TOKYO J. MATH.
VOL. 35, NO. 1, 2012

Continued Fractions and Gauss' Class Number Problem for Real Quadratic Fields

Fuminori KAWAMOTO and Koshi TOMITA

Gakushuin University and Meijo University

Abstract. The main purpose of this article is to present a numerical data which shows relations between real quadratic fields of class number 1 and a mysterious behavior of the period of simple continued fraction expansion of certain quadratic irrationals. For that purpose, we define a class number, a fundamental unit, a discriminant and a Yokoi invariant for a non-square positive integer, and then see that a generalization of theorems of Siegel and of Yokoi holds. These and a theorem of Friesen and Halter-Koch imply several interesting conjectures for solving Gauss' class number problem for real quadratic fields.

1. Introduction

The main purpose of this article is to present a numerical data which shows relations between real quadratic fields of class number 1 and a mysterious behavior of the period of simple continued fraction expansion of certain quadratic irrationals. We shall pose several conjectures which are based on the data. These imply that Gauss' class number problem (Gauss [8, Article 304]) has an affirmative answer, namely, there exist infinitely many real quadratic fields of class number 1. They are very simple. Therefore they are very beautiful. Let d be a non-square positive integer which is not divisible by 4, and we denote by $\ell = \ell(d)$ the period of simple continued fraction expansion of $(1+\sqrt{d})/2$ or $\sqrt{d}$ according to whether $d \equiv 1$ or $d \equiv 2,\ 3 \bmod 4$. We arrange some values of d in ascending order of size in each period ℓ on Table 1.1 below. We take notice of the minimum value of d that is the first column in each period. If it is square-free then it gives a real quadratic field $\mathbf{Q}(\sqrt{d})$. In this paper we shall expect from a numerical experiment that the class number h_d is equal to 1 except for six values of d (Conjecture 4.1).

Let $\mathbf{Q}(\sqrt{d})$ be a real quadratic field where d is a square-free positive integer with $d > 1$. Let $\varepsilon_d > 1$ be the fundamental unit of it and then we can write uniquely $\varepsilon_d = (t + u\sqrt{d})/2$ with positive integers t, u. We define an integer $m_d := [u^2/t]\,(\geq 0)$ and call it the *Yokoi invariant* of a real quadratic field. Here, $[x]$ is the largest integer $\leq x$. This invariant was introduced by Yokoi [25] and he proved that if $d > 13$ then $m_d d < \varepsilon_d < (m_d + 1)d$, so

Received October 28, 2010
2000 *Mathematics Subject Classification*: 11R29 (Primary), 11A55, 11R11, 11R27 (Secondary)
Key words and phrases: Continued fractions, real quadratic fields, fundamental units, class numbers

TABLE 1.1. Some values of d in each period ℓ

ℓ	d					
1	2	5	10	13	26	...
2	3	6	11	15	18	...
3	17	37	61	65	101	...
4	7	14	23	33	34	...
5	41	74	149	157	181	...
6	19	22	54	57	59	...
7	58	89	109	113	137	...
8	31	71	91	135	153	...
9	73	97	106	233	277	...
10	43	67	86	115	118	...
11	265	298	541	554	593	...
12	46	103	127	177	209	...
13	421	746	757	778	1021	...
14	134	179	190	201	251	...
15	193	281	481	1066	1417	...
16	94	191	217	249	302	...
$\vdots$	...	...	...	...	...	...

that the quantity m_d gives a size of the fundamental unit for d. When the value of m_d is large, we may consider that the fundamental unit is large. On the other hand, we see by a very deep result of Siegel [21], concerning the approximate behavior of the product of class number and regulator, that the fundamental unit of a real quadratic field of class number 1 is relatively large. Hence we have to study a real quadratic field whose Yokoi invariant is large in order to find such a field. In [12] we introduced the notion of a *real quadratic field of minimal type* in terms of continued fractions, and provided a possibility that the Yokoi invariant of it is relatively large ([12, Proposition 4.4]). Also, the Yokoi invariant of a real quadratic field that is not of minimal type is equal to at most 3 (cf. [12, Proposition 4.2] and Proposition 4.2). We put $\omega(d) := (1+\sqrt{d})/2$ or $\sqrt{d}$ according to whether $d \equiv 1$ or $d \equiv 2,\ 3 \bmod 4$. The canonical integral basis of $\mathbf{Q}(\sqrt{d})$ is given by $\{1, \omega(d)\}$. We classify the set of all real quadratic fields by using the period $\ell(d)$ of simple continued fraction expansion of $\omega(d)$, and then study the fundamental unit and the class number of $\mathbf{Q}(\sqrt{d})$ in each fixed period $\ell = \ell(d)$. The notion of a real quadratic field of minimal type was born under such a point of view. Though it is known that the fundamental unit is calculated by using the simple continued fraction expansion of $\omega(d)$, the Yokoi invariant is also calculated by using it. In [13], for any non-square positive integer d with $4 \nmid d$, we extended a Yokoi invariant m_d in terms of continued fractions. By using it, we studied an infinite family of real quadratic fields with large even period of minimal type, and then obtained an information on the class number of them ([13, Theorem 1.1]). Next, for any non-square positive integer d, we shall extend a Yokoi invariant m_d and furthermore extend a class number h_d. Then we shall utilize these invariants in order to study a real quadratic field of class number 1. For the application of them to a real quadratic field, we need only the case where $4 \nmid d$.

This paper is organized as follows. In Section 2.1, we let d be any non-square positive integer and associate it with a certain order of a real quadratic field. Then we define a conductor f_d, a discriminant D_d, a fundamental unit E_d, a Yokoi invariant m_d and a class number h_d of d. This new Yokoi invariant m_d is calculated again by using continued fractions (Proposition 3.3). We shall also show theorems of Siegel (Proposition 2.2) and of Yokoi (Theorem 2.1) for a non-square positive integer. Hence we can see by these theorems that if h_d is small for a non-square positive integer d then m_d is relatively large. Though Proposition 2.2 can be shown by using a theorem of Siegel (Hua [10, Theorem 12.15.4]) on an integral binary quadratic form, we will prove it in terms of ideals to make the paper readable and self-contained. Proofs of Theorem 2.1 and Proposition 2.2 are given in Section 3. The notion of a real quadratic field of minimal type stated as above was introduced by using an improved theorem of Friesen and Halter-Koch ([12, Theorem 3.1] and [13, Remark 2.2]). In Section 4.1, this theorem and Proposition 3.3 yield another representation of the Yokoi invariant (Lemma 4.1). In Section 4.2, a rough guess is given by using Lemma 4.1. From this, we deduce a characterization of non-square positive integers whose Yokoi invariant is large. So, we consider a numerical experiment based on this deduction which is carried out by using PARI-GP [2]. In Section 4.2.1, we report the result and pose Conjecture 4.1 mentioned in the beginning and Conjectures 4.2 and 4.3. In Section 4.2.2 we state an approach for solving our conjectures.

For an irrational number ω, we denote by $\omega = [a_0, a_1, \dots]$ the simple continued fraction expansion of it. For a real number x, $[x]$ denotes the largest integer $\leq x$. We denote by $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$ the set of positive integers, the ring of rational integers and the field of rational numbers, respectively. For a set S, $|S|$ denotes the cardinal of S.

2. Definition of invariants and basic properties

Throughout Section 2, we let d be a non-square positive integer. We shall associate it with a certain order of a real quadratic field and define a conductor, a discriminant, a fundamental unit, a Yokoi invariant and a class number of d.

2.1. Definition of invariants. We consider a factorization of d such that

$$d = d_1 d_2^2\,, \quad d_1, d_2 \in \mathbf{N}\,, \text{ and } d_1 \text{ is square-free} \tag{2.1}$$

(therefore, $d_1 > 1$). Then we define a positive integer f_d by putting

$$f = f_d := \begin{cases} d_2/2\,, & \text{if } 4 \mid d \text{ and } d_1 \equiv 2\,,\ 3 \mod 4\,, \\ d_2\,, & \text{otherwise,} \end{cases} \tag{2.2}$$

and call it the *conductor of d* briefly. Now we consider a real quadratic field $K = \mathbf{Q}(\sqrt{d_1})$ and let $\mathcal{O}_f$ be the order of conductor $f = f_d$ in K, that is, a subring of the ring $\mathcal{O}_K$ of integers in K, containing 1, with finite index $(\mathcal{O}_K : \mathcal{O}_f) = f$, so that it is a free $\mathbf{Z}$-module of rank 2 (cf. Cox [5, p.133] and Borevich and Shafarevich [3, p.88, Definition]). If we put

$$\tilde{\omega} = \tilde{\omega}(d) := \begin{cases} (1+\sqrt{d_1})/2\,, & \text{if } d_1 \equiv 1 \mod 4\,, \\ \sqrt{d_1}\,, & \text{if } d_1 \equiv 2,\ 3 \mod 4 \end{cases} \tag{2.3}$$

then we can explicitly write

$$\mathcal{O}_f = \mathbf{Z} + \mathbf{Z} f\tilde{\omega}$$

(cf. [5, Lemma 7.2]). Let D_d be the discriminant of $\mathcal{O}_f$:

$$D_d := \begin{vmatrix} 1 & f\tilde{\omega} \\ 1 & (f\tilde{\omega})' \end{vmatrix}^2 = D_{d_1} f^2\,. \tag{2.4}$$

Here, $(f\tilde{\omega})'$ denotes the non-trivial conjugate of $f\tilde{\omega}$ and D_{d_1} is the discriminant of K. We call D_d the *discriminant of d* briefly. For a ring R, we denote by $R^\times$ the group of units in R. The unit group $\mathcal{O}_f^\times$ becomes a finitely generated $\mathbf{Z}$-module of rank 1 ([3, Chap.2, §3, Theorem 5]). Let $E_d > 1$ be the fundamental unit of $\mathcal{O}_f$, that is, a free $\mathbf{Z}$-basis of $\mathcal{O}_f^\times$, and we put

$$e_f := (\mathcal{O}_K^\times : \mathcal{O}_f^\times)\,.$$

We call E_d the *fundamental unit of d* briefly. If d is square-free then, as $4 \nmid d$ and $d_2 = 1$, we have $f = 1$. In particular, since d_1 is square-free, E_{d_1} is the fundamental unit of $\mathcal{O}_K = \mathcal{O}_1$. Hence we see by the definition of e_f that

$$E_d = E_{d_1}^{e_f}\,. \tag{2.5}$$

We let $\alpha \in \mathcal{O}_K$ and write uniquely $\alpha = x + y\tilde{\omega}$ with some integers x, y. Since $\{1, \tilde{\omega}\}$ is linearly independent over $\mathbf{Z}$, we have

$$\alpha \in \mathcal{O}_f \iff y \equiv 0 \mod f \iff \alpha \equiv a \mod f\mathcal{O}_K \text{ for some } a \in \mathbf{Z}\,. \tag{2.6}$$

DEFINITION 2.1. Since E_d is an element of $\mathcal{O}_K$, we can write uniquely $E_d = (t + u\sqrt{d_1})/2$ with some integers t, u satisfying $t \equiv u \mod 2$. As $E_d > 1$, t and u are positive. (Note that $E_d + E_d' = t$, $E_d - E_d' = u\sqrt{d_1}$. Here, E_d' denotes the non-trivial conjugate of E_d over $\mathbf{Q}$.) If $d_1 \equiv 1 \mod 4$ then, as $E_d = ((t-u)/2) + u\tilde{\omega}$, we have $f \mid u$ by (2.6). If $d_1 \equiv 2,\ 3 \mod 4$ then both t and u are even and $E_d = (t/2) + (u/2)\tilde{\omega}$. We see by (2.6) that $f \mid u$. Thus we can also write

$$E_d = \frac{t + (u/f)f\sqrt{d_1}}{2}\,.$$

Then we define an integer m_d by putting

$$m_d := \left[\frac{(u/f)^2}{t}\right] = \left[\frac{u^2}{tf^2}\right] (\geq 0)\,,$$

and call it the *Yokoi invariant of d* briefly. This definition works for an arbitrary positive integer f, and then we also call the value $m(\mathcal{O}_f) := [u^2/(tf^2)]$ the *Yokoi invariant of the order $\mathcal{O}_f$*.

REMARK 2.1. If d is square-free then, as $f = 1$, m_d coincides with the (original) Yokoi invariant of $\mathcal{O}_K = \mathcal{O}_1$.

If $\mathcal{O}_f \subsetneq \mathcal{O}_K$, that is, $f > 1$ then, since K is the quotient field of $\mathcal{O}_f$, the order $\mathcal{O}_f$ is not integrally closed so that it is not a Dedekind domain. So, we consider the set $I(\mathcal{O}_f)$ of all invertible fractional ideals $\mathfrak{a}$ of $\mathcal{O}_f$. This means that there exists a fractional ideal $\mathfrak{b}$ of $\mathcal{O}_f$ such that $\mathfrak{ab} = \mathcal{O}_f$ (cf. [5, Proposition 7.4]). Then, $I(\mathcal{O}_f)$ becomes an abelian group under multiplication and the set $P(\mathcal{O}_f)$ of all principal ideals is a subgroup of it. The quotient group

$$Cl(\mathcal{O}_f) := I(\mathcal{O}_f)/P(\mathcal{O}_f)$$

is called the ideal class group of $\mathcal{O}_f$, and it is a finite abelian group with order $h_d := |Cl(\mathcal{O}_f)|$. We call h_d the *class number of d* briefly and see by the class number formula for an order ([5, Theorem 7.24], Lang [14, Chap.8, Theorem 7]) that fa_f/e_f is a positive integer and

$$h_d = h_{d_1} \cdot \frac{fa_f}{e_f} \tag{2.7}$$

holds. Here, h_{d_1} is the class number (in the wide sense) of real quadratic field $\mathbf{Q}(\sqrt{d_1})$, and we put

$$a_f := \prod_{p|f}\left(1 - \frac{\chi_{d_1}(p)}{p}\right)\,,$$

where p ranges over all distinct prime divisors of f and χ_{d_1} is the Kronecker character corresponding to $\mathbf{Q}(\sqrt{d_1})$. (Though it is assumed in [5, Theorem 7.24] that K is an imaginary quadratic field, the proof also works for a real quadratic field.)

2.2. Basic properties. Under the above setting, we put $T := t$ and $U := u/f$. Then, T and U are positive integers and the definition of conductor $f = f_d$ yields that

$$E_d = \begin{cases} \dfrac{T + U\sqrt{d/4}}{2}\,, & \text{if } 4 \mid d \text{ and } d_1 \equiv 2,\ 3 \mod 4\,, \\[2ex] \dfrac{T + U\sqrt{d}}{2}\,, & \text{otherwise.} \end{cases} \tag{2.8}$$

We also put

$$\omega = \omega(d) := \begin{cases} \sqrt{d/4}\,, & \text{if } d \equiv 0 \mod 4\,, \\ (1+\sqrt{d})/2\,, & \text{if } d \equiv 1 \mod 4\,, \\ \sqrt{d}\,, & \text{if } d \equiv 2,\ 3 \mod 4\,. \end{cases} \tag{2.9}$$

Then it is known that the simple continued fraction expansion of ω is periodic:

$$\omega = [a_0, \overline{a_1, \dots, a_{\ell-1}, a_\ell}]\,.$$

Here, ℓ is the (minimal) period of ω. Furthermore, if $\ell \geq 2$ then $a_1, \dots, a_{\ell-1}$ is a symmetric string of $\ell - 1$ positive integers. And it holds that $a_\ell = 2a_0$ in the case where $\omega = \sqrt{d/4}$ or $\sqrt{d}$, and that $a_\ell = 2a_0 - 1$ in the case where $\omega = (1 + \sqrt{d})/2$ (cf. [12, Theorem 3.1]). A theorem of Yokoi for a non-square positive integer holds:

THEOREM 2.1. *Under the above setting, the following are true.*

[A] *In the case where* $4 \mid d$ *and* $d_1 \equiv 2,\ 3 \mod 4$, *we have* $T \geq 5$ *if* $d > 12$. *Furthermore*,

$$m_d = [T/(d/4)] = [U/\sqrt{d/4}] = [E_d/(d/4)]\,.$$

[B] *Otherwise, the following hold.*

(i) *We have* $T \geq 5$ *if* $d > 13$ *and* $d \neq 20$.

We assume $d > 13$ *from now on.*

(ii) *When* ℓ *is even, we have*

$$m_d d \leq T - 1 < U\sqrt{d} < E_d < T < (m_d + 1)d\,. \tag{2.10}$$

(iii) *When* ℓ *is odd, we have*

$$m_d d < T < E_d < U\sqrt{d} < T + 1 \leq (m_d + 1)d\,. \tag{2.11}$$

So, dividing both sides of (2.10) *and* (2.11) *by* d *implies that* $m_d = [T/d] = [U/\sqrt{d}] = [E_d/d]$.

REMARK 2.2. Though Theorem 2.1 can be described in terms of discriminants of orders, it is complicated a little. If d is square-free then $f = 1$. And when $d > 13$, we see by Theorem 2.1 [B] that $m_d = [E_d/d]$, hence, $m_d d < E_d < (m_d + 1)d$. Consequently, we obtain a result [25, Theorem 1.1] of Yokoi. Thus we know by Theorem 2.1 that the quantity m_d gives a size of the fundamental unit E_d for $d/4$ or d according to whether "$4 \mid d$ and $d_1 \equiv 2,\ 3 \mod 4$" or not. The value of m_d gives a rough size of E_d instead of the regulator $\log E_d$. When this value is large, since E_d goes away from the origin 0, we may consider that E_d is large.

A theorem of Siegel for a non-square positive integer holds:

PROPOSITION 2.2. *Under the above setting, we have*

$$\lim_{d\to\infty} \frac{\log(h_d \log E_d)}{\log D_d} = \frac{1}{2}\,.$$

Here, d ranges over all non-square positive integers for which $d \to \infty$.

REMARK 2.3. If h_d is small then we know by Proposition 2.2 that E_d is relatively large. Proposition 2.2 can be proved by using a theorem of Siegel (Proposition 3.4) on an integral binary quadratic form. For a generalization of a theorem of Siegel, see Sands [20] and Louboutin [16, Corollary 5].

Proofs of these propositions are given in Section 3. If $h_d = 1$ then, since the fundamental unit E_d is relatively large by Proposition 2.2, we may consider by Theorem 2.1 that the value of m_d is relatively large. A due consideration for this is given in Section 4.2. The definition of the discriminant D_d yields the following:

LEMMA 2.3. *If $d \equiv 0,\ 1 \mod 4$ then $D_d = d$. If $d \equiv 2,\ 3 \mod 4$ then we have $D_d = 4d$.*

PROOF.
(i) If $4 \mid d$ and $d_1 \equiv 2,\ 3 \mod 4$ then $D_{d_1} = 4d_1$. As $f = d_2/2$, (2.4) yields that $D_d = d$.
(ii) If $4 \mid d$ and $d_1 \equiv 1 \mod 4$ then $D_{d_1} = d_1$. As $f = d_2$, we have $D_d = d$.
(iii) If $d \equiv 1$ mod 4 then the factorization $d = d_1 d_2^2$ implies that $d_1 \equiv 1$ mod 4 so that $D_{d_1} = d_1$. As $f = d_2$, we have $D_d = d$.
(iv) If $d \equiv 2,\ 3$ mod 4 then $4 \nmid d$ and $d = d_1 d_2^2$ imply that d_2 is odd. Since $d_1 \equiv 2,\ 3$ mod 4, we have $D_{d_1} = 4d_1$ so that $D_d = 4d$ by $f = d_2$. □

Thus we associate a non-square positive integer d with the order of discriminant d (resp. $4d$) when $d \equiv 0,\ 1$ (resp., $\equiv 2,\ 3$) mod 4. Throughout the remainder of the present paper, we let NS be the set of all non-square positive integers and DS the set of all D's in NS satisfying $D \equiv 0$ or $1 \mod 4$.

REMARK 2.4. In [13, Tables 2 and 3], we constructed sequences $\{d'(t)\}_{t\geq 1}$ of NS such that the calculated values of $m_{d'(t)}$ are constant although $d'(t)$ has a square factor. These examples motivated that for any d in NS, we define the Yokoi invariant m_d of d. As we shall see in Proposition 3.3, it coincides with the Yokoi invariant defined in terms of continued fractions in [13]. So, for any d in NS, we can observe how the value of m_d (and h_d) has changed. Under this point of view, we will consider a numerical experiment in Section 4.2.1. In order to briefly give several remarks on the above invariants of d, we put

$$\mathrm{NS}_0 := \{d \in \mathrm{NS} \mid\ 4 \mid d \text{ and } d/4 \equiv 2,\ 3 \mod 4\}\,,$$

and then we easily see by the definition of f_d that if $d \in \mathrm{NS}_0$ then $f_d = f_{d/4}$. Consequently, $D_d = D_{d/4}$, $E_d = E_{d/4}$, $m_d = m_{d/4}$ and $h_d = h_{d/4}$. (Also, $d_1 \equiv 2,\ 3 \mod 4$ and $d_2/2$ is odd.)

(1) Let OD be the set of all orders in all real quadratic fields. If $D \in \mathrm{DS}$ then there exist uniquely a fundamental discriminant D_K and a positive integer f such that $D = D_K f^2$ (see [10, Theorem 12.11.1]). Here, D_K is the discriminant of a real quadratic field $K = \mathbf{Q}(\sqrt{D})$. So, if $\mathcal{O} = \mathcal{O}_f$ is the order of conductor f (of discriminant D) in K, we know that there exists a bijection: $\mathrm{DS} \cong \mathrm{OD}$, $D \mapsto \mathcal{O}_f$. When $D \in \mathrm{NS}_0$, since $f_D = f_{D/4}$ by the above, we note that $\mathcal{O}_{f_D} = \mathcal{O}_{f_{D/4}}$ and $m_D = m_{D/4}$. Then we can see by the definition of f_d that $m(\mathcal{O}) = m_D$ always holds. (For $D \in \mathrm{DS}$, if $D \in \mathrm{NS}_0$ then take $d := D/4$ and otherwise $d := D$.) Thus the Yokoi invariant of the order $\mathcal{O}$ of discriminant D coincides with that of D. The authors are very grateful to a certain person for carefully reading the earlier manuscript and pointing out this.

(2) For any d in NS, we define $\varphi(d) := D_d$, and then we can prove that φ induces a bijection from the complement $\mathrm{NS} - \mathrm{NS}_0$ of the subset NS_0 of NS into DS. Thus we obtain a bijection: $\mathrm{NS} - \mathrm{NS}_0 \cong \mathrm{DS} \cong \mathrm{OD}$, $d \mapsto D_d \mapsto \mathcal{O}_{f_d}$. Let SF be the set of all square-free positive integers $d > 1$ and DS_{fd} ($\subset$ DS) the set of all fundamental discriminants. Since an element d of SF corresponds bijectively to a field $\mathbf{Q}(\sqrt{d})$, the set SF can be identified with the set of all real quadratic fields. The map φ induces a bijection from SF into DS_{fd}. Since an element d of NS_0 is divisible by 2^2, we have $\mathrm{SF} \subset \mathrm{NS} - \mathrm{NS}_0$.

(3) Let $\kappa_d := \log(h_d \log E_d)/\log D_d$ which gives a sequence of Proposition 2.2. If $d \in \mathrm{NS}_0$ then we have $\kappa_d = \kappa_{d/4}$ by the above. Therefore the limit of the sequence $\{\kappa_d\}_{d \in \mathrm{NS}}$ coincides with that of the sequence $\{\kappa_d\}_{d \in \mathrm{NS}-\mathrm{NS}_0}$.

3. Proofs

We recall a method for calculating the fundamental unit of an order by using continued fractions. First, let Δ be an element of DS and $\mathcal{O}$ the (real quadratic) order of discriminant Δ. We let (x_1, y_1) be the least solution of a Pell equation $x^2 - \Delta y^2 = 4$, or $= -4$, that is, a solution in positive integers which $x_1 + y_1\sqrt{\Delta}$ is the least value of $x + y\sqrt{\Delta}$, and put $\varepsilon_\Delta := (x_1 + y_1\sqrt{\Delta})/2 > 1$. It is known that ε_Δ gives the fundamental unit of $\mathcal{O}$ (see Buchmann and Vollmer [4, Theorem 8.3.5] and Jacobson and Williams [11, pp.81–82]).

LEMMA 3.1. *Under the above setting,* $\mathcal{O}^\times = \langle -1, \varepsilon_\Delta \rangle$ *holds. Namely,* ε_Δ *is the fundamental unit of* $\mathcal{O}$.

Next, we let D be an element of NS and put

$$\delta = \delta(D) := \begin{cases} \sqrt{D}/2\,, & \text{if } D \equiv 0 \mod 4\,, \\ (1+\sqrt{D})/2\,, & \text{if } D \equiv 1 \mod 4\,, \\ \sqrt{D}\,, & \text{if } D \equiv 2,\ 3 \mod 4\,. \end{cases}$$

It is known that if (x_1, y_1) is the least solution of a Pell equation $x^2 - Dy^2 = \pm 4$ then it is calculated from the simple continued fraction expansion of δ with period ℓ: $\delta =$

$[a_0, \overline{a_1, \dots, a_{\ell-1}, a_\ell}]$ as follows. We calculate positive integers p_ℓ, q_ℓ from partial quotients $a_0, a_1, \dots, a_{\ell-1}$ by using recurrence equations:

$$\begin{cases} p_0 = 1\,,\ p_1 = a_0\,,\ p_n = a_{n-1}p_{n-1} + p_{n-2}\,, \\ q_0 = 0\,,\ q_1 = 1\,,\ q_n = a_{n-1}q_{n-1} + q_{n-2}\,, \\ r_0 = 1\,,\ r_1 = 0\,,\ r_n = a_{n-1}r_{n-1} + r_{n-2}\,, \end{cases} \qquad n \geq 2\,. \tag{3.1}$$

(Recurrence equations and partial quotients of a continued fraction are both numbered beginning with 0; The recurrence equation on r_n shall be used below.)

PROPOSITION 3.2 ([11], pp.57–59). *Under the above setting, if we put $\varepsilon_D = (x_1 + y_1\sqrt{D})/2$ then $\varepsilon_D = (sp_\ell - qq_\ell + q_\ell\sqrt{D})/s$, where integers s, q are defined by using an equation $\delta(D) = (q + \sqrt{D})/s$. Namely, $x_1 = 2(sp_\ell - qq_\ell)/s$ and $y_1 = 2q_\ell/s$.*

From now on, let d be a non-square positive integer. We let $d = d_1 d_2^2$ be a factorization of d as in (2.1) into positive integers with d_1 square-free, and consider a real quadratic field $K = \mathbf{Q}(\sqrt{d_1})$. Let $f = f_d$ be a positive integer as in (2.2) and $E_d > 1$ the fundamental unit of the order $\mathcal{O}_{f_d}$ in K.

3.1. Proof of Theorem 2.1. We let $\omega = \omega(d)$ be a quadratic irrational as in (2.9) and consider the simple continued fraction expansion of ω with period ℓ. We calculate nonnegative integers p_ℓ, q_ℓ, r_ℓ from partial quotients $a_0, a_1, \dots, a_{\ell-1}$ by using recurrence equations (3.1). We define integers P_0 and Q_0 as in Table 3.1 below and put

$$G_\ell := Q_0 p_\ell - P_0 q_\ell\,.$$

If $\omega = (1 + \sqrt{d})/2$ then, since $p_\ell = a_0 q_\ell + r_\ell$, we have $G_\ell = (2a_0 - 1)p_\ell + 2r_\ell > 0$. Thus, G_ℓ is always a positive integer.

TABLE 3.1. Definition of P_0, Q_0 and G_ℓ

d (d_1) mod 4	$\omega(d)$	P_0	Q_0	G_ℓ	T	U	U^2/T
0 (1 mod 4)	$\sqrt{d}/2$	0	2	$2p_\ell$	$2p_\ell$	q_ℓ	$q_\ell^2/(2p_\ell)$
0 (2, 3 mod 4)	$\sqrt{d/4}$	0	1	p_ℓ	$2p_\ell$	$2q_\ell$	$2q_\ell^2/p_\ell$
1	$(1+\sqrt{d})/2$	1	2	$2p_\ell - q_\ell$	G_ℓ	q_ℓ	q_ℓ^2/G_ℓ
2, 3	$\sqrt{d}$	0	1	p_ℓ	$2p_\ell$	$2q_\ell$	$2q_\ell^2/p_\ell$

PROPOSITION 3.3. *Under the above setting, the fundamental unit E_d and the Yokoi invariant m_d of d are calculated by using the simple continued fraction expansion of the quadratic irrational $\omega(d)$ as in (2.9). Namely, we have*

$$T = 2G_\ell/Q_0\,, \quad U = 2q_\ell/Q_0\,, \quad m_d = \left[\frac{U^2}{T}\right] = \left[\frac{2q_\ell^2}{G_\ell Q_0}\right].$$

PROOF. If (x_1, y_1) is the least solution of a Pell equation $x^2 - D_d y^2 = \pm 4$ then by Lemma 3.1, we have $E_d = (x_1 + y_1\sqrt{D_d})/2$. We see by Lemma 2.3 and the definition of $\omega(d)$ that

$$\begin{cases} D_d \equiv 0 \mod 4\,, & \omega(d) = \sqrt{D_d}/2\,, \qquad \text{if } d \not\equiv 1 \mod 4\,, \\ D_d \equiv 1 \mod 4\,, & \omega(d) = (1+\sqrt{D_d})/2\,, \qquad \text{if } d \equiv 1 \mod 4\,. \end{cases}$$

Therefore, $\omega(d) = \delta(D_d)$. It follows from Proposition 3.2 that $x_1 = 2p_\ell$ and $y_1 = q_\ell$ if $d \not\equiv 1 \mod 4$, and that $x_1 = 2p_\ell - q_\ell$ and $y_1 = q_\ell$ if $d \equiv 1 \mod 4$. Hence,

$$E_d = \begin{cases} (2p_\ell + q_\ell\sqrt{D_d})/2\,, & \text{if } d \not\equiv 1 \mod 4\,, \\ (2p_\ell - q_\ell + q_\ell\sqrt{D_d})/2\,, & \text{if } d \equiv 1 \mod 4\,. \end{cases} \tag{3.2}$$

By the definition of T and U, we have $m_d = [U^2/T]$. We distinguish four cases to show Proposition.

(i) The case where $4 \mid d$ and $d_1 \equiv 1 \mod 4$. As $D_d = d$, we have $E_d = (2p_\ell + q_\ell\sqrt{d})/2$ by (3.2). Therefore, $T = 2p_\ell$ and $U = q_\ell$. By the definition of G_ℓ and Q_0, we get $T = 2G_\ell/Q_0$ and $U = 2q_\ell/Q_0$ so that $m_d = [2q_\ell^2/(G_\ell Q_0)]$.

(ii) The case where $4 \mid d$ and $d_1 \equiv 2, 3 \mod 4$. As $D_d = d$, we have $E_d = (2p_\ell + 2q_\ell\sqrt{d/4})/2$. Therefore, $T = 2p_\ell = 2G_\ell/Q_0$ and $U = 2q_\ell = 2q_\ell/Q_0$.

(iii) The case where $d \equiv 1 \mod 4$. As $D_d = d$, we have $E_d = (2p_\ell - q_\ell + q_\ell\sqrt{d})/2$. Therefore, $T = 2p_\ell - q_\ell = 2G_\ell/Q_0$ and $U = q_\ell = 2q_\ell/Q_0$.

(iv) The case where $d \equiv 2, 3 \mod 4$. As $D_d = 4d$, we have $E_d = (2p_\ell + 2q_\ell\sqrt{d})/2$. Therefore, $T = 2p_\ell = 2G_\ell/Q_0$ and $U = 2q_\ell = 2q_\ell/Q_0$. This proves our proposition. □

REMARK 3.1. In [13, Definition 3.1], when $4 \nmid d$, we introduced the Yokoi invariant of d in terms of continued fractions. By Proposition 3.3, we see that it coincides with our m_d. Proposition 3.3 can also be proved by using Takagi [22, Theorem 3.9], which for the maximal order $\mathcal{O}_K = \mathcal{O}_1$ is found in Ono [19, Proposition 4.16] (however, the proof also works for any order).

REMARK 3.2. In order to calculate the class number h_d from the class number formula (2.7) for an order, we briefly explain a method for calculating the group index e_f $(= (\mathcal{O}_K^\times : \mathcal{O}_f^\times))$. Let $K = \mathbf{Q}(\sqrt{d_1})$ be a real quadratic field and f a positive integer. Here, d_1 is a square-free positive integer with $d_1 > 1$ $(d = d_1)$. We recall that E_{d_1} is the fundamental unit of $\mathcal{O}_K$ with $E_{d_1} > 1$ and let $\tilde{\omega} = \tilde{\omega}(d_1)$ be a quadratic irrational as in (2.3). Then it is known that E_{d_1} is calculated from the simple continued fraction expansion of $\tilde{\omega}$ with period ℓ (see Proposition 3.2). Let D be the discriminant of $\mathcal{O}_K$ and we write uniquely $E_{d_1} = (t + u\sqrt{D})/2$ with positive integers t, u. For each positive integer k, we define integers u_k's by a formula: $(t_k + u_k\sqrt{D})/2 = E_{d_1}^k$, that is, a recurrence equation:

$$t_k = (tt_{k-1} + Duu_{k-1})/2\,, \quad u_k = (ut_{k-1} + tu_{k-1})/2$$

($t_0 = 2$, $u_0 = 0$). We see by (2.6) that $E_{d_1}^k \in \mathcal{O}_f$ if and only if $u_k \equiv 0 \mod f$. Hence we have

$$e_f = \min\{k \in \mathbf{N} \mid u_k \equiv 0 \mod f\}.$$

On the other hand, we can also calculate this value by using only the simple continued fraction expansion of $\tilde{\omega}$: $\tilde{\omega} = [a_0, a_1, \dots]$. We calculate nonnegative integers p_n, q_n from these partial quotients by using the recurrence equations (3.1). For brevity, we write $\tilde{\omega} = (P_0 + \sqrt{d_1})/Q_0$ with some integers P_0, Q_0 and put $G_n := Q_0 p_n - P_0 q_n$ for all integers $n \geq 0$. Then we see by Proposition 3.3 that $E_{d_1} = ((2G_\ell/Q_0) + (2q_\ell/Q_0)\sqrt{d_1})/2$, and can prove that for any positive integer k, we have $E_{d_1}^k = ((2G_{k\ell}/Q_0) + (2q_{k\ell}/Q_0)\sqrt{d_1})/2$. It follows from this, (2.6) and the definition of e_f that e_f is equal to the least positive integer k such that $q_{k\ell} \equiv 0 \mod f$. Thus, e_f is calculated from the simple continued fraction expansion of $\tilde{\omega}$.

We calculate several values of T by using Proposition 3.3 which are needed in the proof of Theorem 2.1:

EXAMPLE 3.1.

d	2	3	5	6	7	8	10	11	12	13	14	15
T	2	4	1	10	16	2	6	20	4	3	30	8
U	2	2	1	4	6	2	2	6	2	1	8	2

d	17	18	19	20	21	22	23	24	26	27	28
T	8	34	340	4	5	394	48	10	10	52	16
U	2	8	78	1	1	84	10	4	2	10	6

PROOF OF THEOREM 2.1. It is known that $p_\ell^2 - (d/4)q_\ell^2 = (-1)^\ell$ holds in the case where $\omega = \sqrt{d/4}$, that $G_\ell^2 - dq_\ell^2 = (-1)^\ell Q_0^2$ holds in the case where $\omega = (1+\sqrt{d})/2$, and that $p_\ell^2 - dq_\ell^2 = (-1)^\ell$ holds in the case where $\omega = \sqrt{d}$ (cf. [13, Lemma 2.7]). Therefore, if we put

$$E'_d := \begin{cases} \dfrac{T - U\sqrt{d/4}}{2}, & \text{if } 4 \mid d \text{ and } d_1 \equiv 2, 3 \mod 4, \\ \dfrac{T - U\sqrt{d}}{2}, & \text{otherwise,} \end{cases}$$

then it follows from Proposition 3.3, the definition of G_ℓ and Q_0, and (2.8) that $E_d E'_d = (-1)^\ell$, so that

$$\text{[A]:} \quad T^2 - \frac{d}{4}U^2 = (-1)^\ell 4, \tag{3.3}$$

$$\text{[B]:} \quad T^2 - dU^2 = (-1)^\ell 4. \tag{3.4}$$

For brevity, we put $m := m_d$, $E := E_d$ and $E' := E'_d$. First, we show the assertion [B].

[B-i] If $d \geq 29$ then (3.4) yields that

$$29 \leq d \leq dU^2 = T^2 - (-1)^\ell 4 \leq T^2 + 4\,.$$

Therefore, $T \geq 5$. If $13 < d < 29$ and $d \neq 20$ then we obtain $T \geq 5$ from Example 3.1.

[B-ii, iii] We assume $d > 13$. If $d = 20$ then, since $\sqrt{d/4} = \sqrt{5} = [2, \overline{4}]$, we see by Proposition 3.3 that $T = 2p_1 = 4$ and $U = q_1 = 1$, so that $E = (4 + \sqrt{20})/2 = 2 + \sqrt{5}$ and $m_{20} = [1^2/4] = 0$. Hence we have

$$m_{20} \cdot 20 = 0 < T < E = 4.2\cdots < U\sqrt{20} = 4.4\cdots < T + 1 \leq 20 = (m_{20} + 1) \cdot 20\,,$$

which proves (2.11):

$$md < T < E < U\sqrt{d} < T + 1 \leq (m + 1)d\,.$$

So, we assume $d \neq 20$ from now on. Then we have $T \geq 5$ by the assertion [B-i]. Let r be the remainder of the division of U^2 by T. As $m = [U^2/T]$, we have $U^2 = Tm + r$ and $0 \leq r < T$. If we assume $r = 0$ then $U^2 = Tm$. By substituting this for (3.4), we obtain $T(T - dm) = (-1)^\ell 4$, so that $T \mid 4$. This contradicts $T \geq 5$. Hence, $r > 0$. Since

$$T - md = \frac{1}{T}(T^2 - mdT) = \frac{1}{T}\{T^2 - d(U^2 - r)\} = \frac{1}{T}(dr + (-1)^\ell 4)$$

by (3.4), we have $T - md \geq \frac{1}{T}(d + (-1)^\ell 4) > 0$. Therefore, $md < T$ which implies the first inequality of (2.10):

$$md \leq T - 1 < U\sqrt{d} < E < T < (m + 1)d$$

and that of (2.11). Furthermore, since

$$\begin{aligned}(m + 1)d - T &= \frac{1}{T}(dT + dTm - T^2) = \frac{1}{T}\{dT + d(U^2 - r) - T^2\}\\ &= \frac{1}{T}\{d(T - r) + dU^2 - T^2\} = \frac{1}{T}(d(T - r) - (-1)^\ell 4)\end{aligned}$$

by (3.4), $T - r \geq 1$ yields that $(m+1)d - T \geq \frac{1}{T}(d - (-1)^\ell 4) > 0$. Therefore, $T < (m+1)d$. This shows the last inequalities of (2.10) and (2.11). To prove the remaining inequalities, we use the following:

$$E - U\sqrt{d} = (T - U\sqrt{d})/2 = E', \tag{3.5}$$

$$E - T = (-T + U\sqrt{d})/2 = -E'. \tag{3.6}$$

(I) The case where ℓ is even. Since $EE' = 1$ and $E > 0$, $E' > 0$. By (3.5) and (3.6), we have $U\sqrt{d} < E$ and $E < T$. Thus the third and fourth inequalities of (2.10) hold. We see by (3.4) that $T^2 - 4 = dU^2$. Since the left hand side of it is positive by $T \geq 5$, we have $\sqrt{T^2 - 4} = U\sqrt{d}$. Since

$$\sqrt{T^2 - 4}^2 - (T-1)^2 = 2T - 5 \geq 5 > 0\,,$$

we obtain $U\sqrt{d} = \sqrt{T^2 - 4} > T - 1$, which proves the second inequality of (2.10).

(II) The case where ℓ is odd. Since $EE' = -1$ and $E > 0$, $E' < 0$. By (3.5) and (3.6), we have $E < U\sqrt{d}$ and $T < E$. Thus the second and third inequalities of (2.11) hold. We see by (3.4) that $T^2 + 4 = dU^2$ so that $\sqrt{T^2 + 4} = U\sqrt{d}$. Since

$$(T+1)^2 - \sqrt{T^2 + 4}^2 = 2T - 3 \geq 7 > 0\,,$$

we have $T + 1 > \sqrt{T^2 + 4} = U\sqrt{d}$, which proves the fourth inequality of (2.11). Thus we obtain the assertion [B].

[A] Replacing d and (3.4) by $d/4$ and (3.3), we make the same argument as in [B]. First, we show that if $d > 12$ then $T \geq 5$. By the argument as in [B-i], we see that if $d/4 \geq 29$ then $T \geq 5$. Let $3 < d/4 \leq 28$. From Proposition 3.3, we can calculate the value of T by using the simple continued fraction expansion of $\omega = \sqrt{d/4}$. Since $d/4$ is non-square, $d/4 = d_1(d_2/2)^2$ and $d_1 \equiv 2, 3 \mod 4$, we may calculate it when

$$d/4 = 6, 7, 8, 10, 11, 12, 14, 15, 18, 19, 22, 23, 24, 26, 27, 28\,.$$

If $4 \nmid (d/4)$ then, as $d_2/2$ is odd, we have $d/4 \equiv d_1 \equiv 2, 3 \mod 4$. Therefore the value of T is already obtained from Proposition 3.3 (Table 3.1) in Example 3.1. Hence, $T \geq 5$. If $4 \mid (d/4)$ then we obtain the following table:

$d/4$	8	12	24	28
$T = 2p_\ell$	6	14	10	254
$U = 2q_\ell$	2	4	2	48

Hence, $T \geq 5$. Thus, if $d > 12$ then we have $T \geq 5$. Next, by replacing d by $d/4$ and using (3.3), we see from the same argument as in [B-ii, iii] that the assertion [A] holds. This completes the proof. □

3.2. Proof of Proposition 2.2. Let D be a non-square positive integers satisfying $D \equiv 0, 1 \mod 4$, that is, $D \in \mathrm{DS}$, and $h(D)$ the number of equivalent classes (defined by $SL_2(\mathbf{Z})$) of primitive integral binary quadratic forms with discriminant D. We also let (x_0, y_0) be the least solution of a Pell equation $x^2 - Dy^2 = 4$ and put $\varepsilon_D := (x_0 + y_0\sqrt{D})/2$.

PROPOSITION 3.4 ([10], Theorem 12.15.4). *Under the above setting, we have*

$$\lim_{D\to\infty} \frac{\log(h(D)\log\varepsilon_D)}{\log D} = \frac{1}{2}\,.$$

Here, D ranges over all elements of DS *for which $D \to \infty$.*

Though Proposition 2.2 can be shown by using Proposition 3.4, we will prove it in terms of ideals to make the paper readable and self-contained. For brevity, we put

$$\kappa_d := \frac{\log(h_d \log E_d)}{\log D_d}\,.$$

By Remark 2.4, the limit of the sequence $\{\kappa_d\}_{d \in \mathrm{NS}}$ coincides with that of the sequence $\{\kappa_d\}_{d \in \mathrm{NS}-\mathrm{NS}_0}$. For a non-square positive integer d, we also put

$$r_d := \frac{2h_d \log E_d}{\sqrt{D_d}}\,.$$

As $E_d = E_{d_1}^{e_f}$ by (2.5), we have $\log E_d = e_f \log E_{d_1}$. Since $h_d = h_{d_1} f a_f / e_f$ by (2.7), we obtain $h_d \log E_d = f a_f h_{d_1} \log E_{d_1}$. As $D_d = D_{d_1} f^2$ by (2.4), we have $f = \sqrt{D_d}/\sqrt{D_{d_1}}$. Therefore, since

$$h_d \log E_d = \frac{\sqrt{D_d}\, a_f h_{d_1} \log E_{d_1}}{\sqrt{D_{d_1}}}\,,$$

we obtain

$$r_d = a_f r_{d_1}\,. \tag{3.7}$$

It is known that $r_{d_1} = L(1, \chi_{d_1})$ (Narkiewicz [18, Theorem 8.6]), where $L(s, \chi_{d_1}) := \sum_{n=1}^{\infty} \chi_{d_1}(n)/n^s$ denotes Dirichlet's L-function. (r_{d_1} is equal to the residue at $s = 1$ of Dedekind's zeta-function $\zeta_{\mathbf{Q}(\sqrt{d_1})}(s)$ for a real quadratic field $\mathbf{Q}(\sqrt{d_1})$.) To prove Proposition 2.2, we need an estimate for r_{d_1} from above (Lemma 3.5) and that for r_{d_1} from below (Proposition 3.6), which are used in the proof of a (original) theorem of Siegel [21] ([18, Theorem 8.14]).

LEMMA 3.5 ([18], Lemma 8.16). $r_{d_1} = L(1, \chi_{d_1}) < 3 \log D_{d_1}$.

PROPOSITION 3.6 ([18], Lemma 8.17). *Let ε be a positive number. Then there is a positive constant $B_1 = B_1(\varepsilon)$ such that for any square-free positive integer $d_1 > 1$, we have $r_{d_1} \geq B_1 D_{d_1}^{-\varepsilon}$.*

Furthermore we need an estimate for a_f from above. For any positive integer n, we define

$$\phi(n) := \prod_{p \mid n} \left(1 + \frac{1}{p}\right)\,,$$

where p ranges over all distinct prime divisors of n. Since $\chi_{d_1}(p) \in \{-1, 0, 1\}$, we have $a_f \leq \phi(f)$. Therefore we need an estimate for $\phi(n)$ from above to obtain that for a_f from above:

PROPOSITION 3.7. *There is a positive number c such that for any positive integer* $n \geq 3$, *we have* $\phi(n) < c \log\log n$.

If $\varphi(n)$ is Euler's function then, since

$$\frac{n}{\varphi(n)} = \prod_{p|n}\left(1-\frac{1}{p}\right)^{-1} = \prod_{p|n}\left(\sum_{k=0}^{\infty}\frac{1}{p^k}\right) \geq \phi(n)\,,$$

Proposition 3.8 below implies Proposition 3.7.

PROPOSITION 3.8 (Apostol [1], Theorem 13.14 (a)). *There is a positive number c such that for any positive integer* $n \geq 3$, *we have* $\varphi(n) > cn/\log\log n$.

PROOF OF PROPOSITION 2.2. It follows from the definition of r_d and (3.7) that

$$h_d \log E_d = r_d\sqrt{D_d}/2 = a_f r_{d_1}\sqrt{D_d}/2\,. \tag{3.8}$$

First, we give an estimate for κ_d from above. As $D_d = D_{d_1} f^2$, $D_d \geq D_{d_1}$. Lemma 3.5 yields that

$$r_{d_1} < 3\log D_{d_1} \leq 3\log D_d\,.$$

Hence we see by (3.8) that

$$h_d \log E_d < (3a_f/2)\cdot(\log D_d)\sqrt{D_d}\,.$$

Note that $D_d \geq 3$. Then we have

$$\log(h_d \log E_d) < \log(3a_f/2) + \log\log D_d + \frac{1}{2}\log D_d\,,$$

so that

$$\kappa_d < \frac{\log(3a_f/2)}{\log D_d} + \frac{\log\log D_d}{\log D_d} + \frac{1}{2}\,.$$

Let c be a positive number as in Proposition 3.7. As $3f \geq 3$, we have $\phi(3f) < c\log\log(3f)$. Since $D_d = D_{d_1} f^2$, we have $D_d > f^2$ so that $f < \sqrt{D_d}$. By the definition of $\phi(n)$, we obtain

$$a_f \leq \phi(f) \leq \phi(3f) < c\log\log(3f) < c\log\log(3\sqrt{D_d})\,. \tag{3.9}$$

Let c' be a positive number as in Proposition 3.8. On the other hand, $\chi_{d_1}(p) \in \{-1, 0, 1\}$ and $f \mid D_d$ imply that

$$a_f \geq \prod_{p|f}\left(1-\frac{1}{p}\right) \geq \prod_{p|D_d}\left(1-\frac{1}{p}\right) = \varphi(D_d)/D_d > \frac{c'}{\log\log D_d}\,. \tag{3.10}$$

By (3.9) and (3.10), we have

$$\frac{\log(3c'/2)}{\log D_d} - \frac{\log\log\log D_d}{\log D_d} < \frac{\log(3a_f/2)}{\log D_d} < \frac{\log\{3c\log\log(3\sqrt{D_d})/2\}}{\log D_d}.$$

Since we see by Lemma 2.3 that $D_d \to \infty$ as $d \to \infty$, we obtain

$$\frac{\log(3a_f/2)}{\log D_d} \longrightarrow 0, \quad \text{and furthermore,} \quad \frac{\log\log D_d}{\log D_d} \longrightarrow 0.$$

Next, we give an estimate for κ_d from below. Let ε be an arbitrary positive number and $B_1 = B_1(\varepsilon)$ a positive constant as in Proposition 3.6. Then, $r_{d_1} \geq B_1 D_{d_1}^{-\varepsilon}$. As $D_d \geq D_{d_1} > 1$, we have $D_{d_1}^{-\varepsilon} \geq D_d^{-\varepsilon}$, so that $r_{d_1} \geq B_1 D_d^{-\varepsilon}$. We see by (3.8) that

$$h_d \log E_d \geq a_f (B_1/2) D_d^{1/2-\varepsilon}.$$

Therefore,

$$\kappa_d \geq \frac{\log a_f}{\log D_d} + \frac{\log(B_1/2)}{\log D_d} + \frac{1}{2} - \varepsilon.$$

As $d \to \infty$, since $\log(3a_f/2)/\log D_d \to 0$, we obtain

$$\frac{\log a_f}{\log D_d} \longrightarrow 0, \quad \text{and furthermore,} \quad \frac{\log(B_1/2)}{\log D_d} \longrightarrow 0.$$

Since ε is chosen arbitrarily, we see by the above that $\kappa_d \to 1/2$ as $d \to \infty$. This completes the proof. □

4. Gauss' class number problem

We introduce another representation (Lemma 4.1) of the Yokoi invariant m_d by using Proposition 3.3. In Section 4.1, we show by using this representation that the Yokoi invariant of d is small for any positive integer d that is not of minimal type. On the other hand, for a non-square positive integer d, we discuss a sufficient condition for the value of m_d to be large in Section 4.2, and give an interesting numerical data.

4.1. Yokoi invariants of positive integers that are not of minimal type. Let d be a non-square positive integer, that is, $d \in \mathrm{NS}$. We let $d = d_1 d_2^2$ be a factorization of d as in (2.1) into positive integers with d_1 square-free, and consider the simple continued fraction expansion of a quadratic irrational as in (2.9): $\omega = \omega(d) = [a_0, \overline{a_1, \ldots, a_{\ell-1}, a_\ell}]$. We calculate

$$A := q_\ell, \quad B := q_{\ell-1}, \quad C := r_{\ell-1}$$

by using the symmetric part $a_1, \ldots, a_{\ell-1}$ and define polynomials $g(x)$, $h(x)$ of degree 1 and a quadratic polynomial $f(x)$ in $\mathbf{Z}[x]$ by putting

$$g(x) := Ax - (-1)^{\ell} BC, \quad h(x) := Bx - (-1)^{\ell} C^2, \quad f(x) := g(x)^2 + 4h(x).$$

Furthermore, we let s_0 be the least integer s for which $g(s) > 0$, that is, $s > (-1)^{\ell} BC/A$. Then, according to whether $d \equiv 0$, $d \equiv 1$ or $d \equiv 2, 3 \mod 4$, there exists uniquely an integer $s \geq s_0$ such that $d/4 = f(s)/4$, $d = f(s)$ or $d = f(s)/4$. This is known by a theorem of Friesen and Halter-Koch (the assertion [B] of [12, Theorem 3.1]) which is an improvement of results of Friesen [7] and of Halter-Koch [9]. Here, if $s = s_0$ then we say that $d/4$, d or d is a *positive integer with period ℓ of minimal type for* $\sqrt{d/4}$, $(1 + \sqrt{d})/2$ or $\sqrt{d}$ ([12, Definition 3.1]). For brevity, we put

$$\lambda := \frac{A^2}{g(s)A + 2B}.$$

LEMMA 4.1. *Under the above setting, if "*$4 \mid d$ *and* $d_1 \equiv 1 \mod 4$*" or* $d \equiv 1 \mod 4$, *then* $m_d = [\lambda]$. *If "*$4 \mid d$ *and* $d_1 \equiv 2, 3 \mod 4$*" or* $d \equiv 2, 3 \mod 4$, *then we have* $m_d = [4\lambda]$.

PROOF. If $d \not\equiv 1 \mod 4$ then the definition of G_ℓ and Q_0 on Table 3.1 implies that $p_\ell = G_\ell / Q_0$. Therefore, as $p_\ell = a_0 q_\ell + r_\ell$, we see that

$$a_\ell q_\ell = 2a_0 q_\ell = 2((G_\ell/Q_0) - r_\ell).$$

If $d \equiv 1 \mod 4$ then we similarly see by $Q_0 = 2$ that

$$a_\ell q_\ell = (2a_0 - 1) q_\ell = 2((G_\ell/Q_0) - r_\ell).$$

It follows from $q_{\ell-1} = r_\ell$ (the equation (2.5) of [12, Lemma 2.1]) that

$$g(s)A + 2B = a_\ell q_\ell + 2q_{\ell-1} = 2((G_\ell/Q_0) - r_\ell) + 2q_{\ell-1} = 2(G_\ell/Q_0).$$

Hence we obtain

$$\lambda = \frac{q_\ell^2}{2(G_\ell/Q_0)}.$$

First, we assume that "$4 \mid d$ and $d_1 \equiv 1 \mod 4$" or $d \equiv 1 \mod 4$. As $Q_0 = 2$, we have $\lambda = q_\ell^2/G_\ell = (2q_\ell^2)/(G_\ell Q_0)$. Hence Proposition 3.3 implies that $[\lambda] = m_d$. Next, we assume that "$4 \mid d$ and $d_1 \equiv 2, 3 \mod 4$" or $d \equiv 2, 3 \mod 4$. As $Q_0 = 1$, we have $4\lambda = (2q_\ell^2)/G_\ell = (2q_\ell^2)/(G_\ell Q_0)$. Hence Proposition 3.3 implies that $[4\lambda] = m_d$. This proves our lemma. □

By using Lemma 4.1, the exact same calculation in [12, Proposition 4.2] yields the following:

PROPOSITION 4.2. *According to whether* $d \equiv 0$, $d \equiv 1$ *or* $d \equiv 2, 3 \mod 4$, *we assume that* $d/4$, d *or* d *is a positive integer with period* ℓ *that is not of minimal type for* $\sqrt{d/4}$, $(1+\sqrt{d})/2$ *or* $\sqrt{d}$. *Then, if "*$4 \mid d$ *and* $d_1 \equiv 1 \mod 4$*" or* $d \equiv 1 \mod 4$, *then* $m_d = 0$. *If "*$4 \mid d$ *and* $d_1 \equiv 2, 3 \mod 4$*" or* $d \equiv 2, 3 \mod 4$, *then we have* $0 \le m_d \le 3$.

4.2. Large values of Yokoi invariants. Under the above setting, we assume that $4 \nmid d$ to study the class number of a real quadratic field. For brevity, we denote by $\ell(d)$ the period of $\omega = \omega(d)$. We assume that the value of A is "close to" that of B and give a rough guess. Then we see by the definition of λ that the value of λ is "close to" that of $A/(g(s)+2)$. Therefore, if the value of $g(s)$ is small then that of λ is large. When $\omega = (1+\sqrt{d})/2$ (resp., $= \sqrt{d}$), we have $g(s) = 2a_0 - 1$ (resp., $= 2a_0$). Here, $a_0 = [\omega] = [(1+\sqrt{d})/2]$ (resp., $= [\sqrt{d}]$). So, if the value of a_0 or d is small then that of $g(s)$ is small. Since the value of m_d is determined by that of λ by Lemma 4.1, hence, we can deduce that when the value of d is small, that of m_d is relatively large. If this is the case then Theorem 2.1 implies that the fundamental unit E_d is relatively large (Remark 2.2). Then we can expect by Proposition 2.2 that the class number h_d is relatively small (Remark 2.3). We shall report the results of a numerical experiment based on this deduction.

For any positive integer ℓ and $\delta = 1, 2, 3$, we consider a set

$$\mathrm{CF}_{\ell,\delta} := \{d \in \mathrm{NS} \mid \ell = \ell(d) \text{ and } d \equiv \delta \mod 4\},$$

that is, the set of all non-square positive integers d such that $\ell = \ell(d)$ and $d \equiv \delta \mod 4$, and put

$$\mathrm{CF}_\ell := \mathrm{CF}_{\ell,1} \cup \mathrm{CF}_{\ell,2} \cup \mathrm{CF}_{\ell,3}.$$

The set CF_ℓ is that of all non-square positive integers d such that $\ell = \ell(d)$ and $4 \nmid d$. It is known that if $d \equiv 3 \mod 4$ then the period of $\sqrt{d}$ is even. (Since $d \equiv 3 \mod 4$, there is a prime divisor p of d such that $p \equiv 3 \mod 4$. It is known that $p_\ell^2 - dq_\ell^2 = (-1)^{\ell(d)}$ holds (cf. [13, Lemma 2.7]). If $\ell(d)$ is odd then this implies that -1 is a quadratic residue modulo p, which is impossible.) Therefore when ℓ is odd, $\mathrm{CF}_{\ell,3}$ is empty: $\mathrm{CF}_{\ell,3} = \emptyset$. So, we assume from now on that ℓ is even if $\delta = 3$. Then, for any positive integer ℓ and $\delta = 1, 2, 3$, we see by Proposition 4.3 below that

$$\mathrm{CF}_{\ell,\delta} \neq \emptyset, \text{ and hence, } \mathrm{CF}_\ell \neq \emptyset.$$

We define sequences $\{c_n\}_{n\ge0}$, $\{e_n\}_{n\ge0}$ and $\{f_n\}_{n\ge0}$ by putting

$$\begin{cases} c_0 = 0, \ c_1 = 1, \ c_n = 4c_{n-1} + c_{n-2}, \\ e_0 = 0, \ e_1 = 1, \ e_n = 2e_{n-1} + e_{n-2}, \\ f_0 = 0, \ f_1 = 1, \ f_n = f_{n-1} + f_{n-2}, \end{cases} \qquad n \ge 2,$$

where $\{f_n\}_{n\ge0}$ is the Fibonacci sequence. Proposition 4.3 [I-ii], [II-iii] and [III-ii] were proved in Tomita and Yamamuro [24, Theorems 2 and 3]. Though the other assertions are shown by using a theorem of Friesen and Halter-Koch, we state them without proofs. (Each positive integer $N_{\ell,\delta}$ is not of minimal type and is relatively large. Cf. [12, Example 3.5]).

PROPOSITION 4.3. *Let ℓ be a positive integer. For $\delta = 1, 2, 3$, we give a positive integer $N = N_{\ell,\delta}$ below.*

[I] *We can obtain the simple continued fraction expansion of $(1+\sqrt{N})/2$ with period ℓ such that $N \equiv 1 \mod 4$, if we put a positive integer $N = N_{\ell,1}$ as follows.*

(i) *When $3 \mid \ell$, if we put $N := (f_\ell + 1)^2 + 4(f_{\ell-1} + 1)$ then*

$$(1+\sqrt{N})/2 = [(f_\ell + 2)/2, \overline{\underbrace{1, 1, \ldots, 1}_{\ell-1 \text{ times}}, f_\ell + 1}\,].$$

(ii) *When $3 \nmid \ell$, if we put $N := (2f_\ell + 1)^2 + 4(2f_{\ell-1} + 1)$ then*

$$(1+\sqrt{N})/2 = [f_\ell + 1, \overline{\underbrace{1, 1, \ldots, 1}_{\ell-1 \text{ times}}, 2f_\ell + 1}\,].$$

[II] *We can obtain the simple continued fraction expansion of $\sqrt{N}$ with period ℓ such that $N \equiv 2 \mod 4$, if we put a positive integer $N = N_{\ell,2}$ as follows.*

(i) *When $\ell \equiv 0 \mod 4$, or $\ell \equiv 3, 7, 9 \mod 12$, if we put $N := (2e_\ell+1)^2+4e_{\ell-1}+1$ then*

$$\sqrt{N} = [2e_\ell + 1, \overline{\underbrace{2, 2, \ldots, 2}_{\ell-1 \text{ times}}, 4e_\ell + 2}\,].$$

(ii) *When $\ell \equiv 2 \mod 4$, if we put $N := ((e_\ell + 2)/2)^2 + e_{\ell-1} + 1$ then*

$$\sqrt{N} = [(e_\ell + 2)/2, \overline{\underbrace{2, 2, \ldots, 2}_{\ell-1 \text{ times}}, e_\ell + 2}\,].$$

(iii) *When $\ell \equiv 1 \mod 12$, if we put $N := ((f_\ell + 1)/2)^2 + f_{\ell-1} + 1$ then*

$$\sqrt{N} = [(f_\ell + 1)/2, \overline{\underbrace{1, 1, \ldots, 1}_{\ell-1 \text{ times}}, f_\ell + 1}\,].$$

(iv) *When $\ell \equiv 5, 11 \mod 12$, if we put $N := ((3f_\ell + 1)/2)^2 + 3f_{\ell-1} + 1$ then*

$$\sqrt{N} = [(3f_\ell + 1)/2, \overline{\underbrace{1, 1, \ldots, 1}_{\ell-1 \text{ times}}, 3f_\ell + 1}\,].$$

[III] *Let $\ell \geq 2$ be an even integer. Then we can obtain the simple continued fraction expansion of $\sqrt{N}$ with period ℓ such that $N \equiv 3 \mod 4$, if we put a positive integer $N = N_{\ell,3}$ as follows.*

(i) *When* $\ell \equiv 0 \mod 4$, *if we put* $N := ((e_\ell + 2)/2)^2 + e_{\ell-1} + 1$ *then*

$$\sqrt{N} = [(e_\ell + 2)/2, \overline{\underbrace{2, 2, \ldots, 2}_{\ell-1 \text{ times}}, e_\ell + 2}\,].$$

(ii) *When* $\ell \equiv 2,\ 10 \mod 12$, *if we put* $N := ((f_\ell + 1)/2)^2 + f_{\ell-1} + 1$ *then*

$$\sqrt{N} = [(f_\ell + 1)/2, \overline{\underbrace{1, 1, \ldots, 1}_{\ell-1 \text{ times}}, f_\ell + 1}\,].$$

(iii) *When* $\ell \equiv 6 \mod 12$, *if we put* $N := (c_\ell + 2)^2 + 2c_{\ell-1} + 1$ *then*

$$\sqrt{N} = [c_\ell + 2, \overline{\underbrace{4, 4, \ldots, 4}_{\ell-1 \text{ times}}, 2c_\ell + 4}\,].$$

4.2.1. Conjectures and questions. From now on, we put

$$X := \{d \in \mathrm{NS} \mid 4 \nmid d \text{ and } 2 \le d \le 5 \times 10^8\}$$

which has 374988820 elements and first consider the set $\mathrm{CF}_\ell \cap X$. If we put $\alpha_X := \max\{\ell(d) \mid d \in X\}$ then $\alpha_X = 69342$ and $\min \mathrm{CF}_{\alpha_X} = 487067494$. Also, if we let β_X be the maximum of consecutive periods $\ell(d)$, $d \in X$ then $\beta_X = 50394$ and $\min \mathrm{CF}_{\beta_X} = 351665659$. This means that $\mathrm{CF}_\ell \cap X \neq \emptyset$ for all ℓ, $1 \le \ell \le \beta_X$ but $\mathrm{CF}_{\beta_X+1} \cap X = \emptyset$. So, when $1 \le \ell \le \beta_X$, let d_ℓ be the minimal element of $\mathrm{CF}_\ell \cap X$: $d_\ell = \min(\mathrm{CF}_\ell \cap X)$. From Table 1.1 we have $d_1 = 2$, $d_2 = 3$, $d_3 = 17$, and $d_4 = 7, \ldots$. First, let $1 \le \ell \le \beta_X = 50394$. Then, if $\ell \neq 7, 11, 49, 225, 299$ then $h_{d_\ell} = 1$ always holds. (According to whether $\ell = 7$, 11, 49, 225 or 299, we have $d_\ell = 58$, 265, 2746, 40954 or 64234 and then $h_{d_\ell} = 2$.) Next, let $\beta_X < \ell \le \alpha_X = 69342$. Then, if $\mathrm{CF}_\ell \cap X \neq \emptyset$ then $h_{d_\ell} = 1$ holds. Furthermore, when $1 \le \ell \le \alpha_X$, d_ℓ has a square factor only for $\ell = 1032$ (see Table 4.2 below). Also, if $d \in \mathrm{CF}_\ell$ is close to $d_\ell = \min \mathrm{CF}_\ell$ then we can observe that the class number h_d is relatively small, as we have expected as above (cf. Tables 4.1 and 4.2 below). Hence we pose the following conjecture:

CONJECTURE 4.1. *Let* ℓ *be any positive integer and* d_ℓ *the minimal element of* CF_ℓ*:* $d_\ell = \min \mathrm{CF}_\ell$. *Then, if* $\ell \neq 7, 11, 49, 225, 299, 1032$ *then* d_ℓ *is square-free and* $h_{d_\ell} = 1$.

REMARK 4.1. Let $d > 1$ be a square-free positive integer. If h_d is odd then we see by genus theory that d is of the form $d = q, 2q, q_1q_2, p$, or 2. Here, q and q_i are prime numbers that are congruent to 3 modulo 4, and p is a prime number that is congruent to 1 modulo 4. Furthermore, the period $\ell(d)$ for the former three d's is even and that for the latter two d's is odd.

We let ℓ be a positive integer such that $\ell \neq 1, 7, 11, 49, 225, 299, 1032$, and assume that Conjecture 4.1 is true. Then, when ℓ is even, if $d_\ell \equiv 1$ (resp., $\equiv 2, 3$) mod 4 then d_ℓ becomes of the form $d_\ell = q_1q_2$ (resp., $= 2q$, $= q$). When ℓ is odd, d_ℓ becomes a prime number satisfying $d_\ell \equiv 1 \mod 4$.

In the exceptional period 7 of Conjecture 4.1, we arrange 10 values of d in ascending order of size on Table 4.1 below. Here, a factorization of d into prime numbers is given and we put `mtype` := 1 or 0 according to whether d is of minimal type for $\omega(d)$ or not (cf. the beginning of Section 4.1). We see from Table 4.1 that $h_d = 1$ holds for the second value $d\ (= d_{7,1}) = 89$, and if d is close to $d_7 = 58$ then h_d is equal to 1.

TABLE 4.1. 10 values of d in the period 7

d	d mod 4	Factorization of d	h_d	`mtype`
58	2	$2 \cdot 29$	2	0
89	1	89	1	1
109	1	109	1	1
113	1	113	1	1
137	1	137	1	1
202	2	$2 \cdot 101$	2	1
250	2	$2 \cdot 5^3$	2	1
274	2	$2 \cdot 137$	4	1
314	2	$2 \cdot 157$	2	0
373	1	373	1	1

Next, we arrange 10 values of d in ascending order of size in the exceptional period 1032 on Table 4.2 below. From Table 4.2, we see that $d_{1032}\ (= d_{1032,1}) = 366961$ has the square factor 7^2. But the next value $d\ (= d_{1032,2}) = 403246$ is square-free and $h_d = 1$. The Yokoi invariant is very large:

$$
\begin{aligned}
m_{403246} = 1\ &62793\ 60335\ 99180\ 76952\ 88204\ 64544\ 20442\ 27732\ 06006\ 04259\\
&65416\ 44980\ 73794\ 66124\ 47884\ 16186\ 40252\ 39519\ 45266\ 26749\\
&94986\ 44504\ 75511\ 47967\ 85096\ 23563\ 32602\ 07459\ 75536\ 13724\\
&49552\ 70729\ 65969\ 04972\ 08277\ 40104\ 61701\ 17084\ 53985\ 21474\\
&54878\ 15367\ 26338\ 26755\ 80060\ 83049\ 04511\ 10308\ 51090\ 61575\\
&80688\ 43203\ 41690\ 06988\ 21530\ 12092\ 19125\ 74688\ 77831\ 79918\\
&18320\ 05044\ 72939\ 08906\ 28470\ 03942\ 80567\ 64469\ 65824\ 14862\\
&49423\ 84874\ 89808\ 99019\ 31536\ 94516\ 76127\ 44621\ 82669\ 63906\\
&55972\ 12059\ 43036\ 93386\ 55223\ 58242\ 44979\ 77344\ 04205\ 31124\\
&76494\ 52594\ 74003\ 78750\ 10597\ 90709\ 52238\ 83805\ 16124\ 41650\\
&26181\ 43134\ 00803\ 46806\ 95746\ 64378\ 67965\ 01773\ 74498.
\end{aligned}
$$

Also, if d is close to d_{1032} then h_d is equal to 1.

TABLE 4.2. 10 values of d in the period 1032

d	d mod 4	Factorization of d	h_d	mtype
366961	1	$7^2 \cdot 7489$	1	1
403246	2	$2 \cdot 201623$	1	1
409561	1	$23 \cdot 17807$	1	1
419721	1	$3 \cdot 139907$	1	1
421873	1	$43 \cdot 9811$	1	1
434446	2	$2 \cdot 217223$	1	1
453457	1	$467 \cdot 971$	1	1
466558	2	$2 \cdot 233279$	1	1
478041	1	$3 \cdot 159347$	1	1
497121	1	$3 \cdot 165707$	1	1

Next, we consider the set $\mathrm{CF}_{\ell,\delta} \cap X$ and put

$$\alpha_{X,\delta} := \max\{\ell(d) \mid d \in X \text{ and } d \equiv \delta \mod 4\}.$$

We denote by $\beta_{X,\delta}$ the maximum of consecutive periods $\ell(d)$, $d \in X$, $d \equiv \delta \mod 4$. First, we let $\delta = 1$. Then, $\alpha_{X,1} = 64006$ and $\min \mathrm{CF}_{\alpha_{X,1},1} = 487606729$. Also, $\beta_{X,1} = 49635$ and $\min \mathrm{CF}_{\beta_{X,1},1} = 465230089$. When $1 \leq \ell \leq \beta_{X,1}$, as $\mathrm{CF}_{\ell,1} \cap X \neq \emptyset$, let $d_{\ell,1}$ be the minimal element of $\mathrm{CF}_{\ell,1} \cap X$: $d_{\ell,1} = \min(\mathrm{CF}_{\ell,1} \cap X)$. Then, $d_\ell \leq d_{\ell,1}$. From Table 1.1 we have $d_5 = d_{5,1} = 41$ and $d_6 = 19$, $d_{6,1} = 57$. First, let $1 \leq \ell \leq \beta_{X,1} = 49635$. Then, if $\ell \neq 11, 20, 49$ then $h_{d_{\ell,1}} = 1$ always holds. (According to whether $\ell = 11, 20$ or 49, we have $d_{\ell,1} = 265, 1065$ or 3649 and then $h_{d_{\ell,1}} = 2$.) Next, let $\beta_{X,1} < \ell \leq \alpha_{X,1} = 64006$. Then, if $\mathrm{CF}_{\ell,1} \cap X \neq \emptyset$ then $h_{d_{\ell,1}} = 1$ holds. Furthermore, when $1 \leq \ell \leq \alpha_{X,1}$, there are 45 periods ℓ such that $d_{\ell,1}$ has a square factor, and all of them are even. The minimum of such 45 periods is equal to $\ell = 8$ ($d_{\ell,1} = 153 = 3^2 \cdot 17$). For the maximum of them, see Table 4.4 below. Hence we pose the following conjecture:

CONJECTURE 4.2. *Let ℓ be any positive integer and $d_{\ell,1}$ the minimal element of $\mathrm{CF}_{\ell,1}$: $d_{\ell,1} = \min \mathrm{CF}_{\ell,1}$ ($\geq d_\ell$). Then, if $\ell \neq 11, 20, 49$ then $h_{d_{\ell,1}} = 1$. Furthermore, if ℓ is odd then $d_{\ell,1}$ is square-free.*

REMARK 4.2. Let ℓ be an odd integer such that $\ell \neq 1, 7, 11, 49, 225, 299$. If both Conjectures 4.1 and 4.2 are true then we see by genus theory that $d_{\ell,1} = d_\ell$ and $d_{\ell,1}$ becomes a prime number satisfying $d_{\ell,1} \equiv 1 \mod 4$.

In the exceptional period 49 of Conjecture 4.2, we arrange 10 values of d in ascending order of size on Table 4.3 below. We see from Table 4.3 that $h_d = 1$ holds for the third value $d = 4337$.

TABLE 4.3. 10 values of d in the period 49

d	d mod 4	Factorization of d	h_d	mtype
3649	1	$41 \cdot 89$	2	1
3961	1	$17 \cdot 233$	2	1
4337	1	4337	1	1
4789	1	4789	1	1
5581	1	5581	1	1
6421	1	6421	1	1
6473	1	6473	1	1
6569	1	6569	1	1
7433	1	7433	1	1
8081	1	8081	1	1

For the above 45 periods ℓ arranged in ascending order of size, of which $d_{\ell,1}$ has a square factor, we put the last 10 values of them on Table 4.4 below.

TABLE 4.4. 10 values of $d_{\ell,1}$ which has a square factor

ℓ	$d_{\ell,1}$	$d_{\ell,1}$ mod 4	Factorization of $d_{\ell,1}$	$h_{d_{\ell,1}}$	mtype
19524	73114129	1	$11^2 \cdot 604249$	1	1
20304	81837889	1	$7^2 \cdot 1670161$	1	1
26376	136299361	1	$11^2 \cdot 1126441$	1	1
26508	158520769	1	$11^3 \cdot 119099$	1	1
26780	135195841	1	$11^2 \cdot 1117321$	1	1
28424	168055969	1	$19^2 \cdot 465529$	1	1
38752	263471329	1	$11^2 \cdot 2177449$	1	1
43896	384693169	1	$7^2 \cdot 7850881$	1	1
50656	477126841	1	$19^2 \cdot 1321681$	1	1
51188	475887889	1	$19^2 \cdot 1318249$	1	1

Next, we let $\delta = 2$. Then, $\alpha_{X,2} = 69342$ and $\min \mathrm{CF}_{\alpha_{X,2},2} = 487067494$. Also, $\beta_{X,2} = 25904$ and $\min \mathrm{CF}_{\beta_{X,2},2} = 101036686$. When $1 \leq \ell \leq \beta_{X,2}$, as $\mathrm{CF}_{\ell,2} \cap X \neq \emptyset$, let $d_{\ell,2}$ be the minimal element of $\mathrm{CF}_{\ell,2} \cap X$: $d_{\ell,2} = \min(\mathrm{CF}_{\ell,2} \cap X)$. First, let $1 \leq \ell \leq \beta_{X,2} = 25904$.

Then, if ℓ is even and $\ell \neq 18, 20, 30, 42, 62, 90, 92, 120, 204$ then $h_{d_{\ell,2}} = 1$ always holds. (According to whether $\ell = 18, 20, 30, 42, 62, 90, 92, 120$ or 204, we have $d_{\ell,2} = 562$, 606, 946, 1786, 3886, 6526, 7294, 13066 or 30286 and then $h_{d_{\ell,2}} = 2$.) On the other hand, if ℓ is odd and $\ell \neq 1, 3, 15, 117$ then $h_{d_{\ell,2}} = 2$ always holds. (We have $d_{1,2} = 2$ and $h_{d_{1,2}} = 1$. According to whether $\ell = 3, 15$ or 117, we have $d_{\ell,2} = 130$, 1066 or 57586 and then $h_{d_{\ell,2}} = 4$.) Furthermore, $d_{\ell,2}$ is always square-free. Next, let $\beta_{X,2} < \ell \leq \alpha_{X,2} = 69342$ and $\mathrm{CF}_{\ell,2} \cap X \neq \emptyset$. Then, if ℓ is even then $h_{d_{\ell,2}} = 1$. On the other hand, if ℓ is odd then $h_{d_{\ell,2}} = 2$. Furthermore, $d_{\ell,2}$ is always square-free. Therefore we have the following question:

QUESTION 1. Let ℓ be any positive integer and $d_{\ell,2}$ the minimal element of $\mathrm{CF}_{\ell,2}$: $d_{\ell,2} = \min \mathrm{CF}_{\ell,2}$ $(\geq d_\ell)$.

(i) Assume that ℓ is even and $\ell \neq 18, 20, 30, 42, 62, 90, 92, 120, 204$. Is $d_{\ell,2}$ always square-free? Does $h_{d_{\ell,2}} = 1$ always hold?

(ii) Assume that ℓ is odd and $\ell \neq 1, 3, 15, 117$. Is $d_{\ell,2}$ always square-free? Does $h_{d_{\ell,2}} = 2$ always hold?

REMARK 4.3. We let ℓ be an even integer such that $\ell \neq 18, 20, 30, 42, 62, 90, 92, 120, 204$ and assume that the answer to Question 1 (i) is yes. Then we see by genus theory that $d_{\ell,2}$ becomes of the form $d_{\ell,2} = 2q$.

In the exceptional period 15 of Question 1 (ii), we arrange 10 values of d in ascending order of size on Table 4.5 below. We see from Table 4.5 that $h_d = 2$ holds for the second value $d = 1466$.

TABLE 4.5. 10 values of d in the period 15

d	d mod 4	Factorization of d	h_d	`mtype`
1066	2	$2 \cdot 13 \cdot 41$	4	1
1466	2	$2 \cdot 733$	2	1
2290	2	$2 \cdot 5 \cdot 229$	4	1
2738	2	$2 \cdot 37^2$	2	1
2858	2	$2 \cdot 1429$	2	1
3314	2	$2 \cdot 1657$	4	1
3562	2	$2 \cdot 13 \cdot 137$	4	1
4498	2	$2 \cdot 13 \cdot 173$	4	1
4538	2	$2 \cdot 2269$	2	1
4570	2	$2 \cdot 5 \cdot 457$	4	1

In each exceptional period of Conjectures 4.1, 4.2, Question 1 (i) (resp. Question 1 (ii)), $h_d = 1$ (resp. $h_d = 2$) holds for a certain value of d.

Finally, we let $\delta = 3$. Then, $\alpha_{X,3} = 68836$ and $\min \mathrm{CF}_{\alpha_{X,3},3} = 475477759$. Also, $\beta_{X,3} = 55276$ and $\min \mathrm{CF}_{\beta_{X,3},3} = 447639151$. When ℓ is even and $2 \le \ell \le \beta_{X,3}$, as $\mathrm{CF}_{\ell,3} \cap X \ne \emptyset$, let $d_{\ell,3}$ be the minimal element of $\mathrm{CF}_{\ell,3} \cap X$: $d_{\ell,3} = \min(\mathrm{CF}_{\ell,3} \cap X)$. First, we assume that ℓ is even and $2 \le \ell \le \beta_{X,3} = 55276$. Then, $d_{\ell,3}$ is always square-free and $h_{d_{\ell,3}} = 1$ always holds. Next, we assume that ℓ is even and $\beta_{X,3} < \ell \le \alpha_{X,3} = 68836$. Then, if $\mathrm{CF}_{\ell,3} \cap X \ne \emptyset$ then $d_{\ell,3}$ is square-free and $h_{d_{\ell,3}} = 1$. Hence we pose the following conjecture:

CONJECTURE 4.3. *Let ℓ be any positive even integer and $d_{\ell,3}$ the minimal element of* $\mathrm{CF}_{\ell,3}$*: $d_{\ell,3} = \min \mathrm{CF}_{\ell,3}$ ($\ge d_\ell$). Then, $d_{\ell,3}$ is square-free and $h_{d_{\ell,3}} = 1$.*

REMARK 4.4. We let ℓ be a positive even integer and assume that Conjecture 4.3 is true. Then we see by genus theory that $d_{\ell,3}$ becomes a prime number.

We finally supplement the distribution of positive integers with class number 1 based on periods of continued fraction expansions. For each period ℓ, we arrange all elements d of CF_ℓ in order of size:

$$\mathrm{CF}_\ell = \{d_\ell = d_\ell^{(0)} < d_\ell^{(1)} < \cdots < d_\ell^{(i)} < \cdots < d_\ell^{(2000)} < \cdots\}.$$

Let $h = 1, 2$ and for any integer i, $0 \le i \le 2000$, we let $\varphi_h(i)$ be the number of periods $\ell \le 4000$ such that the class number of the ith positive integer $d_\ell^{(i)}$ is equal to h. Then, $\varphi_1(0) = 3995$ and $\varphi_2(0) = 5$. Figure 4.1 indicates the graph of the functions $\varphi_1(i)$ and $\varphi_2(i)$. We see from this graph that in each period, positive integers with class number 1 concentrate on about the minimal element.

4.2.2. Approach for solving our conjectures. In view of Conjectures 4.1, 4.2, 4.3 and Question 1, we may expect that there exist infinitely many real quadratic fields of class number 1. We collect known results to solve our conjectures. Let $N_{\ell,\delta}$ be a positive integer as in Proposition 4.3. Then we have rough estimates from above

$$\min \mathrm{CF}_\ell \le \min\{N_{\ell,1},\ N_{\ell,2},\ N_{\ell,3}\} \text{ and } \min \mathrm{CF}_{\ell,\delta} \le N_{\ell,\delta}\ (\delta = 1,\ 2,\ 3)\,.$$

By [12, Proposition 4.4], it is known that we have to construct real quadratic fields of minimal type (or positive integers of minimal type) to find many real quadratic fields of class number 1. In principle, all positive integers d with period ℓ of minimal type for $\omega(d)$ are calculated by using a theorem of Friesen and Halter-Koch (the assertion [A] of [12, Theorem 3.1]). We think that it is possible to get a better estimate from above of the above minimal elements by this method. On the other hand, Louboutin [15, Theorem 3] gave a necessary and sufficient condition for the class number of a real quadratic field $\mathbf{Q}(\sqrt{d})$ to be equal to 1, involving the simple continued fraction expansion of $\omega(d)$ and a certain finite set of prime numbers (less than Minkowski's constant). For other conditions, see Lu [17] and Dubois and Levesque [6, Section 1]. The authors hope that these facts are useful to solve Conjectures 4.1 4.2 and 4.3 in future. There is a work of Takhtajan and Vinogradov [23] as another approach to Gauss' class number problem.

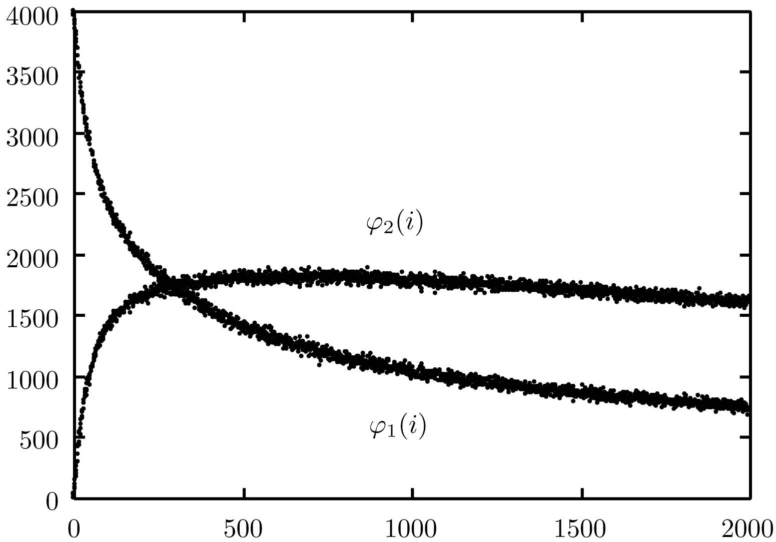

FIGURE 4.1. The graph of the functions $\varphi_1(i)$ and $\varphi_2(i)$

ACKNOWLEDGMENTS. The authors are most grateful to Yoshiyuki Kitaoka for pointing out the result [10, Theorem 12.15.4], and to Humio Ichimura for pointing out the paper [23]. They heartily express their appreciation to Yoshitaka Odai, Katsuya Miyake and Keiichi Komatsu for their encouragement. The first author was partially supported by Grant-in-Aid for Scientific Research (C), No. 22540030, Japan Society for the Promotion of Science.

References

[1] T. APOSTOL, *Introduction to analytic number theory*, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[2] C. BATUT, K. BELABAS, D. BERNARDI, H. COHEN and M. OLIVIER, GP/PARI calculator, Version 2.1.6.

[3] Z. I. BOREVICH and I. R. SHAFAREVICH, *Number theory*, Academic Press, New York, 1966.

[4] J. BUCHMANN and U. VOLLMER, *Binary quadratic forms: An algorithmic approach*, Algorithms and Computation in Mathematics Vol. 20, Springer-Verlag, Berlin, Heidelberg, 2007.

[5] D. A. COX, *Primes of the form* $x^2 + ny^2$, John Wiley & Sons, New York, 1989.

[6] E. DUBOIS and C. LEVESQUE, On determining certain real quadratic fields with class number one and relating this property to continued fractions and primality properties, Nagoya Math. J. **124** (1991), 157–180.

[7] C. FRIESEN, On continued fractions of given period, Proc. Amer. Math. Soc. **103** (1988), 9–14.

[8] C. F. GAUSS, *Disquisitiones Arithmeticae*, Göttingen 1801. (The Springer edition in 1986 of English translation by A. A. Clarke.)

[9] F. HALTER-KOCH, Continued fractions of given symmetric period, Fibonacci Quart. **29** (1991), 298–303.

[10] L. K. HUA, *Introduction to number theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[11] M. J. JACOBSON, JR. and H. C. WILLIAMS, *Solving the Pell equation*, CMS Books in Mathematics, Springer Science+Business Media, LLC, 2009.

[12] F. KAWAMOTO and K. TOMITA, Continued fractions and certain real quadratic fields of minimal type, J. Math. Soc. Japan **60** (2008), 865–903.

[13] F. KAWAMOTO and K. TOMITA, Continued fractions with even period and an infinite family of real quadratic fields of minimal type, Osaka J. Math. **46** No. 4 (2009), 949–993.

[14] S. LANG, *Elliptic functions*, Graduate Texts in Mathematics Vol. 112, Springer-Verlag, New York, 2nd ed. 1987.

[15] S. LOUBOUTIN, Continued fractions and real quadratic fields, J. Number Theory **30** (1988), 167–176.

[16] S. LOUBOUTIN, The Brauer-Siegel theorem, J. London Math. Soc. **72** (2005), 40–52.

[17] H. LU, On the class number of real quadratic fields, Scientia Sinica, Special Issue **2** (1979), 118–130.

[18] W. NARKIEWICZ, *Elementary and analytic theory of algebraic numbers*, Springer-Verlag, Berlin, Heidelberg, 3rd ed. 2004.

[19] T. ONO, *An introduction to algebraic number theory*, Plenum Press, New York-London, 1990.

[20] J. W. SANDS, Generalization of a theorem of Siegel, Acta Arith. **58** (1991), 47–57.

[21] C. L. SIEGEL, Über die Classenzahl quadratischer Zahlkörper, Acta Arith. **1** (1935), 83–86.

[22] T. TAKAGI, *Lectures on elementary number theory* (*Shoto Seisuron Kogi*), Kyoritsu, Tokyo, 1st ed. 1931, 2nd ed. 1971 (in Japanese).

[23] L. A. TAKHTAJAN and A. I. VINOGRADOV, The Gauss-Hasse hypothesis on real quadratic fields with class number one, J. Reine Angew. Math. **335** (1982), 40–86.

[24] K. TOMITA and K. YAMAMURO, Lower bounds for fundamental units of real quadratic fields, Nagoya Math. J. **166** (2002), 29–37.

[25] H. YOKOI, New invariants and class number problem in real quadratic fields, Nagoya Math. J. **132** (1993), 175–197.

Present Addresses:
FUMINORI KAWAMOTO
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,
GAKUSHUIN UNIVERSITY,
1–5–1 MEJIRO, TOSHIMA-KU, TOKYO, 171–8588 JAPAN.
e-mail: fuminori.kawamoto@gakushuin.ac.jp

KOSHI TOMITA
DEPARTMENT OF MATHEMATICS,
MEIJO UNIVERSITY,
TENPAKU-KU, NAGOYA, 468–8502 JAPAN.
e-mail: tomita@meijo-u.ac.jp