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The Effect of External Fields in the Theory of Liquid Crystals

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Abstract. We consider the response of external field to the theory of liquid crystals. We treat the Landau-de Gennes functional with the Dirichlet boundary condition for the director field which may be non-constant. We show that there exist two families of critical points such that one carries out the superheating fields of superconductors and the other one carries out strong stability. We also show that under some conditions, strong field does not bring the pure nematic state which is different response from superconductors.

1. Introduction

In this paper, we consider the change of stability of liquid crystals under applied external fields (electric or magnetic fields). Let $\mathcal{F}_N(\boldsymbol{n}, \nabla\boldsymbol{n})$ be the classical Oseen-Frank density of nematic liquid crystals. Then we must add an external density $-\chi(\boldsymbol{H}\cdot\boldsymbol{n})^2$ to $\mathcal{F}_N(\boldsymbol{n}, \nabla\boldsymbol{n})$, and consider a modified energy functional:

$$\int_\Omega \{\mathcal{F}_N(\boldsymbol{n}, \nabla\boldsymbol{n}) - \chi(\boldsymbol{H}\cdot\boldsymbol{n})^2\}dx\,.$$

Here $\boldsymbol{H}$ is an applied field, χ is a real parameter and $\boldsymbol{n} : \overline{\Omega} \to \mathbb{S}^2$ is a director field of the nematic crystals. See de Gennes and Prost [5, p. 287]. Though there are many article on liquid crystals without external field (for example, Aramaki [1], [2], Bauman et al. [3], Hardt et al. [9], Pan [11], [14]), there are few references which treat applied field. See Lin and Pan [10] and Pan [12], [13].

According to the Landau-de Gennes theory, phase transitions of nematic states to smectic states can be described by the minimizer $(\psi, \boldsymbol{n})$ of the Landau-de Gennes functional:

$$\begin{aligned}(1.1)\quad \mathcal{E}[\psi, \boldsymbol{n}] = \int_\Omega \Big\{ &|\nabla_{q\boldsymbol{n}}\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 \\ &+ K_1|\operatorname{div}\boldsymbol{n}|^2 + K_2|\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}|^2 + K_3|\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2 - \chi(\boldsymbol{H}\cdot\boldsymbol{n})^2\Big\}dx\end{aligned}$$

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where κ, K_1, K_2, K_3 and χ are positive constants, and q is a real number. Here we denoted $\nabla_{q\boldsymbol{n}}\psi = \nabla\psi - iq\boldsymbol{n}\psi$. Without loss of generality, we may assume that $q \geq 0$. For brevity, we write

$$\mathcal{E}[\psi, \boldsymbol{n}] = \mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - \int_\Omega \chi(\boldsymbol{H}\cdot\boldsymbol{n})^2 dx$$

where

$$\mathcal{F}[\boldsymbol{n}] = \int_\Omega \{K_1|\operatorname{div}\boldsymbol{n}|^2 + K_2|\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}|^2 + K_3|\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2\}dx$$

is the simplified Oseen-Frank energy for nematics and

$$\mathcal{G}[\psi, \boldsymbol{n}] = \int_\Omega \left\{|\nabla_{q\boldsymbol{n}}\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2\right\}dx$$

is the Ginzburg-Landau energy for smectics.

We consider the functional $\mathcal{E}$ under the Dirichlet boundary condition for the director field:

$$\boldsymbol{n} = \boldsymbol{e}_0 \quad \text{on } \partial\Omega$$

where $\boldsymbol{e}_0 \in C^2(\partial\Omega, \mathbb{S}^2)$. Thus we treat $\mathcal{E}$ on the space $W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ where

$$W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0) = \{\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{R}^3);\ |\boldsymbol{n}(x)| = 1 \text{ a.e. in } \Omega, \boldsymbol{n} = \boldsymbol{e}_0 \text{ on } \partial\Omega\}.$$

Here and from now, for some Euclidean space E ($= \mathbb{R}, \mathbb{C}, \mathbb{R}^3, \mathbb{C}^3$ or the space $\mathbb{R}^9$ of real 3×3 matrices), $W^{1,2}(\Omega, E)$ denotes the usual Sobolev space and we briefly denote $W^{1,2}(\Omega, \mathbb{R})$ by $W^{1,2}(\Omega)$.

Throughout this paper, we assume that Ω is a simply connected bounded domain with smooth boundary in $\mathbb{R}^3$ and $\boldsymbol{H} = \sigma\boldsymbol{h}$ where $\boldsymbol{h}$ is a unit constant vector and σ is a positive number denoting the intensity of the applied field, and assume that there exists $\boldsymbol{e} \in C^2(\overline{\Omega}, \mathbb{S}^2)$ such that

$$\text{(H.1)} \qquad \operatorname{curl}\boldsymbol{e} = 0, \quad \boldsymbol{h}\cdot\boldsymbol{e} = 0, \quad -\Delta\boldsymbol{e} = |\nabla\boldsymbol{e}|^2\boldsymbol{e} \quad \text{in } \Omega, \quad \boldsymbol{e} = \boldsymbol{e}_0 \text{ on } \partial\Omega$$

and $\boldsymbol{e}$ is a unique minimizer of

$$\inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)} \mathcal{F}[\boldsymbol{n}].$$

Here we note that there are many situations where (H.1) holds. For example, choose the coordinate system $x = (x_1, x_2, x_3)$ such that $\boldsymbol{h} = (0, 0, 1)$. Take some point $a = (a_1, a_2, a_3) \in \mathbb{R}^3 \setminus \overline{\Omega}$ so that

$$\boldsymbol{e}(x_1, x_2, x_3) = \left(\cos\left(\arctan\left(\frac{x_2 - a_2}{x_1 - a_1}\right)\right), \sin\left(\arctan\left(\frac{x_2 - a_2}{x_1 - a_1}\right)\right), 0\right)$$

is well defined. Then $\boldsymbol{e}$ satisfies (H.1). There are a lot of choices of a.

(H.2)
$$\min\{K_1, K_2, K_3\} > K_1 c(\Omega) \max_{x\in\overline{\Omega}} |\nabla \boldsymbol{e}|^2$$

where $c(\Omega) > 0$ is the best constant such that the following Poincaré inequality holds:

$$\int_\Omega |\boldsymbol{w}|^2 dx \le c(\Omega) \int_\Omega |\nabla \boldsymbol{w}|^2 dx$$

for any $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$. Moreover assume that

(H.3) For any $p \in \Omega$, the integral curve of $\boldsymbol{e}$ through p intersects with $\partial\Omega$.

[10] treated the case where $\boldsymbol{h}$ and $\boldsymbol{e}_0$ are constant unit vectors such that $\boldsymbol{h} \cdot \boldsymbol{e}_0 = 0$. Of course the conditions (H.1), (H.2) and (H.3) hold for this case.

By the hypothesis (H.1), Ω is simply connected and $\operatorname{curl} \boldsymbol{e} = 0$ in Ω, there exists a unique function $\varphi \in C^3(\Omega)$ such that

$$\nabla\varphi = \boldsymbol{e} \quad \text{in } \Omega\,, \qquad \int_\Omega \varphi dx = 0\,. \tag{1.2}$$

Our purpose is an extension of their result to the case where $\boldsymbol{e}_0$ is non-constant and the condition (H.1), (H.2) and (H.3) hold.

In our case, the energy functional can be rewritten by

$$\mathcal{E}[\psi, \boldsymbol{n}] = \mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - \chi\sigma^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n})^2 dx\,. \tag{1.3}$$

We also write

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = \mathcal{F}[\boldsymbol{n}] - \chi\sigma^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n})^2 dx\,.$$

Now we can see that the energy functional $\mathcal{E}$ has two families of critical points:

$$\psi = 0\,, \quad \boldsymbol{n} = \boldsymbol{n}_\sigma \tag{1.4}$$

where $\boldsymbol{n}_\sigma$ is a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$:

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma] = \inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)} \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]\,,$$

and

$$\psi = ce^{iq\varphi}\,, \quad \boldsymbol{n} = \boldsymbol{e} \tag{1.5}$$

where φ is as in (1.2) and c is an arbitrary complex number such that $|c| = 1$.

By the analogies between superconductors and liquid crystals (cf. [12] and [13]), we call the family in (1.4) pure nematic states corresponding to the normal states of superconductor, and the family in (1.5) pure smectic states corresponding to the Meissner states of superconductor. We shall see that there exists a critical field $H_n(0) > 0$ such that for $0 \le \sigma < H_n(0)$, the only pure nematic state is $(0, \boldsymbol{e})$. Moreover, we shall show that there exist critical fields H_{sh} and H_s where the pure smectic states change their weak stability (local minimality) at H_{sh} and change their strong stability (global minimality) at H_s. Thus the critical field H_{sh} look like the superheating field of superconductors. We shall also show that in the case of $K_1 = K_2 = K_3$, a liquid crystal under very strong external field may not be in a pure nematic state. On the other hand, in the theory of superconductor, the breakdown of superconductivity occurs under strong external magnetic fields. See Giorgi and Phillips [7]. Thus liquid crystals and superconductors have very different response in strong field.

The plan of this paper is as follows. In section 2, we state the weak stability of a critical point of $\mathcal{E}$. In section 3, we define a critical value H_{sh} and show that when σ increases, the pure smectic states change their weak stabilities at H_{sh}. In section 4, we define a critical value H_s, and show that if $\sigma > H_s$, the global minimizers of $\mathcal{E}$ are not pure smectic states and if $\sigma < H_s$, the only global minimizers of $\mathcal{E}$ are pure smectic states. Finally in section 5, we show that the instabilities in pure nematic states. In the particular case of $K_1 = K_2 = K_3$, when σ is sufficiently large, the pure nematic states are not global minimizers of $\mathcal{E}$. This phenomena clarifies the difference between the liquid crystals and superconductors.

2. Weak stability of critical points

In this section, we give the definition of the weak stability of critical points and a necessary condition for weak stability for a general applied field $\boldsymbol{H}$ and a boundary data $\boldsymbol{u}_0$.

DEFINITION 2.1. (1) We say that $(\psi_0, \boldsymbol{n}_0) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{u}_0)$ is a critical point of $\mathcal{E}$, if and only if for any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}[\psi_t, \boldsymbol{n}_t] = 0$$

where

$$\psi_t = \psi_0 + t\phi\,, \quad \boldsymbol{n}_t = \frac{\boldsymbol{n}_0 + t\boldsymbol{v}}{|\boldsymbol{n}_0 + t\boldsymbol{v}|}\,. \tag{2.1}$$

(2) We say that a critical point $(\psi_0, \boldsymbol{n}_0)$ of $\mathcal{E}$ is weakly stable (local minimizer), if for any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, there exists $T = T(\phi, \boldsymbol{v}) > 0$ such that for any $0 < t < T$,

$$\mathcal{E}[\psi_0, \boldsymbol{n}_0] \le \mathcal{E}[\psi_t, \boldsymbol{n}_t]\,.$$

By computations, we can write

$$\boldsymbol{n}_t = \boldsymbol{n}_0 + t\boldsymbol{n}_1 + t^2\boldsymbol{n}_2 + O(t^3) \tag{2.2}$$

where

$$\boldsymbol{n}_1 = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n}_0)\boldsymbol{n}_0\,,$$

$$\boldsymbol{n}_2 = -(\boldsymbol{v} \cdot \boldsymbol{n}_0)\boldsymbol{v} + \frac{1}{2}[3(\boldsymbol{v} \cdot \boldsymbol{n}_0)^2 - |\boldsymbol{v}|^2]\boldsymbol{n}_0$$

and

$$\nabla_{q\boldsymbol{n}_t}\psi_t = \nabla_{q\boldsymbol{n}_0}\psi_0 + t\Phi_1 + t^2\Phi_2 + O(t^3) \tag{2.3}$$

where

$$\begin{aligned} \Phi_1 &= \nabla_{q\boldsymbol{n}_0}\phi - iq\boldsymbol{n}_1\psi_0\,, \\ \Phi_2 &= -iq(\boldsymbol{n}_1\phi + \boldsymbol{n}_2\psi_0)\,. \end{aligned}$$

Using these formulas, for small t, we can write

$$\begin{aligned} \mathcal{G}[\psi_t, \boldsymbol{n}_t] = \mathcal{G}[\psi_0, \boldsymbol{n}_0] & \\ + 2t \int_\Omega \big\{ & \Re[\overline{\nabla_{q\boldsymbol{n}_0}\phi}\nabla_{q\boldsymbol{n}_0}\psi_0 - \kappa^2\overline{\phi}(1 - |\psi_0|^2)\psi_0] \\ & - q\boldsymbol{n}_1 \cdot \Im(\overline{\psi}_0\nabla_{q\boldsymbol{n}_0}\psi_0)\big\}dx \\ + t^2 \int_\Omega \big\{ & |\Phi_1|^2 - \kappa^2(1 - |\psi_0|^2)|\phi|^2 + 2\kappa^2(\Re(\overline{\phi}\psi_0))^2 \\ & - 2q\Im[(\boldsymbol{n}_1\overline{\phi} + \boldsymbol{n}_2\overline{\psi}_0) \cdot \nabla_{q\boldsymbol{n}_0}\psi_0]\big\}dx + O(t^3)\,. \end{aligned}$$

Here and from now, we denote the real part and imaginary part of a complex number z by $\Re[z]$ and $\Im[z]$, respectively.

$$\begin{aligned} \mathcal{F}[\boldsymbol{n}_t] = \mathcal{F}[\boldsymbol{n}_0] & \\ +2t \int_\Omega \big\{ & K_1(\operatorname{div}\boldsymbol{n}_0)(\operatorname{div}\boldsymbol{n}_1) \\ & + K_2(\boldsymbol{n}_0 \cdot \operatorname{curl}\boldsymbol{n}_0)(\boldsymbol{n}_1 \cdot \operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0 \cdot \operatorname{curl}\boldsymbol{n}_1) \\ & + K_3(\boldsymbol{n}_0 \times \operatorname{curl}\boldsymbol{n}_0) \cdot (\boldsymbol{n}_1 \times \operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0 \times \operatorname{curl}\boldsymbol{n}_1)\big\}dx \\ +t^2 \int_\Omega \big\{ & K_1\big\{(\operatorname{div}\boldsymbol{n}_1)^2 + 2(\operatorname{div}\boldsymbol{n}_0)(\operatorname{div}\boldsymbol{n}_2)\big\} \\ +K_2\big\{ & (\boldsymbol{n}_1 \cdot \operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0 \cdot \operatorname{curl}\boldsymbol{n}_1)^2 \\ & + 2(\boldsymbol{n}_0 \cdot \operatorname{curl}\boldsymbol{n}_0)(\boldsymbol{n}_2 \cdot \operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_1 \cdot \operatorname{curl}\boldsymbol{n}_1 + \boldsymbol{n}_0 \cdot \operatorname{curl}\boldsymbol{n}_2)\big\} \\ +K_3\{ & |\boldsymbol{n}_1 \times \operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0 \times \operatorname{curl}\boldsymbol{n}_1|^2 \\ & + 2(\boldsymbol{n}_0 \times \operatorname{curl}\boldsymbol{n}_0) \cdot (\boldsymbol{n}_2 \times \operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_1 \times \operatorname{curl}\boldsymbol{n}_1 \\ & + \boldsymbol{n}_0 \times \operatorname{curl}\boldsymbol{n}_2)\}dx + O(t^3)\,. \end{aligned}$$

$$\begin{aligned} \int_\Omega (\boldsymbol{H} \cdot \boldsymbol{n}_t)^2 dx = & \int_\Omega (\boldsymbol{H} \cdot \boldsymbol{n}_0)^2 dx \\ & +2t \int_\Omega (\boldsymbol{H} \cdot \boldsymbol{n}_0)(\boldsymbol{H} \cdot \boldsymbol{n}_1)dx \end{aligned}$$

$$+t^2 \int_\Omega \{(\boldsymbol{H}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_2)\}dx + O(t^3)\,.$$

Therefore, we can write

$$\begin{aligned}
\mathcal{E}[\psi_t, \boldsymbol{n}_t] &= \mathcal{E}[\psi_0, \boldsymbol{n}_0] \\
&\quad + 2t\left\{\mathcal{A}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) - \chi \int_\Omega (\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_1)dx\right\} \\
&\quad + t^2\left\{\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v})\right. \\
&\qquad \left. - \chi \int_\Omega \{(\boldsymbol{H}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_2)\}dx\right\} \\
&\quad + O(t^3)
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
\mathcal{A}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) = \int_\Omega &\big\{\Re[\overline{\nabla_{q\boldsymbol{n}_0}\phi}\cdot\nabla_{q\boldsymbol{n}_0}\psi_0 - \kappa^2\overline{\phi}(1-|\psi_0|^2)\psi_0] \\
&- q\boldsymbol{n}_1\cdot\Im(\overline{\psi}_0\nabla_{q\boldsymbol{n}_0}\psi_0) + K_1(\operatorname{div}\boldsymbol{n}_0)(\operatorname{div}\boldsymbol{n}_1) \\
&+ K_2(\boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_0)(\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_1) \\
&+ K_3(\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_0)\cdot(\boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_1)\big\}dx\,,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) = \int_\Omega &\big\{|\nabla_{q\boldsymbol{n}_0}\phi - iq\boldsymbol{n}_1\psi_0|^2 - \kappa^2(1-|\psi_0|^2)|\phi|^2 \\
&+ 2\kappa^2(\Re(\overline{\phi}\psi_0))^2 - 2q\Im[(\boldsymbol{n}_1\overline{\phi} + \boldsymbol{n}_2\psi_0)\nabla_{q\boldsymbol{n}_0}\psi_0] \\
&+ K_1\{(\operatorname{div}\boldsymbol{n}_1)^2 + 2(\operatorname{div}\boldsymbol{n}_0)(\operatorname{div}\boldsymbol{n}_2)\} \\
&+ K_2\{(\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_1)^2 \\
&+ 2(\boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_0)(\boldsymbol{n}_2\cdot\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_1 \\
&+ \boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_2)\} + K_3\{|\boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_1|^2 \\
&+ 2(\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_0)\cdot(\boldsymbol{n}_2\times\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_1 \\
&+ \boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_2)\}dx\,.
\end{aligned} \tag{2.6}$$

Therefore, we have the following lemma.

LEMMA 2.2. (i) $(\psi_0, \boldsymbol{n}_0) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{u}_0)$ *is a critical point of* $\mathcal{E}$ *if and only if for any* $\phi \in W^{1,2}(\Omega, \mathbb{C})$ *and any* $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$,

$$\mathcal{A}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) - \chi \int_\Omega (\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_1)dx = 0.$$

(ii) *If a critical point* $(\psi_0, \boldsymbol{n}_0) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{u}_0)$ *is weakly stable, then for any* $\phi \in W^{1,2}(\Omega, \mathbb{C})$ *and any* $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$,

$$\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) \geq \chi \int_\Omega \{(\boldsymbol{H} \cdot \boldsymbol{n}_1)^2 + 2(\boldsymbol{H} \cdot \boldsymbol{n}_0)(\boldsymbol{H} \cdot \boldsymbol{n}_2)\} dx .$$

REMARK 2.3. (i) If $(\psi, \boldsymbol{n})$ is a critical point of $\mathcal{E}$, then the Euler-Lagrange equation for ψ is the following.

$$\begin{cases} -\nabla_{q\boldsymbol{n}}^2 \psi = \kappa^2(1 - |\psi|^2)\psi & \text{in } \Omega , \\ \nabla_{q\boldsymbol{n}} \psi \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\boldsymbol{\nu}$ denotes the unit outer normal vector to $\partial\Omega$.

(ii) We note that under the hypothesis (H.1), $(\psi_0, \boldsymbol{n}_0) = (ce^{iq\varphi}, \boldsymbol{e})$ where φ is as in (1.2) is a critical point of $\mathcal{E}$. In fact, since $\nabla_{q\boldsymbol{n}_0} \psi_0 = 0$ and $|\psi_0| = 1$, for any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, it follows from (H.1) that

$$\begin{aligned} \mathcal{A}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) - \chi\sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{e})(\boldsymbol{h} \cdot \boldsymbol{n}_0) dx &= K_1 \int_\Omega (\operatorname{div} \boldsymbol{e})(\operatorname{div} \boldsymbol{n}_1) dx \\ &= K_1 \int_\Omega -\nabla(\operatorname{div} \boldsymbol{e}) \cdot (\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}) dx . \end{aligned}$$

Since $\operatorname{curl} \boldsymbol{e} = 0$, it follows from the formula

$$\operatorname{curl}^2 \boldsymbol{e} = -\Delta \boldsymbol{e} + \nabla(\operatorname{div} \boldsymbol{e})$$

that the last line of the above equality is equal to

$$-K_1 \int_\Omega \Delta \boldsymbol{e} \cdot (\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}) dx = K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 \boldsymbol{e} \cdot (\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}) dx = 0$$

from (H.1). Thus $(\psi_0, \boldsymbol{n}_0)$ is a critical point of $\mathcal{E}$.

3. Loss of local minimality of pure smectic states

In this section we shall examine weak stability (local minimality) of pure smectic state $(\psi_0, \boldsymbol{n}_0) = (ce^{iq\varphi}, \boldsymbol{e})$ where $c \in \mathbb{C}$ and $|c| = 1$ and φ is as in (1.2).

For any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, define ψ_t and $\boldsymbol{n}_t$ as in (2.1) with $\boldsymbol{n}_0 = \boldsymbol{e}$. Then we see from $\boldsymbol{h} \cdot \boldsymbol{e}(x) = 0$, that $\boldsymbol{n}_1 = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e}$ and $\sigma \boldsymbol{h} \cdot \boldsymbol{n}_1 = \sigma(\boldsymbol{v} \cdot \boldsymbol{h})$. Thus if the critical point $(\psi_0, \boldsymbol{e})$ is weakly stable, then we see from Lemma 2.2 (ii) that

$$\mathcal{B}(\psi_0, \boldsymbol{e}; \phi, \boldsymbol{v}) \geq \chi\sigma^2 \int_\Omega (\boldsymbol{v} \cdot \boldsymbol{h})^2 dx . \tag{3.1}$$

Since $W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ is dense in $W_0^{1,2}(\Omega, \mathbb{R}^3)$, (3.1) holds for any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$. Since $\nabla_{q\boldsymbol{n}_0} \psi_0 = \nabla_{q\boldsymbol{e}}(ce^{iq\varphi}) = 0$ and $|\psi_0| = 1$, we have, from (2.6),

$$\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) = \int_\Omega \{|\nabla_{qe}\phi - iq\boldsymbol{n}_1\psi_0|^2 + 2\kappa^2(\Re(\overline{\phi}\psi_0))^2 + K_1((\operatorname{div}\boldsymbol{n}_1)^2 + 2(\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)) + K_2(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{n}_1)^2 + K_3|\boldsymbol{e}\times\operatorname{curl}\boldsymbol{n}_1|^2\}dx\,.$$

For any $\phi \in W^{1,2}(\Omega, \mathbb{C})$, we can write $\phi = icqe^{iq\varphi}u$, $u \in W^{1,2}(\Omega, \mathbb{C})$. Therefore,

$$\nabla_{qe}\phi - iq\boldsymbol{n}_1\psi_0 = icqe^{iq\varphi}(\nabla u - \boldsymbol{n}_1)$$

and $\Re(\overline{\phi}\psi_0) = |c|^2\Re(iqu) = -q\Im(u)$. Here since $\boldsymbol{n}_2 \in W_0^{1,2}(\Omega, \mathbb{R}^3)$,

$$2\int_\Omega (\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)dx = -2\int_\Omega \nabla(\operatorname{div}\boldsymbol{e})\cdot\boldsymbol{n}_2 dx\,.$$

By the formula: $\operatorname{curl}^2\boldsymbol{e} = -\Delta\boldsymbol{e} + \nabla(\operatorname{div}\boldsymbol{e})$ and the hypothesis (H.1), we have $\nabla(\operatorname{div}\boldsymbol{e}) = \Delta\boldsymbol{e} = -|\nabla\boldsymbol{e}|^2\boldsymbol{e}$. Moreover, we have $2\boldsymbol{e}\cdot\boldsymbol{n}_2 = (\boldsymbol{v}\cdot\boldsymbol{e})^2 - |\boldsymbol{v}|^2 = -|\boldsymbol{n}_1|^2$. If we write $\boldsymbol{n}_1 = \boldsymbol{w}$, then $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ and $\boldsymbol{w}(x)\cdot\boldsymbol{e}(x) = 0$ a.e. in Ω. Hence we can rewrite

$$\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) = \int_\Omega \{q^2|\nabla u - \boldsymbol{w}|^2 + 2\kappa^2 q^2(\Im(u))^2 - K_1|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2 + K_1(\operatorname{div}\boldsymbol{w})^2 + K_2(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w})^2 + K_3|\boldsymbol{e}\times\operatorname{curl}\boldsymbol{w}|^2\}dx\,.$$

If $(\phi, \boldsymbol{v})$ minimizes $\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v})/\|\boldsymbol{v}\cdot\boldsymbol{h}\|^2_{L^2(\Omega)}$, then u is real valued. Thus we may assume that $u = -\frac{i}{cq}e^{-iq\varphi}\phi$ is a real valued function. We write $\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v})$ by $B(u, \boldsymbol{w})$. That is to say,

$$B(u, \boldsymbol{w}) = \int_\Omega \{q^2|\nabla u - \boldsymbol{w}|^2 - K_1|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2\}dx + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] \tag{3.2}$$

where

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] = \int_\Omega \{K_1(\operatorname{div}\boldsymbol{w})^2 + K_2(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w})^2 + K_3|\boldsymbol{e}\times\operatorname{curl}\boldsymbol{w}|^2\}dx\,.$$

Here we note that under the hypothesis (H.2), we can show the following.

LEMMA 3.1. *Assume that* (H.2) *holds. Then there exists a constant* $c > 0$ *such that*

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2 dx \geq c\|\boldsymbol{w}\|^2_{W^{1,2}(\Omega,\mathbb{R}^3)} \tag{3.3}$$

for all $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$.

PROOF. Since $(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w})^2+|\boldsymbol{e}\times\operatorname{curl}\boldsymbol{w}|^2=|\operatorname{curl}\boldsymbol{w}|^2$ and $\boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^3)$, we have

$$\begin{aligned}
&\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}]-K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx\\
&\quad\ge \min\{K_1,K_2,K_3\}\int_\Omega\{|\operatorname{div}\boldsymbol{w}|^2+|\operatorname{curl}\boldsymbol{w}|^2\}dx-K_1\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2\int_\Omega|\boldsymbol{w}|^2dx\\
&\quad\ge \min\{K_1,K_2,K_3\}\int_\Omega|\nabla\boldsymbol{w}|^2dx-K_1\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2c(\Omega)\int_\Omega|\nabla\boldsymbol{w}|^2dx\\
&\quad\ge \{\min\{K_1,K_2,K_3\}-K_1c(\Omega)\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2\}\int_\Omega|\nabla\boldsymbol{w}|^2dx\\
&\quad\ge c_1\{\min\{K_1,K_2,K_3\}-K_1c(\Omega)\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2\}\|\boldsymbol{w}\|^2_{W_0^{1,2}(\Omega,\mathbb{R}^3)}
\end{aligned}$$

for some positive constant c. Thus (3.3) holds with

$$c=c_1(\min\{K_1,K_2,K_3\}-K_1c(\Omega)\max_{x\in\overline{\Omega}}|\nabla\boldsymbol{e}|^2)\,. \qquad \square$$

DEFINITION 3.2. For $q\ge 0,\kappa>0,K_1>0,K_2>0$ and $K_3>0$, define $H_{sh}=H_{sh}(q,\kappa,K_1,K_2,K_3,\Omega,\boldsymbol{h},\boldsymbol{e})$ by

$$H_{sh}^2=\frac{1}{\chi}\inf\left\{\frac{B(u,\boldsymbol{w})}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}};\ (u,\boldsymbol{w})\in W^{1,2}(\Omega)\times W_0^{1,2}(\Omega,\mathbb{R}^3),\ \boldsymbol{w}(x)\cdot\boldsymbol{e}(x)=0\ \text{a.e. in}\ \Omega,\boldsymbol{h}\cdot\boldsymbol{w}(x)\not\equiv 0\ \text{in}\ \Omega\right\}.$$

From the above arguments, we have the following lemma.

LEMMA 3.3. *If $\sigma<H_{sh}$, then we see that the pure smectic state is weakly stable and if $\sigma>H_{sh}$, then the pure smectic state is not weakly stable.*

PROOF. If $\sigma<H_{sh}$, then

$$\sigma^2\chi\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}<B(u,\boldsymbol{w}) \tag{3.4}$$

for all $(u,\boldsymbol{w})\in W^{1,2}(\Omega)\times W_0^{1,2}(\Omega,\mathbb{R}^3)$ with $\boldsymbol{w}(x)\cdot\boldsymbol{e}(x)=0$ a.e. in Ω and $\boldsymbol{h}\cdot\boldsymbol{w}(x)\not\equiv 0$. For any $\phi\in W^{1,2}(\Omega,\mathbb{C})$ and any $\boldsymbol{v}\in W_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega,\mathbb{R}^3)$, we define ψ_t and $\boldsymbol{n}_t$ by (2.1). We show that

$$\mathcal{E}[\psi_0,\boldsymbol{n}_0]\le\mathcal{E}[\psi_t,\boldsymbol{n}_t] \tag{3.5}$$

for small $|t|>0$.

In the case where $\boldsymbol{v}=0$, we see that $\boldsymbol{n}_t=\boldsymbol{e}$. Thus we have

$$\begin{aligned}
\mathcal{E}[\psi_t,\boldsymbol{n}_t]&=\mathcal{G}[\psi_t,\boldsymbol{n}_t]+\mathcal{F}[\boldsymbol{n}_t]-\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_t)^2dx\\
&=\mathcal{G}[\psi_t,\boldsymbol{e}]+\mathcal{F}[\boldsymbol{e}]
\end{aligned} \tag{3.6}$$

$$\geq K_1 \int_\Omega |\operatorname{div} \boldsymbol{e}|^2 dx = \mathcal{E}[\psi_0, \boldsymbol{n}_0] \,.$$

Thus we see that (3.5) holds for small t.

In the case where $\boldsymbol{v} \neq 0$, we may assume that $|\boldsymbol{v}| = 1$. When $\boldsymbol{v} = \pm \boldsymbol{e}$, then $\boldsymbol{w} = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e} = 0$ and $\boldsymbol{n}_t = \boldsymbol{e}$. Thus since (3.6) holds, we see that (3.5) holds. When $\boldsymbol{v} \neq \pm \boldsymbol{e}$, $\boldsymbol{w} = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e} \neq 0$. If $\boldsymbol{h} \cdot \boldsymbol{w}(x) \equiv 0$, putting $u = -i \frac{1}{cq} e^{-iq\varphi}\phi$, it follows from (3.3) and the Poincaré inequality that

$$B(u, \boldsymbol{w}) \geq c \int_\Omega |\nabla \boldsymbol{w}|^2 dx \geq c_1 \int_\Omega |\boldsymbol{w}|^2 dx > 0 = \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{w})^2 dx \,.$$

Thus it follows from (2.4) that (3.5) holds. If $\boldsymbol{h} \cdot \boldsymbol{w}(x) \neq 0$, using (3.4) we can see that (3.5) also holds.

If $\sigma > H_{sh}$, there exists $(u, \boldsymbol{w}) \in W^{1,2}(\Omega) \times W_0^{1,2}(\Omega, \mathbb{R}^3)$ with $\boldsymbol{w}(x) \cdot \boldsymbol{h} \not\equiv 0$ and $\boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0$ in Ω such that

$$B(u, \boldsymbol{w}) < \chi \sigma^2 \|\boldsymbol{h} \cdot \boldsymbol{w}\|^2_{L^2(\Omega)} \,.$$

It follows from Lemma 2.2 (ii) that $(\psi_0, \boldsymbol{n}_0)$ is not weakly stable. □

For a further simple expression of H_{sh}, let $(u, \boldsymbol{w})$ be a minimizer of H_{sh}. Then u satisfies the equation

$$\begin{cases} \Delta u = \operatorname{div} \boldsymbol{w} & \text{in } \Omega \,, \\ \frac{\partial u}{\partial \boldsymbol{\nu}} = \boldsymbol{w} \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{3.7}$$

If we impose $\int_\Omega u dx = 0$, the solution of (3.7) is unique. We write $u = \xi_{\boldsymbol{w}}$. Then it is clear that $\xi_{\boldsymbol{w}}$ is a minimizer of

$$\omega(\boldsymbol{w}) = \inf_{\xi \in W^{1,2}(\Omega)} \frac{1}{|\Omega|} \int_\Omega |\nabla \xi - \boldsymbol{w}|^2 dx \,.$$

Hence $\xi_{\boldsymbol{w}}$ satisfies

$$\int_\Omega |\nabla \xi_{\boldsymbol{w}} - \boldsymbol{w}|^2 dx = \omega(\boldsymbol{w}) |\Omega| \,.$$

Write $B(\boldsymbol{w}) = B(\xi_{\boldsymbol{w}}, \boldsymbol{w})$. It is clear that for any $b > 0$, $\xi_{b\boldsymbol{w}} = b\xi_{\boldsymbol{w}}$, and so $B(b\boldsymbol{w}) = b^2 B(\boldsymbol{w})$. Therefore we can write

$$\begin{aligned} H_{sh}^2 = \frac{1}{\chi} \inf\{B(\boldsymbol{w}); \boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3), \boldsymbol{w}(x) \cdot \boldsymbol{e}(x) \\ = 0 \text{ a.e. in } \Omega \,, \|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)} = 1\} \,. \end{aligned}$$

Here we note that the pure smectic state involves a complex number c, but $B(u, \boldsymbol{w})$ and $B(\boldsymbol{w})$ are independent of c. From now we write H_{sh} by $H_{sh}(q)$. Then we see that

$$H_{sh}^2(0) = \frac{1}{\chi} \inf\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx;\ \boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3),$$
$$\boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0 \text{ a.e. in } \Omega,\ \|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)} = 1\}\,.$$

PROPOSITION 3.4. *Assume that Ω is simply connected domain with smooth boundary, and* (H.1), (H.2) *and* (H.3) *hold. Then $H_{sh}(q) > 0$ and it is achieved. For fixed κ, K_1, K_2, K_3, Ω, $\boldsymbol{h}$ and $\boldsymbol{e}$, we have*

$$\lim_{q \to +\infty} H_{sh}(q) = +\infty\,.$$

PROOF. Step 1. Let $\boldsymbol{w}_j \in W_0^{1,2}(\Omega, \mathbb{R}^3)$, $\xi_j = \xi_{\boldsymbol{w}_j}$ satisfy $\boldsymbol{w}_j(x) \cdot \boldsymbol{e}(x) = 0$ a.e. in Ω, $\|\boldsymbol{h} \cdot \boldsymbol{w}_j\|_{L^2(\Omega)} = 1$ and $B(\boldsymbol{w}_j) \to \chi H_{sh}^2(q)$ as $j \to \infty$. Since $\boldsymbol{w}_j \in W_0^{1,2}(\Omega, \mathbb{R}^3)$, it follows from (3.3) that there exist constants $c, C > 0$ such that $\|\boldsymbol{w}_j\|^2_{W_0^{1,2}(\Omega,\mathbb{R}^3)} \leq cB(\boldsymbol{w}_j) \leq C$. Thus passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \boldsymbol{w}_0$ weakly in $W_0^{1,2}(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Hence $\boldsymbol{w}_0(x) \cdot \boldsymbol{e}(x) = 0$ a.e. in Ω. Since

$$\int_\Omega |\nabla \xi_j - \boldsymbol{w}_j|^2 dx \leq \frac{1}{q^2} B(\boldsymbol{w}_j) \leq C\,,$$

we see that $\|\nabla \xi_j\|_{L^2(\Omega,\mathbb{R}^3)}$ is bounded. Since $\int_\Omega \xi_j dx = 0$, again applying the Poincaré inequality, we see that $\{\xi_j\}$ is bounded in $W^{1,2}(\Omega, \mathbb{R})$. Moreover, since $\|\operatorname{div} \boldsymbol{w}_j\|_{L^2(\Omega)} \leq C\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \leq C$, $\boldsymbol{w}_j = 0$ on $\partial\Omega$ and $\xi_j = \xi_{\boldsymbol{w}_j}$ is a unique solution of (3.7), it follows from the elliptic estimate that $\{\xi_j\}$ is bounded in $W^{2,2}(\Omega)$. After passing to a subsequence, we may assume that $\xi_j \to \xi_0$ weakly in $W^{2,2}(\Omega)$ and strongly in $W^{1,2}(\Omega)$. Then ξ_0 satisfies (3.7) for $\boldsymbol{w} = \boldsymbol{w}_0$, i.e., $\xi_0 = \xi_{\boldsymbol{w}_0}$. Therefore,

$$B(\boldsymbol{w}_0) = B(\xi_{\boldsymbol{w}_0}, \boldsymbol{w}_0) \leq \liminf_{j \to \infty} B(\xi_j, \boldsymbol{w}_j) = \chi H_{sh}^2(q)\,.$$

Since $\|\boldsymbol{w}_0 \cdot \boldsymbol{h}\|_{L^2(\Omega)} = \lim_{j\to\infty} \|\boldsymbol{w}_j \cdot \boldsymbol{h}\|_{L^2(\Omega)} = 1$, we see that $B(\boldsymbol{w}_0) \geq \chi H_{sh}^2(q)$. Thus $\boldsymbol{w}_0$ is a minimizer of $B(\boldsymbol{w})$, so $(\xi_{\boldsymbol{w}_0}, \boldsymbol{w}_0)$ achieves $H_{sh}(q)$.

We show that $H_{sh}(q) > 0$. If $H_{sh}(q) = 0$, then $B(\boldsymbol{w}_0) = 0$ and so $\nabla \xi_{\boldsymbol{w}_0} = \boldsymbol{w}_0$ and $\operatorname{div} \boldsymbol{w}_0 = 0$. By the uniqueness of the solution of (3.7) with $\int_\Omega u dx = 0$, we have $\xi_{\boldsymbol{w}_0} = 0$, and so $\boldsymbol{w}_0 = 0$. This contradicts the fact that $\|\boldsymbol{h} \cdot \boldsymbol{w}_0\|_{L^2(\Omega)} = 1$.

Step 2. Suppose that $H_{sh}(q) \leq c$ for all $q \geq 0$. Choose $q_j \to \infty$ and choose $u_j \in W^{1,2}(\Omega)$ and $\boldsymbol{w}_j \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ such that

$$\int_\Omega u_j dx = 0, \boldsymbol{e}(x) \cdot \boldsymbol{w}_j(x) = 0 \text{ a.e. in } \Omega,\ \|\boldsymbol{h} \cdot \boldsymbol{w}_j\|_{L^2(\Omega)} = 1\,,$$

and $(u_j, \boldsymbol{w}_j)$ achieves $H_{sh}(q_j)$. Then

$$\int_\Omega q_j^2|\nabla u_j - \boldsymbol{w}_j|^2 dx + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla \boldsymbol{e}|^2|\boldsymbol{w}_j|^2 dx$$
$$\le \chi c^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{w}_j)^2 dx = \chi c^2 .$$

Thus from (3.3) , $\{\boldsymbol{w}_j\}$ is bounded in $W_0^{1,2}(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ weakly in $W_0^{1,2}(\Omega,\mathbb{R}^3)$, strongly in $L^4(\Omega,\mathbb{R}^3)$ and a.e. in Ω. Hence this implies that $\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|_{L^2(\Omega)} = 1$ and $\widehat{\boldsymbol{w}}(x)\cdot\boldsymbol{e}(x) = 0$ a.e. in Ω. Since $\|\nabla u_j - \boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)} = O(q_j^{-1})$ and $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$, $\{\nabla u_j\}$ is bounded in $L^2(\Omega,\mathbb{R}^3)$. Since $\int_\Omega u_j dx = 0$, it follows from the Poincaré inequality that $\{u_j\}$ is bounded in $W^{1,2}(\Omega)$. Passing to a subsequence, we may assume that $u_j \to \widehat{u}$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. Since $\nabla u_j \to \nabla\widehat{u}$ weakly in $L^2(\Omega,\mathbb{R}^3)$ and $\nabla u_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$, we have $\nabla\widehat{u} = \widehat{\boldsymbol{w}}$ and $\nabla u_j \to \nabla\widehat{u}$ strongly in $L^2(\Omega,\mathbb{R}^3)$. Thus we see that $u_j \to \widehat{u}$ strongly in $W^{1,2}(\Omega)$. Moreover, $\nabla\widehat{u} = \widehat{\boldsymbol{w}} = 0$ on $\partial\Omega$ and $\nabla\widehat{u}\cdot\boldsymbol{e} = 0$ in Ω. By the hypothesis (H.3), we see that $\nabla\widehat{u} = 0$ in Ω. In fact, assume that $\nabla\widehat{u}(p) \neq 0$ for some point $p \in \Omega$. Let $\boldsymbol{x} = \boldsymbol{x}(t)$ be the integral curve of $\boldsymbol{e}$ through p. Then since

$$\frac{d}{dt}\nabla\widehat{u}(\boldsymbol{x}(t)) = \nabla(\nabla\widehat{u}(\boldsymbol{x}(t))\cdot\boldsymbol{e}(\boldsymbol{x}(t)) = 0 ,$$

$\nabla\widehat{u}(\boldsymbol{x}(t))$ is independent of t. By the hypothesis (H.3), $\boldsymbol{x}(t)$ intersects with $\partial\Omega$. This contradicts the fact that $\nabla\widehat{u} = 0$ on $\partial\Omega$. Thus $\nabla\widehat{u} = 0$ in Ω and so $\widehat{\boldsymbol{w}} = \nabla\widehat{u} = 0$ in Ω. This contradicts $\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|_{L^2(\Omega)} = 1$. □

We shall derive the Euler-Lagrange equation for the minimizer of $H_{sh}(q)$. Let $(\xi_{\boldsymbol{w}}, \boldsymbol{w})$ be a minimizer of $H_{sh}(q)$. Then we see that $\boldsymbol{w}$ satisfies

$$H_{sh}^2(q) = \frac{1}{\chi}\frac{q^2\|\nabla\xi_{\boldsymbol{w}} - \boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2 dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}} .$$

For $\boldsymbol{v} \in W_0^{1,2}(\Omega,\mathbb{R}^3)$ with $\boldsymbol{v}(x)\cdot\boldsymbol{e}(x) = 0$, since $\xi_{\boldsymbol{w}+t\boldsymbol{v}} = \xi_{\boldsymbol{w}} + t\xi_{\boldsymbol{v}}$ and

$$\int_\Omega (\boldsymbol{h}\cdot(\boldsymbol{w}+t\boldsymbol{v}))^2 dx = \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{w})^2 dx + 2t\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{w})(\boldsymbol{h}\cdot\boldsymbol{v}) dx + O(t^2) ,$$

we have

$$\|\nabla\xi_{\boldsymbol{w}+t\boldsymbol{v}} - (\boldsymbol{w}+t\boldsymbol{v})\|^2_{L^2(\Omega,\mathbb{R}^3)} = \|\xi_{\boldsymbol{w}} - \boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)}$$
$$+2t\int_\Omega (\nabla\xi_{\boldsymbol{w}} - \boldsymbol{w})\cdot(\nabla\xi_{\boldsymbol{v}} - \boldsymbol{v}) dx + O(t^2),$$
$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}+t\boldsymbol{v}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}+t\boldsymbol{v}|^2 dx = \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}]$$

$$-K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx$$
$$+2t \int_\Omega \{K_1 (\mathrm{div}\, \boldsymbol{w})(\mathrm{div}\, \boldsymbol{v}) + K_2 (\boldsymbol{e} \cdot \mathrm{curl}\, \boldsymbol{w})(\boldsymbol{e} \cdot \mathrm{curl}\, \boldsymbol{v})$$
$$+K_3 (\boldsymbol{e} \times \mathrm{curl}\, \boldsymbol{w}) \cdot (\boldsymbol{e} \times \mathrm{curl}\, \boldsymbol{v}) - K_1 |\nabla \boldsymbol{e}|^2 (\boldsymbol{w} \cdot \boldsymbol{v})\} dx + O(t^2).$$

Here we note that

$$(\boldsymbol{e} \times \mathrm{curl}\, \boldsymbol{w}) \cdot (\boldsymbol{e} \times \mathrm{curl}\, \boldsymbol{v}) = (\mathrm{curl}\, \boldsymbol{w} \cdot \mathrm{curl}\, \boldsymbol{v}) - (\mathrm{curl}\, \boldsymbol{w} \cdot \boldsymbol{e})(\mathrm{curl}\, \boldsymbol{v} \cdot \boldsymbol{e})\,.$$

Thus if we define $\boldsymbol{k}(x) = \boldsymbol{h} \times \boldsymbol{e}(x)$ and an orthogonal projection P onto the space $[\boldsymbol{k}(x), \boldsymbol{h}]$ spanned by $\boldsymbol{k}(x)$ and $\boldsymbol{h}$, we get the Euler-Lagrange equation

$$\begin{cases} P\big[-K_1 \nabla(\mathrm{div}\, \boldsymbol{w}) - K_2 \boldsymbol{e} \times \nabla(\boldsymbol{e} \cdot \mathrm{curl}\, \boldsymbol{w}) \\ +K_3 \big(\mathrm{curl}^2 \boldsymbol{w} + \boldsymbol{e} \times \nabla(\boldsymbol{e} \cdot \mathrm{curl}\, \boldsymbol{w})\big) \\ +q^2 (\boldsymbol{w} - \nabla \xi_{\boldsymbol{w}}) - K_1 |\nabla \boldsymbol{e}|^2 \boldsymbol{w}\big] \\ = \chi H_{sh}^2(q)(\boldsymbol{h} \cdot \boldsymbol{w}) \boldsymbol{h} & \text{in } \Omega\,, \\ \boldsymbol{w} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

In particular case where $K_1 = K_2 = K_3 = K$, since $\mathrm{curl}^2 \boldsymbol{w} = -\Delta \boldsymbol{w} + \nabla \mathrm{div}\, \boldsymbol{w}$, we have

$$\begin{cases} P[-K \Delta \boldsymbol{w} + q^2 (\boldsymbol{w} - \nabla \xi_{\boldsymbol{w}}) - K |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2] = \chi H_{sh}^2 (\boldsymbol{h} \cdot \boldsymbol{w}) \boldsymbol{h} & \text{in } \Omega\,, \\ \boldsymbol{w} = 0 \quad \text{on } \partial\Omega & . \end{cases}$$

4. Loss of global minimality of pure smectic states

In this section, we examine loss of global minimality of pure smectic states. In order to do so, let $(\psi_0, \boldsymbol{n}_0)$ be a pure smectic state. That is to say, $\psi_0 = c e^{iq\varphi}, \boldsymbol{n}_0 = \boldsymbol{e}$ with $c \in \mathbb{C}$, $|c| = 1$ and φ is as in (1.2).

If a global minimizer $(\psi, \boldsymbol{n}) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ is not a pure smectic state, then we claim that

$$\boldsymbol{h} \cdot \boldsymbol{n} \not\equiv 0 \quad \text{in } \Omega\,. \tag{4.1}$$

In fact, if $\boldsymbol{h} \cdot \boldsymbol{n} \equiv 0$ in Ω,

$$\mathcal{E}[\psi, \boldsymbol{n}] = \mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] \leq \mathcal{E}[\psi_0, \boldsymbol{n}_0] = \int_\Omega K_1 |\mathrm{div}\, \boldsymbol{e}|^2 dx\,.$$

Hence since $\mathcal{F}[\boldsymbol{n}] \leq K_1 \|\mathrm{div}\, \boldsymbol{e}\|_{L^2(\Omega)}^2 = \mathcal{F}[\boldsymbol{e}]$, we have $\boldsymbol{n} = \boldsymbol{e}$ from (H.1), Moreover, we have

$$0 = \mathcal{G}[\psi, \boldsymbol{n}] = \int_\Omega \left\{ |\nabla \psi - iq\boldsymbol{n}\psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 \right\} dx\,.$$

Therefore, since $|\psi| = 1$, we can write $\psi = ce^{iq\widetilde{\varphi}(x)}$ with $|c| = 1$ locally for some function $\widetilde{\varphi}$. Therefore, $0 = \nabla\psi - iq\boldsymbol{e}\psi = icq(\nabla\widetilde{\varphi} - \boldsymbol{e})e^{iq\widetilde{\varphi}}$. Thus $\nabla\widetilde{\varphi} = \boldsymbol{e}$, so we can write $\psi = ce^{iq\varphi}$ with $|c| = 1$ locally where φ is as in (1.2). Since Ω is connected, $\psi = ce^{iq\varphi}$ in Ω. Then $(\psi, \boldsymbol{n}) = (\psi_0, \boldsymbol{n}_0)$ is a pure smectic state. Hence (4.1) holds.

Thus if $(\psi, \boldsymbol{n})$ is a global minimizer of $\mathcal{E}$ which is not a pure smectic state, we have

$$\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\mathrm{div}\,\boldsymbol{e}\|^2_{L^2(\Omega)} \leq \chi\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}\,.$$

DEFINITION 4.1. For given $q \geq 0, \kappa > 0, K_1, K_2, K_3 > 0$ and $\boldsymbol{h}, \boldsymbol{e}_0$, define $H_s = H_s(q, \kappa, K_1, K_2, K_3, \Omega, \boldsymbol{h}, \boldsymbol{e}_0)$ by

$$H_s^2 = \frac{1}{\chi}\inf\left\{\frac{\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\mathrm{div}\,\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};\right. \qquad (4.2)$$

$$\left.(\psi, \boldsymbol{n}) \in W^{1,2}(\Omega; \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0), \boldsymbol{h}\cdot\boldsymbol{n}(x) \not\equiv 0 \text{ in } \Omega\right\}$$

Then we have the following lemma.

LEMMA 4.2. *Under the assumptions* (H.1), (H.2) *and* (H.3), *we have following.*

(i) *If there exists a global minimizer* $(\psi, \boldsymbol{n})$ *of* $\mathcal{E}$ *which is not a pure smectic state, then* $\sigma \geq H_s$.

(ii) *If* $\sigma > H_s$, *then the global minimizers of* $\mathcal{E}$ *are not pure smectic states.*

(iii) *If* $0 \leq \sigma < H_s$, *then the only global minimizers of* $\mathcal{E}$ *are pure smectic states.*

PROOF. (i) If $(\psi, \boldsymbol{n})$ is a global minimizer of $\mathcal{E}$ which is not a pure smectic state, then from (4.1) $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$ in Ω and $\mathcal{E}[\psi, \boldsymbol{n}] \leq \mathcal{E}[\psi_0, \boldsymbol{n}_0] = K_1\|\mathrm{div}\,\boldsymbol{e}\|^2_{L^2(\Omega)}$. Therefore, we have

$$\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\mathrm{div}\,\boldsymbol{e}\|^2_{L^2(\Omega)} \leq \chi\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n})^2dx\,.$$

This implies that $H_s \leq \sigma$.

(ii) If $\sigma > H_s$, there exists $(\psi, \boldsymbol{n}) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ with $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$ in Ω and

$$\frac{1}{\chi}\frac{\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\mathrm{div}\,\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}} < \sigma^2\,.$$

Thus we have

$$\mathcal{E}[\psi, \boldsymbol{n}] = \mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - \chi\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)} < \mathcal{E}[\psi_0, \boldsymbol{n}_0] = K_1\|\mathrm{div}\,\boldsymbol{e}\|^2_{L^2(\Omega)}\,.$$

This implies that global minimizers of $\mathcal{E}$ are not pure smectic states.

(iii) If $0 \leq \sigma < H_s$ and there exists a global minimizer which is not a pure smectic state, it follows from (i) that $\sigma \geq H_s$. $\square$

In the following we write H_s by $H_s(\kappa, q)$. Since pure smectic states lose the global minimality at $H_s(\kappa, q)$ and lose local minimality at $H_{sh}(q)$, we see that

$$H_s(\kappa, q) \leq H_{sh}(q)\,. \tag{4.3}$$

We define a number H_n which is closely related to H_s.

DEFINITION 4.3. $H_n = H_n(q) = H_n(q, \kappa, K_1, K_2, K_3, \Omega, \boldsymbol{h}, \boldsymbol{e}_0)$ is defined by

$$H_n^2 = \frac{1}{\chi}\inf\left\{\frac{q^2\|\nabla u - \boldsymbol{n}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};\right.$$

$$\left.(u, \boldsymbol{n}) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0),\, \boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0 \text{ in } \Omega\right\}$$

We note that

$$H_n^2(0) = \frac{1}{\chi}\inf\left\{\frac{\mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};\, \boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0),\, \boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0 \text{ in } \Omega\right\}.$$

LEMMA 4.4. *For any* $\kappa > 0$, *we have* $H_s(\kappa, 0) = H_n(0)$, *and for any* $\kappa > 0$ *and* $q \geq 0$, *we have* $H_s(\kappa, q) \geq H_n(0)$.

PROOF. We choose a test field $\psi = 1$ and any $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ with $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$ in Ω, we see that $\mathcal{G}[\psi, \boldsymbol{n}]\big|_{q=0} = 0$. Thus $H_s^2(\kappa, 0) \leq H_n^2(0)$. On the other hand, for any $\kappa > 0$ and $q \geq 0$, it is easily seen that

$$\begin{aligned} H_s^2(\kappa, q) &\geq \frac{1}{\chi}\inf\left\{\frac{\mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};\, \boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0),\, \boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0\right\} \\ &= H_n^2(0)\,. \end{aligned}$$

□

THEOREM 4.5. *Let* Ω *be a simply connected bounded domain in* $\mathbb{R}^3$ *with smooth boundary and assume that* (H.1), (H.2) *and* (H.3) *hold. Then we can get the following.*

(i) *For any* $\kappa > 0$ *and any* $q \geq 0$,

$$0 < H_s(\kappa, q) \leq H_n(q) \leq H_{sh}(q)\,.$$

(ii) *For any* $\kappa > 0$ *and any* $q \geq 0$, *if* $H_s(\kappa, q) < H_{sh}(q)$, *then* $H_s(\kappa, q)$ *is achieved.*

(iii) *For any* $q \geq 0$, *if* $H_n(q) < H_{sh}(q)$, *then* $H_n(q)$ *is achieved. In this case, we have* $H_s(\kappa, q) < H_{sh}(q)$, *so* $H_s(\kappa, q)$ *is achieved.*

PROOF. From now, we denote various constants by c, C, C_1 which may vary from line to line.

(i) Step 1. We show that $H_s(\kappa, q) \leq H_n(q)$.

For any $\phi \in W^{1,2}(\Omega)$ and any $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ with $\boldsymbol{n} \cdot \boldsymbol{h} \not\equiv 0$ in Ω, we take $(e^{iq\phi}, \boldsymbol{n})$ as a test function of $H_s(\kappa, q)$. Then we have

$$H_s^2(\kappa, q) \le \frac{1}{\chi} \frac{q^2\|\nabla\phi - \boldsymbol{n}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}}.$$

This implies that $H_s^2(\kappa, q) \le H_n^2(q)$.

Step 2. We show that $H_n(q) \le H_{sh}(q)$.

Let $u \in W^{1,2}(\Omega)$, $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ with $\boldsymbol{e} \cdot \boldsymbol{w} \equiv 0$ in Ω and $\boldsymbol{h} \cdot \boldsymbol{w} \not\equiv 0$ in Ω and put

$$\phi_t = \varphi + tu\,, \quad \boldsymbol{n}_t = \frac{\boldsymbol{e} + t\boldsymbol{w}}{|\boldsymbol{e} + t\boldsymbol{w}|} = \boldsymbol{e} + t\boldsymbol{w} + O(t^2)$$

where φ is as in (1.2). Then we have

$$q^2\|\nabla\phi_t - \boldsymbol{n}_t\|^2_{L^2(\Omega,\mathbb{R}^3)} = t^2q^2\|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)} + O(t^3)\,,$$

$$\mathcal{F}[\boldsymbol{n}_t] = \mathcal{F}[\boldsymbol{e}] + t^2\left\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] + 2K_1\int_\Omega (\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)dx\right\} + O(t^3)\,,$$

$$\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_t)^2dx = t^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{w})^2dx + O(t^3)\,.$$

Since $\boldsymbol{n}_2 = -\frac{1}{2}|\boldsymbol{w}|^2\boldsymbol{e}$, it follows from the hypothesis (H.1) that

$$2K_1\int_\Omega(\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)dx = K_1\int_\Omega \nabla(\operatorname{div}\boldsymbol{e})\cdot|\boldsymbol{w}|^2\boldsymbol{e}dx = -K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx.$$

Therefore,

$$\begin{aligned}
&\frac{q^2\|\nabla\phi_t - \boldsymbol{n}_t\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_t] - \mathcal{F}[\boldsymbol{e}]}{\|\boldsymbol{h}\cdot\boldsymbol{n}_t\|^2_{L^2(\Omega)}} \\
&\quad= \frac{q^2\|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] + 2K_1\int_\Omega(\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t) \\
&\quad= \frac{q^2\|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t)\,.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\chi H_n^2(q) &\le \frac{q^2\|\nabla\phi_t - \boldsymbol{n}_t\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_t] - \mathcal{F}[\boldsymbol{e}]}{\|\boldsymbol{h}\cdot\boldsymbol{n}_t\|^2_{L^2(\Omega)}} \\
&= \frac{q^2\|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t)\,.
\end{aligned}$$

Letting $t \to 0$, we have

$$\chi H_n^2(q) \leq \frac{q^2\|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega |\nabla \boldsymbol{e}|^2|\boldsymbol{w}|^2 dx}{\|\boldsymbol{h}\cdot\boldsymbol{w}\|^2_{L^2(\Omega)}}\,.$$

Since $W_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega,\mathbb{R}^3)$ is dense in $W_0^{1,2}(\Omega,\mathbb{R}^3)$, we get $H_n(q) \leq H_{sh}(q)$.

Step 3. We show that $H_s(\kappa,q) > 0$.

Since $H_{sh}(q) > 0$ from Proposition 3.4, if $H_s(\kappa,q) = H_{sh}(q)$, the result is trivial. So we assume that $H_s(\kappa,q) < H_{sh}(q)$. We borrow the result of (ii) which is proved independently of (i). Since $H_s(\kappa,q)$ is achieved, let $(\psi,\boldsymbol{n})$ be a minimizer of $H_s(\kappa,q)$. Assume that $H_s(\kappa,q) = 0$. Then we have

$$\begin{aligned} 0 &\leq \mathcal{G}[\psi,\boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} \\ &= \chi H_s^2(\kappa,q)\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)} = 0\,, \quad \boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0 \text{ in } \Omega\,. \end{aligned}$$

This implies that $\mathcal{F}[\boldsymbol{n}] = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$. By the hypothesis (H.1), we see that $\boldsymbol{n} = \boldsymbol{e}$ in Ω. This contradicts the fact that $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$. Thus we see that (i) holds if (ii) is proved independently.

Proof of (ii). We assume that $H_s(\kappa,q) < H_{sh}(q)$.

Step 4. Let $\{(\psi_j,\boldsymbol{n}_j)\}$ be a minimizing sequence of $H_s(\kappa,q)$. Then

$$\mathcal{G}[\psi_j,\boldsymbol{n}_j] + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} = (\chi H_s(\kappa,q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{n}_j\|^2_{L^2(\Omega)} \tag{4.4}$$

Since $|\boldsymbol{h}\cdot\boldsymbol{n}_j| \leq 1$, the right hand side of (4.4) is bounded. Thus $\{\operatorname{div}\boldsymbol{n}_j\}$ is bounded in $L^2(\Omega)$, $\{\operatorname{curl}\boldsymbol{n}_j\}$ is bounded in $L^2(\Omega,\mathbb{R}^3)$ and $\boldsymbol{n}_j = \boldsymbol{e}_0$ on $\partial\Omega$. It follows from Dautray and Lions [4] (or Girault and Raviart [8], Temam [15]) that $\{\boldsymbol{n}_j\}$ is bounded in $W^{1,2}(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{n}_j \to \widehat{\boldsymbol{n}}$ weakly in $W^{1,2}(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$ and a.e. in Ω. Thus we have $|\widehat{\boldsymbol{n}}| = 1$ a.e. in Ω and $\widehat{\boldsymbol{n}} = \boldsymbol{e}_0$ on $\partial\Omega$, so $\widehat{\boldsymbol{n}} \in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$. On the other hand, we see from (4.4) that $\{\nabla_{q\boldsymbol{n}_j}\psi_j\}$ is bounded in $L^2(\Omega,\mathbb{C}^3)$ and $\{\psi_j\}$ is bounded in $L^4(\Omega,\mathbb{C})$. Since

$$\|\nabla\psi_j\|_{L^2(\Omega,\mathbb{C}^3)} \leq \|\nabla_{q\boldsymbol{n}_j}\psi_j\|_{L^2(\Omega,\mathbb{C}^3)} + \|q\boldsymbol{n}_j\psi_j\|_{L^2(\Omega,\mathbb{C}^3)}\,,$$

we see that $\{\psi_j\}$ is bounded in $W^{1,2}(\Omega,\mathbb{C})$. After passing to a subsequence, we may assume that $\psi_j \to \widehat{\psi}$ weakly in $W^{1,2}(\Omega,\mathbb{C})$ and strongly in $L^4(\Omega,\mathbb{C})$. Thus we have

$$\begin{aligned} &\mathcal{G}[\widehat{\psi},\widehat{\boldsymbol{n}}] + \mathcal{F}[\widehat{\boldsymbol{n}}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} \\ &\quad\leq \liminf_{j\to\infty}\{\mathcal{G}[\psi_j,\boldsymbol{n}_j] + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}\} \\ &\quad= \chi H_s^2(\kappa,q)\|\boldsymbol{h}\cdot\widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.5}$$

If we can show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$ in Ω, we see that $H_s(\kappa,q)$ is achieved.

Step 5. Assume that $\boldsymbol{h} \cdot \widehat{\boldsymbol{n}} \equiv 0$ in Ω. Then it follows from (4.5) and the hypothesis (H.1) that $\widehat{\boldsymbol{n}} = \boldsymbol{e}$ in Ω. Moreover, $\nabla\widehat{\psi} - iq\boldsymbol{e}\widehat{\psi} = 0$, $|\widehat{\psi}| = 1$. Then we can write $\widehat{\psi} = ce^{iq\varphi}$ for some $c \in \mathbb{C}$ with $|c| = 1$ where φ is as in (1.2). For brevity, we assume that $c = 1$.

Since $\boldsymbol{h} \cdot \boldsymbol{n}_j \not\equiv 0$, we have $\boldsymbol{n}_j \not\equiv \boldsymbol{e}$. We write

$$\boldsymbol{n}_j = \boldsymbol{e} + \varepsilon_j \boldsymbol{w}_j\,, \quad \psi_j = e^{iq\varphi}(1 + iq\varepsilon_j g_j) \tag{4.6}$$

such that $\varepsilon_j = \|\boldsymbol{n}_j - \boldsymbol{e}\|_{W^{1,2}(\Omega,\mathbb{R}^3)} > 0$, $\boldsymbol{w}_j \in W_0^{1,2}(\Omega,\mathbb{R}^3)$ and $\boldsymbol{w}_j$ satisfies that $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$. Using the Poincaré inequality and the formula $|\boldsymbol{e} \cdot \operatorname{curl} \boldsymbol{w}|^2 + |\boldsymbol{e} \times \operatorname{curl} \boldsymbol{w}|^2 = |\operatorname{curl} \boldsymbol{w}|^2$, we have

$$\begin{aligned} \mathcal{F}[\boldsymbol{n}_j] - \mathcal{F}[\boldsymbol{e}] &= \varepsilon_j^2 \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] \\ &= \varepsilon_j^2 \int_\Omega \{K_1 |\operatorname{div} \boldsymbol{w}_j|^2 + K_2 |\boldsymbol{e} \cdot \operatorname{curl} \boldsymbol{w}_j|^2 + K_3 |\boldsymbol{e} \times \operatorname{curl} \boldsymbol{w}_j|^2\} dx \\ &\geq \varepsilon_j^2 \min\{K_1, K_2, K_3\} \int_\Omega \{|\operatorname{div} \boldsymbol{w}_j|^2 + |\operatorname{curl} \boldsymbol{w}_j|^2\} dx \\ &= C\varepsilon_j^2 \min\{K_1, K_2, K_3\} \int_\Omega |\nabla \boldsymbol{w}_j|^2 dx \\ &\geq c\varepsilon_j^2 \min\{K_1, K_2, K_3\}\,. \end{aligned} \tag{4.7}$$

Thus we have

$$\varepsilon_j^2 \leq \frac{1}{c \min\{K_1, K_2, K_3\}} (\mathcal{F}[\boldsymbol{n}_j] - \mathcal{F}[\boldsymbol{e}]) = o(1)$$

as $j \to \infty$. Since $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$, after passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ weakly in $W^{1,2}(\Omega, \mathbb{R}^3)$ and strongly in $L^4(\Omega, \mathbb{R}^3)$. Since

$$1 = |\boldsymbol{n}_j|^2 = |\boldsymbol{e} + \varepsilon_j \boldsymbol{w}_j|^2 = 1 + 2\varepsilon_j \boldsymbol{e} \cdot \boldsymbol{w}_j + \varepsilon_j^2 |\boldsymbol{w}_j|^2\,,$$

we have $\boldsymbol{e} \cdot \boldsymbol{w}_j = -\frac{\varepsilon_j}{2} |\boldsymbol{w}_j|^2 \to 0$ strongly in $L^2(\Omega)$, so $\boldsymbol{e} \cdot \widehat{\boldsymbol{w}} = 0$ a.e. in Ω. Since

$$\begin{aligned} |\nabla_{q\boldsymbol{n}_j} \psi_j|^2 &= q^2 \varepsilon_j^2 |\nabla g_j - (1 + iq\varepsilon_j g_j) \boldsymbol{w}_j|^2\,, \\ |\psi_j|^2 &= 1 + q\varepsilon_j(-2\Im(g_j) + q\varepsilon_j |g_j|^2)\,, \end{aligned}$$

using (4.4), we have

$$\begin{aligned} (\chi H_s^2(\kappa, q) &+ o(1)) \|\boldsymbol{h} \cdot \boldsymbol{w}_j\|_{L^2(\Omega)}^2 \\ &= \frac{1}{\varepsilon_j^2} (\chi H_s^2(\kappa, q) + o(1)) \|\boldsymbol{h} \cdot \boldsymbol{n}_j\|_{L^2(\Omega)}^2 \\ &= \frac{1}{\varepsilon_j^2} \mathcal{G}[\psi_j, \boldsymbol{n}_j] + \frac{1}{\varepsilon_j^2} (\mathcal{F}[\boldsymbol{n}_j] - K_1 \|\operatorname{div} \boldsymbol{e}\|_{L^2(\Omega)}^2) \end{aligned} \tag{4.8}$$

$$= \int_\Omega \left\{ q^2 |\nabla g_j - (1 + iq\varepsilon_j g_j)\boldsymbol{w}_j|^2 + \frac{\kappa^2 q^2}{2}(-2\Im(g_j) + q\varepsilon_j |g_j|^2)^2 \right\} dx$$
$$+ \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] .$$

Thus we have

$$\int_\Omega |\nabla g_j - \psi_j e^{-iq\varphi}\boldsymbol{w}_j|^2 dx = \int_\Omega |\nabla g_j - (1 + iq\varepsilon_j g_j)\boldsymbol{w}_j|^2 dx \le C_1 .$$

Therefore,

$$\begin{aligned}\|\nabla g_j\|_{L^2(\Omega,\mathbb{C}^3)} &\le \|\nabla g_j - \psi_j e^{-iq\varphi}\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} + \|\psi_j \boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} \\ &\le C + \|\psi_j\|_{W^{1,2}(\Omega,\mathbb{C})}\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le C_1 .\end{aligned}$$

Put $\widetilde{g}_j = g_j - b_j$ where $b_j = \frac{1}{|\Omega|}\int_\Omega g_j dx$. Since $\int_\Omega \widetilde{g}_j dx = 0$, it follows from the Poincaré inequality that $\|\widetilde{g}_j\|_{L^2(\Omega,\mathbb{C})} \le c(\Omega)\|\nabla g_j\|_{L^2(\Omega,\mathbb{C}^3)} \le C$, so $\|\widetilde{g}_j\|_{W^{1,2}(\Omega,\mathbb{C})} \le C$. By the Sobolev lemma,

$$\|\widetilde{g}_j\|_{L^4(\Omega,\mathbb{C})} \le C\|\widetilde{g}_j\|_{W^{1,2}(\Omega,\mathbb{C})} \le C_1 .$$

Hence,

$$\|\widetilde{g}_j \boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} \le \|\widetilde{g}_j\|_{L^4(\Omega,\mathbb{C})}\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le C .$$

Now we estimate b_j.

Since

$$\psi_j = e^{iq\varphi}(1 + iq\varepsilon_j g_j) = \widehat{\psi} + iq\varepsilon_j g_j e^{iq\varphi}$$

and $\psi_j \to \widehat{\psi}$ in $L^4(\Omega,\mathbb{C})$, we have $\varepsilon_j g_j \to 0$ in $L^4(\Omega,\mathbb{C})$. Thus $\varepsilon_j b_j = \varepsilon_j g_j - \varepsilon_j \widetilde{g}_j = o(1)$. On the other hand, we have

$$\begin{aligned} C &\ge \|\nabla g_j - (1 + iq\varepsilon_j g_j)\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} \\ &= \|\nabla \widetilde{g}_j - (1 + iq\varepsilon_j b_j)\boldsymbol{w}_j - iq\varepsilon_j \widetilde{g}_j \boldsymbol{w}\|_{L^2(\Omega,\mathbb{C}^3)} \\ &\ge \|\nabla \widetilde{g}_j - (1 + iq\varepsilon_j b_j)\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} - O(\varepsilon_j \|\widetilde{g}_j \boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)}) \\ &= \|\nabla \widetilde{g}_j - (1 + iq\varepsilon_j b_j)\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} - O(\varepsilon_j) .\end{aligned}$$

Put $u_j = \widetilde{g}_j/(1 + iq\varepsilon_j b_j)$, then

$$\|\nabla \widetilde{g}_j - (1 + iq\varepsilon_j b_j)\boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{C}^3)} = |1 + iq\varepsilon_j b_j|^2 \|\nabla u_j - \boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} .$$

Therefore, we have $\|\nabla u_j - \boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} \le C$, so $\|\nabla u_j\|_{L^2(\Omega,\mathbb{C}^3)} \le C_1$. Since $\int_\Omega u_j dx = 0$, it follows from the Poincaré inequality that $\{u_j\}$ is bounded in $W^{1,2}(\Omega,\mathbb{C})$. Passing to a subsequence, we may assume that $u_j \to \widehat{u}$ weakly in $W^{1,2}(\Omega,\mathbb{C})$ and strongly in $L^2(\Omega,\mathbb{C})$. From (4.8), it follows that

$$(\chi H_s^2(\kappa, q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|^2_{L^2(\Omega)}$$

$$= \int_{\Omega} \left\{ q^2(1+o(1))|\nabla u_j - \boldsymbol{w}_j|^2 + \frac{\kappa^2 q^2}{2}(-2\Im(g_j) + q\varepsilon_j |g_j|^2)^2 \right\} dx$$
$$+\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j]\,.$$

This implies that

$$q^2\|\nabla u_j - \boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{C}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] \le (\chi H_s^2(\kappa,q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|^2_{L^2(\Omega)}. \tag{4.9}$$

Letting $j \to \infty$, we get

$$q^2\|\nabla \widehat{u} - \widehat{\boldsymbol{w}}\|^2_{L^2(\Omega,\mathbb{C}^3)} + \mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}] \le \chi H_s^2(\kappa,q)\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}\,. \tag{4.10}$$

We note that we may take $\widehat{u}$ to be a real valued function.

Step 6. We show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \not\equiv 0$ in Ω.

Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \equiv 0$ in Ω. Since $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$, we have $\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|_{L^2(\Omega)} \to 0$. Since $\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] \to 0$ from (4.8), we see that $\operatorname{div}\boldsymbol{w}_j \to 0$ in $L^2(\Omega)$ and $\operatorname{curl}\boldsymbol{w}_j \to 0$ in $L^2(\Omega,\mathbb{R}^3)$. Since $\operatorname{div}\widehat{\boldsymbol{w}} = 0$, $\operatorname{curl}\widehat{\boldsymbol{w}} = 0$, $\nabla\widehat{u} = \widehat{\boldsymbol{w}}$ and $\widehat{\boldsymbol{w}} = 0$ on $\partial\Omega$ from (4.10), we see that $\Delta\widehat{u} = 0$ in Ω and $\nabla\widehat{u} = 0$ on $\partial\Omega$. Applying the maximum principal, we see that $\widehat{u}$ is a constant in Ω, so $\widehat{\boldsymbol{w}} = 0$. Thus $\boldsymbol{w}_j \to 0$ strongly in $L^2(\Omega,\mathbb{R}^3)$. Therefore, it follows from [4] that

$$\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le C(\|\operatorname{div}\boldsymbol{w}_j\|_{L^2(\Omega)} + \|\operatorname{curl}\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)} + \|\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)}) \to 0$$

as $j \to \infty$. This contradicts the fact that $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$.

Thus from (4.10),

$$\chi H_s^2(\kappa,q) \ge \frac{q^2\|\nabla\widehat{u} - \widehat{\boldsymbol{w}}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}]}{\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}} \ge \chi H_{sh}^2(q)\,.$$

Hence we get $H_s(\kappa,q) = H_{sh}(q)$. This contradicts our hypothesis. Thus we get $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$ in Ω. By Step 4, we see that $H_s(\kappa,q)$ is achieved. Therefore (ii) holds, so (i) also holds.

Proof of (iii). Assume that $H_n(q) < H_{sh}(q)$.

Step 7. Let $\{(u_j,\boldsymbol{n}_j)\} \subset W^{1,2}(\Omega) \times W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ with $\boldsymbol{h}\cdot\boldsymbol{n}_j \not\equiv 0$ in Ω be a minimizing sequence of $H_n(q)$. Then we have

$$\begin{aligned} q^2\|\nabla u_j - \boldsymbol{n}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} &+ \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} \\ &= (\chi H_n^2(q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{n}_j\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.11}$$

Since $|\boldsymbol{h}\cdot\boldsymbol{n}_j| \le 1$, the right hand side of (4.11) is bounded. Thus we see that $\{\operatorname{div}\boldsymbol{n}_j\}$ is bounded in $L^2(\Omega)$, $\{\operatorname{curl}\boldsymbol{n}_j\}$ is bounded in $L^2(\Omega,\mathbb{R}^3)$, $|\boldsymbol{n}_j| = 1$ a.e. in Ω and $\boldsymbol{n}_j = \boldsymbol{e}$ on $\partial\Omega$. Therefore, it follows from [4] that $\{\boldsymbol{n}_j\}$ is bounded in $W^{1,2}(\Omega,\mathbb{R}^3)$. After passing to a subsequence, we may assume that $\boldsymbol{n}_j \to \widehat{\boldsymbol{n}}$ weakly in $W^{1,2}(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$ and a.e. in Ω. As in (ii), we get $\widehat{\boldsymbol{n}} \in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$. When $q > 0$, it follows from (4.11) that $\{\nabla u_j\}$ is bounded in $L^2(\Omega,\mathbb{R}^3)$. Put $\widehat{u}_j = u_j - d_j$ where $d_j = \frac{1}{|\Omega|}\int_\Omega u_j dx$. Applying the Poincaré inequality, we see that $\{\widehat{u}_j\}$ is bounded in $W^{1,2}(\Omega)$. Passing to a subsequence, we may assume that $\widehat{u}_j \to \widehat{u}$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^4(\Omega)$. Letting $j \to \infty$ in (4.11), we have

$$
\begin{aligned}
(4.12) \qquad & q^2\|\nabla\widehat{u} - \widehat{\boldsymbol{n}}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\widehat{\boldsymbol{n}}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} \\
&\le \liminf_{j\to\infty}\{q^2\|\nabla\widehat{u}_j - \boldsymbol{n}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}\} \\
&= \chi H_n^2(q)\|\boldsymbol{h}\cdot\widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}\,.
\end{aligned}
$$

If we show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$ in Ω, we see that $H_n(q)$ is achieved. When $q = 0$, if we show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$, by the definition of $H_n(0)$, we also see that $H_n(0)$ is achieved.

Step 8. Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \equiv 0$ in Ω.

Then from (4.12), $\mathcal{F}[\widehat{\boldsymbol{n}}] \le K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$. Thus we have $\widehat{\boldsymbol{n}} = \boldsymbol{e}$ and $\nabla\widehat{u} = \widehat{\boldsymbol{n}} = \boldsymbol{e}$ if $q > 0$. Hence if we write $\boldsymbol{n}_j = \boldsymbol{e} + \varepsilon_j\boldsymbol{w}_j$, $\widehat{u}_j = \widehat{u} + \varepsilon_j g_j$, where $\varepsilon_j = \|\boldsymbol{n}_j - \boldsymbol{e}\|_{W^{1,2}(\Omega,\mathbb{R}^3)} > 0$, then $\boldsymbol{w}_j \in W_0^{1,2}(\Omega,\mathbb{R}^3)$ and $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$. According to Lemma 3.1, we have

$$
\begin{aligned}
\mathcal{F}[\boldsymbol{n}_j] - \mathcal{F}[\boldsymbol{e}] &= \varepsilon_j^2\left\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2dx\right\} \\
&\ge c\varepsilon_j^2\|\boldsymbol{w}_j\|^2_{W^{1,2}(\Omega,\mathbb{R}^3)}\,.
\end{aligned}
$$

As the proof of (ii) we get $\varepsilon_j^2 = o(1)$ as $j \to \infty$. Since $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$, after passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ weakly in $W_0^{1,2}(\Omega,\mathbb{R}^3)$ and strongly in $L^4(\Omega,\mathbb{R}^3)$. Since $\boldsymbol{e}\cdot\boldsymbol{w}_j = -\frac{\varepsilon_j}{2}|\boldsymbol{w}_j|^2 \to 0$ strongly in $L^2(\Omega)$, we see that $\boldsymbol{e}\cdot\widehat{\boldsymbol{w}} = 0$ a.e. in Ω. Since $\nabla\widehat{u}_j = \nabla\widehat{u} + \varepsilon_j\nabla g_j = \boldsymbol{e} + \varepsilon_j\nabla g_j$, we have

$$
\begin{aligned}
(4.13) \qquad & (\chi H_n^2(q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|^2_{L^2(\Omega)} \\
&= \frac{1}{\varepsilon_j^2}(\chi H_n^2(q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{n}_j\|^2_{L^2(\Omega)} \\
&= \frac{1}{\varepsilon_j^2}\{q^2\|\nabla\widehat{u}_j - \boldsymbol{n}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}\} \\
&= \frac{1}{\varepsilon_j^2}q^2\|\nabla\widehat{u}_j - \boldsymbol{n}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2dx \\
&= q^2\|\nabla g_j - \boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2dx\,.
\end{aligned}
$$

Since $\int_\Omega g_j dx = 0$, $\|g_j\|_{L^2(\Omega)} \le C\|\nabla g_j\|_{L^2(\Omega,\mathbb{R}^3)} \le C_1$. After passing to a subsequence, we may assume that $g_j \to \widehat{g}$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^4(\Omega)$. By (4.13),

$$
(4.14) \qquad q^2\|\nabla\widehat{g} - \widehat{\boldsymbol{w}}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\widehat{\boldsymbol{w}}|^2dx
$$

$$\leq \liminf_{j\to\infty}\{q^2\|\nabla g_j - \boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla \boldsymbol{e}|^2|\boldsymbol{w}_j|^2dx\}$$
$$= \chi H_n^2(q)\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}\,.$$

Step 9. We shall show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \not\equiv 0$ in Ω.

Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \equiv 0$ in Ω. Since $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$, we have $\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|_{L^2(\Omega)} \to 0$. Since $\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] \to 0$ from (4.13), we see that $\operatorname{div}\boldsymbol{w}_j \to 0$ in $L^2(\Omega)$ and $\operatorname{curl}\boldsymbol{w}_j \to 0$ in $L^2(\Omega,\mathbb{R}^3)$. Thus $\operatorname{div}\widehat{\boldsymbol{w}} = 0$, $\operatorname{curl}\widehat{\boldsymbol{w}} = 0$, $\nabla\widehat{g} = \widehat{\boldsymbol{w}}$ in Ω and $\widehat{\boldsymbol{w}} = 0$ on $\partial\Omega$. Therefore, $\Delta\widehat{g} = 0$ in Ω and $\nabla\widehat{g} = 0$ on $\partial\Omega$. By the maximum principle, we see that $\widehat{g}$ is a constant and so $\widehat{\boldsymbol{w}} = 0$ in Ω. Thus $\boldsymbol{w}_j \to 0$ strongly in $L^2(\Omega,\mathbb{R}^3)$. According to [4], $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \to 0$. This is a contradiction. Hence we have

$$\chi H_{sh}^2(q) \leq \frac{q^2\|\nabla\widehat{g} - \widehat{\boldsymbol{w}}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx}{\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}} \leq \chi H_n^2(q)\,.$$

This completes the proof. □

5. Instabilities in pure nematic states

In this section we examine the local minimality as well as global minimality of the pure nematic states. Let $\psi = 0$ and $\boldsymbol{n} = \boldsymbol{n}_\sigma$ where $\boldsymbol{n}_\sigma$ is a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$:

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma] = \inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)} \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]$$

and $\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = \mathcal{F}[\boldsymbol{n}] - \chi\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}$. We note that $(0,\boldsymbol{n})$ is a critical point of $\mathcal{E}$ if and only if $\boldsymbol{n}$ is a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$. Define

$$C(\sigma) = C(\sigma,\kappa,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e}_0) = \inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)} \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]$$

and

$$\begin{aligned}\mathcal{M}(\sigma) &= \mathcal{M}(\sigma,\kappa,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e}_0)\\ &= \{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0);\, \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = C(\sigma)\}\,.\end{aligned}$$

If $\boldsymbol{n}\in\mathcal{M}(\sigma)$, then $(0,\boldsymbol{n})$ is a critical point of $\mathcal{E}$. When $\boldsymbol{n}$ is a minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$, we look for the Euler-Lagrange equation for $\boldsymbol{n}$. For any $\boldsymbol{v}\in W_0^{1,2}(\Omega,\mathbb{R}^3)$, we compute

$$\left.\frac{d}{dt}\right|_{t=0}\left\{\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}+t\boldsymbol{v}] - \int_\Omega \lambda(|\boldsymbol{n}+t\boldsymbol{v}|^2-1)dx\right\} = 0$$

where λ is the Lagrange multiplier which depends on x. By the standard arguments, we get the Euler-Lagrange equation for $\boldsymbol{n}$:

$$(5.1)\qquad \begin{cases} -K_1\nabla(\operatorname{div}\boldsymbol{n}) + K_2\{(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\operatorname{curl}\boldsymbol{n} + \operatorname{curl}\big((\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\boldsymbol{n}\big)\} \\ +K_3\{|\operatorname{curl}\boldsymbol{n}|^2\boldsymbol{n} - (\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\operatorname{curl}\boldsymbol{n} + \operatorname{curl}^2\boldsymbol{n} \\ -\operatorname{curl}\big((\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\boldsymbol{n}\big)\} \\ -\chi\sigma^2(\boldsymbol{h}\cdot\boldsymbol{n})\boldsymbol{h} - \lambda\boldsymbol{n} = 0 & \text{in } \Omega\,, \\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega\,. \end{cases}$$

We can compute the Lagrange multiplier λ:

$$\begin{aligned} \lambda = \lambda(x) = \boldsymbol{n}\cdot\big[&-K_1\nabla(\operatorname{div}\boldsymbol{n}) + K_2\{(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\operatorname{curl}\boldsymbol{n} \\ &+ \operatorname{curl}\big((\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\boldsymbol{n}\big)\} + K_3\{|\operatorname{curl}\boldsymbol{n}|^2\boldsymbol{n} - (\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\operatorname{curl}\boldsymbol{n} \\ &+ \operatorname{curl}^2\boldsymbol{n} - \operatorname{curl}\big((\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\boldsymbol{n}\big)\} - \chi\sigma^2(\boldsymbol{h}\cdot\boldsymbol{n})\boldsymbol{h}\big] \end{aligned}$$

In the particular case where $K_1 = K_2 = K_3 = K$, we use the formulas: $\operatorname{curl}^2\boldsymbol{n} = -\Delta\boldsymbol{n} + \nabla\operatorname{div}\boldsymbol{n}$ and $-\Delta\boldsymbol{n}\cdot\boldsymbol{n} = |\nabla\boldsymbol{n}|^2$ which follows from $\boldsymbol{n}\cdot\boldsymbol{n} = 1$. In this case, we have

$$(5.2)\qquad \begin{cases} -K\Delta\boldsymbol{n} = K|\nabla\boldsymbol{n}|^2\boldsymbol{n} + \chi\sigma^2\big((\boldsymbol{h}\cdot\boldsymbol{n})\boldsymbol{h} - (\boldsymbol{h}\cdot\boldsymbol{n})^2\boldsymbol{n}\big) & \text{in } \Omega\,, \\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega. \end{cases}$$

Since $\boldsymbol{h}\cdot\boldsymbol{e} = 0$, $\boldsymbol{e}$ is a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$ for any σ. Recall that

$$\begin{aligned} H_{sh}^2(0) = \frac{1}{\chi}\inf\Big\{&\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2 dx; \\ &\boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^3),\ \boldsymbol{w}(x)\cdot\boldsymbol{e}(x) = 0 \text{ in } \Omega,\ \|\boldsymbol{h}\cdot\boldsymbol{w}\|_{L^2(\Omega)} = 1\Big\}, \end{aligned}$$

$$H_n^2(0) = \frac{1}{\chi}\inf\left\{\frac{\mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};\ \boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0),\ \ \boldsymbol{h}\cdot\boldsymbol{n}\not\equiv 0 \text{ in } \Omega\right\}$$

and $0 < H_n(0) \le H_{sh}(0)$ from Theorem 4.5 (i).

We give a simple criterion for $\boldsymbol{n} = \boldsymbol{e}$ to be a global minimizer.

LEMMA 5.1. (i) *If $0 \le \sigma < H_n(0)$, then $\boldsymbol{n} = \boldsymbol{e}$ is the only global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$ in $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$.*

(ii) *If $H_n(0) < H_{sh}(0)$ and $H_n(0) < \sigma < H_{sh}(0)$, then $\boldsymbol{n} = \boldsymbol{e}$ is not a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$ in $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$, but it is weakly stable (a local minimizer).*

(iii) *If $\sigma > H_{sh}(0)$, $\boldsymbol{n} = \boldsymbol{e}$ is not weakly stable.*

PROOF. First note that $\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$, so if $\boldsymbol{n}$ is a global minimizer, then $\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$. If $\boldsymbol{h}\cdot\boldsymbol{n}\equiv 0$, then $\mathcal{F}[\boldsymbol{n}] = \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}] =$

$K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$. Hence $\boldsymbol{n}=\boldsymbol{e}$. Therefore, a global minimizer $\boldsymbol{n}$ with $\boldsymbol{n}\not\equiv\boldsymbol{e}$ satisfies $\boldsymbol{h}\cdot\boldsymbol{n}\not\equiv 0$ in Ω.

(i) When $0\le\sigma<H_n(0)$, if $\mathcal{F}_{\sigma\boldsymbol{h}}$ has a global minimizer $\boldsymbol{n}$ which is not $\boldsymbol{e}$, then $\boldsymbol{h}\cdot\boldsymbol{n}\not\equiv 0$ in Ω. Thus

$$K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}=\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}]\ge\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]=\mathcal{F}[\boldsymbol{n}]-\chi\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}\,.$$

This implies

$$\sigma^2\ge\frac{1}{\chi}\frac{\mathcal{F}[\boldsymbol{n}]-K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}}\ge H_n^2(0)$$

which is a contradiction. Hence only the global minimizer is $\boldsymbol{e}$.

When $\sigma>H_n(0)$, choose $\widehat{\boldsymbol{n}}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ such that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}}\not\equiv 0$ in Ω and

$$\frac{1}{\chi}\frac{\mathcal{F}[\widehat{\boldsymbol{n}}]-K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}}<H_n^2(0)+\delta<\sigma^2\,.$$

Then for some $\delta>0$

$$\begin{aligned}\mathcal{F}_{\sigma\boldsymbol{h}}[\widehat{\boldsymbol{n}}]&=\mathcal{F}[\widehat{\boldsymbol{n}}]-\chi\sigma^2\|\boldsymbol{h}\cdot\widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}\\&<\mathcal{F}[\widehat{\boldsymbol{n}}]-\chi(H_n^2(0)+\delta)\|\boldsymbol{h}\cdot\widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}\\&<K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}=\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}]\,.\end{aligned}$$

Thus we see that

$$\inf\{\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}];\,\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)\}<\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}]\,.$$

Hence $\boldsymbol{e}$ is not a global minimizer.

In order to examine weak stability (local minimality), for any $\boldsymbol{v}\in W_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega,\mathbb{R}^3)$, if we put $\psi_0=\phi=0$, $\boldsymbol{n}_0=\boldsymbol{e}$ in (2.1), we have

$$\begin{aligned}&\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_t]-\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}]\\&\quad=t^2\left\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{n}_1]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{n}_1|^2dx-\chi\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}_1\|^2_{L^2(\Omega)}\right\}+O(t^3)\end{aligned}$$

where $\boldsymbol{n}_1=\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}$. Since $\boldsymbol{h}\cdot\boldsymbol{e}=0$, if

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}|^2dx-\chi\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{v}\|^2_{L^2(\Omega)}>0\,,$$

$(0,\boldsymbol{e})$ is weakly stable. If $\sigma>H_{sh}(0)$, there exists $\boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^3)$ such that $\boldsymbol{w}(x)\cdot\boldsymbol{e}(x)=0$ in Ω, $\|\boldsymbol{h}\cdot\boldsymbol{w}\|_{L^2(\Omega)}=1$ and

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx<\chi\sigma^2\,.$$

If we take $\boldsymbol{v} = \boldsymbol{w}$,

$$\mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}_t] - \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}] = t^2 \left\{ \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx - \chi\sigma^2 \right\} + O(t^3) .$$

Since $\mathcal{F}(\boldsymbol{e})[\boldsymbol{n}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx - \chi\sigma^2 < 0$, $\boldsymbol{n} = \boldsymbol{e}$ is not weakly stable. Thus (iii) holds.

When $H_n(0) < \sigma < H_{sh}(0)$, for any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$,

$$\begin{aligned} \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}_t] - \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}] = t^2 \Big\{ & \mathcal{F}(\boldsymbol{e})[\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{v} \\ & - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}|^2 dx - \chi\sigma^2 \|\boldsymbol{h}\cdot(\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e})\|^2_{L^2(\Omega)} \Big\} + O(t^3) . \end{aligned}$$

If $\sigma < H_{sh}(0)$,

$$\begin{aligned} & \chi\sigma^2 \int_\Omega \big(\boldsymbol{h}\cdot(\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}\big)^2 dx \\ & \quad < \mathcal{F}(\boldsymbol{e})[\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}|^2 dx . \end{aligned}$$

Thus $(0, \boldsymbol{e})$ is weakly stable. This completes the proof. □

Next, we consider a question: When $\boldsymbol{n}_\sigma \in \mathcal{M}(\sigma, \kappa, K_1, K_2, K_3, \boldsymbol{h}, \boldsymbol{e}_0)$ is a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$, is $(0, \boldsymbol{n}_\sigma)$ a global minimizer of $\mathcal{E}$?

Let $\mu = \mu(q\boldsymbol{n})$ be the lowest eigenvalue of the magnetic Neumann problem

$$\begin{cases} -\nabla^2_{q\boldsymbol{n}}\phi = \mu\phi & \text{in } \Omega , \\ \nabla_{q\boldsymbol{n}}\phi \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega . \end{cases} \tag{5.3}$$

That is to say,

$$\mu(q\boldsymbol{n}) = \inf_{0\neq\phi\in W^{1,2}(\Omega,\mathbb{C})} \frac{\|\nabla_{q\boldsymbol{n}}\phi\|^2_{L^2(\Omega,\mathbb{C}^3)}}{\|\phi\|^2_{L^2(\Omega,\mathbb{C})}} .$$

Define

$$\mu_*(q, \sigma) = \mu_*(q, \sigma, K_1, K_2, K_3, \boldsymbol{h}, \boldsymbol{e}_0) = \inf_{\boldsymbol{n}\in\mathcal{M}(\sigma,\kappa,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e})} \mu(q\boldsymbol{n}) .$$

LEMMA 5.2. (i) *If $(\psi, \boldsymbol{n})$ is a global minimizer of $\mathcal{E}$ which is not a pure nematic state, then $\mu(q\boldsymbol{n}) < \kappa^2$.*

(ii) *If $\mu_*(q, \sigma) < \kappa^2$, then pure nematic states are not global minimizers of $\mathcal{E}$.*

PROOF. (i) Let $(\psi, \boldsymbol{n})$ be a global minimizer of $\mathcal{E}$. If $\psi = 0$, $\boldsymbol{n} \in \mathcal{M}(\sigma)$. Hence $(0, \boldsymbol{n})$ is a pure nematic state. Therefore, if $(\psi, \boldsymbol{n})$ is not a pure nematic state, then $\psi \not\equiv 0$. Choose $\boldsymbol{n}_\sigma \in \mathcal{M}(\sigma)$. Then since

$$\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \leq \mathcal{G}[0, \boldsymbol{n}_\sigma] + \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma] ,$$

$$\int_\Omega \left\{ |\nabla_{q\boldsymbol{n}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right\} dx \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma] - \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \le 0\,.$$

Thus

$$\int_\Omega \{ |\nabla_{q\boldsymbol{n}}\psi|^2 - \kappa^2|\psi|^2 \} dx \le -\frac{\kappa^2}{2}\int_\Omega |\psi|^4 dx < 0\,.$$

This implies that $\mu(q\boldsymbol{n}) < \kappa^2$.

(ii) For a pure nematic state $(0, \boldsymbol{n}_\sigma)$,

$$\mathcal{E}[0, \boldsymbol{n}_\sigma] = \frac{\kappa^2}{2}|\Omega| + \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma]\,.$$

If $\mu_*(q, \sigma) < \kappa^2$, there exists $\boldsymbol{n} \in \mathcal{M}(\sigma)$ such that $\mu(q\boldsymbol{n}) < \kappa^2$. Let ϕ be an eigenfunction of (5.3) associated with $\mu(q\boldsymbol{n})$. Then

$$\mathcal{E}[t\phi, \boldsymbol{n}] = \int_\Omega \{ t^2(|\nabla_{q\boldsymbol{n}}\phi|^2 - \kappa^2|\phi|^2) + t^4|\phi|^4 \} dx + \frac{\kappa^2}{2}|\Omega| + \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]\,.$$

Since $\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma] = C(\sigma)$, therefore, we have

$$\mathcal{E}[t\phi, \boldsymbol{n}] - \mathcal{E}[0, \boldsymbol{n}_\sigma] = t^2(\mu(q\boldsymbol{n}) - \kappa^2)\int_\Omega |\phi|^2 dx + t^4\int_\Omega |\phi|^4 dx < 0$$

for small $t > 0$. Thus $(0, \boldsymbol{n}_\sigma)$ is not a global minimizer of $\mathcal{E}$. □

PROPOSITION 5.3. *If $0 \le \sigma \le H_n(0)$ and $\kappa > 0$, or $\sigma > H_n(0)$ and $\mu_*(q, \sigma) < \kappa^2$, then the pure nematic states are not global minimizer of $\mathcal{E}$.*

PROOF. When $0 \le \sigma < H_n(0)$, it follows from Lemma 5.1 (i) that $\mathcal{M}(\sigma) = \{\boldsymbol{e}\}$. When $\sigma = H_n(0)$,

$$\sigma^2 \le \frac{1}{\chi}\frac{\mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}}$$

for any $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ with $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$ in Ω. Therefore,

$$\mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} - \chi\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)} \ge 0\,. \tag{5.4}$$

This inequality holds even in the case $\boldsymbol{h}\cdot\boldsymbol{n} = 0$. Therefore, (5.4) hold for any $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$. This implies that $C(\sigma) \ge K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$.

On the other hand, $C(\sigma) \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$. Therefore, we have $C(\sigma) = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} = \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{e}]$, so $\boldsymbol{e} \in \mathcal{M}(\sigma)$. If we put $\psi = e^{iq\varphi}$, then $\nabla_{q\boldsymbol{e}}\psi = 0$. So $\mu(q\boldsymbol{e}) = 0$. Thus we have

$$\mu_*(q, \sigma) \le \mu(q\boldsymbol{e}) = 0 < \kappa^2$$

for any $\kappa > 0$. Therefore, from Lemma 5.2 (ii), we see that pure nematic states are not global minimizers of $\mathcal{E}$. If $\sigma > H_n(0)$ and $\mu_*(q, \sigma) < \kappa^2$, it suffices to apply Lemma 5.2 (ii). $\square$

Now define

$$\sigma_*(\kappa, q) = \inf\{\sigma > 0;\ \mu_*(q, \sigma) \geq \kappa^2\},$$
$$Q_*(\kappa, q) = \inf\{q > 0;\ \mu_*(q, \sigma) \geq \kappa^2\}.$$

Let $\sigma > H_n(0)$. Summing up the above, the pure nematic states are not global minimizers in the following cases.

(1) $0 < \sigma < \sigma_*(\kappa, q)$.

(2) $\mu_*(q, \sigma) < \kappa^2$.

(3) $0 \leq q < Q_*(q, \sigma)$.

The following theorem indicates the difference between liquid crystals and superconductors under strong external field.

THEOREM 5.4. *Let $q, \kappa, K_1, K_2, K_3, \boldsymbol{h}, \boldsymbol{e}_0$ with $K_1 = K_2 = K_3 = K$ be given. Assume that* (H.1), (H.2) *and* (H.3) *hold. Then if σ is sufficiently large, the pure nematic states are not global minimizers.*

In order to prove the theorem, we need a lemma.

LEMMA 5.5. (i) *For large σ, $C(\sigma) \leq -\chi\sigma^2|\Omega| + C_1\sigma$ where $C_1 > 0$ depends only on $K_1, K_2, K_3, \boldsymbol{h}, \boldsymbol{e}_0$ and Ω.*

(ii) *Let $\boldsymbol{n}_\sigma$ be a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$. Then $|\boldsymbol{h}\cdot\boldsymbol{n}_\sigma| \to 1$ in $L^2(\Omega)$ as $\sigma \to +\infty$.*

(iii) *Assume that $K_1 = K_2 = K_3$. Then $\boldsymbol{h}\cdot\boldsymbol{n}_\sigma \to 1$ or -1 in $L^2(\Omega)$ as $\sigma \to +\infty$.*

PROOF. After rotating the coordinate system, we may assume that $\boldsymbol{h} = \boldsymbol{e}_3$. Define $\boldsymbol{k}(x) = \boldsymbol{h} \times \boldsymbol{e}(x)$. Then $(\boldsymbol{e}(x), \boldsymbol{k}(x), \boldsymbol{h})$ is a orthonormal basis in $\mathbb{R}^3$. For $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$, we can write

$$\boldsymbol{n} = n_{\boldsymbol{e}}\boldsymbol{e} + n_{\boldsymbol{k}}\boldsymbol{k} + n_{\boldsymbol{h}}\boldsymbol{h}, \quad n_{\boldsymbol{e}}^2 + n_{\boldsymbol{k}}^2 + n_{\boldsymbol{h}}^2 = 1 \text{ a.e. in } \Omega.$$

We see that

$$\begin{aligned}\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] &= \mathcal{F}[\boldsymbol{n}] - \chi\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n})^2dx\\ &= \mathcal{F}[\boldsymbol{n}] - \chi\sigma^2\int_\Omega n_{\boldsymbol{h}}^2dx\\ &= \mathcal{F}[\boldsymbol{n}] - \chi\sigma^2\int_\Omega(1 - n_{\boldsymbol{e}}^2 - n_{\boldsymbol{k}}^2)dx\\ &= \mathcal{F}_\sigma[\boldsymbol{n}] - \chi\sigma^2|\Omega|\end{aligned}$$

where

$$\mathcal{F}_\sigma[\boldsymbol{n}] = \mathcal{F}[\boldsymbol{n}] + \chi\sigma^2\int_\Omega(n_{\boldsymbol{e}}^2 + n_{\boldsymbol{k}}^2)dx.$$

Proof of (i). Choose a test field

$$\begin{aligned}\boldsymbol{n} &= (\cos\phi)\boldsymbol{e} + (\sin\phi)\boldsymbol{h}\\ &= (\cos\phi)e_1(x)\boldsymbol{e}_1 + (\cos\phi)e_2(x)\boldsymbol{e}_2 + (\sin\phi)\boldsymbol{e}_3\end{aligned}$$

where $\boldsymbol{e}(x) = e_1(x)\boldsymbol{e}_1 + e_2(x)\boldsymbol{e}_2$. Then since

$$\begin{aligned}\operatorname{div}\boldsymbol{n} = &-(\sin\phi)(\partial_1\phi)e_1 + (\cos\phi)(\partial_1 e_1) - (\sin\phi)(\partial_2\phi)e_2\\ &+ (\cos\phi)(\partial_2 e_2) + (\cos\phi)(\partial_3\phi)\end{aligned}$$

and $\boldsymbol{e} \in C^2(\overline{\Omega}, \mathbb{R}^3)$, we see that $|\operatorname{div}\boldsymbol{n}|^2 \le C(|\nabla\phi|^2 + 1)$. Similarly we have

$$|\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}|^2 + |\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2 \le C_1(|\nabla\phi|^2 + 1)\,.$$

Thus if we write $\mathcal{F}_\sigma[\boldsymbol{n}] = \int_\Omega f_{\sigma,\boldsymbol{h}}(\phi)dx$, we have

$$|f_{\sigma,\boldsymbol{h}}(\phi)| \le C\max\{K_1, K_2, K_3\}(|\nabla\phi|^2 + 1) + \chi\sigma(\cos\phi)^2\,.$$

For any $\varepsilon > 0$, define $\Omega_\varepsilon = \{x \in \Omega; d(x, \partial\Omega) < \varepsilon\}$ and $\Omega^\varepsilon = \{x \in \Omega; d(x.\partial\Omega) \ge \varepsilon\}$, and decompose $\mathcal{F}_\sigma[\boldsymbol{n}]$ as follows: $\mathcal{F}_\sigma[\boldsymbol{n}] = \mathcal{F}_{\sigma,1}[\boldsymbol{n}] + \mathcal{F}_{\sigma,2}[\boldsymbol{n}]$ where

$$\mathcal{F}_{\sigma,1}[\boldsymbol{n}] = \int_{\Omega_\varepsilon} f_{\sigma,\boldsymbol{h}}(\phi)dx\,,\quad \mathcal{F}_{\sigma,2}[\boldsymbol{n}] = \int_{\Omega^\varepsilon} f_{\sigma,\boldsymbol{h}}(\phi)dx\,.$$

Choose ϕ such that

$$\phi = \begin{cases}\frac{\pi}{2} & \text{in } \Omega^\varepsilon\,,\\ 0 & \text{on } \partial\Omega\,,\\ |\nabla\phi| \le \frac{C}{\varepsilon} & \text{in } \Omega\,.\end{cases}$$

Then $\mathcal{F}_{\sigma,2}[\boldsymbol{n}] \le C\max\{K_1, K_2, K_3\}$ and

$$\begin{aligned}\mathcal{F}_{\sigma,1}[\boldsymbol{n}] &\le \int_{\Omega_\varepsilon}\{C\max\{K_1, K_2, K_3\}(|\nabla\phi|^2 + 1) + \chi\sigma^2(\cos\phi)^2\}dx\\ &\le \Big[C\max\{K_1, K_2, K_3\}\Big(\frac{C_2^2}{\varepsilon^2} + 1\Big) + \chi\sigma^2\Big]|\Omega_\varepsilon|\,.\end{aligned}$$

Since $\partial\Omega$ is smooth, there exists $C_0 > 0$ depending only on $\partial\Omega$ such that $|\Omega_\varepsilon| \le C_0\varepsilon$ for any small $\varepsilon > 0$. For large σ, choose $\varepsilon > 0$ so that

$$\varepsilon = \min\left\{\frac{\sqrt{C}C_2}{\sigma}\sqrt{\frac{\max\{K_1, K_2, K_3\}}{\chi}}, 1\right\}.$$

Then we have $\mathcal{F}_{\sigma,1}[\boldsymbol{n}] \le C_3 + C_4\sigma$. Therefore, we have

$$C(\sigma) \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \le -\chi\sigma^2|\Omega| + C_4\sigma + C_5\,.$$

Thus (i) holds.

Proof of (ii). From (i), we see that

$$\mathcal{F}_\sigma[\boldsymbol{n}_\sigma] = \mathcal{F}[\boldsymbol{n}_\sigma] + \chi\sigma^2 \int_\Omega (n_{\boldsymbol{e}}^2 + n_{\boldsymbol{k}}^2)dx \le C_1\sigma \,.$$

This implies that

$$\int_\Omega (n_{\boldsymbol{e}}^2 + n_{\boldsymbol{k}}^2)dx \le \frac{C_1}{\chi\sigma} \to 0$$

as $\sigma \to +\infty$. Thus $\int_\Omega (1 - |n_{\sigma,\boldsymbol{h}}|^2)dx \to 0$ as $\sigma \to +\infty$. Since $|n_{\sigma,\boldsymbol{h}}| \le 1$, for any $1 < p < +\infty$,

$$\int_\Omega (1 - |\boldsymbol{n}_{\sigma,\boldsymbol{h}}|)^p dx \le 2^{p-1} \int_\Omega (1 - |\boldsymbol{n}_{\sigma,\boldsymbol{h}}|)dx \to 0 \,.$$

Since $\boldsymbol{h} \cdot \boldsymbol{n}_\sigma = n_{\sigma,\boldsymbol{h}}$, we see that $|\boldsymbol{h} \cdot \boldsymbol{n}_\sigma| \to 1$ in $L^2(\Omega)$ as $\sigma \to +\infty$.

Proof of (iii). When $K_1 = K_2 = K_3 = K$, we shall show that $\boldsymbol{n}_\sigma$ has the following property: $n_{\sigma,\boldsymbol{h}} > 0$ in Ω or $n_{\sigma,\boldsymbol{h}} < 0$ in Ω or $n_{\sigma,\boldsymbol{h}} \equiv 0$ in Ω.

In fact, $\boldsymbol{n}_\sigma$ satisfies the Euler-Lagrange equation (5.2). That is to say, if we write $\boldsymbol{n}_\sigma = \boldsymbol{n}$ for brevity,

$$\begin{cases} -\Delta\boldsymbol{n} = |\nabla\boldsymbol{n}|^2\boldsymbol{n} + b^2\sigma^2[n_{\boldsymbol{h}}\boldsymbol{h} - n_{\boldsymbol{h}}^2\boldsymbol{n}] & \text{in } \Omega\,, \\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega \end{cases} \tag{5.5}$$

where $b^2 = \chi/K$. Since $n_{\boldsymbol{h}} \in W^{1,2}(\Omega)$, it is well known that $|n_{\boldsymbol{h}}| \in W^{1,2}(\Omega)$ and $|\nabla|n_{\boldsymbol{h}}|| = |\nabla n_{\boldsymbol{h}}|$ a.e. in Ω. This fact implies that if we define $\boldsymbol{u} = n_{\boldsymbol{e}}\boldsymbol{e} + n_{\boldsymbol{k}}\boldsymbol{k} + u_{\boldsymbol{h}}\boldsymbol{h}$ where $u_{\boldsymbol{h}} = |n_{\boldsymbol{h}}|$, $\boldsymbol{u}$ is also a minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$. In fact, we can write

$$C(\sigma) = \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = \int_\Omega |\nabla\boldsymbol{n}|^2 dx + C(\boldsymbol{e}_0) - \chi\sigma^2 \int_\Omega (\boldsymbol{n}\cdot\boldsymbol{h})^2 dx$$

where $C(\boldsymbol{e}_0)$ is a constant depending only on $\boldsymbol{e}_0$ (cf. [3]). Therefore, $\boldsymbol{u}$ satisfies (5.5). We rewrite the equation (5.5) for $u_{\boldsymbol{h}} = |n_{\boldsymbol{h}}|$ into the form

$$\Delta u_{\boldsymbol{h}} = -|\nabla\boldsymbol{n}|^2 u_{\boldsymbol{h}} - b^2\sigma^2\big(u_{\boldsymbol{h}}(1 - u_{\boldsymbol{h}}^2)\big) \le 0 \text{ in } \Omega\,, \tag{5.6}$$

$$n_{\boldsymbol{h}} = 0 \text{ on } \partial\Omega\,.$$

By the weak Harnack inequality for non-negative superharmonic function (cf. Gilbarg and Trudinger [6, Theorem 9.22]), for any $B_{2R}(y) \subset \Omega$,

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} u_{\boldsymbol{h}}^p dx\right)^{1/p} \le C \operatorname*{ess\,inf}_{B_R(y)} u_{\boldsymbol{h}}$$

for some $p > 0$. This implies that if $u_{\boldsymbol{h}} \not\equiv 0$ in Ω, then $u_{\boldsymbol{h}} > 0$ in Ω.

End of proof of (iii)

By (ii), $n_{\sigma,\boldsymbol{h}} \not\equiv 0$ in Ω for large σ. Therefore, $n_{\sigma,\boldsymbol{h}} > 0$ in Ω or $n_{\sigma\boldsymbol{h}} < 0$ in Ω. Assume that $n_{\sigma,\boldsymbol{h}} > 0$ in Ω. By (ii), $n_{\sigma,\boldsymbol{h}} \to 1$ in $L^2(\Omega)$ as $\sigma \to \infty$. Hence $(n_{\boldsymbol{e}}^2 + n_{\boldsymbol{k}}^2)^{1/2} \to 0$ in $L^2(\Omega)$. Thus we have $\boldsymbol{n}_\sigma \to \boldsymbol{h}$ in $L^2(\Omega, \mathbb{R}^3)$, that is to say, $\boldsymbol{h} \cdot \boldsymbol{n}_\sigma \to 1$ in $L^2(\Omega)$. □

PROOF OF THEOREM 5.4.

Let $\boldsymbol{n}_\sigma$ be a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$. We shall estimate $\mu(q\boldsymbol{n}_\sigma)$ for large σ. By Lemma 5.5, we may assume that $\boldsymbol{n}_\sigma \to \boldsymbol{h}$ strongly in $L^2(\Omega, \mathbb{R}^3)$. If we define $\phi(x) = e^{iq\boldsymbol{h}\cdot\boldsymbol{x}}$, then $\nabla_{q\boldsymbol{n}_\sigma}\phi = iq(\boldsymbol{h} - \boldsymbol{n}_\sigma)e^{iq\boldsymbol{h}\cdot\boldsymbol{x}}$. Therefore, we have

$$\int_\Omega |\nabla_{q\boldsymbol{n}_\sigma}\phi|^2 dx = q^2 \int_\Omega |\boldsymbol{h} - \boldsymbol{n}_\sigma|^2 dx \to 0$$

as $\sigma \to +\infty$. Hence

$$\begin{aligned}\mu(q\boldsymbol{n}_\sigma) &\le \frac{\|\nabla_{q\boldsymbol{n}_\sigma}\phi\|^2_{L^2(\Omega,\mathbb{C}^3)}}{\|\phi\|^2_{L^2(\Omega,\mathbb{C})}} \\ &= \frac{q^2\|\boldsymbol{h} - \boldsymbol{n}_\sigma\|^2_{L^2(\Omega,\mathbb{R}^3)}}{|\Omega|} \to 0\end{aligned}$$

as $\sigma \to +\infty$. Thus for any $\kappa > 0$, $\mu(q\boldsymbol{n}_\sigma) < \kappa^2$ for large σ. Thus from Lemma 5.2 (ii), we see that pure nematic states are not global minimizers of $\mathcal{E}$.

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