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Covers in 4-uniform Intersecting Families with Covering Number Three

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Abstract. Let k be an integer. In [3, 4], Frankl, Ota and Tokushige proved that the maximum number of three-covers of a k-uniform intersecting family with covering number three is $k^3 - 3k^2 + 6k - 4$ for $k = 3$ or $k \geq 9$, but the case $4 \leq k \leq 8$ remained open. In this paper, we prove that the same holds for $k = 4$, and show that a 4-uniform family with covering number three which has 36 three-covers is uniquely determined.

1. Introduction

Throughout this paper, we let X denote a finite set. We let 2^X denote the family of all subsets of X and, for an integer $k \geq 1$, we let $\binom{X}{k}$ denote the family of those subsets of X which have cardinality k. A family $\mathcal{F} \subseteq 2^X$ is said to be k-uniform if $\mathcal{F} \subseteq \binom{X}{k}$. Let $\mathcal{F} \subseteq 2^X$ be a k-uniform family. We say that $\mathcal{F}$ is *intersecting* if $F \cap G \neq \emptyset$ for all $F, G \in \mathcal{F}$. A set $C \subseteq X$ is called a *cover* of $\mathcal{F}$ if it intersects with every member of $\mathcal{F}$, i.e., $C \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Let $\mathcal{C}(\mathcal{F}) := \{C : C \text{ is a cover of } \mathcal{F}\}$. The *covering number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is defined by $\tau(\mathcal{F}) := \min_{C \in \mathcal{C}(\mathcal{F})} |C|$. Note that if $\mathcal{F}$ is intersecting, then we have $\tau(\mathcal{F}) \leq k$ because $\mathcal{F} \subseteq \mathcal{C}(\mathcal{F})$. For an integer $t \geq 1$, we define $\mathcal{C}_t(\mathcal{F}) := \mathcal{C}(\mathcal{F}) \cap \binom{X}{t}$. Note that if $t < \tau(\mathcal{F})$, then $\mathcal{C}_t(\mathcal{F}) = \emptyset$. Also it is easy to see that if $t = \tau(\mathcal{F})$, then $|\mathcal{C}_t(\mathcal{F})| \leq k^t$ (see, for example, the proof of Lemma 2.1 (ii) (a) in Section 2).

Let t, k be integers with $k \geq t \geq 1$, and assume that $|X|$ is sufficiently large compared with t and k. Define

$$p_t(k) := \max\left\{|\mathcal{C}_t(\mathcal{F})| : \mathcal{F} \subseteq 2^X \text{is } k\text{-uniform and intersecting, and } \tau(\mathcal{F}) = t\right\}$$

(from the fact that every k-uniform family $\mathcal{F}$ with $\tau(\mathcal{F}) = t$ satisfies $|\mathcal{C}_t(\mathcal{F})| \leq k^t$, which is mentioned at the end of the preceding paragraph, it follows that if $|X|$ is sufficiently large, then the value of $p_t(k)$ does not depend on $|X|$). This paper is concerned with $p_3(k)$. However, the definition of $p_t(k)$ looks somewhat technical. Thus we here state a result of Frankl, Ota

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and Tokushige [4], which shows the importance of the function $p_{t-1}(k)$ in the study of the following more natural function $f_{k,t}(n)$. Define

$$f_{k,t}(n) := \max\left\{|\mathcal{F}| : \mathcal{F} \subseteq 2^X \text{is } k\text{-uniform and intersecting, and } \tau(\mathcal{F}) = t\right\},$$

where $n = |X|$. Note that the famous theorem of Erdős, Ko and Rado [1] shows that if $n \geq 2k \geq 4$ and $t \geq 1$, then $f_{k,t}(n) \leq \binom{n-1}{k-1}$. Now clearly $f_{k,1}(n) = \binom{n-1}{k-1}$. For $t \geq 2$, it is shown in [4] that if k is sufficiently large compared with t, then, as n tends to infinity, we have $f_{k,t}(n) \leq p_{t-1}(k)\binom{n}{k-t} + O(n^{k-t-1})$ (in fact, it is expected, though not yet proved, that equality holds). This shows the role of the function $p_{t-1}(k)$ in the determination of $f_{k,t}(n)$ (for a more precise result concerning the case where $2 \leq t \leq 4$, see [7], [2] and [3]).

We turn to $p_t(k)$. Clearly $p_1(k) = k$ for every $k \geq 1$. For $t \geq 2$, in Frankl, Ota and Tokushige [5], it is conjectured that $p_t(k) = k^t - \binom{t}{2}k^{t-1} + O(k^{t-2})$ $(k \to \infty)$, and the conjecture is settled affirmatively for $t = 4, 5$ (for $t \geq 6$, it is proved in the same paper that $p_t(k) \leq k^t - \frac{1}{\sqrt{2}}\lfloor\frac{t-1}{2}\rfloor^{\frac{3}{2}}k^{t-1} + O(k^{t-2})$). For $t = 2, 3$, the following precise results are proved in [2], [3] and [4].

THEOREM A (Frankl [2]). *Let $k \geq 2$. Then $p_2(k) = k^2 - k + 1$.*

THEOREM B (Frankl, Ota and Tokushige [3, 4]). *Let $k = 3$ or $k \geq 9$. Then $p_3(k) = k^3 - 3k^2 + 6k - 4$.*

We now describe examples related to Theorems A and B.

EXAMPLE 1. Let $k \geq 2$. Fix $2k-1$ elements y_i, z_j $(1 \leq i \leq k$ and $1 \leq j \leq k-1)$ of X. Set $Y := \{y_1, y_2, \ldots, y_k\}$, $Z_1 := \{z_1, z_2, \ldots, z_{k-1}, y_1\}$ and $Z_2 := \{z_1, z_2, \ldots, z_{k-1}, y_2\}$, and define $\mathcal{F}_1^{(k)} := \{Y, Z_1, Z_2\}$. Then $\mathcal{F}_1^{(k)}$ is k-uniform and intersecting, $\tau(\mathcal{F}_1^{(k)}) = 2$, and $|\mathcal{C}_2(\mathcal{F}_1^{(k)})| = k^2 - k + 1$.

EXAMPLE 2. Let $k \geq 3$. Fix $3(k-1)$ elements x_i, y_i, z_i $(1 \leq i \leq k-1)$ of X. For each $i = 1, 2$, set $X_i := \{x_1, x_2, \ldots, x_{k-1}, y_i\}$, $Y_i := \{y_1, y_2, \ldots, y_{k-1}, z_i\}$ and $Z_i := \{z_1, z_2, \ldots, z_{k-1}, x_i\}$, and define $\mathcal{F}_2^{(k)} := \{X_1, X_2, Y_1, Y_2, Z_1, Z_2\}$. Then $\mathcal{F}_2^{(k)}$ is k-uniform and intersecting, $\tau(\mathcal{F}_2^{(k)}) = 3$, and $|\mathcal{C}_3(\mathcal{F}_2^{(k)})| = (k-1)^3 + 3(k-1) = k^3 - 3k^2 + 6k - 4$.

In [2], [3] and [4], the following two theorems, which are stronger than Theorems A and B, are actually proved.

THEOREM C (Frankl [2]). *Let $k \geq 2$, and let $\mathcal{F} \subseteq \binom{X}{k}$ be an intersecting family with $\tau(\mathcal{F}) = 2$. Then $|\mathcal{C}_2(\mathcal{F})| \leq k^2 - k + 1$, with equality if and only if $\mathcal{F}$ is isomorphic to $\mathcal{F}_1^{(k)}$.*

THEOREM D (Frankl et al. [3, 4]). *Let $k = 3$ or $k \geq 9$, and let $\mathcal{F} \subseteq \binom{X}{k}$ be an intersecting family with $\tau(\mathcal{F}) = 3$. Then $|\mathcal{C}_3(\mathcal{F})| \leq k^3 - 3k^2 + 6k - 4$, with equality if and only if $\mathcal{F}$ is isomorphic to $\mathcal{F}_2^{(k)}$.*

It is natural to conjecture that Theorems B and D hold for $4 \le k \le 8$ as well. In this paper, as an initial step toward the determination of $p_3(k)$ for $4 \le k \le 8$, we prove the following theorem.

THEOREM 1. *We have* $p_3(4) = 36$.

We actually prove the following stronger result, which is an analogue of Theorems C and D;

THEOREM 2. *Let* $\mathcal{F} \subseteq \binom{X}{4}$ *be an intersecting family with* $\tau(\mathcal{F}) = 3$. *Then* $|\mathcal{C}_3(\mathcal{F})| \le 36$, *with equality if and only if* $\mathcal{F}$ *is isomorphic to* $\mathcal{F}_2^{(4)}$.

Our notation is standard except for the following. Let $\mathcal{A} \subseteq 2^X$ and $Y, Z \subseteq X$ with $Y \cap Z = \emptyset$, and write $Y = \{y_1, y_2, \ldots, y_l\}$ and $Z = \{z_1, z_2, \ldots, z_m\}$. We define $\mathcal{A}(y_1 y_2 \cdots y_l) = \mathcal{A}(Y) := \{A \in \mathcal{A} : Y \subseteq A\}$, $\mathcal{A}(\bar{y_1}\bar{y_2} \cdots \bar{y_l}) = \mathcal{A}(\bar{Y}) := \{A \in \mathcal{A} : Y \cap A = \emptyset\}$ and $\mathcal{A}(y_1 y_2 \cdots y_l \bar{z_1}\bar{z_2} \cdots \bar{z_l}) = \mathcal{A}(Y\bar{Z}) := \{A \in \mathcal{A} : Y \subseteq A \text{ and } Z \cap A = \emptyset\}$.

2. Preliminaries

Throughout the rest of this paper, let $\mathcal{F} \subseteq \binom{X}{4}$ be an intersecting family with $\tau(\mathcal{F}) = 3$, and let $\mathcal{C} := \mathcal{C}_3(\mathcal{F})$. We start with two easy lemmas.

LEMMA 2.1. *Let* $x, y \in X$ *with* $x \ne y$. *Then the following hold.*

(i) *We have* $|\mathcal{F}(\bar{x})| \ge 3$ *and* $|\mathcal{F}(\bar{x}\bar{y})| \ge 1$.

(ii) (a) *We have* $|\mathcal{C}(xy)| \le 4$.

(b) *Suppose that* $|\mathcal{C}(xy)| = 4$. *Then* $|\mathcal{F}(\bar{x}\bar{y})| = 1$ *and, if we write* $\mathcal{F}(\bar{x}\bar{y}) = \{F\}$, *then* $\mathcal{C}(xy) = \big\{\{x, y, z\} : z \in F\big\}$.

PROOF. Suppose that $|\mathcal{F}(\bar{x})| \le 2$. Then since $\mathcal{F}$ is intersecting, there exists $v \in X - \{x\}$ such that $v \in F$ for each $F \in \mathcal{F}(\bar{x})$. This means that $\{x, v\}$ is a cover of $\mathcal{F}$, which contradicts the assumption that $\tau(\mathcal{F}) = 3$. Thus $|\mathcal{F}(\bar{x})| \ge 3$. Similarly if $\mathcal{F}(\bar{x}\bar{y}) = \emptyset$, then $\{x, y\}$ is a cover of $\mathcal{F}$, a contradiction. Thus $|\mathcal{F}(\bar{x}\bar{y})| \ge 1$. This proves (i). To prove (ii), having (i) in mind, take $F \in \mathcal{F}(\bar{x}\bar{y})$. Then by the definition of $\mathcal{C}$, $\mathcal{C}(xy) \subseteq \big\{\{x, y, z\} : z \in F\big\}$. Hence $|\mathcal{C}(xy)| \le 4$. Suppose that $|\mathcal{C}(xy)| = 4$. Then $\mathcal{C}(xy) = \big\{\{x, y, z\} : z \in F\big\}$. Since F is an arbitrary member of $\mathcal{F}(\bar{x}\bar{y})$, this also implies $\mathcal{F}(\bar{x}\bar{y}) = \{F\}$. Thus (ii) is proved. □

LEMMA 2.2. *Let* v, w, x *and* y *be four distinct elements of* X. *Suppose that* $|\mathcal{C}(xy\bar{v}\bar{w})| = 4$, *and write* $\mathcal{F}(\bar{x}\bar{y}) = \{F\}$. *Then* $F \cap \{v, w\} = \emptyset$.

PROOF. In view of Lemma 2.1 (ii) (a), we have $\mathcal{C}(xy) = \mathcal{C}(xy\bar{v}\bar{w})$ and $|\mathcal{C}(xy)| = 4$. Hence by Lemma 2.1 (ii) (b), $\mathcal{C}(xy) = \big\{\{x, y, z\} : z \in F\big\}$. Since $\mathcal{C}(xy) = \mathcal{C}(xy\bar{v}\bar{w})$, this implies $F \cap \{v, w\} = \emptyset$. □

LEMMA 2.3. *Let $Y \subseteq X$ with $1 \le |Y| \le 2$. Let $F_1, F_2, F_3 \in \mathcal{F}$, and suppose that $F_i \cap F_j = Y$ for any i, j with $1 \le i < j \le 3$. Then the following hold.*

(i) *If $|Y| = 2$, then $|\mathcal{C}(\bar{Y})| \le 8$.*
(ii) *If $|Y| = 1$, and $|F \cap G| = 1$ for all $F, G \in \mathcal{F}$ with $F \ne G$, then $|\mathcal{C}(\bar{Y})| \le 19$.*

PROOF. Since $F_i \cap F_j = Y$ for any i, j with $1 \le i < j \le 3$ and since $C \cap (F_i - Y) \ne \emptyset$ for any $C \in \mathcal{C}(\bar{Y})$ and any i with $1 \le i \le 3$,

$$\mathcal{C}(\bar{Y}) \subseteq \left\{ \{\alpha, \beta, \gamma\} : \alpha \in F_1 - Y, \beta \in F_2 - Y, \gamma \in F_3 - Y \right\}. \tag{2.1}$$

Hence if $|Y| = 2$, then $|\mathcal{C}(\bar{Y})| \le (4 - |Y|)^3 = 8$.

Suppose that $|Y| = 1$, and $|F \cap G| = 1$ for all $F, G \in \mathcal{F}$ with $F \ne G$. By Lemma 2.1 (i), we can take $G \in \mathcal{F}(\bar{Y})$. Then by assumption, $|F_i \cap G| = 1$ for each $1 \le i \le 3$. Write $(F_i - Y) \cap G = \{a_i\}$ for each $1 \le i \le 3$. Then by (2.1), $C \cap \{a_1, a_2, a_3\} \ne \emptyset$ for all $C \in \mathcal{C}(\bar{Y})$, and hence $\mathcal{C}(\bar{Y}) \subseteq \left\{ \{\alpha, \beta, \gamma\} : \alpha \in F_1 - Y, \beta \in F_2 - Y, \gamma \in F_3 - Y \right\} - \left\{ \{\alpha, \beta, \gamma\} : \alpha \in F_1 - (Y \cup \{a_1\}), \beta \in F_2 - (Y \cup \{a_2\}), \gamma \in F_3 - (Y \cup \{a_3\}) \right\}$. Consequently $|\mathcal{C}(\bar{Y})| \le 27 - 8 = 19$. □

In the following three lemmas, Lemma 2.4 through 2.6, we fix the following notation. Let $F_1, F_2 \in \mathcal{F}$ with $|F_1 \cap F_2| = 2$, and write $F_1 =: \{a_1, a_2, a_3, a_4\}$ and $F_2 =: \{b_1, b_2, b_3, b_4\}$ so that $a_i = b_i$ for each $1 \le i \le 2$ and $a_i \ne b_i$ for each $3 \le i \le 4$. Set $\mathcal{G}_1 := \mathcal{C}(a_3 b_3 \bar{a}_1 \bar{a}_2)$, $\mathcal{G}_2 := \mathcal{C}(a_3 b_4 \bar{a}_1 \bar{a}_2)$, $\mathcal{G}_3 := \mathcal{C}(a_4 b_4 \bar{a}_1 \bar{a}_2)$ and $\mathcal{G}_4 := \mathcal{C}(a_4 b_3 \bar{a}_1 \bar{a}_2)$. Note that $\mathcal{C}(\bar{a}_1 \bar{a}_2) = \bigcup_{l=1}^{4} \mathcal{G}_l$.

LEMMA 2.4. *Let l be an integer with $1 \le l \le 4$, and suppose that $|\mathcal{G}_l| = 4$. Then $\mathcal{G}_l \cap \mathcal{G}_{l-1} \ne \emptyset$ and $\mathcal{G}_l \cap \mathcal{G}_{l+1} \ne \emptyset$, where indices are to be read modulo 4.*

PROOF. By the cyclic symmetry of $\{a_3, b_3\}, \{a_3, b_4\}, \{a_4, b_4\}$ and $\{a_4, b_3\}$, we may assume $l = 1$. Then by Lemma 2.1 (ii) (a), $\mathcal{G}_1 = \mathcal{C}(a_3 b_3)$. Having Lemma 2.1 (ii) (b) in mind, write $\mathcal{F}(\bar{a}_3 \bar{b}_3) = \{F\}$. Then by Lemma 2.2, $F \cap \{a_1, a_2\} = \emptyset$. Since $\mathcal{F}$ is intersecting, it follows that $F \cap F_1 = \{a_4\}$ and $F \cap F_2 = \{b_4\}$. Hence by Lemma 2.1 (ii) (b), $\{a_3, b_3, a_4\}, \{a_3, b_3, b_4\} \in \mathcal{C}(a_3 b_3)$. This implies $\{a_3, b_3, a_4\} \in \mathcal{G}_1 \cap \mathcal{G}_4$ and $\{a_3, b_3, b_4\} \in \mathcal{G}_1 \cap \mathcal{G}_2$, and hence we have $\mathcal{G}_1 \cap \mathcal{G}_4 \ne \emptyset$ and $\mathcal{G}_1 \cap \mathcal{G}_2 \ne \emptyset$. □

LEMMA 2.5. *We have $|\mathcal{C}(\bar{a}_1 \bar{a}_2)| \le 12$. Furthermore, if equality holds, then one of the following holds:*

(i) *$|\mathcal{G}_l| = 4$ for each $1 \le l \le 4$, and $\binom{\{a_3, a_4, b_3, b_4\}}{3} \subseteq \mathcal{C}(\bar{a}_1 \bar{a}_2)$; or*
(ii) *$|\mathcal{G}_l| = 3$ for each $1 \le l \le 4$, and $\mathcal{G}_l \cap \mathcal{G}_m = \emptyset$ for any l, m with $1 \le l < m \le 4$.*

PROOF. Since $|C| = 3$ for all $C \in \mathcal{C}$, we clearly have $\mathcal{G}_1 \cap \mathcal{G}_3 = \mathcal{G}_2 \cap \mathcal{G}_4 = \emptyset$. This implies that for each $1 \le m \le 4$, we have $\bigcap_{\substack{1 \le l \le 4 \\ l \ne m}} \mathcal{G}_l = \emptyset$. Hence by the inclusion-exclusion principle, $|\mathcal{C}(\bar{a}_1 \bar{a}_2)| = |\bigcup_{1 \le l \le 4} \mathcal{G}_l| = \sum_{l=1}^{4} (|\mathcal{G}_l| - |\mathcal{G}_l \cap \mathcal{G}_{l+1}|)$, where indices are to be read modulo 4. By Lemmas 2.1 (ii) (a) and 2.4, $|\mathcal{G}_l| - |\mathcal{G}_l \cap \mathcal{G}_{l+1}| \leq 3$ for each $1 \leq l \leq 4$. Consequently $|\mathcal{C}(\bar{a}_1\bar{a}_2)| = \sum_{l=1}^{4}(|\mathcal{G}_l| - |\mathcal{G}_l \cap \mathcal{G}_{l+1}|) \leq 12$. Suppose that equality holds. Then $|\mathcal{G}_l| - |\mathcal{G}_l \cap \mathcal{G}_{l+1}| = 3$ for each $1 \leq l \leq 4$. If $|\mathcal{G}_l| = 4$ for each $1 \leq l \leq 4$, then by Lemma 2.4, $\binom{\{a_3,a_4,b_3,b_4\}}{3} \subseteq \mathcal{C}(\bar{a}_1\bar{a}_2)$, and hence (i) holds. Thus by symmetry, we may assume that $|\mathcal{G}_1| = 3$ and $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$. Since $|\mathcal{G}_2| - |\mathcal{G}_2 \cap \mathcal{G}_3| = 3$ and $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$, it follows from Lemma 2.4 that $|\mathcal{G}_2| = 3$ and $\mathcal{G}_2 \cap \mathcal{G}_3 = \emptyset$. By a similar argument, we get $|\mathcal{G}_3| = |\mathcal{G}_4| = 3$ and $\mathcal{G}_3 \cap \mathcal{G}_4 = \mathcal{G}_1 \cap \mathcal{G}_4 = \emptyset$. Since $\mathcal{G}_1 \cap \mathcal{G}_3 = \mathcal{G}_2 \cap \mathcal{G}_4 = \emptyset$, this means that (ii) holds. □

LEMMA 2.6. *Suppose that* $|\mathcal{F}(a_1a_2)| \geq 3$. *Then* $|\mathcal{C}(\bar{a}_1\bar{a}_2)| \leq 10$. *Furthermore, if* $|\mathcal{C}(\bar{a}_1\bar{a}_2)| = 10$, *then* $|\mathcal{F}(a_1a_2)| = 3$, *and there exist* $x \in \{a_3, a_4\}$ *and* $y \in \{b_3, b_4\}$ *such that* $\mathcal{F}(a_1a_2) = \{F_1, F_2, \{a_1, a_2, x, y\}\}$.

PROOF. Suppose that $|\mathcal{C}(\bar{a}_1\bar{a}_2)| \geq 10$. Let $F_3 \in \mathcal{F}(a_1a_2) - \{F_1, F_2\}$, and write $F_3 = \{a_1, a_2, x, y\}$. Then by Lemma 2.3 (i), $\{x, y\} \cap (F_1 \cup F_2) \neq \emptyset$. Suppose that $x \notin F_1 \cup F_2$ or $y \notin F_1 \cup F_2$. By the symmetry of x and y and the symmetry of a_3, a_4, b_3 and b_4, we may assume that $x = a_3$ and $y \notin F_1 \cup F_2$. Then $\mathcal{G}_3 \cup \mathcal{G}_4 \subseteq \{\{a_3, a_4, b_4\}, \{a_4, b_4, y\}, \{a_3, a_4, b_3\}, \{a_4, b_3, y\}\}$, which implies $\mathcal{C}(\bar{a}_1\bar{a}_2) = \bigcup_{i=1}^{4} \mathcal{G}_i \subseteq \mathcal{G}_1 \cup \mathcal{G}_2 \cup \{\{a_4, b_4, y\}, \{a_4, b_3, y\}\}$. Hence by Lemmas 2.1 (ii) (a) and 2.4, $|\mathcal{C}(\bar{a}_1\bar{a}_2)| \leq |\mathcal{G}_1 \cup \mathcal{G}_2| + 2 = |\mathcal{G}_1| + (|\mathcal{G}_2| - |\mathcal{G}_1 \cap \mathcal{G}_2|) + 2 \leq 4 + 3 + 2 = 9$, a contradiction. Thus $x, y \in F_1 \cup F_2$. By the symmetry of x and y and the symmetry of $\{a_3, b_3\}$, $\{a_3, b_4\}$, $\{a_4, b_3\}$ and $\{a_4, b_4\}$, we may assume that $x = a_3$ and $y = b_3$. Then $\mathcal{G}_3 \subseteq \{\{a_3, a_4, b_4\}, \{a_4, b_3, b_4\}\}$. Hence $\mathcal{C}(\bar{a}_1\bar{a}_2) = \bigcup_{i=1}^{4} \mathcal{G}_i = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_4$. Since $\mathcal{G}_2 \cap \mathcal{G}_4 = \emptyset$, this together with Lemmas 2.1 (ii) (a) and 2.4 implies that $|\mathcal{C}(\bar{a}_1\bar{a}_2)| = |\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_4| = (|\mathcal{G}_1| - |\mathcal{G}_1 \cap \mathcal{G}_2|) + |\mathcal{G}_2| + (|\mathcal{G}_4| - |\mathcal{G}_1 \cap \mathcal{G}_4|) \leq 3 + 4 + 3 = 10$. Since we are assuming that $|\mathcal{C}(\bar{a}_1\bar{a}_2)| \geq 10$, it follows that $|\mathcal{C}(\bar{a}_1\bar{a}_2)| = 10$. Note that this in particular implies that $|\mathcal{G}_l| \geq 3$ for each $1 \leq l \leq 4$ with $l \neq 3$. We also have $|\mathcal{G}_3| \leq |\{\{a_3, a_4, b_4\}, \{a_4, b_3, b_4\}\}| = 2$.

Suppose that there exists $F_4 \in \mathcal{F}(a_1a_2) - \{F_1, F_2, F_3\}$. Arguing as in the first half of the preceding paragraph, we see that there exist $x' \in \{a_3, a_4\}$ and $y' \in \{b_3, b_4\}$ such that $F_4 = \{a_1, a_2, x', y'\}$. Then $\{x', y'\} \neq \{x, y\}$. Hence, arguing as in the second half of the preceding paragraph, we see that for some m $(1 \leq m \leq 4)$ with $m \neq 3$, we have $|\mathcal{G}_l| \geq 3$ for each $l \neq m$. But this contradicts the assertion that $|\mathcal{G}_3| \leq 2$. Therefore $\mathcal{F}(a_1a_2) = \{F_1, F_2, F_3\}$. □

In Lemmas 2.7 through 2.11, we fix the following notation. Let $F, F' \in \mathcal{F}$ with $F \neq F'$, and let $j_0 := |F \cap F'|$. Let $a \in X - (F \cup F')$, and set $\mathcal{H} := \{\{a, v, w\} : v \in F - F', w \in F' - F\}$ (it is not always true that $\mathcal{H} \subseteq \mathcal{C}$). The following lemma follows from the definition of $\mathcal{H}$.

LEMMA 2.7. (i) *For each* $v \in F - F'$, $|\mathcal{H}(v)| = 4 - j_0$.

(ii) $|\mathcal{H}| = (4 - j_0)^2$.

LEMMA 2.8. (i) $\mathcal{C}(a) \subseteq \left(\bigcup_{u \in F \cap F'} \mathcal{C}(au)\right) \cup \mathcal{H}$.

(ii) $|\mathcal{C}(a)| \le 4j_0 + (4-j_0)^2$.

PROOF. Take $C \in \mathcal{C}(a) - \bigcup_{u\in F\cap F'} \mathcal{C}(au)$. Then since $C \cap F \neq \emptyset$ and $C \cap F' \neq \emptyset$, we get $C \in \mathcal{H}$. Since C is arbitrary, this proves (i). Statement (ii) follows from (i) and Lemmas 2.1 (ii) (a) and 2.7 (ii). □

We here prove three technical lemmas.

LEMMA 2.9. *Suppose that* $j_0 = 1$, *and let* $v_0 \in F - F'$ *and* $w_0 \in F' - F$. *Then* $|\mathcal{C}(a)| - |\mathcal{C}(av_0)| - |\mathcal{C}(aw_0)| + |\mathcal{C}(av_0w_0)| \le 8$.

PROOF. Write $F \cap F' = \{u\}$. Note that $|\mathcal{C}(av_0)| + |\mathcal{C}(aw_0)| - |\mathcal{C}(av_0w_0)| = |\mathcal{C}(av_0) \cup \mathcal{C}(aw_0)|$. Hence by Lemmas 2.8 (i) and 2.1 (ii) (a), $|\mathcal{C}(a)| - (|\mathcal{C}(av_0)| + |\mathcal{C}(aw_0)| - |\mathcal{C}(av_0w_0)|) = |\mathcal{C}(a)| - |\mathcal{C}(av_0) \cup \mathcal{C}(aw_0)| = |\mathcal{C}(a) - \big(\mathcal{C}(av_0) \cup \mathcal{C}(aw_0)\big)| = |\big(\mathcal{C}(au) \cup (\mathcal{C}(a) \cap \mathcal{H})\big) - \big(\mathcal{C}(av_0) \cup \mathcal{C}(aw_0)\big)| \le |\mathcal{C}(au)| + |\big(\mathcal{C}(a) \cap \mathcal{H}\big) - \big(\mathcal{C}(av_0) \cup \mathcal{C}(aw_0)\big)| \le |\mathcal{C}(au)| + |\big\{\{a, v, w\} : v \in F - F' - \{v_0\}, w \in F' - F - \{w_0\}\big\}| \le 4 + (4 - j_0 - 1)^2 = 8$. □

LEMMA 2.10. *Suppose that* $j_0 \le 2$ *and* $|\mathcal{C}(a)| \ge 11$. *Then* $\mathcal{C}(av) \neq \emptyset$ *for each* $v \in F$.

PROOF. Note that $j_0 = 1$ or 2. Take $v \in F$.

First we consider the case where $v \in F \cap F'$. By Lemma 2.8 (i), $\mathcal{C}(a) \subseteq \big(\bigcup_{u\in F\cap F'} \mathcal{C}(au)\big) \cup \mathcal{H} = \mathcal{C}(av) \cup \big(\bigcup_{u\in (F\cap F')-\{v\}} \mathcal{C}(au)\big) \cup \mathcal{H}$. We have $|\bigcup_{u\in (F\cap F')-\{v\}} \mathcal{C}(au)| \le 4(j_0 - 1)$ by Lemma 2.1 (ii) (a) and $|\mathcal{H}| = (4 - j_0)^2$ by Lemma 2.7 (ii). Hence $|\mathcal{C}(a)| \le |\mathcal{C}(av)| + 4(j_0 - 1) + (4 - j_0)^2$. Since $|\mathcal{C}(a)| \ge 11$ by assumption, this implies $|\mathcal{C}(av)| \ge 11 - \big(4(j_0 - 1) + (4 - j_0)^2\big)$. Since $j_0 = 1$ or 2, we get $|\mathcal{C}(av)| \ge 2$.

Next we consider the case where $v \in F - F'$. By Lemma 2.8 (i), $\mathcal{C}(a) = \big(\bigcup_{u\in F\cap F'} \mathcal{C}(au)\big) \cup \big(\mathcal{C}(a) \cap \mathcal{H}\big) = \big(\bigcup_{u\in F\cap F'} \mathcal{C}(au)\big) \cup \big(\mathcal{C}(av) \cap \mathcal{H}\big) \cup \Big(\mathcal{C}(a) \cap \big(\mathcal{H} - \mathcal{H}(v)\big)\Big) \subseteq \big(\bigcup_{u\in F\cap F'} \mathcal{C}(au)\big) \cup \mathcal{C}(av) \cup \big(\mathcal{H} - \mathcal{H}(v)\big)$. We have $|\bigcup_{u\in F\cap F'} \mathcal{C}(au)| \le 4j_0$ by Lemma 2.1 (ii) (a) and $|\mathcal{H} - \mathcal{H}(v)| = (3 - j_0)(4 - j_0)$ by Lemma 2.7. Hence $|\mathcal{C}(a)| \le 4j_0 + |\mathcal{C}(av)| + (3 - j_0)(4 - j_0)$. Since $|\mathcal{C}(a)| \ge 11$ and $j_0 = 1$ or 2, this implies $|\mathcal{C}(av)| \ge 11 - \big(4j_0 + (3 - j_0)(4 - j_0)\big) = 1$, as desired. □

LEMMA 2.11. *Suppose that* $j_0 \le 2$ *and* $|\mathcal{C}(a)| \ge 12$. *Then* $|\mathcal{H} - \mathcal{C}(a)| \le 1$.

PROOF. Note that $|\mathcal{H} - \mathcal{C}(a)| = |\big(\mathcal{C}(a) \cup \mathcal{H}\big) - \mathcal{C}(a)| = |\mathcal{C}(a) \cup \mathcal{H}| - |\mathcal{C}(a)|$. Hence by Lemma 2.8 (i), $|\mathcal{H} - \mathcal{C}(a)| = |(\bigcup_{u\in F\cap F'} \mathcal{C}(au)\big) \cup \mathcal{H}| - |\mathcal{C}(a)|$. Consequently $|\mathcal{H} - \mathcal{C}(a)| \le 4j_0 + (4 - j_0)^2 - |\mathcal{C}(a)|$ by Lemmas 2.1 (ii) (a) and 2.7 (ii). Since $|\mathcal{C}(a)| \ge 12$ and $j_0 = 1$ or 2 by assumption, this implies $|\mathcal{H} - \mathcal{C}(a)| \le 4j_0 + (4 - j_0)^2 - 12 \le 1$. □

The following lemma follows from Theorem C. However, for the convenience of the reader, we include a proof which does not depend on Theorem C.

LEMMA 2.12. *Let $a \in X$. Then $|\mathcal{C}(a)| \leq 13$. Furthermore, if equality holds, then there exist $Y, Z \subseteq X - \{a\}$ with $Y \cap Z = \emptyset$ and $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $|Y| = 4$, $|Z| = 3$ and $\mathcal{F}(\bar{a}) = \{Y, Z \cup \{y_1\}, Z \cup \{y_2\}\}$.*

PROOF. In view of Lemma 2.1 (i), we can take $F, F' \in \mathcal{F}(\bar{a})$ with $F \neq F'$. Let j_0 and $\mathcal{H}$ be as in Lemmas 2.7 and 2.8. By Lemma 2.8 (ii), $|\mathcal{C}(a)| \leq 13$. Suppose that equality holds. Then by Lemma 2.8 (ii), $j_0 = 1$ or 3. Further by Lemmas 2.8 (i), 2.1 (ii) (a) and 2.7 (ii),

$$|\mathcal{C}(au)| = 4 \text{ for each } u \in F \cap F', \tag{2.2}$$

$$\mathcal{C}(au) \cap \mathcal{C}(au') = \emptyset \text{ for any } u, u' \in F \cap F' \text{ with } u \neq u', \tag{2.3}$$

and

$$\mathcal{H} \subseteq \mathcal{C}(a). \tag{2.4}$$

Assume for the moment that $j_0 = 3$. Let $u \in F \cap F'$. By Lemma 2.1 (i), we can take $F'' \in \mathcal{F}(\bar{a}\bar{u})$. If $F'' \cap (F \cap F') \neq \emptyset$, then, letting $u' \in F'' \cap (F \cap F')$, we get $\{a, u, u'\} \in \mathcal{C}(au)$ from (2.2) and Lemma 2.1 (ii) (b), which implies $\{a, u, u'\} \in \mathcal{C}(au) \cap \mathcal{C}(au')$, contradicting (2.3). Thus $F'' \cap (F \cap F') = \emptyset$. Hence $|F'' \cap F| = |F'' \cap F'| = 1$. This means that replacing F' by F'', we may assume $j_0 = 1$.

In the rest of the proof of Lemma 2.12, we assume $j_0 = 1$. Write $F \cap F' = \{y_1\}$. By Lemma 2.1 (i), we can take $G \in \mathcal{F}(\bar{a}) - \{F, F'\}$. By (2.4), we have $G \supseteq F - F'$ or $G \supseteq F' - F$. We may assume $G \supseteq F' - F$. Then $|G \cap F| = 1$. Write $G \cap F = \{y_2\}$. Since $G \neq F'$, $y_2 \neq y_1$. Since G is arbitrary, we get $\mathcal{F}(\bar{a}) - \{F, F'\} \subseteq \mathcal{F}(\bar{a}\bar{y}_1)$. Since $|\mathcal{C}(ay_1)| = 4$ by (2.2), it now follows from Lemma 2.1 (ii) (b) that $\mathcal{F}(\bar{a}) - \{F, F'\} = \{G\}$. Therefore if we let $Y = F$ and $Z = F' - F$, Y, Z, y_1 and y_2 have the required properties. □

3. Proof of Theorem 2

As in Section 2, let $\mathcal{F} \subseteq \binom{X}{4}$ be an intersecting family with $\tau(\mathcal{F}) = 3$, and let $\mathcal{C} = \mathcal{C}_3(\mathcal{F})$. In order to prove Theorem 2, it suffices to show that $\mathcal{F} \cong \mathcal{F}_2^{(4)}$, assuming that $|\mathcal{C}| \geq 36$. First we prove a technical claim, which we use toward the end of the proof.

CLAIM 3.1. *Let $F, F', G \in \mathcal{F}$, and suppose that $|F \cap G| = |F' \cap G| = |F \cap F'| = 1$ and $F \cap G \neq F' \cap G$. Write $G = \{a_1, a_2, a_3, a_4\}$ so that $F \cap G = \{a_1\}$ and $F' \cap G = \{a_2\}$. Then $|\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3)| \geq 2$.*

PROOF. By the inclusion-exclusion principle, $|\mathcal{C}(a_1) \cup \mathcal{C}(a_2) \cup \mathcal{C}(a_3)| = |\mathcal{C}(a_1)| + |\mathcal{C}(a_2)| + |\mathcal{C}(a_3)| - |\mathcal{C}(a_1a_2)| - |\mathcal{C}(a_1a_3)| - |\mathcal{C}(a_2a_3)| + |\mathcal{C}(a_1a_2a_3)| \leq |\mathcal{C}(a_1)| + |\mathcal{C}(a_2)| + (|\mathcal{C}(a_3)| - |\mathcal{C}(a_1a_3)| - |\mathcal{C}(a_2a_3)| + |\mathcal{C}(a_1a_2a_3)|)$. Since $|\mathcal{C}(a_1)| \leq 13$ and $|\mathcal{C}(a_2)| \leq 13$ by Lemma 2.12 and $|\mathcal{C}(a_3)| - |\mathcal{C}(a_1a_3)| - |\mathcal{C}(a_2a_3)| + |\mathcal{C}(a_1a_2a_3)| \leq 8$ by Lemma 2.9, we obtain $|\mathcal{C}(a_1) \cup \mathcal{C}(a_2) \cup \mathcal{C}(a_3)| \leq 34$. Recall that we are assuming $|\mathcal{C}| \geq 36$. Thus $|\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3)| = |\mathcal{C}| - |\mathcal{C}(a_1) \cup \mathcal{C}(a_2) \cup \mathcal{C}(a_3)| \geq 2$. □

Next we show that $\mathcal{F}$ has two members whose intersection has cardinality greater than or equal to two.

CLAIM 3.2. *There exist $F, G \in \mathcal{F}$ with $F \neq G$ such that $|F \cap G| \geq 2$.*

PROOF. Suppose that $|F \cap G| = 1$ for all $F, G \in \mathcal{F}$ with $F \neq G$. For each $x \in X$, we have $|\mathcal{C}(\bar{x})| = |\mathcal{C}| - |\mathcal{C}(x)| \geq 36 - 13 = 23$ by Lemma 2.12, and hence it follows from Lemma 2.3 (ii) that $|\mathcal{F}(x)| \leq 2$. Since $\mathcal{F}$ is intersecting, there exists $x \in X$ such that $|\mathcal{F}(x)| = 2$. Let $\mathcal{F}(x) = \{F_1, F_2\}$, and write $F_1 = \{x, a_2, a_3, a_4\}$ and $F_2 = \{x, b_2, b_3, b_4\}$. By Lemma 2.1 (i), we can take $F_3 \in \mathcal{F}(\bar{x}\bar{b}_2)$. Then $F_3 \cap \{a_2, a_3, a_4\} \neq \emptyset$ and $F_3 \cap \{b_3, b_4\} \neq \emptyset$. By symmetry, we may assume that $F_3 = \{a_2, b_3, y, z\}$ with $y, z \in X - (F_1 \cup F_2)$. By Lemma 2.1 (i), we can take $F_4 \in \mathcal{F}(\bar{x}\bar{a}_2)$. Since $|\mathcal{F}(v)| \leq 2$ for all $v \in X$, it follows that $F_4 \cap \{a_3, a_4\} \neq \emptyset$, $F_4 \cap \{b_2, b_4\} \neq \emptyset$ and $F_4 \cap \{y, z\} \neq \emptyset$. By symmetry, we may assume that $F_4 = \{a_3, b_2, y, w\}$ with $w \in X - (\bigcup_{i=1}^{3} F_i)$. By Lemma 2.1 (i), we can take $F_5 \in \mathcal{F}(\bar{x}\bar{y})$. Then $F_5 \cap \{a_2, a_3, b_2, b_3, x, y\} = \emptyset$, and hence $F_5 = \{a_4, b_4, z, w\}$. By inspection, we now see that $|\mathcal{C}| \leq |\mathcal{C}_3(\{F_1, F_2, F_3, F_4, F_5\})| = 30$, which contradicts the assumption that $|\mathcal{C}| \geq 36$. □

Having Claim 3.2 in mind, take $F_1, F_2 \in \mathcal{F}$ ($F_1 \neq F_2$) with $|F_1 \cap F_2| \geq 2$, and set $i_0 := |F_1 \cap F_2|$. Write $F_1 = \{a_1, a_2, a_3, a_4\}$ and $F_2 = \{b_1, b_2, b_3, b_4\}$ so that $a_i = b_i$ for each $1 \leq i \leq i_0$ and $a_i \neq b_i$ for each $i_0 + 1 \leq i \leq 4$.

We consider the cases where $i_0 = 2$ and $i_0 = 3$ separately. In Case 1, the case where $i_0 = 2$, we obtain a contradiction, which means that $\mathcal{F}$ has the property that there exist no $F, G \in \mathcal{F}$ such that $|F \cap G| = 2$. In Case 2, the case where $i_0 = 3$, based on this property, we show that $\mathcal{F}$ is isomorphic to $\mathcal{F}_2^{(4)}$.

Case 1: $i_0 = 2$.

CLAIM 3.3. *One of the following holds*:

(i) $|\mathcal{C}(a_1)| = |\mathcal{C}(a_2)| = 12$ *and* $\mathcal{C}(a_1a_2) = \emptyset$; *or*

(ii) $|\mathcal{C}(a_i)| = 13$, $|\mathcal{C}(a_{3-i})| \geq 11$ *and* $|\mathcal{C}(a_1a_2)| \leq 2$ *for some i with* $1 \leq i \leq 2$.

PROOF. Since $|\mathcal{C}(a_1)| + |\mathcal{C}(a_2)| - |\mathcal{C}(a_1a_2)| = |\mathcal{C}(a_1) \cup \mathcal{C}(a_2)| = |\mathcal{C}| - |\mathcal{C}(\bar{a}_1\bar{a}_2)| \geq 36 - 12 = 24$ by Lemma 2.5, the desired conclusion follows from Lemma 2.12. □

CLAIM 3.4. *We have* $|\mathcal{C}(a_1)| \geq 13$ *or* $|\mathcal{C}(a_2)| \geq 13$.

PROOF. Suppose that $|\mathcal{C}(a_i)| \leq 12$ for each $i = 1, 2$. Then by Claim 3.3, $|\mathcal{C}(a_i)| = 12$ for each $i = 1, 2$ and $\mathcal{C}(a_1a_2) = \emptyset$, and hence $|\mathcal{C}(\bar{a}_1\bar{a}_2)| = |\mathcal{C}| - |\mathcal{C}(a_1)| - |\mathcal{C}(a_2)| \geq 36 - 24 = 12$. By Lemma 2.5, this implies $|\mathcal{C}(\bar{a}_1\bar{a}_2)| = 12$. Let $\mathcal{G}_1 := \mathcal{C}(a_3b_3\bar{a}_1\bar{a}_2)$, $\mathcal{G}_2 := \mathcal{C}(a_3b_4\bar{a}_1\bar{a}_2)$, $\mathcal{G}_3 := \mathcal{C}(a_4b_4\bar{a}_1\bar{a}_2)$ and $\mathcal{G}_4 := \mathcal{C}(a_4b_3\bar{a}_1\bar{a}_2)$. Then (i) or (ii) of Lemma 2.5 holds.

First we consider the case where Lemma 2.5 (ii) holds; that is to say, $|\mathcal{G}_l| = 3$ for each $1 \leq l \leq 4$, and $\mathcal{G}_l \cap \mathcal{G}_m = \emptyset$ for any l, m with $1 \leq l < m \leq 4$. Write $\mathcal{G}_1 = \big\{\{a_3, b_3, x\}, \{a_3, b_3, y\}, \{a_3, b_3, z\}\big\}$. Since $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$ and $\mathcal{G}_1 \cap \mathcal{G}_4 = \emptyset$, we have $\{x, y, z\} \cap$

$\{a_4, b_4\} = \emptyset$. Hence $x, y, z \in X - \bigcup_{i=1}^{4}\{a_i, b_i\}$. Take $F \in \mathcal{F}(\bar{a}_3\bar{b}_3)$. Then $\{x, y, z\} \subseteq F$. Since $F \cap F_h \neq \emptyset$ for each $h = 1, 2$, $F = \{a_1, x, y, z\}$ or $F = \{a_2, x, y, z\}$. We may assume $F = \{a_1, x, y, z\}$. Take $F' \in \mathcal{F}(\bar{a}_1\bar{a}_2)$. Since $F' \cap F_h \neq \emptyset$ for each $h = 1, 2$, $\{a_i, b_j\} \subseteq F'$ for some $i, j \in \{3, 4\}$. Hence $|F \cap F'| \leq 2$. Also note that $a_2 \notin F \cup F'$ and $a_1 \in F$. Consequently, applying Lemma 2.10 with $a = a_2$ and $v = a_1$, we obtain $\mathcal{C}(a_1a_2) \neq \emptyset$. Therefore we get a contradiction to the earlier assertion that $\mathcal{C}(a_1a_2) = \emptyset$.

Next we consider the case where Lemma 2.5 (i) holds; that is to say, $|\mathcal{G}_l| = 4$ for each $1 \leq l \leq 4$ and $\binom{\{a_3, a_4, b_3, b_4\}}{3} = \mathcal{W} \subseteq \mathcal{C}(\bar{a}_1\bar{a}_2)$. Since $\mathcal{G}_1 \subseteq \mathcal{C}(a_3b_3)$, we have $4 = |\mathcal{G}_1| \leq |\mathcal{C}(a_3b_3)| \leq 4$ by Lemma 2.1 (ii) (a). This forces $|\mathcal{G}_1| = |\mathcal{C}(a_3b_3)| = 4$, and hence $\mathcal{G}_1 = \mathcal{C}(a_3b_3)$. Similarly $\mathcal{G}_2 = \mathcal{C}(a_3b_4)$, $\mathcal{G}_3 = \mathcal{C}(a_4b_4)$ and $\mathcal{G}_4 = \mathcal{C}(a_4b_3)$. Since $|\mathcal{C}(a_3b_3)| = |\mathcal{C}(a_4b_4)| = 4$, it follows form Lemma 2.1 (ii) (b) that $|\mathcal{F}(\bar{a}_3\bar{b}_3)| = |\mathcal{F}(\bar{a}_4\bar{b}_4)| = 1$. Write $\mathcal{F}(\bar{a}_3\bar{b}_3) = \{F\}$ and $\mathcal{F}(\bar{a}_4\bar{b}_4) = \{F'\}$. By Lemma 2.2, $F \cap \{a_1, a_2\} = F' \cap \{a_1, a_2\} = \emptyset$. Since $\{a_3, b_3, a_4\}, \{a_3, b_3, b_4\} \in \mathcal{W} \subseteq \mathcal{C}(\bar{a}_1\bar{a}_2)$, we have $\{a_3, b_3, a_4\}, \{a_3, b_3, b_4\} \in \mathcal{C}(\bar{a}_1\bar{a}_2a_3b_3) = \mathcal{C}(a_3b_3)$. Hence $a_4, b_4 \in F - F'$. Similarly $a_3, b_3 \in F' - F$. Hence $|F \cap F'| \leq 2$. Also note that $a_2 \notin F \cup F'$. Consequently, applying Lemma 2.11 with $a = a_2$, we see that at least one of $\{a_2, a_3, b_4\}$ and $\{a_2, a_4, b_3\}$ belongs to $\mathcal{C}$. If $\{a_2, a_3, b_4\} \in \mathcal{C}$, $\{a_2, a_3, b_4\} \in \mathcal{C}(a_3b_4) - \mathcal{G}_2$; if $\{a_2, a_4, b_3\} \in \mathcal{C}$, $\{a_2, a_4, b_3\} \in \mathcal{C}(a_4b_3) - \mathcal{G}_4$. Therefore we get a contradiction to the fact that we have both $\mathcal{G}_2 = \mathcal{C}(a_3b_4)$ and $\mathcal{G}_4 = \mathcal{G}(a_4b_3)$.

Thus in either case, we get a contradiction. This completes the proof of Claim 3.4. □

By Claim 3.4, (i) of Claim 3.3 does not hold. Hence (ii) of Claim 3.3 holds. By symmetry, we may assume $|\mathcal{C}(a_1)| = 13$, $|\mathcal{C}(a_2)| \geq 11$ and $|\mathcal{C}(a_1a_2)| \leq 2$. By Lemma 2.12 there exist $Y, Z \subseteq X - \{a_1\}$ with $Y \cap Z = \emptyset$ and $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $|Y| = 4$, $|Z| = 3$ and $\mathcal{F}(\bar{a}_1) = \{Y, Z \cup \{y_1\}, Z \cup \{y_2\}\}$. Then

$$\mathcal{C}(a_1) = \big\{\{a_1, y, z\} : y \in Y, z \in Z\big\} \cup \big\{\{a_1, y_1, y_2\}\big\}. \tag{3.1}$$

If $a_2 \in Y \cup Z$, then by (3.1), $|\mathcal{C}(a_1a_2)| \geq 3$, which contradicts the fact that $|\mathcal{C}(a_1a_2)| \leq 2$. Thus $a_2 \notin Y \cup Z$. By (3.1), this implies

$$\mathcal{C}(a_1a_2) = \emptyset\,. \tag{3.2}$$

Set $F_3 := Y$, $F_4 = Z \cup \{y_1\}$ and $F_5 := Z \cup \{y_2\}$. Note that $\mathcal{F}(\bar{a}_1) = \mathcal{F}(\bar{a}_1\bar{a}_2) = \{F_3, F_4, F_5\}$.

CLAIM 3.5. *We have* $\mathcal{F}(a_1) = \mathcal{F}(a_2) = \mathcal{F}(a_1a_2)$ *and* $\mathcal{F}(\bar{a}_1) = \mathcal{F}(\bar{a}_2) = \mathcal{F}(\bar{a}_1\bar{a}_2)$.

PROOF. Since $\mathcal{F}(\bar{a}_1) = \mathcal{F}(\bar{a}_1\bar{a}_2)$, we have $\mathcal{F}(\bar{a}_1) \subseteq \mathcal{F}(\bar{a}_2)$, and hence $\mathcal{F}(a_2) \subseteq \mathcal{F}(a_1)$. By way of contradiction, suppose that $\mathcal{F}(a_1) - \mathcal{F}(a_2) \neq \emptyset$, and take $F \in \mathcal{F}(a_1) - \mathcal{F}(a_2)$. Since $|F_3 \cap F_4| = 1$, at least one of F_3 and F_4, say F_h, satisfies $|F \cap F_h| \leq 2$. Note that $a_2 \notin F \cup F_h$ and $a_1 \in F - F_h$. Consequently, applying Lemma 2.10 with $F' = F_h$, $a = a_2$ and $v = a_1$, we get $\mathcal{C}(a_1a_2) \neq \emptyset$. But this contradicts (3.2). Thus $\mathcal{F}(a_1) = \mathcal{F}(a_2)$. This implies $\mathcal{F}(\bar{a}_1) = \mathcal{F}(\bar{a}_2)$, and hence $\mathcal{F}(a_1) = \mathcal{F}(a_2) = \mathcal{F}(a_1a_2)$ and $\mathcal{F}(\bar{a}_1) = \mathcal{F}(\bar{a}_2) = \mathcal{F}(\bar{a}_1\bar{a}_2)$. □

CLAIM 3.6. *There exist* $v \in \{a_3, a_4\}$ *and* $w \in \{b_3, b_4\}$ *such that* $\{v, w\} \cap Y \neq \emptyset$ *and* $\{v, w\} \cap Z \neq \emptyset$.

PROOF. Recall that $F_3, F_4, F_5 \in \mathcal{F}(\bar{a}_1\bar{a}_2)$. Hence $F_4 \cap \{a_3, a_4\} = F_4 \cap F_1 \neq \emptyset$ and $F_4 \cap \{b_3, b_4\} = F_4 \cap F_2 \neq \emptyset$. This implies that we have $Z \cap \{a_3, a_4\} \neq \emptyset$ or $Z \cap \{b_3, b_4\} \neq \emptyset$. By symmetry, we may assume that $Z \cap \{b_3, b_4\} \neq \emptyset$. Note that $Y \cap \{a_3, a_4\} = F_3 \cap \{a_3, a_4\} = F_3 \cap F_1 \neq \emptyset$. Now if we take $v \in Y \cap \{a_3, a_4\}$ and $w \in Z \cap \{b_3, b_4\}$, then v and w have the required properties. □

Let v and w be as in Claim 3.6. Let $F_6 \in \mathcal{F}(\bar{v}\bar{w})$. Since $\{a_1, v, w\} \in \mathcal{C}(a_1)$ by (3.1), $a_1 \in F_6$. Hence by Claim 3.5, $F_6 \in \mathcal{F}(a_1a_2) - \{F_1, F_2\}$. By Lemma 2.12, $|\mathcal{C}(\bar{a}_1\bar{a}_2)| \geq |\mathcal{C}| - |\mathcal{C}(a_1)| - |\mathcal{C}(a_2)| \geq 36 - 26 = 10$. In view of Lemma 2.6, this implies that $|\mathcal{C}(\bar{a}_1\bar{a}_2)| = 10$, $\mathcal{F}(a_1a_2) = \{F_1, F_2, F_6\}$, and $F_6 = \{a_1, a_2, c, d\}$, where $\{a_3, a_4\} - \{v\} = \{c\}$ and $\{b_3, b_4\} - \{w\} = \{d\}$. Take $F \in \mathcal{F}(\bar{c}\bar{d})$. Since $\mathcal{F}(a_1a_2) = \{F_1, F_2, F_6\}$, $F \notin \mathcal{F}(a_1a_2)$. Therefore by Claim 3.5, $F \in \mathcal{F} - \mathcal{F}(a_1a_2) = \mathcal{F} - \mathcal{F}(a_1) = \mathcal{F}(\bar{a}_1) = \mathcal{F}(\bar{a}_1\bar{a}_2)$. But then $F \cap F_6 = F \cap \{a_1, a_2, c, d\} = \emptyset$, which contradicts the assumption that $\mathcal{F}$ is intersecting. This completes the discussion for Case 1.

Case 2: $i_0 = 3$.

We have shown that Case 1 leads to a contradiction. Thus

$$|F \cap G| = 1 \text{ or } |F \cap G| = 3 \text{ for any } F, G \in \mathcal{F} \text{ with } F \neq G. \tag{3.3}$$

Let $F_3 \in \mathcal{F}(\bar{a}_3\bar{a}_4)$. Then by (3.3), $|F_3 \cap F_1| = |F_3 \cap \{a_1, a_2\}| = 1$. By the symmetry of a_1 and a_2, we may assume that $F_3 \cap F_1 = \{a_1\}$. By (3.3), this implies $F_3 \cap F_2 = F_3 \cap \{a_1, b_4\} = \{a_1\}$. Hence $F_3 \cap (F_1 \cup F_2) = \{a_1\}$. Write $F_3 = \{a_1, c_1, c_2, c_3\}$. Then $c_i \in X - (F_1 \cup F_2)$ for each $1 \leq i \leq 3$. Let $F_4 \in \mathcal{F}(\bar{a}_1\bar{a}_4)$. Then we can argue as above using (3.3), to get $|F_4 \cap (F_1 \cup F_2)| = |F_4 \cap \{a_2, a_3\}| = 1$. By the symmetry of a_2 and a_3, we may assume that $F_4 \cap (F_1 \cup F_2) = \{a_2\}$. By (3.3), either $F_4 \cap F_3 = \{c_1, c_2, c_3\}$ or $|F_4 \cap F_3| = |F_4 \cap \{c_1, c_2, c_3\}| = 1$. Suppose that $|F_4 \cap F_3| = 1$. Then $\{a_4, b_4, c_i\}$ is the only possible member of $\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3)$, where c_i is the unique element of $F_4 \cap F_3$. Hence $|\mathcal{C}(\bar{a}_1\bar{a}_2\bar{a}_3)| \leq 1$. But since $|F_3 \cap F_1| = |F_4 \cap F_1| = |F_4 \cap F_3| = 1$, $F_3 \cap F_1 = \{a_1\}$ and $F_4 \cap F_1 = \{a_2\}$, this contradicts Claim 3.1. Thus $F_4 \cap F_3 = \{c_1, c_2, c_3\}$, and hence $F_4 = \{a_2, c_1, c_2, c_3\}$.

Let $F_5 \in \mathcal{F}(\bar{a}_1\bar{c}_3)$. Then by (3.3), $|F_5 \cap (F_3 \cup F_4)| = |F_5 \cap \{c_1, c_2\}| = 1$. By the symmetry of c_1 and c_2, we may assume that $F_5 \cap (F_3 \cup F_4) = \{c_1\}$. Then by (3.3), $F_5 \cap (F_1 \cup F_2) = \{a_3\}$ or $F_5 \cap (F_1 \cup F_2) = \{a_4, b_4\}$. If $F_5 \cap (F_1 \cup F_2) = \{a_3\}$, then since $|F_5 \cap F_3| = 1$, we get a contradiction to Claim 3.1 by arguing as in the first paragraph with F_4 replaced by F_5. Thus $F_5 \cap (F_1 \cup F_2) = \{a_4, b_4\}$. Hence $F_5 = \{c_1, a_4, b_4, d\}$ with $d \in X - (\bigcup_{h=1}^{4} F_h)$. Let $F_6 \in \mathcal{F}(\bar{a}_1\bar{c}_1)$. Then by (3.3), $|F_6 \cap (F_3 \cup F_4)| = |F_6 \cap \{c_2, c_3\}| = 1$. By the symmetry of c_2 and c_3, we may assume that $F_6 \cap (F_3 \cup F_4) = \{c_2\}$. Then, arguing as in the first paragraph with F_1 and F_2 replaced by F_3 and F_4, and F_3 and F_4 replaced by F_5 and F_6, we obtain $F_6 = \{c_2, a_4, b_4, d\}$.

Now note that $\{F_1, F_2, F_3, F_4, F_5, F_6\} \cong \mathcal{F}_2^{(4)}$. Since $|\mathcal{C}| \geq 36$, this implies $|\mathcal{C}| = 36$ and $\mathcal{C} = \mathcal{C}_3(\{F_1, \ldots, F_6\})$. In particular,

$$\Big\{\{x, y, z\} : x \in \{c_1, c_2, c_3\},\ y \in \{a_1, a_2, a_3\},\ z \in \{a_4, b_4, d\}\Big\} \subseteq \mathcal{C}, \tag{3.4}$$

and

$$\Big\{\{a_1, a_2, z\} : z \in \{a_4, b_4, d\}\Big\} \cup \Big\{\{a_4, b_4, x\} : x \in \{c_1, c_2, c_3\}\Big\}$$
$$\cup \Big\{\{c_1, c_2, y\} : y \in \{a_1, a_2, a_3\}\Big\} \subseteq \mathcal{C}\,. \qquad (3.5)$$

Suppose that there exists $F \in \mathcal{F} - \{F_1, \ldots, F_6\}$. By (3.4), we have $F \supseteq \{c_1, c_2, c_3\}$ or $F \supseteq \{a_1, a_2, a_3\}$ or $F \supseteq \{a_4, b_4, d\}$. By symmetry, we may assume $F \supseteq \{c_1, c_2, c_3\}$. Since $F \cap F_1 \neq \emptyset$, $F \cap F_2 \neq \emptyset$ and $F \neq F_3, F_4$, this forces $F = \{a_3, c_1, c_2, c_3\}$. But then $\{a_1, a_2, d\} \cap F = \emptyset$, which contradicts (3.5). Therefore $\mathcal{F} = \{F_1, \ldots, F_6\} \cong \mathcal{F}_2^{(4)}$.

This completes the proof of Theorem 2. □

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