by the multiplication are all isomorphisms of vector spaces.

For $\gamma \in Q$ we set

$$U^{\pm}_{\gamma} = \{x \in U^{\pm} \mid K_\lambda x K_\lambda^{-1} = q^{(\lambda,\gamma)} x \ (\lambda \in \Lambda)\}\,.$$

We have $U^{\pm}_{\pm\gamma} = \{0\}$ unless $\gamma \in Q^+$, and

$$U^{\pm} = \bigoplus_{\gamma \in Q^+} U^{\pm}_{\pm\gamma}\,, \quad \dim U^{\pm}_{\pm\gamma} < \infty \quad (\gamma \in Q^+)\,.$$

For $i \in I$ we can define an algebra automorphism T_i of U by

$$T_i(K_\mu) = K_{s_i\mu} \quad (\mu \in \Lambda)\,,$$

$$T_i(E_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} E_i^{(-a_{ij}-k)} E_j E_i^{(k)} & (j \in I,\ j \neq i)\,, \\ -F_i K_i & (j = i)\,, \end{cases}$$

$$T_i(F_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{k} F_i^{(k)} F_j F_i^{(-a_{ij}-k)} & (j \in I,\ j \neq i)\,, \\ -K_i^{-1} E_i & (j = i)\,. \end{cases}$$

For $w \in W$ we define an algebra automorphism T_w of U by $T_w = T_{i_1} \cdots T_{i_n}$ where $w = s_{i_1} \cdots s_{i_n}$ is a reduced expression. The automorphism T_w does not depend on the choice of a reduced expression (see Lusztig [10]).

We fix a reduced expression

$$w_0 = s_{i_1} \cdots s_{i_N}$$

of w_0, and set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad (1 \leqq k \leqq N)\,.$$

Then we have $\Delta^+ = \{\beta_k \mid 1 \leqq k \leqq N\}$. For $1 \leqq k \leqq N$ set

(3.16) $$E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k})\,, \quad F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k})\,.$$

Then $\{E_{\beta_N}^{m_N} \cdots E_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$ (resp. $\{F_{\beta_N}^{m_N} \cdots F_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$) is an $\mathbf{F}$-basis of U^+ (resp. U^-), called the PBW-basis (see Lusztig [9]). For $1 \leqq k \leqq N$, $m \geqq 0$ we also set

$$E_{\beta_k}^{(m)} = E_{\beta_k}^m/[m]_{q_{\beta_k}}!, \quad F_{\beta_k}^{(m)} = F_{\beta_k}^m/[m]_{q_{\beta_k}}!, \tag{3.17}$$

where $q_\beta = q^{(\beta,\beta)/2}$ for $\beta \in \Delta^+$.

There exists a bilinear form

$$\tau : U^{\geqq 0} \times U^{\leqq 0} \to \mathbf{F}\,, \tag{3.18}$$

called the Drinfeld paring, which is characterized by

$$\tau(x, y_1 y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \qquad (x \in U^{\geqq 0},\ y_1, y_2 \in U^{\leqq 0})\,, \tag{3.19}$$

$$\tau(x_1 x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \qquad (x_1, x_2 \in U^{\geqq 0},\ y \in U^{\leqq 0})\,, \tag{3.20}$$

$$\tau(K_\lambda, K_\mu) = q^{-(\lambda,\mu)} \qquad (\lambda, \mu \in \Lambda)\,, \tag{3.21}$$

$$\tau(K_\lambda, F_i) = \tau(E_i, K_\lambda) = 0 \qquad (\lambda \in \Lambda,\ i \in I)\,, \tag{3.22}$$

$$\tau(E_i, F_j) = \delta_{ij}/(q_i^{-1} - q_i) \qquad (i, j \in I)\,. \tag{3.23}$$

PROPOSITION 3.2 ([7], [8], [11]). *We have*

$$\tau(E_{\beta_N}^{m_N} \cdots E_{\beta_1}^{m_1} K_\lambda, F_{\beta_N}^{n_N} \cdots F_{\beta_1}^{n_1} K_\mu)$$
$$= q^{-(\lambda,\mu)} \prod_{k=1}^{N} \delta_{m_k,n_k} (-1)^{m_k} [m_k]_{q_{\beta_k}}! q_{\beta_k}^{m_k(m_k-1)/2} (q_{\beta_k} - q_{\beta_k}^{-1})^{-m_k}\,.$$

3.3. Quantized coordinate algebra of G. We denote by C the subspace of $U^* = \mathrm{Hom}_{\mathbf{F}}(U, \mathbf{F})$ spanned by the matrix coefficients of finite dimensional U-modules E such that

$$E = \bigoplus_{\lambda \in \Lambda} E_\lambda \quad \text{with} \quad E_\lambda = \{v \in E \mid K_\mu v = q^{(\lambda,\mu)} v\ (\forall \mu \in \Lambda)\}\,.$$

Then C is endowed with a structure of Hopf algebra via

$$\langle \varphi\psi, u\rangle = \langle \varphi \otimes \psi, \Delta(u)\rangle \qquad (\varphi, \psi \in C,\ u \in U)\,,$$
$$\langle 1, u\rangle = \varepsilon(u) \qquad (u \in U)\,,$$
$$\langle \Delta(\varphi), u \otimes u'\rangle = \langle \varphi, uu'\rangle \qquad (\varphi \in C,\ u, u' \in U)\,,$$
$$\varepsilon(\varphi) = \langle \varphi, 1\rangle, \qquad (\varphi \in C)\,,$$
$$\langle S(\varphi), u\rangle = \langle \varphi, S(u)\rangle \qquad (\varphi \in C,\ u \in U),$$

where $\langle\ ,\ \rangle : C \times U \to \mathbf{F}$ is the canonical paring. C is also endowed with a structure of U-bimodule by

$$\langle u'\varphi u'', u\rangle = \langle \varphi, u''uu'\rangle \quad (\varphi \in C, u, u', u'' \in U)\,.$$

The Hopf algebra C is a q-analogue of the coordinate algebra $\mathbf{C}[G]$ of G (see [9], [15]).

Set

$$(U^{\pm})^{\bigstar} = \bigoplus_{\gamma \in Q^+} \mathrm{Hom}_{\mathbf{F}}(U^{\pm}_{\pm\gamma}, \mathbf{F}) \subset \mathrm{Hom}_{\mathbf{F}}(U, \mathbf{F})\,.$$

For $\lambda \in \Lambda$ define an algebra homomorphism $\chi_\lambda : U^0 \to \mathbf{F}$ by $\chi_\lambda(K_\mu) = q^{(\lambda,\mu)}$. Under the identification $U^- \otimes U^0 \otimes U^+ \cong U$ of vector spaces we have

$$C \subset (U^-)^{\bigstar} \otimes \left(\bigoplus_{\lambda \in \Lambda} \mathbf{F}\chi_\lambda \right) \otimes (U^+)^{\bigstar} \subset U^*\,. \tag{3.24}$$

3.4. Ring of differential operators. In general for a Hopf algebra $\mathcal{H}$ over $\mathbf{C}$ we use the following notation for the comultiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$:

$$\Delta(u) = \sum_{(u)} u_{(0)} \otimes u_{(1)} \quad (u \in \mathcal{H}).$$

We have an $\mathbf{F}$-algebra structure of $D = C \otimes_{\mathbf{F}} U$, called the Heisenberg double of C and U (see e.g. [12]). It is given by

$$(\varphi \otimes u)(\varphi' \otimes u') = \sum_{(u)} \varphi(u_{(0)}\varphi') \otimes u_{(1)}u' \quad (\varphi, \varphi' \in C, u, u' \in U)\,.$$

In our case the algebra D is an analogue of the ring of differential operators on G. We will identify U and C with subalgebras of D by the embeddings $U \ni u \mapsto 1 \otimes u \in D$ and $C \ni \varphi \mapsto \varphi \otimes 1 \in D$ respectively.

3.5. Quantized enveloping algebra of $\mathfrak{k}$. The quantized enveloping algebra $V = U_q(\mathfrak{k})$ of $\mathfrak{k}$ is an associative algebra over $\mathbf{F}$ with identity element 1 generated by the elements Z_λ ($\lambda \in \Lambda$), X_i, Y_i ($i \in I$) satisfying the following defining relations:

$$Z_0 = 1\,, \quad Z_\lambda Z_\mu = Z_{\lambda+\mu} \qquad (\lambda, \mu \in \Lambda)\,, \tag{3.25}$$

$$Z_\lambda X_i Z_\lambda^{-1} = q^{(\lambda,\alpha_i)} X_i \qquad (\lambda \in \Lambda, i \in I)\,, \tag{3.26}$$

$$Z_\lambda Y_i Z_\lambda^{-1} = q^{(\lambda,\alpha_i)} Y_i \qquad (\lambda \in \Lambda, i \in I)\,, \tag{3.27}$$

$$X_i Y_j - Y_j X_i = 0 \qquad (i, j \in I), \tag{3.28}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n X_i^{(1-a_{ij}-n)} X_j X_i^{(n)} = 0 \qquad (i, j \in I,\ i \neq j)\,, \tag{3.29}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n Y_i^{(1-a_{ij}-n)} Y_j Y_i^{(n)} = 0 \qquad (i, j \in I,\ i \neq j)\,, \tag{3.30}$$

where

$$X_i^{(n)} = X_i^n/[n]_{q_i}!\,, \quad Y_i^{(n)} = Y_i^n/[n]_{q_i}!\,.$$

We define subalgebras V^0, $V^{\geqq 0}$, $V^{\leqq 0}$, V^+, V^- of V by

(3.31) $$V^0 = \langle Z_\lambda \mid \lambda \in \Lambda \rangle\,,$$

(3.32) $$V^{\geqq 0} = \langle Z_\lambda, X_i \mid \lambda \in \Lambda, i \in I \rangle\,,$$

(3.33) $$V^{\leqq 0} = \langle Z_\lambda, Y_i \mid \lambda \in \Lambda, i \in I \rangle\,,$$

(3.34) $$V^+ = \langle X_i \mid i \in I \rangle\,,$$

(3.35) $$V^- = \langle Y_i \mid i \in I \rangle\,.$$

Similarly to Proposition 3.1 we have the following.

PROPOSITION 3.3. (i) *$\{Z_\lambda \mid \lambda \in \Lambda\}$ is an* **F**-*basis of* V^0.
(ii) *The linear maps*

$$V^- \otimes V^0 \otimes V^+ \to V \leftarrow V^+ \otimes V^0 \otimes V^-\,,$$
$$V^+ \otimes V^0 \to V^{\geqq 0} \leftarrow V^0 \otimes V^+\,, \quad V^- \otimes V^0 \to V^{\leqq 0} \leftarrow V^0 \otimes V^-$$

induced by the multiplication are all isomorphisms of vector spaces.

Moreover, we have algebra isomorphisms

$$\jmath^{\leqq 0} : V^{\leqq 0} \to U^{\leqq 0} \quad (Y_i \mapsto F_i,\ Z_\lambda \mapsto K_{-\lambda})\,,$$
$$\jmath^{\geqq 0} : V^{\geqq 0} \to U^{\geqq 0} \quad (X_i \mapsto E_i,\ Z_\lambda \mapsto K_\lambda)\,.$$

We define a bilinear form

(3.36) $$\sigma : U \times V \to \mathbf{F}$$

by

$$\sigma(u_+u_0S(u_-), v_-v_+v_0) = \tau(u_+, \jmath^{\leqq 0}(v_-))\tau(u_0, \jmath^{\leqq 0}(v_0))\tau(\jmath^{\geqq 0}(v_+), u_-)$$
$$(u_\pm \in U^\pm, u_0 \in U^0, v_\pm \in V^\pm, v_0 \in V^0)\,.$$

The following result is a consequence of Gavarini [6, Theorem 6.2].

PROPOSITION 3.4. *We have*

$$\sigma(u, vv') = (\sigma \otimes \sigma)(\Delta(u), v \otimes v') \quad (u \in U, v, v' \in V)\,.$$

3.6. A-forms. We fix a subring **A** of **F** containing $\mathbf{C}[q^{\pm 1/|\Lambda/Q|}]$. We denote by $U_{\mathbf{A}}^L$ the Lusztig **A**-form of U, i.e., $U_{\mathbf{A}}^L$ is the **A**-subalgebra of U generated by the elements

$$E_i^{(m)},\ F_i^{(m)},\ K_\lambda \quad (i \in I,\ m \geqq 0,\ \lambda \in \Lambda)\,.$$

Set

$$U_{\mathbf{A}}^{L,\pm} = U_{\mathbf{A}}^{L} \cap U^{\pm}, \quad U_{\mathbf{A}}^{L,0} = U_{\mathbf{A}}^{L} \cap U^{0},$$
$$U_{\mathbf{A}}^{L,\geqq 0} = U_{\mathbf{A}}^{L} \cap U^{L,\geqq 0}, \quad U_{\mathbf{A}}^{L,\leqq 0} = U_{\mathbf{A}}^{L} \cap U^{L,\leqq 0}.$$

Then $U_{\mathbf{A}}^{L}$, $U_{\mathbf{A}}^{L,0}$, $U_{\mathbf{A}}^{L,\geqq 0}$, $U_{\mathbf{A}}^{L,\leqq 0}$ are endowed with structures of Hopf algebras over $\mathbf{A}$ via the Hopf algebra structure of U, and the multiplication of $U_{\mathbf{A}}^{L}$ induces isomorphisms

$$U_{\mathbf{A}}^{L} \simeq U_{\mathbf{A}}^{L,-} \otimes U_{\mathbf{A}}^{L,0} \otimes U_{\mathbf{A}}^{L,+} \simeq U_{\mathbf{A}}^{L,+} \otimes U_{\mathbf{A}}^{L,0} \otimes U_{\mathbf{A}}^{L,-},$$
$$U_{\mathbf{A}}^{L,\geqq 0} \simeq U_{\mathbf{A}}^{L,0} \otimes U_{\mathbf{A}}^{L,+} \simeq U_{\mathbf{A}}^{L,+} \otimes U_{\mathbf{A}}^{L,0},$$
$$U_{\mathbf{A}}^{L,\leqq 0} \simeq U_{\mathbf{A}}^{L,0} \otimes U_{\mathbf{A}}^{L,-} \simeq U_{\mathbf{A}}^{L,-} \otimes U_{\mathbf{A}}^{L,0}$$

of $\mathbf{A}$-modules. Fix a subset Λ_0 of Λ such that $\Lambda_0 \to \Lambda/2Q$ is bijective. Then $U_{\mathbf{A}}^{L,+}$, $U_{\mathbf{A}}^{L,-}$, $U_{\mathbf{A}}^{L,0}$ are free $\mathbf{A}$-modules with bases

$$\{E_{\beta_N}^{(m_N)} \cdots E_{\beta_1}^{(m_1)} \mid m_1, \dots, m_N \geqq 0\},$$
$$\{F_{\beta_N}^{(m_N)} \cdots F_{\beta_1}^{(m_1)} \mid m_1, \dots, m_N \geqq 0\},$$
$$\left\{K_\lambda \prod_{i \in I} \begin{bmatrix} K_i \\ n_i \end{bmatrix} \mid \lambda \in \Lambda_0,\ n_i \geqq 0\right\}$$

respectively, where

$$\begin{bmatrix} K_i \\ m \end{bmatrix} = \prod_{s=0}^{m-1} \frac{q_i^{-s} K_i - q_i^{s} K_i^{-1}}{q_i^{s+1} - q_i^{-s-1}} \quad (m \geqq 0).$$

We denote by $V_{\mathbf{A}}$ the $\mathbf{A}$-subalgebra of V generated by the elements

$$X_i^{(m)},\ Y_i^{(m)},\ Z_\lambda,\ \begin{bmatrix} Z_i \\ m \end{bmatrix} \quad (i \in I,\ m \geqq 0, \lambda \in \Lambda),$$

where $Z_i = Z_{\alpha_i}$ for $i \in I$ and

$$\begin{bmatrix} Z_i \\ m \end{bmatrix} = \prod_{s=0}^{m-1} \frac{q_i^{-s} Z_i - q_i^{s} Z_i^{-1}}{q_i^{s+1} - q_i^{-s-1}} \quad (m \geqq 0).$$

Set

$$V_{\mathbf{A}}^{\pm} = V_{\mathbf{A}} \cap V^{\pm}, \quad V_{\mathbf{A}}^{0} = V_{\mathbf{A}} \cap V^{0},$$
$$V_{\mathbf{A}}^{\geqq 0} = V_{\mathbf{A}} \cap V^{\geqq 0}, \quad V_{\mathbf{A}}^{\leqq 0} = V_{\mathbf{A}} \cap V^{\leqq 0}.$$

Then the multiplication of $V_{\mathbf{A}}$ induces isomorphisms

$$V_{\mathbf{A}} \simeq V_{\mathbf{A}}^{-} \otimes V_{\mathbf{A}}^{0} \otimes V_{\mathbf{A}}^{+} \simeq V_{\mathbf{A}}^{+} \otimes V_{\mathbf{A}}^{0} \otimes V_{\mathbf{A}}^{-},$$

$$V_{\mathbf{A}}^{\geqq 0} \simeq V_{\mathbf{A}}^{0} \otimes V_{\mathbf{A}}^{+} \simeq V_{\mathbf{A}}^{+} \otimes V_{\mathbf{A}}^{0},$$
$$V_{\mathbf{A}}^{\leqq 0} \simeq V_{\mathbf{A}}^{0} \otimes V_{\mathbf{A}}^{-} \simeq V_{\mathbf{A}}^{-} \otimes V_{\mathbf{A}}^{0}$$

of $\mathbf{A}$-modules, and we have

$$J^{\geqq 0}(V_{\mathbf{A}}^{\geqq 0}) = U_{\mathbf{A}}^{L,\geqq 0}, \quad J^{\geqq 0}(V_{\mathbf{A}}^{+}) = U_{\mathbf{A}}^{L,+}, \quad J^{\geqq 0}(V_{\mathbf{A}}^{0}) = U_{\mathbf{A}}^{L,0},$$
$$J^{\leqq 0}(V_{\mathbf{A}}^{\leqq 0}) = U_{\mathbf{A}}^{L,\leqq 0}, \quad J^{\leqq 0}(V_{\mathbf{A}}^{-}) = U_{\mathbf{A}}^{L,-}, \quad J^{\leqq 0}(V_{\mathbf{A}}^{0}) = U_{\mathbf{A}}^{L,0}.$$

Set

(3.37) $$U_{\mathbf{A}} = \{u \in U \mid \sigma(u, V_{\mathbf{A}}) \subset \mathbf{A}\},$$

(3.38) $$U_{\mathbf{A}}^{\pm} = U^{\pm} \cap U_{\mathbf{A}}, \quad U_{\mathbf{A}}^{0} = U^{0} \cap U_{\mathbf{A}},$$

(3.39) $$U_{\mathbf{A}}^{\geqq 0} = U^{\geqq 0} \cap U_{\mathbf{A}}, \quad U_{\mathbf{A}}^{\leqq 0} = U^{\leqq 0} \cap U_{\mathbf{A}}.$$

Then we have

(3.40) $$U_{\mathbf{A}}^{+} = \{x \in U^{+} \mid \tau(x, U_{\mathbf{A}}^{L,-}) \in \mathbf{A}\},$$

(3.41) $$U_{\mathbf{A}}^{-} = \{y \in U^{-} \mid \tau(U_{\mathbf{A}}^{L,+}, y) \in \mathbf{A}\},$$

(3.42) $$U_{\mathbf{A}}^{0} = \sum_{\lambda \in \Lambda} \mathbf{A} K_{\lambda},$$

and the multiplication of U induces isomorphisms

$$U_{\mathbf{A}} \simeq U_{\mathbf{A}}^{+} \otimes U_{\mathbf{A}}^{0} \otimes U_{\mathbf{A}}^{-},$$
$$U_{\mathbf{A}}^{\geqq 0} \simeq U_{\mathbf{A}}^{0} \otimes U_{\mathbf{A}}^{+} \simeq U_{\mathbf{A}}^{+} \otimes U_{\mathbf{A}}^{0},$$
$$U_{\mathbf{A}}^{\leqq 0} \simeq U_{\mathbf{A}}^{0} \otimes U_{\mathbf{A}}^{-} \simeq U_{\mathbf{A}}^{-} \otimes U_{\mathbf{A}}^{0}$$

of $\mathbf{A}$-modules.

For $i \in I$ we set

(3.43) $$A_i = (q_i - q_i^{-1})E_i, \quad B_i = (q_i - q_i^{-1})F_i.$$

For $1 \leqq k \leqq N$ we also set

(3.44) $$A_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})E_{\beta_k}, \qquad B_{\beta_k} = (q_{\beta_k} - q_{\beta_k}^{-1})F_{\beta_k}.$$

By Proposition 3.2 we have the following.

LEMMA 3.5. *$\{A_{\beta_N}^{m_N} \cdots A_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$ (resp. $\{B_{\beta_N}^{m_N} \cdots B_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$) is an $\mathbf{A}$-basis of $U_{\mathbf{A}}^{+}$ (resp. $U_{\mathbf{A}}^{-}$). In particular, we have $U_{\mathbf{A}}^{\pm} \subset U_{\mathbf{A}}^{L,\pm}$, and $U_{\mathbf{A}} \subset U_{\mathbf{A}}^{L}$.*

It follows that $U_{\mathbf{A}}$ coincides with the $\mathbf{A}$-form of U considered in De Concini-Procesi [4]. In particular, we have the following.

PROPOSITION 3.6. (i) *$U_{\mathbf{A}}^0, U_{\mathbf{A}}^+, U_{\mathbf{A}}^-, U_{\mathbf{A}}^{\geqq 0}, U_{\mathbf{A}}^{\leqq 0}, U_{\mathbf{A}}$ are $\mathbf{A}$-subalgebras of U.*

(ii) *$U_{\mathbf{A}}^0, U_{\mathbf{A}}^{\geqq 0}, U_{\mathbf{A}}^{\leqq 0}, U_{\mathbf{A}}$ are Hopf algebras over $\mathbf{A}$.*

Let $\iota : U_{\mathbf{A}} \to U_{\mathbf{A}}^L$ be the inclusion. We denote by

$$\sigma_{\mathbf{A}} : U_{\mathbf{A}} \times V_{\mathbf{A}} \to \mathbf{A}\,. \tag{3.45}$$

the bilinear form induced by $\sigma : U \times V \to \mathbf{F}$.

We set

$$C_{\mathbf{A}} = \{\varphi \in C \mid \langle \varphi, U_{\mathbf{A}}^L \rangle \subset \mathbf{A}\}\,, \tag{3.46}$$

$$D_{\mathbf{A}} = C_{\mathbf{A}} \otimes_{\mathbf{A}} U_{\mathbf{A}} \subset D\,. \tag{3.47}$$

Then $C_{\mathbf{A}}$ is a Hopf algebra over $\mathbf{A}$ as well as a $U_{\mathbf{A}}^L$-bimodule, and $D_{\mathbf{A}}$ is an $\mathbf{A}$-subalgebra of D. It easily follows that

$$\left(\bigoplus_{\lambda \in \Lambda} \mathbf{F}\chi_\lambda\right) \cap \mathrm{Hom}_{\mathbf{A}}(U_{\mathbf{A}}^{L,0}, \mathbf{A}) = \bigoplus_{\lambda \in \Lambda} \mathbf{A}\chi_\lambda\,. \tag{3.48}$$

Hence by (3.24) we have

$$C_{\mathbf{A}} = \left((U_{\mathbf{A}}^{L,-})^{\bigstar} \otimes \left(\bigoplus_{\lambda \in \Lambda} \mathbf{A}\chi_\lambda\right) \otimes (U_{\mathbf{A}}^{L,+})^{\bigstar}\right) \cap C \subset \mathrm{Hom}_{\mathbf{A}}(U_{\mathbf{A}}^L, \mathbf{A})\,, \tag{3.49}$$

where $(U_{\mathbf{A}}^{L,\pm})^{\bigstar} = \mathrm{Hom}_{\mathbf{A}}(U_{\mathbf{A}}^{L,\pm}, \mathbf{A}) \cap (U^{\pm})^{\bigstar}$.

3.7. Specialization. For $z \in \mathbf{C}^\times$ set

$$\mathbf{A}_z = \left\{f/g \mid f, g \in \mathbf{C}[q^{\pm 1/|\Lambda/Q|}],\ g(z) \neq 0\right\} \subset \mathbf{F}\,,$$

and define an algebra homomorphism

$$\pi_z : \mathbf{A}_z \to \mathbf{C}$$

by $\pi_z(q^{1/|\Lambda/Q|}) = z$. We set

$$U_z^L = \mathbf{C} \otimes_{\mathbf{A}_z} U_{\mathbf{A}_z}^L\,, \quad V_z = \mathbf{C} \otimes_{\mathbf{A}_z} V_{\mathbf{A}_z}\,, \quad U_z = \mathbf{C} \otimes_{\mathbf{A}_z} U_{\mathbf{A}_z}\,,$$

$$C_z = \mathbf{C} \otimes_{\mathbf{A}_z} C_{\mathbf{A}_z}\,, \quad D_z = \mathbf{C} \otimes_{\mathbf{A}_z} D_{\mathbf{A}_z}\,.$$

with respect to π_z. Then U_z^L, U_z, C_z are Hopf algebras over $\mathbf{C}$, and V_z, D_z are $\mathbf{C}$-algebras. We denote by

$$\pi_z^{U^L} : U_{\mathbf{A}_z}^L \to U_z^L\,, \quad \pi_z^V : V_{\mathbf{A}_z} \to V_z\,, \quad \pi_z^U : U_{\mathbf{A}_z} \to U_z,$$

$$\pi_z^C : C_{\mathbf{A}_z} \to C_z\,, \quad \pi_z^D : D_{\mathbf{A}_z} \to D_z$$

the natural homomorphisms. We also define $U_z^{L,\pm}$, $U_z^{L,0}$, $U_z^{L,\geqq 0}$, $U_z^{L,\leqq 0}$, $V_z^{\pm}$, V_z^0, $V_z^{\geqq 0}$, $V_z^{\leqq 0}$, $U_z^{\pm}$, U_z^0, $U_z^{\geqq 0}$, $U_z^{\leqq 0}$ similarly. The bilinear form $\sigma_{\mathbf{A}_z} : U_{\mathbf{A}_z} \times V_{\mathbf{A}_z} \to \mathbf{A}_z$ induces a bilinear form

$$\sigma_z : U_z \times V_z \to \mathbf{C}\,. \tag{3.50}$$

Set

$$J_z = \{v \in V_z \mid \sigma_z(U_z, v) = \{0\}\}\,, \quad J_z^0 = J_z \cap V_z^0\,.$$

LEMMA 3.7. *J_z is a two-sided ideal of V_z, and we have $J_z = V_z^- V_z^+ J_z^0$. In particular, we have $J_z \cap V^{\geqq 0} = V_z^+ J_z^0$, and $J_z \cap V^{\leqq 0} = V_z^- J_z^0$.*

PROOF. By Proposition 3.4 J_z is a two-sided ideal. Set $V_z' = V_z / V_z^- V_z^+ J_z^0$. Since the multiplication of V_z induces an isomorphism $V_z \simeq V_z^- \otimes V_z^+ \otimes V_z^0$, we have

$$V_z' \simeq (V_z^- \otimes V_z^+ \otimes V_z^0)/(V_z^- \otimes V_z^+ \otimes J_z^0) \simeq V_z^- \otimes V_z^+ \otimes (V_z^0/J_z^0)\,.$$

Let $\sigma_z' : U_z \times V_z' \to \mathbf{C}$ be the bilinear form induced by σ_z. Then we see easily from the definition of σ and Proposition 3.2 that $\{v \in V_z' \mid \sigma_z'(U_z, v) = \{0\}\} = \{0\}$. Hence $J_z = V_z^- V_z^+ J_z^0$. □

We define an algebra $\overline{V}_z$ by $\overline{V}_z = V_z/J_z$, and denote by $\overline{\pi}_z^V : V_{\mathbf{A}_z} \to \overline{V}_z$ the canonical homomorphism. Let $\overline{\sigma}_z : U_z \times \overline{V}_z \to \mathbf{C}$ be the bilinear form induced by (3.50). Denote the images of V_z^0, $V_z^{\pm}$, $V_z^{\geqq 0}$, $V_z^{\leqq 0}$ under $V_z \to \overline{V}_z$ by $\overline{V}_z^0$, $\overline{V}_z^{\pm}$, $\overline{V}_z^{\geqq 0}$, $\overline{V}_z^{\leqq 0}$ respectively. Then the multiplication of $\overline{V}_z$ induces isomorphisms

$$\overline{V}_z \simeq \overline{V}_z^- \otimes \overline{V}_z^+ \otimes \overline{V}_z^0\,,$$
$$\overline{V}_z^{\geqq 0} \simeq \overline{V}_z^+ \otimes \overline{V}_z^0\,, \quad \overline{V}_z^{\leqq 0} \simeq \overline{V}_z^- \otimes \overline{V}_z^0\,.$$

Let $\lambda \in \Lambda$. By abuse of notation we also denote by $\chi_\lambda : U_z^{L,0} \to \mathbf{C}$ the algebra homomorphism induced by $\chi_\lambda : U \to \mathbf{F}$. We see easily the following

LEMMA 3.8. *$\{\chi_\lambda \mid \lambda \in \Lambda\}$ is a linearly independent subset of $(U_z^{L,0})^*$.*

LEMMA 3.9. *The bilinear form $\overline{\sigma}_z$ is perfect in the sense that*

$$u \in U_z\,, \quad \overline{\sigma}_z(u, \overline{V}_z) = \{0\} \implies u = 0\,, \tag{3.51}$$

$$v \in \overline{V}_z\,, \quad \overline{\sigma}_z(U_z, v) = \{0\} \implies v = 0\,. \tag{3.52}$$

PROOF. (3.52) is clear from the definition. We see easily from the definition of σ and Proposition 3.2 that the proof of (3.51) is reduced to showing

$$u \in U_z^0, \quad \sigma_z(u, V_z^0) = \{0\} \implies u = 0.$$

This follows from Lemma 3.8 in view of

$$U_z^0 = \bigoplus_{\lambda \in \Lambda} \mathbf{C} K_\lambda, \quad V_z^0 \cong U_z^{L,0}.$$

□

Set

$$I_z^0 = \jmath^{\geqq 0}(J_z^0) \subset U_z^{L,0},$$
$$I_z^{\geqq 0} = U_z^{L,+} I_z^0 \subset U_z^{L,\geqq 0}, \qquad I_z^{\leqq 0} = U_z^{L,-} I_z^0 \subset U_z^{L,\leqq 0},$$
$$I_z = U_z^{L,-} U_z^{L,+} I_z^0 \subset U_z^L.$$

The we have

$$I_z^0 = \{u \in U_z^{L,0} \mid \chi_\lambda(u) = 0 \ (\lambda \in \Lambda)\}. \tag{3.53}$$

LEMMA 3.10. *I_z^0, $I_z^{\geqq 0}$, $I_z^{\leqq 0}$, I_z are Hopf ideals of $U_z^{L,0}$, $U_z^{L,\geqq 0}$, $U_z^{L,\leqq 0}$, U_z^L respectively.*

PROOF. From (3.53) we see easily that I_z^0 is a Hopf ideal of $U_z^{L,0}$. It remains to show $I_z^0 U_z^{L,\pm} \subset U_z^{L,\pm} I_z^0$. Using $\jmath^{\geqq 0}$, $\jmath^{\leqq 0}$ we see that this is equivalent to $J_z^0 V_z^{\pm} \subset V_z^{\pm} J_z^0$. This follows from Lemma 3.7. □

We define a Hopf algebra $\overline{U}_z^L$ by $\overline{U}_z^L = U_z^L / I_z$, and denote by $\overline{\pi}_z^{U^L} : U_{\mathbf{A}_z}^L \to \overline{U}_z^L$ the canonical homomorphism. Denote the images of $U_z^{L,0}$, $U_z^{L,\pm}$, $U_z^{L,\geqq 0}$, $U_z^{L,\leqq 0}$ under $U_z^L \to \overline{U}_z^L$ by $\overline{U}_z^{L,0}$, $\overline{U}_z^{L,\pm}$, $\overline{U}_z^{L,\geqq 0}$, $\overline{U}_z^{L,\leqq 0}$ respectively. We also denote by

$$\overline{\jmath}_z^{\geqq 0} : \overline{V}_z^{\geqq 0} \to \overline{U}_z^{L,\geqq 0}, \quad \overline{\jmath}_z^{\leqq 0} : \overline{V}_z^{\leqq 0} \to \overline{U}_z^{L,\leqq 0} \tag{3.54}$$

the algebra isomorphisms induced by $\jmath^{\geqq 0}$ and $\jmath^{\leqq 0}$.

By (3.49) and Lemma 3.8 we have

$$C_z \subset (U_z^{L,-})^\bigstar \otimes \left(\bigoplus_{\lambda \in \Lambda} \mathbf{C} \chi_\lambda \right) \otimes (U_z^{L,+})^\bigstar \subset (U_z^L)^*, \tag{3.55}$$

where $(U_z^{L,\pm})^\bigstar = \mathbf{C} \otimes_{\mathbf{A}} (U_{\mathbf{A}}^{L,\pm})^\bigstar \subset \mathrm{Hom}_{\mathbf{C}}(U_z^{L,\pm}, \mathbf{C})$. Hence the natural paring $\langle\, ,\, \rangle : C_z \times U_z^L \to \mathbf{C}$ descends to

$$\langle\, ,\, \rangle : C_z \times \overline{U}_z^L \to \mathbf{C},$$

by which the canonical map $C_z \to (\overline{U}^L_z)^*$ is injective. Moreover, C_z turns out to be a $\overline{U}^L_z$-bimodule.

3.8. Specialization to 1. For an algebraic groups S over $\mathbf{C}$ with Lie algebra $\mathfrak{s}$ we will identify the coordinate algebra $\mathbf{C}[S]$ of S with a subspace of the dual space $U(\mathfrak{s})^*$ of the enveloping algebra $U(\mathfrak{s})$ by the canonical Hopf paring

$$\langle\ ,\ \rangle : \mathbf{C}[S] \otimes U(\mathfrak{s}) \to \mathbf{C}$$

given by

$$\langle \varphi, u\rangle = (L_u(\varphi))(1) \quad (\varphi \in \mathbf{C}[S], u \in U(\mathfrak{s}))\,.$$

Here, $U(\mathfrak{s}) \ni u \mapsto L_u \in \mathrm{End}_{\mathbf{C}}(\mathbf{C}[S])$ is the algebra homomorphism given by

$$(L_a(\varphi))(g) = \frac{d}{dt}\varphi(g\exp(ta))|_{t=0} \quad (a \in \mathfrak{g}, g \in S, \varphi \in \mathbf{C}[S])\,.$$

We see easily that J_1 is generated by the elements $\pi_1^V(Z_\lambda) - 1 \in V_1$ for $\lambda \in \Lambda$. From this we see easily the following.

LEMMA 3.11. (i) *We have an isomorphism* $\overline{V}_1 \cong U(\mathfrak{k})$ *of algebras satisfying*

$$\overline{\pi}_1^V(X_i) \leftrightarrow x_i\,, \quad \overline{\pi}_1^V(Y_i) \leftrightarrow y_i\,,$$

$$\overline{\pi}_1^V\left(\begin{bmatrix} Z_i \\ m \end{bmatrix}\right) \leftrightarrow \binom{t_i}{m} := t_i(t_i-1)\cdots(t_i-m+1)/m!\,.$$

(ii) *We have an isomorphism* $\overline{U}^L_1 \cong U(\mathfrak{g})$ *of Hopf algebras satisfying*

$$\overline{\pi}_1^{U^L}(E_i) \leftrightarrow e_i, \qquad \overline{\pi}_1^{U^L}(F_i) \leftrightarrow f_i,$$

$$\overline{\pi}_1^{U^L}\left(\begin{bmatrix} K_i \\ m \end{bmatrix}\right) \leftrightarrow \binom{h_i}{m} := h_i(h_i-1)\cdots(h_i-m+1)/m!\,.$$

In the rest of this paper we will occasionally identify $\overline{V}_1$ and $\overline{U}^L_1$ with $U(\mathfrak{k})$ and $U(\mathfrak{g})$ respectively.

From the identification $\overline{U}^L_1 = U(\mathfrak{g})$ we have the following.

LEMMA 3.12. *The canonical paring*

$$\langle\ ,\ \rangle : C_1 \times \overline{U}^L_1 \to \mathbf{C}$$

induces an isomorphism

$$C_1 \cong \mathbf{C}[G]\,(\subset U(\mathfrak{g})^* \cong (\overline{U}^L_1)^*) \tag{3.56}$$

of Hopf algebras.

In [4] De Concini-Procesi proved an isomorphism

$$U_1 \cong \mathbf{C}[K] \tag{3.57}$$

of Poisson Hopf algebras. They established (3.57) by giving a correspondence between generators of both sides and proving the compatibility after a lengthy calculation. Later Gavarini [6] gave a more natural approach to the isomorphism (3.57) using the Drinfeld paring. Namely we have the following.

PROPOSITION 3.13 (Gavarini [6]). *The bilinear form* $\overline{\sigma}_1 : U_1 \times \overline{V}_1 \to \mathbf{C}$ *induces a Hopf algebra isomorphism*

$$\Upsilon : U_1 \to \mathbf{C}[K] \, (\subset U(\mathfrak{k})^* \simeq \overline{V}_1^*) \, . \tag{3.58}$$

The enveloping algebra $U(\mathfrak{k}^{\pm})$ has the direct sum decomposition

$$U(\mathfrak{k}^{\pm}) = \bigoplus_{\beta \in Q^+} U(\mathfrak{k}^{\pm})_{\pm\beta},$$

where

$$U(\mathfrak{k}^{\pm})_{\pm\beta}, = \{x \in U(\mathfrak{k}^{\pm}) \mid [(h, -h), x] = \beta(h)x \;\; (h \in \mathfrak{h})\}$$

for $\beta \in Q^+$ (note that we have an isomorphism $\mathfrak{h} \ni h \leftrightarrow (h, -h) \in \mathfrak{k}^0$). Then we have

$$\mathbf{C}[K^{\pm}] = \bigoplus_{\beta \in Q^+} (U(\mathfrak{k}^{\pm})_{\pm\beta})^* \subset U(\mathfrak{k}^{\pm})^*.$$

Moreover, we have

$$\mathbf{C}[K^0] = \bigoplus_{\lambda \in \Lambda} \mathbf{C}\hat{\chi}_\lambda \subset U(\mathfrak{k}_0)^* \, ,$$

where $\hat{\chi}_\lambda : U(\mathfrak{k}^0) \to \mathbf{C}$ is the algebra homomorphism given by $\hat{\chi}_\lambda(h, -h) = \lambda(h) \; (h \in \mathfrak{h})$. The isomorphism

$$K^+ \times K^- \times K^0 \simeq K \quad ((g_+, g_-, g_0) \leftrightarrow g_+ g_- g_0)$$

of algebraic varieties induced by the product of the group K gives an identification

$$\mathbf{C}[K^+] \otimes \mathbf{C}[K^-] \otimes \mathbf{C}[K^0] \simeq \mathbf{C}[K] \tag{3.59}$$

of vector spaces. On the other hand the multiplication of the algebra $U(\mathfrak{k})$ induces an identification

$$U(\mathfrak{k}^+) \otimes U(\mathfrak{k}^-) \otimes U(\mathfrak{k}^0) \simeq U(\mathfrak{k}) \, .$$

Then the canonical embedding $\mathbf{C}[K] \subset U(\mathfrak{k})^*$ is given by

$$\mathbf{C}[K] \simeq \mathbf{C}[K^+] \otimes \mathbf{C}[K^-] \otimes \mathbf{C}[K^0] \subset U(\mathfrak{k}^+)^* \otimes U(\mathfrak{k}^-)^* \otimes U(\mathfrak{k}^0)^*$$

$$\subset (U(\mathfrak{k}^+)\otimes U(\mathfrak{k}^-)\otimes U(\mathfrak{k}^0))^* = U(\mathfrak{k})^* .$$

For $i \in I$ we define $a_i \in \mathbf{C}[K^-] \subset U(\mathfrak{k}^-)^*$, $b_i \in \mathbf{C}[K^+] \subset U(\mathfrak{k}^+)^*$ by

$$\langle a_i, U(\mathfrak{k}^-)_{-\beta}\rangle = 0 \quad (\beta \neq \alpha_i), \quad \langle a_i, y_i\rangle = -1,$$
$$\langle b_i, U(\mathfrak{k}^+)_{\beta}\rangle = 0 \quad (\beta \neq \alpha_i), \quad \langle b_i, x_i\rangle = 1.$$

We identify $\mathbf{C}[K^\pm]$, $\mathbf{C}[K^0]$ with subalgebras of $\mathbf{C}[K]$ via (3.59), and regard $a_i, b_i, \hat{\chi}_\lambda$ ($i \in I, \lambda \in \Lambda$) as elements of $\mathbf{C}[K]$. By the above argument we see easily the following.

LEMMA 3.14. *Under the identification* (3.58) *we have*

$$\pi_1^U(A_i) \leftrightarrow a_i, \quad \pi_1^U(B_i) \leftrightarrow b_i\hat{\chi}_{-\alpha_i}, \quad \pi_1^U(K_\lambda) \leftrightarrow \hat{\chi}_\lambda \quad (i \in I, \lambda \in \Lambda).$$

Let $\iota_1 : U_1 \to \overline{U}_1^L$ be the homomorphism induced by the inclusion $\iota : U_{\mathbf{A}_1} \to U_{\mathbf{A}_1}^L$. By Lemma 3.5 we see easily the following.

LEMMA 3.15. *For $x \in U_1$ we have $\iota_1(x) = \varepsilon(x)1$.*

From this we obtain the following easily.

LEMMA 3.16. *D_1 is a commutative algebra. In particular, it is identified as an algebra with the coordinate algebra $\mathbf{C}[G]\otimes\mathbf{C}[K]$ of $G \times K$.*

3.9. Specialization to roots of 1. From now on, we fix an integer $\ell > 1$ satisfying

(a) ℓ is odd,
(b) ℓ is prime to 3 if $\mathfrak{g}$ is of type G_2,
(c) ℓ is prime to $|\Lambda/Q|$,

and a primitive ℓ-th root $\zeta \in \mathbf{C}$ of 1. Note that $\pi_\zeta : \mathbf{A}_\zeta \to \mathbf{C}$ sends q to $\zeta^{|\Lambda/Q|}$, which is also a primitive ℓ-th root of 1 by our assumption (c).

REMARK 3.17. Denote by $U^{DK}_{\mathbf{C}[q^{\pm 1/|\Lambda/Q|}]}$ the De Concini-Kac $\mathbf{C}[q^{\pm 1/|\Lambda/Q|}]$-form of U (see [2]). Namely $U^{DK}_{\mathbf{C}[q^{\pm 1/|\Lambda/Q|}]}$ is the $\mathbf{C}[q^{\pm 1/|\Lambda/Q|}]$-subalgebra of U generated by $\{K_\lambda, E_i, F_i \mid \lambda \in \Lambda, i \in I\}$. Then we have $U_\zeta \simeq \mathbf{C}\otimes_{\mathbf{C}[q^{\pm 1/|\Lambda/Q|}]} U^{DK}_{\mathbf{C}[q^{\pm 1/|\Lambda/Q|}]}$ with respect to $q^{1/|\Lambda/Q|} \mapsto \zeta$.

We denote by $\tilde{\xi} : U^L_\zeta \to U^L_1$ Lusztig's Frobenius morphism (see [9]). Namely, $\tilde{\xi}$ is an algebra homomorphism given by

$$\tilde{\xi}(\pi_\zeta^{U^L}(E_i^{(n)})) = \begin{cases} \pi_1^{U^L}(E_i^{(n/\ell)}) & (\ell \mid n) \\ 0 & (\ell \nmid n), \end{cases} \tag{3.60}$$

$$\tilde{\xi}(\pi_\zeta^{U^L}(F_i^{(n)})) = \begin{cases} \pi_1^{U^L}(F_i^{(n/\ell)}) & (\ell \mid n) \\ 0 & (\ell \nmid n), \end{cases} \tag{3.61}$$

$$\tilde{\xi}\left(\pi_\zeta^{U^L}\left(\begin{bmatrix} K_i \\ m \end{bmatrix}\right)\right) = \begin{cases} \pi_1^{U^L}\left(\begin{bmatrix} K_i \\ m/\ell \end{bmatrix}\right) & (\ell \mid m) \\ 0 & (\ell \nmid m)\,, \end{cases} \tag{3.62}$$

$$\tilde{\xi}(\pi_\zeta^{U^L}(K_\lambda)) = \pi_1^{U^L}(K_\lambda) \quad (\lambda \in \Lambda)\,. \tag{3.63}$$

It is a Hopf algebra homomorphism. Moreover, for any $\beta \in \Delta^+$ we have

$$\tilde{\xi}(\pi_\zeta^{U^L}(E_\beta^{(n)})) = \begin{cases} \pi_1^{U^L}(E_\beta^{(n/\ell)}) & (\ell \mid n) \\ 0 & (\ell \nmid n)\,, \end{cases} \tag{3.64}$$

$$\tilde{\xi}(\pi_\zeta^{U^L}(F_\beta^{(n)})) = \begin{cases} \pi_1^{U^L}(F_\beta^{(n/\ell)}) & (\ell \mid n) \\ 0 & (\ell \nmid n)\,. \end{cases} \tag{3.65}$$

LEMMA 3.18. *We have* $\tilde{\xi}(I_\zeta) \subset I_1$.

PROOF. It is sufficient to show $\tilde{\xi}(I_\zeta^0) \subset I_1^0$. For $z \in \mathbf{C}^\times$, $m = (m_i)_{i\in I} \in \mathbf{Z}_{\geqq 0}^I$, and $\nu \in \Lambda_0$ set

$$K_{m,\nu}(z) = \pi_z^{U^L}\left(K_\nu \prod_{i\in I} \begin{bmatrix} K_i \\ m_i \end{bmatrix}\right) \in U_z^{L,0}\,.$$

Any element u of $U_z^{L,0}$ is uniquely written as a finite sum

$$u = \sum_{m,\nu} c_{m,\nu} K_{m,\nu}(z) \quad (c_{m,\nu} \in \mathbf{C})\,.$$

Then we have $u \in I_z^0$ if and only if

$$\sum_{m,\nu} c_{m,\nu} q^{(\lambda,\nu)} \begin{bmatrix} (\lambda, \alpha_i^\vee) \\ m_i \end{bmatrix}_{q_i} \Bigg|_{q^{1/|\Lambda/Q|}=z} = 0 \quad (\forall \lambda \in \Lambda)\,.$$

Hence it is sufficient to show that

$$\sum_{m,\nu} c_{m,\nu} q^{(\lambda,\nu)} \begin{bmatrix} (\lambda, \alpha_i^\vee) \\ m_i \end{bmatrix}_{q_i} \Bigg|_{q^{1/|\Lambda/Q|}=\zeta} = 0 \quad (\forall \lambda \in \Lambda) \tag{3.66}$$

implies

$$\sum_{m,\nu} c_{\ell m,\nu} \binom{(\mu, \alpha_i^\vee)}{m_i} = 0 \quad (\forall \mu \in \Lambda)\,. \tag{3.67}$$

Indeed (3.67) follows by setting $\lambda = \ell\mu$ in (3.66). □

We denote by

$$\xi : \overline{U}^L_\zeta \to \overline{U}^L_1 \ (= U(\mathfrak{g})) \tag{3.68}$$

the Hopf algebra homomorphism induced by $\tilde{\xi}$. By Lusztig [9] we have the following.

PROPOSITION 3.19. *There exists a unique linear map*

$$^t\xi : C_1 \ (= \mathbf{C}[G]) \to C_\zeta \tag{3.69}$$

satisfying

$$\langle {}^t\xi(\varphi), v\rangle = \langle \varphi, \xi(v)\rangle \quad (\varphi \in C_1,\ v \in \overline{U}^L_\zeta)\,. \tag{3.70}$$

It is an injective Hopf algebra homomorphism whose image is contained in the center of C_ζ.

LEMMA 3.20. *There exists an algebra homomorphism*

$$\eta : \overline{V}_\zeta \to \overline{V}_1 \tag{3.71}$$

such that

$$\eta(v) = (\overline{J}_1^{\geqq 0})^{-1}(\xi(\overline{J}_\zeta^{\geqq 0}(v))) \qquad (v \in \overline{V}_\zeta^{\geqq 0})\,,$$
$$\eta(v) = (\overline{J}_1^{\leqq 0})^{-1}(\xi(\overline{J}_\zeta^{\leqq 0}(v))) \qquad (v \in \overline{V}_\zeta^{\leqq 0})\,.$$

PROOF. It is sufficient to show that the linear map $\eta : \overline{V}_\zeta \to \overline{V}_1$ defined by

$$\eta(v_- v_{\geqq 0}) = (\overline{J}_1^{\leqq 0})^{-1}(\xi(\overline{J}_\zeta^{\leqq 0}(v_-)))(\overline{J}_1^{\geqq 0})^{-1}(\xi(\overline{J}_\zeta^{\geqq 0}(v_{\geqq 0})))$$

for $v_- \in \overline{V}_\zeta^-$, $v_{\geqq 0} \in \overline{V}_\zeta^{\geqq 0}$ is an algebra homomorphism. This follows easily from $[\overline{V}_\zeta^+, \overline{V}_\zeta^-] = 0$. □

By Gavarini [6, Theorem 7.9] we have the following.

PROPOSITION 3.21. *There exists a unique linear map*

$$^t\eta : U_1 \to U_\zeta \tag{3.72}$$

satisfying

$$\overline{\sigma}_\zeta({}^t\eta(u), v) = \overline{\sigma}_1(u, \eta(v)) \quad (u \in U_1,\ v \in \overline{V}_\zeta)\,. \tag{3.73}$$

It is an injective Hopf algebra homomorphism whose image is contained in the center of U_ζ. *Moreover, for any* $\beta \in \Delta^+$ *we have*

$$^t\eta(\pi_1^U(A_\beta)) = \pi_\zeta^U(A_\beta^\ell)\,, \quad {}^t\eta(\pi_1^U(B_\beta)) = \pi_\zeta^U(B_\beta^\ell)\,.$$

Let $\iota_\zeta : U_\zeta \to \overline{U}^L_\zeta$ be the homomorphisms induced by $\iota : U_\zeta \to \overline{U}^L_\zeta$. We see easily the following.

LEMMA 3.22. (i) *For $x \in U_\zeta$ we have $\xi(\iota_\zeta(x)) = \varepsilon(x)1$.*
(ii) *For $y \in U_1$ we have $\iota_\zeta({}^t\eta(y)) = \varepsilon(y)1$.*

PROPOSITION 3.23. *The image of the linear map*

$${}^t\xi \otimes {}^t\eta : D_1(= C_1 \otimes U_1) \to D_\zeta(= C_\zeta \otimes U_\zeta)$$

is contained in the center of D_ζ. In particular, ${}^t\xi \otimes {}^t\eta$ is an algebra homomorphism.

PROOF. Let $\varphi \in C_1$ and $x \in U_\zeta$. For $u \in \overline{U}^L_\zeta$ we have

$$\sum_{(x)} \langle \iota_\zeta(x_{(0)}) \cdot {}^t\xi(\varphi), u \rangle x_{(1)} = \sum_{(x)} \langle {}^t\xi(\varphi), u\iota_\zeta(x_{(0)}) \rangle x_{(1)}$$
$$= \sum_{(x)} \langle \varphi, \xi(u\iota_\zeta(x_{(0)})) \rangle x_{(1)} = \langle \varphi, \xi(u) \rangle x = \langle {}^t\xi(\varphi), u \rangle x \,,$$

and hence $x\,{}^t\xi(\varphi) = {}^t\xi(\varphi)x$ in D_ζ. It follows that ${}^t\xi(\varphi)$ is contained in the center for any $\varphi \in C_1$.

Let $y \in U_1$. For $\psi \in C_\zeta$ we have

$$({}^t\eta(y))\psi = \sum_{(y)} (\iota_\zeta({}^t\eta(y_{(0)})) \cdot \psi)\,{}^t\eta(y_{(1)}) = \sum_{(y)} \varepsilon(y_{(0)})\psi\,{}^t\eta(y_{(1)}) = \psi({}^t\eta(y)) \,,$$

and hence ${}^t\eta(y)$ is contained in the center for any $y \in U_1$. □

4. Poisson structure arising from quantized enveloping algebras

The following result is well-known (see [4]).

PROPOSITION 4.1. *Let $\mathbf{B}$ be a commutative algebra over $\mathbf{C}$. We assume that we are given $\hbar \in \mathbf{B}$ such that $\mathbf{B}/\hbar\mathbf{B} \cong \mathbf{C}$.*

Let $\mathcal{R}$ be a (not necessarily commutative) $\mathbf{B}$-algebra such that $\hbar : \mathcal{R} \to \mathcal{R}$ is injective. Then the center $Z(\mathcal{R}/\hbar\mathcal{R})$ of $\mathcal{R}/\hbar\mathcal{R}$ is endowed with a structure of Poisson algebra by

$$\{\overline{b}_1, \overline{b}_2\} = \overline{\left(\frac{b_1b_2 - b_2b_1}{\hbar}\right)} \quad (b_1, b_2 \in \mathcal{R}, \overline{b}_1, \overline{b}_2 \in Z(\mathcal{R}/\hbar\mathcal{R}))\,.$$

Assume moreover that $\mathcal{R}$ is a Hopf algebra and that there exists a Hopf subalgebra H of $\mathcal{R}/\hbar\mathcal{R}$ such that $H \subset Z(\mathcal{R}/\hbar\mathcal{R})$ and $\{H, H\} \subset H$. Then H is naturally a Poisson Hopf algebra.

We will apply this fact to the situation $\mathbf{B} = \mathbf{A}_\zeta$, $\hbar = \ell(q^\ell - q^{-\ell})$, and $\mathcal{R} = C_{\mathbf{A}_\zeta}, U_{\mathbf{A}_\zeta}, D_{\mathbf{A}_\zeta}$. Note that we have $\mathbf{A}_\zeta/\ell(q^\ell - q^{-\ell})\mathbf{A}_\zeta \cong \mathbf{C}$ by

$$\operatorname{Ker}\pi_\zeta = \mathbf{A}_\zeta(q^{1/|\Lambda/Q|} - \zeta) = \mathbf{A}_\zeta\ell(q^\ell - q^{-\ell}).$$

The cases $\mathcal{R} = C_{\mathbf{A}_\zeta}, U_{\mathbf{A}_\zeta}$ is already known. Namely, we have the following.

THEOREM 4.2 ([3]). *The Hopf subalgebra* $\operatorname{Im}{}^t\xi$ *of* $Z(C_\zeta)$ *is closed under the Poisson bracket given in Proposition* 4.1. *Moreover, the isomorphism* $\operatorname{Im}{}^t\xi \cong \mathbf{C}[G]$ *is that of Poisson Hopf algebras, where the Poisson Hopf algebra structure of* $\mathbf{C}[G]$ *is the one for* $\mathbf{C}[\Delta G] \cong \mathbf{C}[G]$ *attached to the Manin triple* $(\mathfrak{g}\oplus\mathfrak{g}, \Delta\mathfrak{g}, \mathfrak{k})$.

THEOREM 4.3 ([4], [6]). *The Hopf subalgebra* $\operatorname{Im}{}^t\eta$ *of* $Z(U_\zeta)$ *is closed under the Poisson bracket given in Proposition* 4.1. *Moreover, the isomorphism* $\operatorname{Im}{}^t\eta \cong \mathbf{C}[K]$ *is that of Poisson Hopf algebras, where the Poisson Hopf algebra structure of* $\mathbf{C}[K]$ *is the one attached to the Manin triple* $(\mathfrak{g}\oplus\mathfrak{g}, \mathfrak{k}, \Delta\mathfrak{g})$.

In the rest of this paper we will deal with the case where $\mathcal{R} = D_{\mathbf{A}_\zeta}$. The following is the main result of this paper.

THEOREM 4.4. *The subalgebra* $\operatorname{Im}({}^t\xi\otimes{}^t\eta)$ *of* $Z(D_\zeta)$ *is closed under the Poisson bracket given in Proposition* 4.1. *Moreover, under the identification*

$$\operatorname{Im}({}^t\xi\otimes{}^t\eta) \cong \mathbf{C}[G]\otimes\mathbf{C}[K] \cong \mathbf{C}[\Delta G]\otimes\mathbf{C}[K]$$

this Poisson algebra structure coincides with the one attached to the Manin triple $(\mathfrak{g}\oplus\mathfrak{g}, \Delta\mathfrak{g}, \mathfrak{k})$ *as in Proposition* 2.3.

Set

$$\mathcal{J} = \operatorname{Ker}(\xi\circ\overline{\pi}_\zeta^{U^L}) \subset U^L_{\mathbf{A}_\zeta},$$
$$\mathcal{I} = \{x \in U_{\mathbf{A}_\zeta} \mid \langle x, V_{\mathbf{A}_\zeta}\rangle \subset \ell(q^\ell - q^{-\ell})\mathbf{A}_\zeta\}.$$

LEMMA 4.5. *Let* $h \in \operatorname{Im}({}^t\xi)$ *and* $\varphi \in \operatorname{Im}({}^t\eta)$. *Take* $p \in \mathbf{C}[G]$ *and* $\Phi \in U_{\mathbf{A}_\zeta}$ *such that* $h = {}^t\xi(p)$ *and* $\varphi = \pi_\zeta^U(\Phi)$ *respectively. Assume*

$$\Phi\otimes 1 - \sum_{(\Phi)}\Phi_{(1)}\otimes\iota(\Phi_{(0)}) \in \ell(q^\ell - q^{-\ell})\sum_r \Psi_r\otimes X_r + \mathcal{I}\otimes\mathcal{J} \subset U_{\mathbf{A}_\zeta}\otimes U^L_{\mathbf{A}_\zeta}$$

with $\Psi_r \in U_{\mathbf{A}_\zeta}$, $X_r \in U^L_{\mathbf{A}_\zeta}$. *Then we have*

$$\{h, \varphi\} = \sum_r {}^t\xi((\xi\circ\overline{\pi}_\zeta^{U^L})(X_r)\cdot p)\otimes\pi_\zeta^U(\Psi_r)$$

with respect to the Poisson structure of $Z(D_\zeta)$ *given in Proposition* 4.1.

PROOF. Take $H \in C_{\mathbf{A}_\zeta}$ such that $h = \pi_\zeta^C(H)$. For $u \in U^L_{\mathbf{A}_\zeta}$, $v \in V_{\mathbf{A}_\zeta}$ we see easily that

$$\begin{aligned}&\langle \{h, \varphi\}, \overline{\pi}_\zeta^{U^L}(u) \otimes \overline{\pi}_\zeta^V(v)\rangle \\ &\quad = \pi_\zeta\Bigl(\langle H, u(\langle \Phi, v\rangle 1 - \sum_{(\Phi)} \langle \Phi_{(1)}, v\rangle \iota(\Phi_{(0)}))\rangle / \ell(q^\ell - q^{-\ell})\Bigr).\end{aligned}$$

Write

$$\Phi \otimes 1 - \sum_{(\Phi)} \Phi_{(1)} \otimes \iota(\Phi_{(0)}) = \ell(q^\ell - q^{-\ell}) \sum_r \Psi_r \otimes X_r + \sum_s \Xi_s \otimes Y_s\,,$$

where $\Xi_s \in \mathcal{I}$, $Y_s \in \mathcal{J}$. Then we have

$$\begin{aligned}&\langle \{h, \varphi\}, \overline{\pi}_\zeta^{U^L}(u) \otimes \overline{\pi}_\zeta^V(v)\rangle \\ &\quad = \sum_r \pi_\zeta(\langle \Psi_r, v\rangle)\pi_\zeta(\langle H, uX_r\rangle) + \sum_s \pi_\zeta\left(\frac{\langle \Xi_s, v\rangle}{\ell(q^\ell - q^{-\ell})}\right)\pi_\zeta(\langle H, uY_s\rangle) \\ &\quad = \sum_r \langle \pi_\zeta^U(\Psi_r), \overline{\pi}_\zeta^V(v)\rangle \langle h, \overline{\pi}_\zeta^{U^L}(u)\overline{\pi}_\zeta^{U^L}(X_r)\rangle \\ &\qquad + \sum_s \pi_\zeta\left(\frac{\langle \Xi_s, v\rangle}{\ell(q^\ell - q^{-\ell})}\right)\langle h, \overline{\pi}_\zeta^{U^L}(u)\overline{\pi}_\zeta^{U^L}(Y_s)\rangle\,.\end{aligned}$$

By $h = {}^t\xi(p)$ we have

$$\begin{aligned}\langle h, \overline{\pi}_\zeta^{U^L}(u)\overline{\pi}_\zeta^{U^L}(X_r)\rangle &= \langle p, (\xi \circ \overline{\pi}_\zeta^{U^L})(u)(\xi \circ \overline{\pi}_\zeta^{U^L})(X_r)\rangle \\ &= \langle (\xi \circ \overline{\pi}_\zeta^{U^L})(X_r) \cdot p, (\xi \circ \overline{\pi}_\zeta^{U^L})(u)\rangle = \langle {}^t\xi((\xi \circ \overline{\pi}_\zeta^{U^L})(X_r) \cdot p), \overline{\pi}_\zeta^{U^L}(u)\rangle\,.\end{aligned}$$

Similarly, we have

$$\langle h, \overline{\pi}_\zeta^{U^L}(u)\overline{\pi}_\zeta^{U^L}(Y_s)\rangle = \langle p \cdot (\xi \circ \overline{\pi}_\zeta^{U^L})(u), (\xi \circ \overline{\pi}_\zeta^{U^L})(Y_s)\rangle = 0\,.$$

Now the assertion is clear. □

Now let us show Theorem 4.4. By Theorem 4.2 and Theorem 4.3 it is sufficient to show that for $h \in \mathrm{Im}({}^t\xi)$, $\varphi \in \mathrm{Im}({}^t\eta)$ our Poisson bracket $\{h, \varphi\}$ defined above coincides with the one coming from the Manin triple. In order to avoid confusion we denote by $\{\ ,\ \}'$ the Poisson bracket of $\mathbf{C}[G] \otimes \mathbf{C}[K]$ coming from the Manin triple. We need to show

$$\{h, \varphi\} = \{h, \varphi\}' \quad (\forall h \in \mathrm{Im}({}^t\xi)) \tag{4.1}$$

for any $\varphi \in \mathrm{Im}({}^t\eta)$. If (4.1) holds for $\varphi \in \mathrm{Im}({}^t\eta)$, we have

$$\{f, \varphi\} = \{f, \varphi\}' \quad (\forall f \in \mathrm{Im}({}^t\xi \otimes {}^t\eta)) \tag{4.2}$$

by

$$\{h\psi,\varphi\} = \{h,\varphi\}\psi + h\{\psi,\varphi\} = \{h,\varphi\}'\psi + h\{\psi,\varphi\}' = \{h\psi,\varphi\}'$$

for $h \in \mathrm{Im}({}^t\xi)$, $\psi \in \mathrm{Im}({}^t\eta)$. Hence for each $\varphi \in \mathrm{Im}({}^t\eta)$ (4.1) is equivalent to (4.2). Then it follows from the definition of the Poisson algebra that (4.1) for $\varphi = \varphi_1, \varphi = \varphi_2$ imply those for $\varphi = \varphi_1\varphi_2, \varphi = \{\varphi_1, \varphi_2\}$. Therefore it is sufficient to show (4.1) in the cases where φ belongs to a generator system of the Poisson algebra $\mathrm{Im}({}^t\eta)$. By [4] the Poisson algebra $\mathbf{C}[K]$ is generated by the elements of the form $\hat{\chi}_\lambda, a_i, b_i$ for $\lambda \in \Lambda, i \in I$. Under the isomorphism $\mathbf{C}[K] \cong \mathrm{Im}({}^t\eta)$ of Poisson algebras we have

$$\begin{aligned} \hat{\chi}_\lambda &\longleftrightarrow \pi_\zeta^U(K_{\ell\lambda}) && (\lambda \in \Lambda)\,, \\ a_i\hat{\chi}_{-\alpha_i} &\longleftrightarrow \pi_\zeta^U((q_i - q_i^{-1})^\ell E_i^\ell K_i^{-\ell}) && (i \in I)\,, \\ b_i\hat{\chi}_{-\alpha_i} &\longleftrightarrow \pi_\zeta^U((q_i - q_i^{-1})^\ell F_i^\ell) && (i \in I)\,. \end{aligned}$$

Hence we have only to show (4.1) in the cases

$$\varphi = \pi_\zeta^U(K_{\ell\lambda})\,, \quad \varphi = \pi_\zeta^U((q_i - q_i^{-1})^\ell E_i^\ell K_i^{-\ell})\,, \quad \varphi = \pi_\zeta^U((q_i - q_i^{-1})^\ell F_i^\ell)$$

for $\lambda \in \Lambda, i \in I$.

For bases $\{X_r\}$ and $\{Y_r\}$ of $\mathfrak{g}$ and $\mathfrak{k}$ respectively such that $\rho((X_r, X_r), Y_s) = \delta_{rs}$ we have

$$\{h,\varphi\}' = \sum_r L_{X_r}(h)R_{Y_r}(\varphi) \quad (h \in \mathbf{C}[G], \varphi \in \mathbf{C}[K]).$$

From this we can easily deduce

$$\{h, \hat{\chi}_\lambda\}' = -\frac{1}{2}L_{H_\lambda}(h)\hat{\chi}_\lambda \qquad (\lambda \in \Lambda)\,, \tag{4.3}$$

$$\{h, a_i\hat{\chi}_{-\alpha_i}\}' = -\frac{(\alpha_i,\alpha_i)}{2}L_{e_i}(h)\hat{\chi}_{-\alpha_i} \qquad (i \in I)\,, \tag{4.4}$$

$$\{h, b_i\hat{\chi}_{-\alpha_i}\}' = -\frac{(\alpha_i,\alpha_i)}{2}L_{f_i}(h)\hat{\chi}_{-\alpha_i} \qquad (i \in I)\,, \tag{4.5}$$

where $H_\lambda \in \mathfrak{h}$ is given by $\kappa(H_\lambda, H) = \lambda(H) \quad (H \in \mathfrak{h})$.

Let us show (4.1) for $\varphi = \pi_\zeta^U((q_i - q_i^{-1})^\ell F_i^\ell)$. For $\Phi = (q_i - q_i^{-1})^\ell F_i^\ell$ we have

$$\begin{aligned} &\Phi \otimes 1 - \sum_{(\Phi)} \Phi_{(1)} \otimes \iota(\Phi_{(0)}) \\ &\quad = (q_i - q_i^{-1})^\ell \left(F_i^\ell \otimes 1 - \sum_{r=0}^{\ell} q_i^{r(\ell-r)} \begin{bmatrix} \ell \\ r \end{bmatrix}_{q_i} [r]!_{q_i} F_i^{\ell-r} K_i^{-r} \otimes F_i^{(r)} \right) \\ &\quad \in (q_i - q_i^{-1})^\ell (F_i^\ell \otimes 1 - (F_i^\ell \otimes 1 + [\ell]!_{q_i} K_i^{-\ell} \otimes F_i^{(\ell)})) + \mathcal{I} \otimes \mathcal{J} \\ &\quad = \ell(q^\ell - q^{-\ell}) \left(-\frac{(q_i - q_i^{-1})^\ell [\ell]!_{q_i}}{\ell(q^\ell - q^{-\ell})} K_i^{-\ell} \otimes F_i^{(\ell)} \right) + \mathcal{I} \otimes \mathcal{J}\,. \end{aligned}$$

Hence the assertion follows from

$$\left.\frac{(q_i - q_i^{-1})^\ell [\ell]!_{q_i}}{\ell(q^\ell - q^{-\ell})}\right|_{q^{1/|\Lambda/Q|}=\zeta} = \frac{(\alpha_i, \alpha_i)}{2},$$

which is easily checked. The verification of (4.1) for $\varphi = \pi_\zeta^U((q_i - q_i^{-1})^\ell E_i^\ell K_i^{-\ell})$ is similar and omitted.

Let us finally show (4.1) for $\varphi = \pi_\zeta^U(K_{\ell\lambda})$. We need to show

$$\{{}^t\xi(p), \varphi\} = -\frac{1}{2}{}^t\xi(L_{H_\lambda}(p)) \otimes \varphi$$

for $p \in \mathbf{C}[G]$. Take $H \in C_{\mathbf{A}_\zeta}$ such that $\pi_\zeta^C(H) = {}^t\xi(p)$. For $z \in \mathbf{C}^\times$ and $\nu \in \Lambda$ we set

$$(C_{\mathbf{A}_z})_\nu = \{\varphi \in C_{\mathbf{A}_z} \mid u \cdot \varphi = \chi_\nu(u)\varphi \ (u \in U_{\mathbf{A}_z}^{L,0})\},$$
$$(C_z)_\nu = \{\varphi \in C_z \mid u \cdot \varphi = \chi_\nu(u)\varphi \ (u \in U_z^{L,0})\}.$$

Then we have

$${}^t\xi(\mathbf{C}[G]_\nu) \subset (C_\zeta)_{\ell\nu} = \pi_\zeta^C((C_{\mathbf{A}_\zeta})_{\ell\nu}) \quad (\nu \in \Lambda),$$

and hence we may assume $p \in \mathbf{C}[G]_\nu$ and $H \in (C_{\mathbf{A}_\zeta})_{\ell\nu}$. For $\Phi = K_{\ell\lambda}$ we have

$$\Phi \otimes 1 - \sum_{(\Phi)} \Phi_{(1)} \otimes \iota(\Phi_{(0)}) = K_\lambda^\ell \otimes 1 - K_\lambda^\ell \otimes \iota(K_\lambda^\ell) = -K_\lambda^\ell \otimes (\iota(K_\lambda^\ell) - 1).$$

Hence for $u \in U_{\mathbf{A}_\zeta}^L$, $v \in V_{\mathbf{A}_\zeta}$ we have

$$\begin{aligned}
&\langle\{{}^t\xi(p), \varphi\}, \overline{\pi}_\zeta^{U^L}(u) \otimes \overline{\pi}_\zeta^V(v)\rangle \\
&= \pi_\zeta\left(\left\langle H, u\left(\langle\Phi, v\rangle 1 - \sum_{(\Phi)}\langle\Phi_{(1)}, v\rangle\iota(\Phi_{(0)})\right)\right\rangle \Big/ \ell(q^\ell - q^{-\ell})\right) \\
&= -\pi_\zeta(\langle K_{\ell\lambda}, v\rangle\langle H, u(\iota(K_{\ell\lambda}) - 1)\rangle/\ell(q^\ell - q^{-\ell})) \\
&= -\pi_\zeta(\langle K_{\ell\lambda}, v\rangle\langle(\iota(K_{\ell\lambda}) - 1) \cdot H, u\rangle/\ell(q^\ell - q^{-\ell})) \\
&= -\pi_\zeta((q^{\ell^2(\lambda,\nu)} - 1)/\ell(q^\ell - q^{-\ell}))\,\pi_\zeta(\langle K_{\ell\lambda}, v\rangle)\pi_\zeta(\langle H, u\rangle) \\
&= -\frac{\ell(\lambda, \nu)}{2\ell}\langle\varphi, \overline{\pi}_\zeta^V(v)\rangle\langle{}^t\xi(p), \overline{\pi}_\zeta^{U^L}(u)\rangle \\
&= -\frac{1}{2}\langle{}^t\xi(L_{H_\lambda}(p)) \otimes \varphi, \overline{\pi}_\zeta^{U^L}(u) \otimes \overline{\pi}_\zeta^V(v)\rangle.
\end{aligned}$$

The proof of Theorem 4.4 is complete.

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