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Gauss Sums on Finite Groups

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Abstract. We shall discuss Gauss sums on finite groups and give several examples including the case of the complex reflection groups $G(m, r, n)$, and hence finite symmetric groups, and also finite Weyl groups.

0. Introduction

For an odd prime p, the classical Gauss sum g_p is given by

$$g_p = \sum_{x=0}^{p-1} e^{2x^2\pi\sqrt{-1}/p} = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e^{2\pi x\sqrt{-1}/p},$$

where $\left(\frac{x}{p}\right)$ is the Legendre symbol. There are several generalizations of this sum, and vast studies have been done. What we shall discuss in this paper starts by regarding the above sum g_p as a sum on the finite cyclic group $\mathbf{F}_p^\times$ of nonzero elements of a finite field with p elements $\mathbf{F}_p$. Generalizing the pair of $\mathbf{F}_p^\times$ and the Legendre symbol to a pair of a finite group G and its complex character χ, we can define a Gauss sum $\tau_G(\chi, \psi_\rho)$ on G associated with a modular representation ρ of G. (In the case of g_p, ρ is the identity.) Precise definition will be given in the next section.

As for this sum $\tau_G(\chi, \psi_\rho)$, T. Kondo firstly determined the values for a finite general linear groups G in [5], for every irreducible character χ and the canonical representation ρ. Also in a series of papers starting with [4], Kim-Lee, D. S. Kim and Kim-Park explicitly described the value $\tau_G(\chi, \psi_\rho)$ for classical groups G and for linear characters χ, i.e. of degree 1. Saito-Shinoda [8, 9] considered $\tau_G(\chi, \psi_\rho)$ for finite reductive groups G and for the Deligne-Lusztig generalized character χ, and applied this result, in particular, to determine $\tau_G(\chi, \psi_\rho)$ for $G = Sp(4, q)$ and for all irreducible characters χ of G.

Thus the Gauss sums on finite groups treated so far are related with finite linear algebraic groups. The purpose of this paper is to consider the Gauss sums on (not necessary algebraic) finite groups, particularly on finite complex reflection groups $G(m, r, n)$.

This paper is organized as follows: after preliminaries in §1, in §2 we determine explicitly $\tau_G(\chi, \psi_\rho)$ for the complex reflection group $G = G(m, 1, n)$ and for all irreducible

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characters χ of G with an n-dimensional modular representation ρ. Main tool in this section is the invariant theory described by I. G. Macdonald in [7]. The results includes the case of finite symmetric groups and also Weyl groups of type B. We also mention the case of alternating groups in this section. In §3, applying Clifford's theorem for this case described by N. Kawanaka in [3], the Gauss sums of the complex reflection groups $G(m, r, n)$ are determined. In §4, Weyl groups of finite exceptional type will be treated. For calculation in this section we used CHEVIE [2].

1. Preliminaries

1.1. Let G be a finite group, $\pi : G \to GL_m(\mathbf{C})$ an ordinary representation, and $\rho : G \to GL_n(\mathbf{F}_q)$ be a modular representation over a finite field with q elements, where q is a power of a prime number p. Throughout this paper we fix a nontrivial additive character e of $\mathbf{F}_q$, $e : \mathbf{F}_q \to \mathbf{C}^\times$. Using this e we define a class function ψ_ρ on G as follows: for $x \in G$, $\psi_\rho(x) = e(\mathrm{Tr}\rho(x))$, where Tr denotes the trace of a matrix.

Let

$$W_G(\pi, \psi_\rho) = \sum_{x\in G} \pi(x)\psi_\rho(x) \in M_m(\mathbf{C}),$$

where $M_m(\mathbf{C})$ is the algebra of all square matrices of degree m with complex coefficients. The trace of $W_G(\pi, \psi_\rho)$ will be called the Gauss sum on G associated with π and ρ, and will be denoted by $\tau_G(\chi_\pi, \psi_\rho)$, where χ_π is the character of π; thus

$$\tau_G(\chi_\pi, \psi_\rho) = \mathrm{Tr} W_G(\pi, \psi_\rho).$$

If there is no afraid of confusion, we shall simply write $\tau_G(\chi_\pi)$ instead of $\tau_G(\chi_\pi, \psi_\rho)$.

If π is irreducible, then by Schur's Lemma $W_G(\pi, \psi_\rho) = w_G(\chi_\pi)I_m$ is a scalar matrix with some $w_G(\chi_\pi) \in \mathbf{C}$, and hence $\tau_G(\chi_\pi) = m \cdot w_G(\chi_\pi)$, where m is the degree of π.

1.2. For $\mathbf{C}$-valued functions f and g on G, let

$$\langle f, g\rangle_G = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}$$

be the usual hermitian inner product on the space of $\mathbf{C}$-valued functions on G, where $\overline{g(x)}$ is the complex conjugate of $g(x)$. Thus for a character χ on G, we have $\tau_G(\chi) = |G|\langle \chi, \overline{\psi_\rho}\rangle_G$. Let $\mathrm{cf}(G)$ be the space of $\mathbf{C}$-valued class functions on G. Then by linearity we can extend τ_G to a linear mapping on $\mathrm{cf}(G)$. It is sometimes useful to consider $|G|^{-1}\tau_G$, so we define a linear map $\tilde{\tau}_G$ on $\mathrm{cf}(G)$ by

$$\tilde{\tau}_G(\chi) = |G|^{-1}\tau_G(\chi) = \langle \chi, \overline{\psi_\rho}\rangle_G .$$

Now the following lemma is an immediate consequence of the Frobenius reciprocity.

LEMMA 1.3. *Let G be a finite group, H its subgroup, χ a character of H, and ρ be a modular representation of G. Then we have*

$$\tilde{\tau}_G(\mathrm{ind}_H^G(\chi), \psi_\rho) = \tilde{\tau}_H(\chi, \psi_{\rho|_H}),$$

where $\mathrm{ind}_H^G(\chi)$ *denotes the induced character of χ from H.*

2. Gauss sums on complex reflection groups $G(m, 1, n)$

2.1. Let m, n be two positive integers, $\omega = e^{2\pi\sqrt{-1}/m}$ and $\langle\omega\rangle^n = \langle\omega\rangle \times \cdots \times \langle\omega\rangle$ be the direct product of n copies of the cyclic group $\langle\omega\rangle$ of order m. The symmetric group $\mathcal{S}_n$ acts on $\langle\omega\rangle^n$ by permuting the factors: for $g = (g_1, g_2, \ldots, g_n) \in \langle\omega\rangle^n$ and $\sigma \in \mathcal{S}_n$, $\sigma(g_1, g_2, \ldots, g_n) = (g_{\sigma^{-1}(1)}, g_{\sigma^{-1}(2)}, \ldots, g_{\sigma^{-1}(n)})$.

Then we define the group $G(m, 1, n)$ as a semidirect product of $\langle\omega\rangle^n$ with $\mathcal{S}_n$ given by this action. Fixing m, we shall write $G_n = G(m, 1, n)$ throughout this section. The elements of G_n may be thought of as permutation matrices with entries in $\langle\omega\rangle$. Thus we identify $(g, \sigma) \in \langle\omega\rangle^n \rtimes \mathcal{S}_n = G_n$ with the $n \times n$ matrix having (i, j)-entry $g_i\delta_{i\sigma(j)}$.

Let p be a prime number such that m divides $p-1$, and a be an element of $\mathbf{F}_p^\times$ which has order m. Then we can define a modular representation $\rho^{(n)} : G_n \to GL_n(\mathbf{F}_p)$ by replacing ω with a. Fixing a nontrivial additive character e of $\mathbf{F}_p$, we also define $\psi^{(n)} = e \circ \mathrm{Tr}\rho^{(n)}$.

In this section for a complex-valued class function χ on G_n, we shall write the Gauss sum $\tau_{G_n}(\chi)$ simply by $\tau_n(\chi)$. Thus

$$\tau_n(\chi) = \sum_{x \in G_n} \chi(x)\psi^{(n)}(x).$$

The purpose of this section is to determine explicitly $\tau_n(\chi)$ associated with any irreducible character χ of G_n.

2.2. Let $\mathcal{P}$ be the set of all partitions and $\mathcal{P}^m$ the direct product of m copies of $\mathcal{P}$. We define

$$\mathcal{P}^m(n) = \{\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m-1)}) \in \mathcal{P}^m \mid \sum_{j=0}^{m-1} |\lambda^{(j)}| = n\}.$$

Since the set of all irreducible characters of G_n is parametrized by $\mathcal{P}^m(n)$ (cf. [7], I, Appendix B), we let χ^λ denote the irreducible character of G_n corresponding to $\lambda \in \mathcal{P}^m(n)$.

Let $R(G_n)$ denote the vector space over $\mathbf{C}$ generated by the irreducible characters of G_n and let $R(G) = \bigoplus_{n \geqslant 0} R(G_n)$. If $u \in R(G_n)$, $v \in R(G_l)$, then we define

$$u \cdot v = \mathrm{ind}_{G_n \times G_l}^{G_{n+l}}(u \times v).$$

With this multiplication, $R(G)$ is a commutative, associative, graded $\mathbf{C}$-algebra with identity element (loc. cit. p. 171).

PROPOSITION 2.3. *Let $\tilde{\tau}$ be a $\mathbf{C}$-linear mapping, $\tilde{\tau} : R(G) \to \mathbf{C}$, defined by*

$$\tilde{\tau}(u) = \sum_n \frac{\tau_n(u_n)}{|G_n|} = \sum_n \langle u_n, \overline{\psi^{(n)}} \rangle_{G_n}, \quad (u = \sum u_n \ \mathit{with}\ u_n \in R(G_n)) .$$

Then $\tilde{\tau}$ is a ring homomorphism.

PROOF. If $u \in R(G_n)$, $v \in R(G_l)$, then by Frobenius reciprocity we have

$$\begin{aligned}
\tilde{\tau}(u \centerdot v) &= \langle \mathrm{ind}_{G_n \times G_l}^{G_{n+l}}(u \times v), \overline{\psi^{(n+l)}} \rangle_{G_{n+l}} \\
&= \langle u \times v, \overline{\psi^{(n+l)}}|_{G_n \times G_l} \rangle_{G_n \times G_l} \\
&= \langle u \times v, \overline{\psi^{(n)}} \times \overline{\psi^{(l)}} \rangle_{G_n \times G_l} \\
&= \langle u, \overline{\psi^{(n)}} \rangle_{G_n} \langle v, \overline{\psi^{(l)}} \rangle_{G_l} \\
&= \tilde{\tau}(u)\tilde{\tau}(v) ,
\end{aligned}$$

which proves (2.3). □

2.4. We define the characters $\eta_n^{(j)}$ $(j = 0, 1, \dots, m-1)$ of G_n of degree 1 as follows:

$$\eta_n^{(j)}(g, \sigma) = (g_1 g_2 \dots g_n)^j \quad (g = (g_1, g_2, \dots, g_n) \in \langle \omega \rangle^n, \sigma \in S_n) .$$

Notice that $\eta_n^{(0)}$ is the trivial character of G_n. Let $\zeta = e(1) \in \mathbf{C}$. So ζ is a pth root of unity. We can determine the Gauss sums associated with $\eta_n^{(j)}$ in the following two theorems.

THEOREM 2.5. *Let $W^{(0)}(t)$ be the generating function of $\left(\tilde{\tau}(\eta_n^{(0)})\right)_{n \geqslant 0}$: $W^{(0)}(t) = \sum_{n \geq 0} \tilde{\tau}(\eta_n^{(0)}) t^n$. Then*

$$W^{(0)}(t) = \frac{1}{1-t} \exp\left(\frac{\sum_{i=0}^{m-1} \zeta^{a^i} - m}{m} t \right) .$$

PROOF. Although we took a from $\mathbf{F}_p^\times$, in this proof we consider a to be a positive integer, $0 < a < p$, and define a function f, $f : \langle \omega \rangle \to \mathbf{Z}$, by

$$f(\omega^i) = a^i \quad (0 \leqslant i \leqslant m-1) .$$

Using this function f, let $F(x) = (f(g_i)\delta_{i\sigma(j)})_{1 \leqslant i, j \leqslant n} \in M_n(\mathbf{Z})$ for each $x = (g, \sigma) \in G_n$. Then $\psi^{(n)}(x) = \zeta^{\mathrm{Tr} F(x)}$ for all $x \in G_n$, and we have $\tau_n(\eta_n^{(0)}) = \sum_{x \in G_n} \zeta^{\mathrm{Tr} F(x)}$.

Now, for a nonnegative integer r, we define

$$A_n(r) = \{x \in G_n \mid \mathrm{Tr} F(x) = r\} ,$$

$$B_n(r) = \{\sigma \in S_n \mid m(\sigma) = r\},$$

where $m(\sigma)$ is the number of fixed points of σ.

Applying Möbius inversion formula for a poset of finite set with inclusion as its partial ordering, we can obtain

$$|B_n(r)| = \sum_{l=0}^{n-r} (-1)^l \frac{n!}{r!l!}.$$

Hence

$$\begin{aligned} |A_n(r)| &= \sum_{\substack{k_0,\ldots,k_{m-1}\geqslant 0,\\ \sum_{i=0}^{m-1} a^i k_i = r}} \left| B_n\left(\sum_{i=0}^{m-1} k_i\right)\right| \frac{\left(\sum_{i=0}^{m-1} k_i\right)!}{k_0!\cdots k_{m-1}!} m^{n-\sum_{i=0}^{m-1} k_i} \\ &= \sum_{\substack{k_0,\ldots,k_{m-1}\geqslant 0,\\ \sum_{i=0}^{m-1} a^i k_i = r}} \sum_{l=0}^{n-\sum_{i=0}^{m-1} k_i} \frac{n!}{k_0!\cdots k_{m-1}!l!} m^{n-\sum_{i=0}^{m-1} k_i} (-1)^l . \end{aligned}$$

Therefore

$$\begin{aligned} \tau_n(\eta_n^{(0)}) &= \sum_{r=0}^{a^{m-1}n} |A_n(r)|\, \zeta^r \\ &= \sum_{r=0}^{a^{m-1}n} \sum_{\substack{k_0,\ldots,k_{m-1}\geqslant 0,\\ \sum_{i=0}^{m-1} a^i k_i = r}} \sum_{l=0}^{n-\sum_{i=0}^{m-1} k_i} \frac{n!}{k_0!\cdots k_{m-1}!l!} m^{n-\sum_{i=0}^{m-1} k_i} (-1)^l \zeta^r . \end{aligned}$$

Thus

$$\begin{aligned} &\tau_n(\eta_n^{(0)}) - mn\tau_{n-1}(\eta_{n-1}^{(0)}) \\ &\quad = \sum_{r=0}^{a^{m-1}n} \sum_{\substack{k_0,\ldots,k_{m-1}\geqslant 0,\\ \sum_{i=0}^{m-1} a^i k_i = r,\\ \sum_{i=0}^{m-1} k_i \leqslant n}} \frac{n!}{k_0!\cdots k_{m-1}!(n-\sum_{i=0}^{m-1} k_i)!} (-m)^{n-\sum_{i=0}^{m-1} k_i} \zeta^r \\ &\quad = \sum_{\substack{k_0,\ldots,k_m\geqslant 0,\\ \sum_{i=0}^{m} k_i = n}} \frac{n!}{k_0!\cdots k_{m-1}!k_m!} \left(\prod_{i=0}^{m-1} \zeta^{a^i k_i}\right) (-m)^{k_m} \\ &\quad = \left(\sum_{i=0}^{m-1} \zeta^{a^i} - m\right)^n . \end{aligned}$$

Consequently

$$\begin{aligned}(1-t)W^{(0)}(t) &= 1+\sum_{n\geqslant 1}(\tilde{\tau}(\eta_n^{(0)})-\tilde{\tau}(\eta_{n-1}^{(0)}))t^n \\ &= \sum_{n\geqslant 0}\frac{1}{n!}\left(\frac{\sum_{i=0}^{m-1}\zeta^{a^i}-m}{m}t\right)^n \\ &= \exp\left(\frac{\sum_{i=0}^{m-1}\zeta^{a^i}-m}{m}t\right),\end{aligned}$$

which proves (2.5). □

THEOREM 2.6. *For $j=1,\dots,m-1$, let $W^{(j)}(t)$ be the generating function of $\big(\tilde{\tau}(\eta_n^{(j)})\big)_{n\geqslant 0}$: $W^{(j)}(t)=\sum_{n\geq 0}\tilde{\tau}(\eta_n^{(j)})t^n$. Then*

$$W^{(j)}(t)=\exp\left(\frac{\sum_{i=0}^{m-1}\omega^{ij}\zeta^{a^i}}{m}t\right).$$

PROOF. Let d be the greatest common divisor of m and j. Using the same notation as in the preceding theorem, we have

$$\begin{aligned}\tau_n(\eta_n^{(j)}) &= \sum_{x\in G_n}\eta_n^{(j)}(x)\zeta^{\mathrm{Tr}F(x)} \\ &= \sum_{r=0}^{a^{m-1}n}\sum_{x\in A_n(r)}\eta_n^{(j)}(x)\zeta^r \\ &= \sum_{r=0}^{a^{m-1}n}\sum_{l=0}^{m/d-1}\sum_{x\in A_n(r)\cap(\eta_n^{(j)})^{-1}(\omega^{jl})}\omega^{jl}\zeta^r \\ &= \sum_{r=0}^{a^{m-1}n}\sum_{\substack{k_0,\dots,k_{m-1}\geqslant 0,\\ \sum_{i=0}^{m-1}a^ik_i=r,\\ \sum_{i=0}^{m-1}k_i\leqslant n-1}}\sum_{l=0}^{m/d-1}d\omega^{jl}\left|B_n\left(\sum_{i=0}^{m-1}k_i\right)\right|\frac{\left(\sum_{i=0}^{m-1}k_i\right)!}{k_0!\cdots k_{m-1}!}m^{n-\sum_{i=0}^{m-1}k_i-1}\zeta^r \\ &\quad+\sum_{r=0}^{a^{m-1}n}\sum_{l=0}^{m/d-1}\sum_{\substack{k_0,\dots,k_{m-1}\geqslant 0,\\ \sum_{i=0}^{m-1}a^ik_i=r,\\ \sum_{i=0}^{m-1}k_i=n,\\ \sum_{i=0}^{m-1}ik_i\equiv l\ (\mathrm{mod}\ m/d)}}\frac{n!}{k_0!\cdots k_{m-1}!}\omega^{jl}\zeta^r.\end{aligned}$$

Since $1 \leqslant j \leqslant m-1$, it follows that $\sum_{l=0}^{m/d-1} \omega^{jl} = 0$. Hence

$$\tau_n(\eta_n^{(j)}) = \sum_{\substack{k_0,\dots,k_{m-1} \geqslant 0, \\ \sum_{i=0}^{m} k_i = n}} \frac{n!}{k_0! \cdots k_{m-1}!} \prod_{i=0}^{m-1} (\omega^{ij} \zeta^{a^i})^{k_i}$$
$$= \left(\sum_{i=0}^{m-1} \omega^{ij} \zeta^{a^i} \right)^n .$$

Therefore $W^{(j)}(t) = \exp\left(m^{-1} \sum_{i=0}^{m-1} \omega^{ij} \zeta^{a^i} t\right)$. □

2.7. We shall use the following notation on partitions and symmetric functions (cf. [7], I). Let λ be a partition. Then $n(\lambda) = \sum_{i \geqslant 1} (i-1)\lambda_i$, and for each point $x = (i, j)$ in the diagram of λ, $c(x) = j - i$, $h(x)$ is the hook-length of x, and $h(\lambda)$ is the product of the hook-lengths of λ. Moreover let Λ be the ring of symmetric functions in countably many independent variables $x = (x_1, x_2, \dots)$. Following functions are defined in Λ: the rth elementary symmetric function e_r, the rth complete symmetric function h_r, the rth power sum p_r, and the Schur function s_λ corresponding to each $\lambda \in \mathcal{P}$.

Let n be a nonnegative integer, z a nonzero complex number and let $f \in \Lambda$. We consider the following specialization of x_i:

$$x_i = \frac{z}{n}\left(1 + \frac{z}{n}\right)^{i-1} \; (1 \leqslant i \leqslant n), \quad x_{n+1} = x_{n+2} = \cdots = 0 .$$

The specialized value of f in this way is denoted by $f^{(n)}$.

LEMMA 2.8. *Let r be a nonnegative integer and $\lambda \in \mathcal{P}$. Then*

$$\lim_{n\to\infty} (e_r^{(n)}) = \lim_{n\to\infty} (h_r^{(n)}) = \frac{(e^z-1)^r}{r!}, \quad \lim_{n\to\infty} (s_\lambda^{(n)}) = \frac{(e^z-1)^{|\lambda|}}{h(\lambda)} .$$

PROOF. Let $y = 1 + z/n$. Then

$$e_r^{(n)} = \left(\frac{z}{n}\right)^r y^{\frac{r(r-1)}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_y ,$$
$$h_r^{(n)} = \left(\frac{z}{n}\right)^r \begin{bmatrix} n+r-1 \\ r \end{bmatrix}_y ,$$
$$s_\lambda^{(n)} = \left(\frac{z}{n}\right)^{|\lambda|} y^{n(\lambda)} \prod_{x \in \lambda} \frac{1 - y^{n+c(x)}}{1 - y^{h(x)}} ,$$

(cf. [7], I, 2. Ex.1 and 3. Ex.1) where $\begin{bmatrix} n \\ r \end{bmatrix}_y$ is the Gaussian polynomial in the variable y.

Hence

$$e_r^{(n)} = z^r y^{\frac{r(r-1)}{2}} \left(\prod_{k=0}^{r-1} \left(\left(1 + \frac{z}{n} \right)^{n-k} - 1 \right) \right) \left(\prod_{k=1}^{r} \left(kz + \sum_{i=2}^{k} \binom{k}{i} \frac{z^i}{n^{i-1}} \right) \right)^{-1}$$
$$\to \frac{(e^z - 1)^r}{r!} \quad (n \to \infty) .$$

Similarly we have $h_r^{(n)} \to \dfrac{(e^z - 1)^r}{r!}$, $s_\lambda^{(n)} \to \dfrac{(e^z - 1)^{|\lambda|}}{h(\lambda)}$ $(n \to \infty)$. □

2.9. In this subsection, we shall follow the notation in ([7], I. Appendix B, 5 and 6). Let $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots)$ $(j = 0, 1, \dots, m-1)$ be independent variables over $\mathbf{C}$ and $\Lambda(G)$ denote the graded $\mathbf{C}$-algebra generated by $p_r(x^{(j)})$ $(r \geqslant 1,\ j = 0, 1, \dots, m-1)$:

$$\Lambda(G) = \mathbf{C}[p_r(x^{(j)}) \ :\ r \geqslant 1,\ j = 0, 1, \dots, m-1] .$$

An isomorphism c of graded $\mathbf{C}$-algebras, $c : \Lambda(G) \to \Lambda(G)$, is defined by

$$c(p_r(x^{(j)})) = \frac{1}{m} \left(\sum_{k=0}^{m-1} \omega^{jk} p_r(x^{(k)}) \right) .$$

For $\nu = (\nu_1, \nu_2, \dots) \in \mathcal{P}$, let $p_\nu(x^{(j)}) = p_{\nu_1}(x^{(j)}) p_{\nu_2}(x^{(j)}), \dots$, $l(\nu) = \#\{j \mid \nu_j > 0\}$ and $z_\nu = \prod_{i \geqslant 1} i^{m_i(\nu)} m_i(\nu)!$, where $m_i(\nu) = \#\{j \mid \nu_j = i\}$. Also for $\rho = (\rho^{(0)}, \dots, \rho^{(m-1)}) \in \mathcal{P}^m$, let

$$P_\rho = \prod_{j=0}^{m-1} p_{\rho^{(j)}}(x^{(j)}),\ \ Z_\rho = \prod_{j=0}^{m-1} z_{\rho^{(j)}} m^{l(\rho^{(j)})} .$$

Then $\{P_\rho\}_{\rho \in \mathcal{P}^m}$ is a $\mathbf{C}$-basis of $\Lambda(G)$. For $f = \sum_{\rho \in \mathcal{P}^m} a_\rho P_\rho \in \Lambda(G)$, $(a_\rho \in \mathbf{C})$, let

$$\overline{f} = \sum_{\rho \in \mathcal{P}^m} \overline{a_\rho} P_\rho .$$

Then we can define a hermitian inner product on $\Lambda(G)$ as follows:
if $f = \sum_\rho a_\rho P_\rho,\ g = \sum_\rho b_\rho P_\rho,\ (a_\rho, b_\rho \in \mathbf{C})$, then

$$\langle f, g \rangle = \sum_{\rho \in \mathcal{P}^m} a_\rho \overline{b_\rho} Z_\rho .$$

Moreover, a $\mathbf{C}$-linear mapping ch : $R(G) \to \Lambda(G)$ is defined by

$$\mathrm{ch}(u) = \frac{1}{|G_n|} \sum_{x \in G_n} u(x) P_{\rho(x)} ,$$

where $\rho(x)$ is the type of x, which generalizes the cycle type of the symmetric group $\mathcal{S}_n$ and parametrizes the conjugacy classes of G_n (cf. [7], I. Appendix B, 3).

It is known that ch is an isometric isomorphism of graded $\mathbf{C}$-algebras and satisfies

$$\mathrm{ch}(\chi^\lambda) = \prod_{j=0}^{m-1} c(s_{\lambda^{(j)}}(x^{(j)}))$$

(cf. loc. cit., (6.3)). In particular $\mathrm{ch}(\eta_n^{(j)}) = c(h_n(x^{(j)}))$.

By (2.3), (2.5), (2.6) and (2.8), we can obtain explicit expressions of Gauss sums on $G(m, 1, n)$ as follows:

THEOREM 2.10. *Let χ^λ be the irreducible character corresponding to $\lambda \in \mathcal{P}^m$. Then*

$$\tilde{\tau}(\chi^\lambda) = \sum_{\mu} \frac{1}{h(\mu)} \left(\frac{\sum_{i=0}^{m-1} \zeta^{a^i} - m}{m} \right)^{|\mu|} \prod_{j=1}^{m-1} \frac{1}{h(\lambda^{(j)})} \left(\frac{\sum_{i=0}^{m-1} \omega^{ij} \zeta^{a^i}}{m} \right)^{|\lambda^{(j)}|},$$

where the summation is over all partitions μ such that $\lambda^{(0)} - \mu$ is a horizontal strip.

PROOF. We define an endomorphism ϕ of $\mathbf{C}$-algebras, $\phi : \Lambda(G) \to \Lambda(G)$, by

$$\phi(h_r(x^{(0)})) = \sum_{k=0}^{r} h_k(x^{(0)}),$$
$$\phi(h_r(x^{(j)})) = h_r(x^{(j)}) \quad (j = 1, \dots, m-1).$$

Let

$$z^{(0)} = \log \left(m^{-1} \sum_{i=0}^{m-1} \zeta^{a^i} \right),$$
$$z^{(j)} = \log \left(m^{-1} \left(\sum_{i=0}^{m-1} \omega^{ij} \zeta^{a^i} + m \right) \right), \quad (j = 1, \dots, m-1),$$

and as in 2.7, we consider the following specialization of $f \in \Lambda(G)$ obtained by putting

$$x_i^{(j)} = \begin{cases} \dfrac{z^{(j)}}{n} \left(1 + \dfrac{z^{(j)}}{n} \right)^{i-1} & (1 \leqslant i \leqslant n), \\ 0 & (i > n), \end{cases} \quad (j = 0, 1, \dots, m-1).$$

The specialized value of f in this way is denoted by $f^{(n)}$. Moreover we define a homomorphism of algebras Φ, $\Phi : \Lambda(G) \to \mathbf{C}$, by $\Phi(f) = \lim_{n\to\infty} f^{(n)}$. Then by (2.5), (2.6) and

(2.8), we have the following commutative diagram:

$$\begin{array}{ccccc} & & \phi & & \\ & \Lambda(G) & \rightarrow & \Lambda(G) & \\ \mathrm{ch}^{-1}\circ c & \downarrow & & \downarrow & \Phi \\ & R(G) & \rightarrow & \mathbf{C} & \\ & & \tilde{\tau} & & \end{array}$$

Hence $\tilde{\tau}(\chi^{\lambda}) = \Phi \circ \phi(\prod_{j=0}^{m-1} s_{\lambda^{(j)}}(x^{(j)}))$.

From $\sum_{k=0}^{r}(-1)^k e_k h_{r-k} = 0$ $(r \geqslant 1)$, we obtain

$$\phi(e_r(x^{(0)})) = e_r(x^{(0)}) + e_{r-1}(x^{(0)})$$

by induction on r. Hence, for each $\nu \in \mathcal{P}$, we have

$$\begin{aligned} \phi(s_\nu(x^{(0)})) &= \phi(\det(e_{\nu_i{}'-i+j}(x^{(0)}))_{1\leqslant i,j\leqslant n}) \\ &= \det(e_{\nu_i{}'-i+j}(x^{(0)}) + e_{\nu_i{}'-1-i+j}(x^{(0)}))_{1\leqslant i,j\leqslant n} \\ &= \sum_{\varepsilon\in\{0,1\}^n} \det(e_{\nu_i{}'-\varepsilon_i-i+j}(x^{(0)}))_{1\leqslant i,j\leqslant n} \\ &= \sum_{\substack{\varepsilon\in\{0,1\}^n, \\ \nu'-\varepsilon\in\mathcal{P}}} s_{(\nu'-\varepsilon)'}(x^{(0)}) \\ &= \sum_{\substack{\mu\subset\nu, \\ \nu-\mu:\text{horizontal strip}}} s_\mu(x^{(0)}), \end{aligned}$$

where ν' is the conjugate of ν. Therefore

$$\begin{aligned} \tilde{\tau}(\chi^\lambda) &= \prod_{j=0}^{m-1} \Phi\circ\phi(s_{\lambda^{(j)}}(x^{(j)})) \\ &= \sum_{\substack{\mu\subset\lambda^{(0)}, \\ \lambda^{(0)}-\mu:\text{horizontal strip}}} \Phi(s_\mu(x^{(0)})) \prod_{j=1}^{m-1} \Phi(s_{\lambda^{(j)}}(x^{(j)})) \\ &= \sum_{\substack{\mu\subset\lambda^{(0)}, \\ \lambda^{(0)}-\mu:\text{horizontal strip}}} \frac{(e^{z^{(0)}}-1)^{|\mu|}}{h(\mu)} \prod_{j=1}^{m-1} \frac{(e^{z^{(j)}}-1)^{|\lambda^{(j)}|}}{h(\lambda^{(j)})} \qquad \text{(by (2.8))} \\ &= \sum_{\substack{\mu\subset\lambda^{(0)}, \\ \lambda^{(0)}-\mu:\text{horizontal strip}}} \frac{1}{h(\mu)}\left(\frac{\sum_{i=0}^{m-1}\zeta^{a^i}-m}{m}\right)^{|\mu|} \prod_{j=1}^{m-1} \frac{1}{h(\lambda^{(j)})}\left(\frac{\sum_{i=0}^{m-1}\omega^{ij}\zeta^{a^i}}{m}\right)^{|\lambda^{(j)}|}, \end{aligned}$$

which proves the assertion of our theorem. □

COROLLARY 2.11. *Let $G = \mathcal{S}_n$ be the symmetric group of degree n and ρ be the modular representation over $\mathbf{F}_p$ induced by the permutation representation of G on n letters. Then the Gauss sum associated with the irreducible character $\lambda \in \mathcal{P}(n)$ is*

$$\tilde{\tau}_{\mathcal{S}_n}(\chi^\lambda) = \sum_\mu \frac{1}{h(\mu)}(\zeta - 1)^{|\mu|},$$

where the summation is over all partitions μ such that $\lambda - \mu$ is a horizontal strip.

PROOF. This is the special case of $m = 1$ in (2.10). □

REMARK 2.12. Since the permutation representation $\rho^{(n)}$ of $\mathcal{S}_n$ is the sum of the trivial representation and the reflection representation ρ', we have

$$\mathrm{Tr}\rho^{(n)}(\sigma) = \mathrm{Tr}\rho'(\sigma) + 1 \quad (\sigma \in \mathcal{S}_n).$$

Thus, if we use the reflection representation ρ' instead of the permutation representation $\rho^{(n)}$ to determine the Gauss sums on $\mathcal{S}_n$, then we have

$$\tilde{\tau}_{\mathcal{S}_n}(\chi^\lambda, \psi_{\rho'}) = \zeta^{-1}\,\tilde{\tau}_{\mathcal{S}_n}(\chi^\lambda, \psi^{(n)}) = \sum_\mu \frac{\zeta^{-1}}{h(\mu)}(\zeta - 1)^{|\mu|},$$

where the summation is over all partitions μ such that $\lambda - \mu$ is a horizontal strip.

COROLLARY 2.13. *Let $G = W(B_n)$ be the finite Weyl group of type B_n with $n \geq 2$, and ρ be the modular representation of G over $\mathbf{F}_p$ induced by the reflection representation of G. Then the Gauss sum associated with the irreducible character $\lambda \in \mathcal{P}^2(n)$ is*

$$\tilde{\tau}_{W(B_n)}(\chi^\lambda) = \sum_\mu \frac{1}{h(\mu)h(\lambda^{(1)})}\left(\frac{\zeta + \zeta^{-1} - 2}{2}\right)^{|\mu|}\left(\frac{\zeta - \zeta^{-1}}{2}\right)^{|\lambda^{(1)}|}$$

where the summation is over all partitions μ such that $\lambda^{(0)} - \mu$ is a horizontal strip.

PROOF. This is the special case of $m = 2$ in (2.10). □

REMARK 2.14. Applying (1.3) and Clifford's theorem to $\mathcal{A}_n$, the alternating group of degree n, as a normal subgroup of $\mathcal{S}_n$, we can obtain the Gauss sums on $\mathcal{A}_n$ too. In fact, if λ is a partition of n which is not self-dual, then the restriction of the irreducible character χ^λ of $\mathcal{S}_n$ to $\mathcal{A}_n$ is irreducible and

$$\chi^\lambda|_{\mathcal{A}_n} = \chi^{\lambda'}|_{\mathcal{A}_n},$$

which is denoted by $\chi^\lambda_{\mathcal{A}_n}$. If λ is self-dual, the restriction of the character χ^λ to $\mathcal{A}_n$ splits into two distinct irreducible characters which are denoted by $\chi^{\lambda^+}_{\mathcal{A}_n}$ and $\chi^{\lambda^-}_{\mathcal{A}_n}$. Then the Gauss sums for the irreducible character of $\mathcal{A}_n$ are given as follows:

$$\tilde{\tau}_{\mathcal{A}_n}(\chi^{\lambda}_{\mathcal{A}_n}) = \tilde{\tau}_{\mathcal{S}_n}(\chi^{\lambda}) + \tilde{\tau}_{\mathcal{S}_n}(\chi^{\lambda'}) , \quad \text{if } \lambda \text{ is not self-dual,}$$
$$\tilde{\tau}_{\mathcal{A}_n}(\chi^{\lambda^+}_{\mathcal{A}_n}) = \tilde{\tau}_{\mathcal{A}_n}(\chi^{\lambda^-}_{\mathcal{A}_n}) = \tilde{\tau}_{\mathcal{S}_n}(\chi^{\lambda}) , \quad \text{if } \lambda \text{ is self-dual.}$$

3. Gauss sums on complex reflection groups $G(m, r, n)$

3.1. First, we summarize the character theory of $G(m, r, n)$, following Kawanaka [3]. Let $\eta = \eta_n^{(1)}$ be the linear character of $G(m, 1, n)$ given in (2.4). For a natural number r dividing m, the complex reflection group $G(m, r, n)$ is defined by

$$G(m, r, n) = \mathrm{Ker}\, \eta^{m/r} .$$

For each irreducible character χ^{λ} of $G(m, 1, n)$, $\chi^{\lambda} \otimes \eta$ is again irreducible. So the cyclic group $\langle\eta\rangle$ of order m acts on the set of irreducible characters of $G(m, 1, n)$, hence on the parameter space $\mathcal{P}^m(n)$. More explicitly, for $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(m-1)}) \in \mathcal{P}^m(n)$,

$$\chi^{\lambda} \otimes \eta = \chi^{\mu} , \quad \lambda,\ \mu \in \mathcal{P}^m(n) ,$$

is equivalent with

$$\mu = (\lambda^{(m-1)}, \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(m-2)}) .$$

By Clifford's theorem, we obtain the irreducible characters of $G(m, r, n)$ (cf.[3] Proposition 2.5).

PROPOSITION 3.2. *Let r be a positive integer dividing m, and $s(\lambda, r)$ the order of the stabilizer of $\lambda \in \mathcal{P}^m(n)$ in $\langle\eta^{m/r}\rangle$.*

(1) *The restriction $\chi^{\lambda}|G(m, r, n)$ is a sum of $s(\lambda, r)$ distinct irreducible characters χ_i^{λ} $(i = 1, 2, \dots, s(\lambda, r))$ of $G(m, r, n)$, which are mutually conjugate under $G(m, 1, n)$.*

(2) *For $\lambda, \mu \in \mathcal{P}^m(n)$, $\chi^{\lambda}|G(m, r, n)$ and $\chi^{\mu}|G(m, r, n)$ have a common constituent of irreducible characters if and only if λ and μ are in the same orbit under $\langle\eta^{m/r}\rangle$. In this case, we have*

$$\chi^{\lambda}|G(m, r, n) = \chi^{\mu}|G(m, r, n) .$$

(3) *Every irreducible character of $G(m, r, n)$ is uniquely obtained as*

$$\chi_i^{\lambda} \ (i = 1, 2, \dots, s(\lambda, r)) ,$$

where λ runs over a complete set of representatives of the $\langle\eta^{m/r}\rangle$-orbits in $\mathcal{P}^m(n)$.

PROPOSITION 3.3. *Let $[\lambda]_r$ denote the $\langle\eta^{m/r}\rangle$-orbit containing λ in $\mathcal{P}^m(n)$. Then for an irreducible character χ_i^{λ} of $G(m, r, n)$, we have*

$$\tilde{\tau}_{G(m,r,n)}(\chi_i^{\lambda}) = \sum_{\mu \in [\lambda]_r} \tilde{\tau}_{G(m,1,n)}(\chi^{\mu}) .$$

PROOF. By (3.2), we have

$$\operatorname{ind}_{G(m,r,n)}^{G(m,1,n)} \chi_i^{\lambda} = \sum_{\mu \in [\lambda]_r} \chi^{\mu},$$

and hence by (1.3) and (2.10), we obtain the Gauss sums on $G(m,r,n)$. □

REMARK 3.4. Above (3.3) includes the case of the dihedral group of order $2m$, $I_2(m)$, and the Weyl group of type D_n, since $G(m,m,2)$ is the dihedral group and $G(2,2,n)$ is the Weyl group of type D_n.

4. Gauss sums on finite exceptional Weyl groups

4.1. Let W be an exceptional finite Weyl group. If we take simple roots as a basis of the reflection representation space, then all components of representation matrices are integers and the representation matrices can be considered as elements in $GL_n(\mathbf{F}_p)$. So we define Gauss sums on W, using this representation as a modular representation. We also fix a non-trivial additive character e of $\mathbf{F}_p$, $e : \mathbf{F}_p \to \mathbf{C}^\times$, and put $\zeta = e(1)$: hence $\zeta = e^{2a\pi\sqrt{-1}/p}$ with an integer a prime to p.

The values of Gauss sums are obtained directly from the character table. We used CHEVIE [2], in GAP. In case of type F_4 and G_2, we will show lists of all Gauss sums $\tau_W(\chi)$, but in case of type E_6, E_7 and E_8, we show Gauss sums only for the trivial representation, the sign representation and the reflection representation. The other values can be found in the web page below:

http://pweb.sophia.ac.jp/y-gomi/Gauss-sum.html

In the table below,

$$y = \zeta + \zeta^{-1} - 2 = 2\left(\cos\frac{2a\pi}{p} - 1\right), \quad w = \zeta - \zeta^{-1} = 2\sqrt{-1}\sin\frac{2a\pi}{p}.$$

For the notation of irreducible characters of type F_4 and G_2 we follow [6].

4.2. TABLES OF $\tau_W(\chi)$

Type E_6

triv.	$\zeta^6 + 36\zeta^4 + 240\zeta^3 + 2430\zeta^2 + 13104\zeta + 20820 + 11664\zeta^{-1} + 3465\zeta^{-2} + 80\zeta^{-3}$
sign	$\zeta^6 - 36\zeta^4 + 240\zeta^3 - 810\zeta^2 + 1584\zeta - 1860 + 1296\zeta^{-1} - 495\zeta^{-2} + 80\zeta^{-3}$
ref.	$6\zeta^6 + 144\zeta^4 + 720\zeta^3 + 4860\zeta^2 + 13104\zeta - 11664\zeta^{-1} - 6930\zeta^{-2} - 240\zeta^{-3}$

Type E_7

triv.	$y^7 + 14y^6 + 140y^5 + 1512y^4 + 20160y^3 + 241920y^2 + 1451520y + 2903040$
sign	$w\left(y^6 + 12y^5 - 8y^4 + 288y^3\right)$
ref.	$w\left(7y^6 + 84y^5 + 700y^4 + 6048y^3 + 60480y^2 + 483840y + 1451520\right)$

Type E_8

triv.	$y^8+16y^7+224y^6+4032y^5+120960y^4+3870720y^3+58060800y^2+348364800y+696729600$
sign	$y^8+16y^7-16y^6+1152y^5+17280y^4$
ref.	$w\left(8y^7+112y^6+1344y^5+20160y^4+483840y^3+11612160y^2+116121600y+348364800\right)$
ϕ	$w\left(8y^7+112y^6-96y^5+5760y^4+69120y^3\right)$

(ϕ is the product of the sign character and the reflection character.)

Type F_4

$\phi_{1,0}$	$y^4+8y^3+96y^2+576y+1152$
$\phi_{1,12'},\ \phi_{1,12''}$	y^4+8y^3
$\phi_{1,24}$	$y^4+8y^3+48y^2$
$\phi_{2,4'},\ \phi_{2,4''}$	$2y^4+16y^3+48y^2$
$\phi_{2,16'},\ \phi_{2,16''}$	$2y^4+16y^3$
$\phi_{4,8}$	$4y^4+32y^3+96y^2$
$\phi_{9,2}$	$9y^4+72y^3+288y^2+576y$
$\phi_{9,6'},\ \phi_{9,6''},\ \phi_{9,10}$	$9y^4+72y^3+144y^2$
$\phi_{6,6'}$	$6y^4+48y^3+96y^2$
$\phi_{6,6''}$	$6y^4+48y^3+240y^2+576y$
$\phi_{12,4}$	$12y^4+96y^3+192y^2$
$\phi_{4,1}$	$w\left(4y^3+24y^2+192y+576\right)$
$\phi_{4,7'},\ \phi_{4,7''}$	$w\left(4y^3+24y^2\right)$
$\phi_{4,13}$	$w\left(4y^3+24y^2+96y\right)$
$\phi_{8,3'},\ \phi_{8,3''}$	$w\left(8y^3+48y^2+96y\right)$
$\phi_{8,9'},\ \phi_{8,9''}$	$w\left(8y^3+48y^2\right)$
$\phi_{16,5}$	$w\left(16y^3+96y^2+192y\right)$

Type G_2

$\phi_{1,0}$	$y^2+6y+12$
$\phi_{1,6}$	y^2+6y
$\phi_{1,3'},\ \phi_{1,3''}$	wy
$\phi_{2,1}$	$2w(y+3)$
$\phi_{2,2}$	$2y^2+6y$

REMARK 4.3. If the scalar transformation -1 lies in W, then it is easy to show that $\tau_W(\chi)$ is real (resp. pure imaginary) when $\chi(-1)=\chi(1)$ (resp. $\chi(-1)=-\chi(1)$), for an irreducible character χ. This phenomena is similar to the original Gauss sum g_p.

Moreover in the case of type E_7, E_8, F_4 or G_2, we can observe that if $\chi(-1)=\chi(1)$ (resp. $\chi(-1)=-\chi(1)$), then $\tau_W(\chi)$ (resp. $\tau_W(\chi)/(\zeta-\zeta^{-1})$) is written as a polynomial in

$y(=\zeta+\zeta^{-1}-2)$. Furthermore the coefficients of these polynomials in y are non-negative integers except the case when χ is the sign character of type E_7, or the sign character or ϕ, the product of the sign character and the reflection character, of type E_8.

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