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Rational Solutions of Difference Painlevé Equations

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Abstract. We capture all the rational solutions of some difference Painlevé equations of P_I and P_{II} types. For non-autonomous cases, it is shown that all the rational solutions of the difference P_{II} are ones generated by successive application of auto-Bäcklund transformations to the seed solution vanishing identically, and that the other equations of P_I type admit no rational solutions. For autonomous cases, all the nontrivial rational solutions are obtained, and they exist under a certain condition on a fixed point of the equation. If such a condition is not satisfied, there exist solutions that are rational in an exponential function.

1. Introduction

The Painlevé equations are characterised by the Painlevé property: that, for every solution, all the movable singularities are poles. Discrete analogues of Painlevé equations, which are called discrete Painlevé equations, were discovered in various problems in mathematical physics. The non-autonomous mapping

$$\mathrm{dP_{II}} \qquad y_{n+1} + y_{n-1} = \frac{(\alpha n + \beta) y_n + \gamma}{1 - y_n^2},$$

which appears in connection with unitary matrix models of two-dimensional quantum gravity [7], is known as the discrete P_{II} [1, 11, 12]. Indeed, the continuous limit $n = \varepsilon^{-1} t$, $y_n = \varepsilon u(t)$, $\alpha = \varepsilon^3$, $\beta = 2$, $\gamma = a\varepsilon^3$ ($\varepsilon \to 0$) maps $\mathrm{dP_{II}}$ to the second Painlevé equation P_{II}: $u'' = 2u^3 + tu + a$. Furthermore, as in the case of continuous Painlevé equations, by the degeneration procedure $y_n = z_n/\delta$, $\alpha n + \beta = -(\alpha' n + \beta')/\delta^2$, $\gamma = -\gamma'/\delta^3$ ($\delta \to 0$), equation $\mathrm{dP_{II}}$ is reduced to the discrete P_I

$$\mathrm{dP_{I}} \qquad z_{n+1} + z_{n-1} = \frac{\alpha' n + \beta'}{z_n} + \frac{\gamma'}{z_n^2},$$

which is also obtained from Bäcklund transformations for the third Painlevé equation [1]. If $\alpha' = -1$, the continuous limit $n = \varepsilon^{-1} t$, $z_n = -\varepsilon^{-5/2} + \varepsilon^{-1/2} u(t)$, $\beta' = 6\varepsilon^{-5}$, $\gamma' = 4\varepsilon^{-15/2}$ ($\varepsilon \to 0$) yields the first Painlevé equation P_I: $u'' = 6u^2 + t$. In general, discrete Painlevé

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equations admit the singularity confinement property [12], which has been considered to correspond to the Painlevé property.

Particular solutions of dP_{II} including rational solutions were presented by [4, 5, 13]. In particular, if $\gamma/\alpha \in \mathbf{Z}$, successive application of auto-Bäcklund transformations to the seed solution $y_n \equiv 0$ of dP_{II} with $\gamma = 0$ yields a sequence of rational solutions of dP_{II}. These rational solutions are expressible in terms of τ-functions constructed based on the bilinear formalism of dP_{II} [5, 13]. In view of the fact that, for the continuous P_{II}, all the rational solutions are known [6], it seems important to check whether all the rational solutions of dP_{II} are obtained in this way.

For the nature of our problem, instead of discrete equations, we consider the difference versions of them that are obtained by replacing n with the complex variable x. The purpose of this paper is to capture all the rational solutions of the difference Painlevé equations

$$y(x+1) + y(x-1) = \frac{\alpha x + \beta}{y(x)} + \gamma\,, \tag{1.1}$$

$$y(x+1) + y(x) + y(x-1) = \frac{\alpha x + \beta}{y(x)} + \gamma\,, \tag{1.2}$$

$$y(x+1) + y(x-1) = \frac{\alpha x + \beta}{y(x)} + \frac{\gamma}{y(x)^2}\,, \tag{1.3}$$

$$y(x+1) + y(x-1) = \frac{(\alpha x + \beta)y(x) + \gamma}{1 - y(x)^2} \tag{1.4}$$

with $\alpha, \beta, \gamma \in \mathbf{C}$. Equation (1.3) (respectively, (1.4)) corresponds to dP_I (respectively, dP_{II}) mentioned above, and the discrete versions of (1.1) and (1.2) are known as other types of discrete P_I [11]. Our results are stated in Section 2. For non-autonomous cases, all the rational solutions of (1.4) coincide with ones mentioned above, and the other equations admit no rational solutions. For autonomous cases, we present all the nontrivial rational solutions, and they exist under a certain condition on a fixed point of the equation. If such a condition is not satisfied, we may find exact solutions that are rational in an exponential function.

2. Results

2.1. Non-autonomous cases. For non-autonomous cases we have the following results.

THEOREM 2.1. *Suppose that* $\alpha \neq 0$. *Then* (1.1), (1.2) *and* (1.3) *admit no rational solutions.*

THEOREM 2.2. *Suppose that* $\alpha \neq 0$. *Then* (1.4) *admits rational solutions if and only if* $\gamma/\alpha \in \mathbf{Z}$. *If* $\gamma = 0$, *then* (1.4) *admits the unique rational solution* $y(x) \equiv 0$. *If* $N = \gamma/\alpha \in \mathbf{Z}\setminus\{0\}$, *then* (1.4) *admits a unique rational solution* $P(x)/Q(x)$ *such that* $\deg P + 1 = \deg Q = N^2$, *where the polynomials* $P(x)$ *and* $Q(x)$ *are relatively prime.*

REMARK 2.1. As will be shown in the proof, these rational solutions of (1.4) are generated by successive application of auto-Bäcklund transformations to the solution $y(x) \equiv 0$.

2.2. Autonomous cases. Suppose that $\alpha = 0$, and write (1.l) ($1 \le l \le 4$) in the form

$$y(x+1) + y(x-1) = R_l(y(x))$$

with

$$R_1(y) := \frac{\beta}{y} + \gamma\,, \quad R_2(y) := -y + \frac{\beta}{y} + \gamma\,, \quad R_3(y) := \frac{\beta}{y} + \frac{\gamma}{y^2}\,, \quad R_4(y) := \frac{\beta y + \gamma}{1 - y^2}\,.$$

To exclude the cases where the equation is linear or is reducible to another equation, we impose the conditions:

$$\begin{aligned} &\beta \ne 0 \text{ on } R_1(y) \text{ and } R_2(y)\,;\\ &\gamma \ne 0 \text{ on } R_3(y)\,;\\ &(\beta, \gamma) \ne (0,0)\,, \quad \beta \pm \gamma \ne 0 \text{ on } R_4(y) \end{aligned} \tag{2.1}$$

(for example, if $\beta \pm \gamma = 0$, then (1.4) is reducible to (1.1)). Then all the nontrivial rational solutions are given as follows.

THEOREM 2.3. (1) *Each equation* (1.l) ($1 \le l \le 4$) *admits nontrivial rational solutions if and only if $R_l(y)$ satisfies condition* (C.l) *given in the list*:

(C.1) $\gamma = \pm\, 2\sqrt{-2\beta}$ *in* $R_1(y)$;

(C.2) $\gamma = \pm\, 2\sqrt{-3\beta}$ *in* $R_2(y)$;

(C.3) $\gamma = \pm\, \sqrt{6}\beta^{3/2}/9$ *in* $R_3(y)$;

(C.4) $\gamma = \pm\, \sqrt{6}(2-\beta)^{3/2}/9, \quad \beta \ne -4, 1/2$ *in* $R_4(y)$.

(2) *Under condition* (C.l), *all the nontrivial rational solutions of* (1.l) *are given by* $\{\phi_l^{\pm}(x-c) \mid c \in \mathbf{C}\}$ *with*

$$\phi_1^{\pm}(x) = \pm\sqrt{-\beta/2}\left(1 - \frac{3}{x^2-1}\right),$$

$$\phi_2^{\pm}(x) = \pm\sqrt{-\beta/3}\left(1 - \frac{8}{4x^2-1}\right),$$

$$\phi_3^{\pm}(x) = \mp\sqrt{\beta/6}\left(1 - \frac{1}{x^2}\right),$$

$$\phi_4^{\pm}(x) = \begin{cases} \pm\sqrt{(2-\beta)/6}\left(1 + \dfrac{2(\beta+4)}{2(2-\beta)x^2 - 3}\right) & \text{if } \beta \ne 2\,,\\ \pm\dfrac{1}{x} & \text{if } \beta = 2\,. \end{cases}$$

Let us call $y_0 \in \mathbf{C}$ a *fixed point of* $R_l(y)/2$ if $R_l(y_0)/2 = y_0$. As will be mentioned in Lemma 4.1 the function $R_l(y)/2$ possesses a fixed point y_0 such that $R'_l(y_0) = 2$ if and only if condition (C.l) is satisfied. In the complementary cases, there exist exact solutions that are rational in an exponential function with its period not equal to 1.

THEOREM 2.4. *Suppose that* $R_l(y)$ $(1 \le l \le 4)$ *does not satisfy condition* (C.l). *Then* $R_l(y)/2$ *always possesses a fixed point* y_0 *satisfying*

$$R'_l(y_0) \neq \begin{cases} -1 & \text{if } l = 1\,, \\ -2 & \text{if } l = 2,\ 4 \end{cases}$$

in addition to $R'_l(y_0) \neq 2$ $(1 \le l \le 4)$. *For each fixed point with this property*, *equation* (1.l) *admits a family of solutions* $\{F_l(y_0; e^{-\sigma(x-c)}) \mid c \in \mathbf{C}\}$. *Here* $\sigma = \sigma(y_0)$ *is a complex number such that* $e^{-\sigma} + e^{\sigma} = R'_l(y_0)$, *and* $F_l(y_0; X)$ *is a rational function expressed as follows*:

(1) *for* $1 \le l \le 3$,

$$F_l(y_0; X) = y_0\left(1 + \frac{X}{(X - a_l)(X - b_l)}\right)$$

with a_l, b_l *given by*

$$a_1 = -\frac{e^{-\sigma}}{(e^{-\sigma} - 1)(e^{-3\sigma} - 1)}\,, \quad b_1 = -\frac{e^{-3\sigma}}{(e^{-\sigma} - 1)(e^{-3\sigma} - 1)}\,,$$
$$a_2 = -\frac{1}{(e^{-\sigma} + 1)(e^{-\sigma} + e^{\sigma} - 2)}\,, \quad b_2 = -\frac{e^{-\sigma}}{(e^{-\sigma} + 1)(e^{-\sigma} + e^{\sigma} - 2)}\,,$$
$$a_3 = b_3 = -\frac{1}{e^{-\sigma} + e^{\sigma} - 2}\,;$$

(2) *for* $l = 4$,

$$F_4(y_0; X) = \begin{cases} y_0\left(1 + \dfrac{X}{(X - a_4)(X - b_4)}\right) & \text{if } y_0 \neq 0\,, \\ \dfrac{X}{(X - a_4^0)(X - b_4^0)} & \text{if } y_0 = 0\,, \end{cases}$$

with a_4, b_4, a_4^0, b_4^0 *given by*

$$a_4 + b_4 = \frac{2Y_0}{e^{-\sigma} + e^{\sigma} - 2}\,, \quad a_4 b_4 = \frac{Y_0(4Y_0 + 2 - e^{-\sigma} - e^{\sigma})}{(e^{-\sigma} + e^{\sigma} - 2)(e^{-2\sigma} + e^{2\sigma} - 2)}\,, \quad Y_0 = \frac{y_0^2}{1 - y_0^2}\,,$$
$$a_4^0 = -\frac{1}{e^{-\sigma} - e^{\sigma}}\,, \quad b_4^0 = \frac{1}{e^{-\sigma} - e^{\sigma}}\,.$$

REMARK 2.2. These exponential type solutions and the rational solutions in Theorem 2.3 appear to be degenerate cases of exact solutions expressed in terms of elliptic functions (cf. [2, 8, 9]).

3. Proofs of Theorems 2.1 and 2.2

3.1. Proof of Theorem 2.2. Suppose that $\alpha \neq 0$. Let $\phi(x)$ be a rational solution of (1.4). Supposing that $\phi(x) = C_0 x^m + O(x^{m-1})$ around $x = \infty$, where $C_0 \neq 0$, $m \in \mathbf{Z}$, and substituting $y(x) = \phi(x)$ into (1.4), we easily have $m = -1$ and $\alpha C_0 + \gamma = 0$. This fact implies the following:

LEMMA 3.1. *Any rational solution $\phi(x)$ of* (1.4) *satisfies $\phi(x) = -(\gamma/\alpha)x^{-1} + O(x^{-2})$ around $x = \infty$. In particular, if $\gamma = 0$, then* (1.4) *admits the unique rational solution $y(x) \equiv 0$.*

Furthermore we have

LEMMA 3.2. *If $\phi(x_0 - 1) = \infty$ and if $\phi(x_0 - 3) \neq \infty$, then $x = x_0 + 1$ is not a pole of $\phi(x)$.*

PROOF. This lemma is nothing but the singularity confinement property. Supposing $\phi(x_0 - 1) = \infty$ and $\phi(x_0 - 3) \neq \infty$, from (1.4) with $y(x) = \phi(x_0 - 2 + t)$ we derive

$$\phi(x_0 - 2 + t) = \pm 1 + \varepsilon(t)$$

and

$$\phi(x_0 - 1 + t) = -\frac{g(t)}{2}\varepsilon(t)^{-1} + O(1)\,, \tag{3.1}$$

$$g(t) := \alpha(x_0 - 2 + t) + \beta \pm \gamma\,,$$

where $\varepsilon(t) = O(t)$ as $t \to 0$. Similarly we have

$$\phi(x_0 + t) = \mp 1 + \frac{\alpha(x_0 + t) + \beta \mp \gamma}{g(t)}\varepsilon(t) + O(\varepsilon(t)^2)\,. \tag{3.2}$$

From (3.1) and (3.2) it follows that $\phi(x_0 + 1 + t) = O(1)$ as $t \to 0$, which implies the lemma. □

Suppose that $\gamma \neq 0$, and that (1.4) admits a rational solution written in the form $\phi(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are relatively prime. By Lemma 3.1, we may write

$$P(x) = -(\gamma/\alpha)x^{q-1} + b_1 x^{q-2} + \cdots\,,$$

$$Q(x) = x^q + c_1 x^{q-1} + c_2 x^{q-2} + \cdots$$

with $q = \deg Q \in \mathbf{N}$. Substituting $P(x)/Q(x)$ into (1.4), we have

$$\frac{P(x+1)Q(x-1) - P(x-1)Q(x+1)}{Q(x-1)Q(x+1)} = \frac{Q(x)((\alpha x + \beta)P(x) + \gamma Q(x))}{Q(x)^2 - P(x)^2}\,.$$

Then the numerator and the denominator on the left-hand side are relatively prime. Indeed, if this is not the case, there exists x_0 satisfying $Q(x_0 - 1) = Q(x_0 + 1) = 0$ and $\phi(x_0 - 3) \neq \infty$, which contradicts Lemma 3.2. Comparing the degrees of denominators on both sides, we have

$$Q(x-1)Q(x+1) = Q(x)^2 - P(x)^2 ,$$

which yields

$$(2c_2 + c_1^2 - q)x^{2q-2} + \cdots = (2c_2 + c_1^2 - \gamma^2/\alpha^2)x^{2q-2} + \cdots ,$$

and hence $\gamma^2/\alpha^2 = q$. Let $y_\gamma(x)$ be a solution of (1.4). Recall the auto-Bäcklund transformations with respect to the parameter γ:

$$y_{\gamma+\alpha}(x) = T_\gamma^+ y_\gamma(x) := -y_\gamma(x) + \frac{(2\gamma+\alpha)(1+y_\gamma(x))}{2(y_\gamma(x)+1)(1-y_\gamma(x+1)) - \alpha x - \beta + \gamma} ,$$

$$y_{-\gamma}(x) = S_\gamma y_\gamma(x) := -y_\gamma(x)$$

and

$$y_{\gamma-\alpha}(x) = T_\gamma^- y_\gamma(x) := (S_{-\gamma+\alpha} \circ T_{-\gamma}^+ \circ S_\gamma) y_\gamma(x)$$

(see [10, 13], and note that γ corresponds to $-a$ of [13, equation (8)]). The denominator of $T_\gamma^+\phi(x)$ does not vanish identically, since, by Lemma 3.1, it is $-\alpha x + O(1)$ near $x = \infty$. Hence $T_\gamma^+\phi(x)$ is also a rational solution of (1.4) with $\gamma+\alpha$ instead of γ. Under the supposition $\gamma + \alpha \neq 0$, by the same argument as above, we obtain $(\gamma+\alpha)^2/\alpha^2 = q' \in \mathbf{N}$. If $\gamma/\alpha = \pm\sqrt{q} \notin \mathbf{Z}$, then $(\pm\sqrt{q}+1)^2 = q + 1 \pm 2\sqrt{q} = q'$, implying that $q - 1/4 \in \mathbf{Z}$, which is a contradiction. Thus we conclude that $\gamma/\alpha = N \in \mathbf{Z}$, and that $\deg Q = \deg P + 1 = N^2$.

Conversely suppose that $\gamma/\alpha = N \in \mathbf{Z}$. If $N > 0$, starting from the seed solution $\phi_0(x) \equiv 0$ of (1.4) with $\gamma = 0$, we get the rational solution

$$\phi_N(x) = (T_{(N-1)\alpha}^+ \circ T_{(N-2)\alpha}^+ \circ \cdots \circ T_\alpha^+ \circ T_0^+)\phi_0(x)$$

of (1.4) with $\gamma = N\alpha$. Furthermore let $\phi(x)$ be a rational solution of (1.4) with $\gamma = N\alpha$, $N > 0$. By Lemma 3.1,

$$(T_\alpha^- \circ T_{2\alpha}^- \circ \cdots \circ T_{(N-1)\alpha}^- \circ T_{N\alpha}^-)\phi(x) = \phi_0(x) \equiv 0 ,$$

and hence $\phi(x) = \phi_N(x)$, since $T_{\gamma-\alpha}^+ \circ T_\gamma^- = \mathrm{id}$. This fact implies the uniqueness of the rational solution $\phi_N(x)$. The case $N < 0$ is treated in a similar way. Thus Theorem 2.2 is proved.

REMARK 3.1. The uniqueness property above also follows from the fact that the coefficients of the Laurent series expansion of $\phi(x)$ around $x = \infty$ are uniquely determined.

REMARK 3.2. For a solution of difference equations, it is not easy to know its local behaviour around a pole located in $\mathbf{C}$. In our arguments, we have used the singularity confinement property in place of the series expansion around a movable pole for continuous Painlevé equations (cf. [6]).

3.2. Proof of Theorem 2.1. Suppose that $\alpha \neq 0$. Considering the Laurent series expansion around $x = \infty$, we easily see that (1.1) and (1.2) admits no rational solutions. Similarly equation (1.3) with $\gamma = 0$ admits no rational solutions. If $\alpha\gamma \neq 0$, a rational solution of (1.3) may be written as $P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are relatively prime polynomials satisfying $\deg Q = \deg P + 1 \geq 1$. Substitution of this into (1.3) yields

$$\frac{P(x+1)Q(x-1) - P(x-1)Q(x+1)}{Q(x-1)Q(x+1)} = \frac{Q(x)((\alpha x + \beta)P(x) + \gamma Q(x))}{P(x)^2} .$$

The numerator and the denominator on the left-hand side are relatively prime, since this rational solution has the same property as in Lemma 3.2. Indeed, supposing $y(x_0 - 3) \neq \infty$ and $y(x_0 - 1) = \infty$, we have

$$\begin{aligned}
&y(x_0 - 2 + t) = \varepsilon(t) , \\
&y(x_0 - 1 + t) = \gamma\varepsilon(t)^{-2} + \alpha(x_0 - 2 + t)\varepsilon(t)^{-1} + O(1) , \\
&y(x_0 + t) = -\varepsilon(t) + (\alpha/\gamma)(x_0 - 1 + t)\varepsilon(t)^2 + O(\varepsilon(t)^3) , \\
&y(x_0 + 1 + t) = O(1) ,
\end{aligned}$$

where $\varepsilon(t) \to 0$ as $t \to 0$. Hence $\deg P^2 \geq 2 \deg Q$, which is a contradiction. This implies that (1.3) admits no rational solutions provided that $\alpha \neq 0$.

4. Proof of Theorem 2.3

To prove Theorem 2.3, we examine a fixed point of $R_l(y)/2$ $(1 \leq l \leq 4)$.

LEMMA 4.1. *Suppose that $R_l(y)$ satisfies* (2.1). *There exists a fixed point y_0 of $R_l(y)/2$ such that $R_l'(y_0) = 2$, if and only if $R_l(y)$ satisfies condition* (C.*l*) *in Theorem* 2.3. *In each case all the fixed points such that $R_l'(y_0) = 2$ are listed as follows*:

$$\begin{aligned}
&y_0^{\pm} = \pm\sqrt{-\beta/2} && \textit{for } R_1(y) \textit{ under } (C.1) ; \\
&y_0^{\pm} = \pm\sqrt{-\beta/3} && \textit{for } R_2(y) \textit{ under } (C.2) ; \\
&y_0^{\pm} = \mp\sqrt{\beta/6} && \textit{for } R_3(y) \textit{ under } (C.3) ; \\
&y_0^{\pm} = \pm\sqrt{(2-\beta)/6} && \textit{for } R_4(y) \textit{ under } (C.4) .
\end{aligned}$$

PROOF. If $R_l(y_0)/2 = y_0$ and if $R_l'(y_0) = 2$, then y_0 is at least a double zero of the function $R_l(y) - 2y$. For example, suppose that y_0 is a double zero of $R_4(y) - 2y$. Then y_0 is a zero of the polynomials $\beta y + \gamma - 2y(1 - y^2)$ and $6y^2 + (\beta - 2)$. They have a common zero if and only if $\gamma = \pm\sqrt{6}(2-\beta)^{3/2}/9$; and then $y_0 = \pm\sqrt{(2-\beta)/6}$. Noting that $\beta \pm \gamma = 0$ (cf. (2.1)) holds if and only if $\beta = -4, 1/2$, we obtain condition (C.4). In a similar way, for $R_l(y)$ $(1 \leq l \leq 3)$, we have the conditions and the corresponding fixed points. □

Suppose that $\phi(x)$ is a nontrivial rational solution of (1.4) with $\alpha = 0$. Then the Laurent series expansion around $x = \infty$ satisfies $\phi(x) = y_0 + O(x^{-1})$, where y_0 is a fixed point of $R_4(y)/2$.

Suppose that $R_4'(y_0) = 2$. Then γ satisfies (C.4) and $y_0 = y_0^{\pm}$ are as in Lemma 4.1. Let $\phi^{\pm}(x)$ denote the rational solutions corresponding to the fixed points $y_0 = y_0^{\pm}$, respectively. Then they satisfy

$$y(x+1) + y(x-1) = \frac{\beta y(x) \pm \sqrt{6}(2-\beta)^{3/2}/9}{1 - y(x)^2}.$$

Under the condition $\beta \neq 2$, set $\phi^{\pm}(x) = y_0^{\pm}(1 + \psi(x))$. Then $\psi(x)$ satisfies

$$\psi(x+1) + \psi(x-1) = \frac{2\psi(x)((2-\beta)\psi(x) + \beta + 4)}{-(2-\beta)\psi(x)^2 - 2(2-\beta)\psi(x) + \beta + 4}.$$

Supposing $\psi(x) = \lambda/(x^2 - \mu)$ and substituting this into the equation above, we obtain $\lambda = (\beta+4)/(2-\beta)$, $\mu = 3/(2(2-\beta))$. In this way we find the rational solutions $\phi_4^{\pm}(x)$. If $\beta = 2$, then (1.4) is

$$y(x+1) + y(x-1) = \frac{2y(x)}{1 - y(x)^2}.$$

Supposing $y(x) = \lambda/x$, we obtain $\phi_4^{\pm}(x) = \pm 1/x$ as in the theorem.

To show the uniqueness of $\phi_4^{\pm}(x)$ for each fixed point, we set $\phi^{\pm}(x) = y_0^{\pm} + \Psi^{\pm}(x)$ with $\Psi^{\pm}(x) = \sum_{j\geq 1} c_j^{\pm} x^{-j}$ around $x = \infty$. Since $R_4'(y_0^{\pm}) = 2$, $z(x) = \Psi^{\pm}(x)$ satisfy

$$z(x+1) + z(x-1) - 2z(x) = \sum_{j\geq 2} a_j^{\pm} z(x)^j \tag{4.1}$$

with

$$a_2^{\pm} = \frac{R_4''(y_0^{\pm})}{2} = \frac{36 y_0^{\pm}}{\beta + 4} = \frac{\pm 36\sqrt{(2-\beta)/6}}{\beta + 4},$$

where the right-hand side is convergent if $|z(x)|$ is sufficiently small. Here we note that $(y_0^{\pm})^2 - 1 \neq 0$ for $R_4(y)$, because the case $(\beta, \gamma) = (-4, \pm 4)$ is excluded by (2.1).

Suppose that $\beta \neq 2$, namely $a_2^{\pm} \neq 0$. Substitute $\Psi^{\pm}(x)$ into (4.1) and observe that

$$(x+1)^{-j} - 2x^{-j} + (x-1)^{-j} = j(j+1)x^{-j-2}\left(1 + \sum_{k\geq 1} \frac{2\,(j+2)_{2k}}{(2k+2)!} x^{-2k}\right)$$

with $(j+2)_{2k} = (j+2)(j+3)\cdots(j+2k+1)$. Comparing the coefficients of x^{-2}, x^{-3}, x^{-4} and x^{-5} on both sides of (4.1), we have $c_1^{\pm} = 0$, $6c_2^{\pm} = a_2^{\pm}(c_2^{\pm})^2$ and $12c_3^{\pm} = a_2^{\pm} \cdot 2c_2^{\pm} c_3^{\pm}$, which imply that $c_2^{\pm} = 6(a_2^{\pm})^{-1}$ and that $c_3^{\pm} = c$ is arbitrary. For $j \geq 4$, observing the coefficients of x^{-j-2} on both sides, we have the relations

$$j(j+1)c_j^{\pm} = Q_j^{\pm}(c_2^{\pm}, c_3^{\pm}, \dots, c_{j-1}^{\pm}),$$

by which $c_j^{\pm}$ $(j \geq 4)$ are uniquely determined, where $Q_j^{\pm}$ are polynomials in $c_2^{\pm}, \dots, c_{j-1}^{\pm}$. Since

$$\phi_4^{\pm}(x-c) = y_0^{\pm} + 6(a_2^{\pm})^{-1}(x^{-2} + 2cx^{-3} + \cdots),$$

this fact implies that $\phi^{\pm}(x) = y_0^{\pm} + \Psi^{\pm}(x)$ coincide with $\phi_4^{\pm}(x-c)$, respectively, up to the arbitrary constant c.

If $\beta = 2$, then $\gamma = 0$, $y_0^{\pm} = 0$, and hence $a_2^{\pm} = 0$, $a_3^{\pm} = a_3 = 2$, $a_j^{\pm} = a_j$ $(j \geq 4)$ in (4.1). Substitution of $\Psi(x) = \sum_{j\geq 1} c_j x^{-j}$ into (4.1) yields the relations $2c_1 = 2c_1^3$ and $6c_2 = 2\cdot 3c_1^2 c_2$, implying that $c_1 = \pm 1$ and that $c_2 = c$ is arbitrary. By the same argument as above, we conclude that $\phi^{\pm}(x)$ coincide with $\pm 1/(x \mp c)$.

Consider the remaining case where the fixed point y_0 satisfies $R_4'(y_0) \neq 2$. By (2.1) we have $y_0 \neq \pm 1$. Then $z(x) = \phi(x) - y_0$ satisfies

$$z(x+1) + z(x-1) - R_4'(y_0)z(x) = \sum_{j\geq 2} a_j z(x)^j$$

around $x = \infty$. Substituting $z(x) = \sum_{j\geq 1} c_j x^{-j}$ into this, we may inductively get $c_j = 0$ for every $j \geq 1$, which implies that (1.4) admits no rational solutions other than $y(x) \equiv y_0$ if $R_4'(y_0) \neq 2$. Thus we arrive at the conclusion for (1.4) of Theorem 2.3. The other equations are treated in a similar manner.

5. Proof of Theorem 2.4

Suppose that condition (C.l) is not satisfied. By Lemma 4.1, every fixed point of $R_l(y)/2$ $(1 \leq l \leq 4)$ satisfies $R_l'(y_0) \neq 2$. For $l = 4$, since the degree of the denominator of $R_4(y)$ is 2, by [3], there exists a fixed point y_0^* of $R_4(y)/2$ such that either $|R_4'(y_0^*)| > 2$ or $R_4'(y_0^*) = 2$. Then $|R_4'(y_0^*)| > 2$, and hence $R_4'(y_0^*) \neq -2$. For $l = 1$, supposing $R_1(y_0)/2 = y_0$ and $R_1'(y_0) = -1$, we have $\beta = \gamma^2$, $y_0 = \gamma$, and then there exists another fixed point $y_0^* = -\gamma/2$ such that $R_1'(y_0^*) = -4$. For $l = 2$ as well we may show the existence of y_0 such that $R_2'(y_0) \neq -2$ by direct computation or by using the result of [3]. At any rate the existence of y_0 as in Theorem 2.4 is guaranteed.

For (1.4), suppose that $y_0 \neq 0$, and put $y(x) = y_0(1 + z(x))$. Since $R_4(y_0)/2 = y_0$, we have, from (1.4), that

$$z(x+1) + z(x-1) = \frac{R_4'(y_0)z(x) + 2Y_0 z(x)^2}{1 - 2Y_0 z(x) - Y_0 z(x)^2}, \quad Y_0 = \frac{y_0^2}{1 - y_0^2}. \tag{5.1}$$

Substitute $z(x) = X(X-a_4)^{-1}(X-b_4)^{-1}$ with $X = e^{-\sigma x}$ into (5.1). Then the left-hand side is

$$\frac{(e^{-\sigma}+e^{\sigma})X^3 - 2(a_4+b_4)X^2 + (e^{-\sigma}+e^{\sigma})a_4b_4X}{(X^2-(a_4e^{-\sigma}+b_4e^{\sigma})X+a_4b_4)(X^2-(a_4e^{\sigma}+b_4e^{-\sigma})X+a_4b_4)},$$

and the right-hand side is $P_4(X)/Q_4(X)$ with

$$P_4(X) = R_4'(y_0)X^3 + (2Y_0-(a_4+b_4)R_4'(y_0))X^2 + R_4'(y_0)a_4b_4X,$$

$$Q_4(X) = X^4 - 2(a_4+b_4+Y_0)X^3 + (a_4^2+b_4^2+4a_4b_4+2Y_0(a_4+b_4)-Y_0)X^2 - 2a_4b_4(a_4+b_4+Y_0)X + a_4^2b_4^2.$$

Comparing the coefficients on both sides, we obtain the relation $e^{-\sigma}+e^{\sigma} = R_4'(y_0) \neq \pm 2$ and the desired expressions of a_4+b_4 and a_4b_4 with $e^{-\sigma} \neq \pm 1$.

Suppose that $y_0 = 0$ and that $R_4'(0) = \beta \neq \pm 2$. Substitute $y(x) = F_4(0; X) = X(X-a_4^0)^{-1}(X-b_4^0)^{-1}$ with $X = e^{-\sigma x}$ into (1.4). Since $\gamma = 0$, the right-hand side of (1.4) is $P_4^0(X)/Q_4^0(X)$ with

$$P_4^0(X) = R_4'(0)X(X^2-(a_4^0+b_4^0)X+a_4^0b_4^0),$$

$$Q_4^0(X) = X^4 - 2(a_4^0+b_4^0)X^3 + ((a_4^0)^2+(b_4^0)^2+4a_4^0b_4^0-1)X^2 - 2a_4^0b_4^0(a_4^0+b_4^0)X + (a_4^0)^2(b_4^0)^2.$$

Using this, we obtain the expressions of a_4^0 and b_4^0. The other equations are treated in a similar way.

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