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# Weighted Composition Operators on $C(X)$ and $\mathrm{Lip}_c(X, \alpha)$

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**Abstract.** Let $A$ and $B$ be subalgebras of $C(X)$ and $C(Y)$, respectively, for some topological spaces $X$ and $Y$. An arbitrary map $T : A \to B$ is said to be multiplicatively range-preserving if for every $f, g \in A$, $(fg)(X) = (TfTg)(Y)$, and $T$ is said to be separating if $TfTg = 0$ whenever $fg = 0$.

For a given metric space $X$ and $\alpha \in (0, 1]$, let $\mathrm{Lip}_c(X, \alpha)$ be the algebra of all complex-valued functions on $X$ satisfying the Lipschitz condition of order $\alpha$ on each compact subset of $X$. In this note we first investigate the general form of multiplicatively range-preserving maps from $C(X)$ onto $C(Y)$ for realcompact spaces $X$ and $Y$ (not necessarily compact or locally compact) and then we consider such preserving maps from $\mathrm{Lip}_c(X, \alpha)$ onto $\mathrm{Lip}_c(Y, \beta)$ for metric spaces $X$ and $Y$ and $\alpha, \beta \in (0, 1]$. We show that in both cases multiplicatively range-preserving maps are weighted composition operators which induce homeomorphisms between $X$ and $Y$. We also give a description of a linear separating map $T : A \to C(Y)$, where $A$ is either $C(X)$ for a normal space $X$ or $\mathrm{Lip}_c(X, \alpha)$ for a metric space $X$ and $0 < \alpha \leq 1$ and $Y$ is an arbitrary Hausdorff space.

## 1. Introduction

Given two subalgebras $A$ and $B$ of continuous functions on topological spaces $X$ and $Y$, respectively, a (not necessarily linear) map $T : A \to B$ is called *multiplicatively range-preserving* if $(fg)(X) = (TfTg)(Y)$ holds for all $f, g \in A$. For Banach algebras $A$ and $B$, a map $T : A \to B$ is said to be *multiplicatively spectrum-preserving* if $\sigma(fg) = \sigma(TfTg)$, $f, g \in A$, where $\sigma(.)$ denotes the spectrum of an element in a Banach algebra.

There is a vast literature concerning the maps, not assumed to be linear, between certain Banach algebras of functions preserving some structures such as norm, range, spectrum or particular subsets of the range and the spectrum. Multiplicatively spectrum-preserving maps were first studied by Molnár in [18]. He proved that if $X$ is a first countable compact Hausdorff space, then each surjective multiplicatively spectrum-preserving map $T$ on the supremum norm Banach algebra $C(X)$ of all continuous complex-valued functions on $X$, is "almost" automorphism; more precisely, $T$ is a weighted composition operator of the form

$$Tf(x) = h(x)f(\varphi(x)) \quad (f \in C(X), x \in X),$$

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where $h$ is a continuous function on $X$ taking its value in {-1,1} and $\varphi$ is a homeomorphism on $X$. Then in [19] Rao and Roy generalized Molnár's result to the case where $C(X)$ is replaced by a uniform algebra $A$ on a compact Hausdorff space $X$ such that $X$ is the maximal ideal space of $A$. They also extended the result to the case where $A$ is a (not necessarily unital) uniform algebra on a locally compact Hausdorff space $X$ whose maximal ideal space is the same as $X$ (see [20]). Simultaneously, in [8], Hatori, Miura and Takagi characterized the general form of surjective multiplicatively range-preserving maps between uniform algebras on compact Hausdorff spaces. They also proved in [9] that if $T$ is a multiplicatively spectrum-preserving map from a unital semisimple commutative Banach algebra $A$ onto a unital commutative Banach algebra $B$ with $T(1_A) = 1_B$, then $B$ is semisimple and $T$ is an algebra isomorphism. In [11] the authors obtained similar results for multiplicatively range-preserving maps between certain (not necessarily unital) Banach function algebras. In [17], introducing the peripheral range $\mathrm{Ran}_\pi(f) = \{z \in f(X) : |z| = \sup_{x\in X} |f(x)|\}$ of a function $f \in C(X)$, where $X$ is a compact Hausdorff space, Luttman and Tonev studied surjective maps $T : A \to B$ between unital uniform algebras $A$ and $B$ satisfying the following condition

$$\mathrm{Ran}_\pi(fg) = \mathrm{Ran}_\pi(TfTg) \quad (f, g \in A)\,.$$

Recently their results have been generalized in [10] for uniformly closed subalgebras of $C_0(X)$ for a locally compact Hausdorff space $X$. Similar results can be found in [13] and [14] for Lipschitz algebras of functions.

In the first part of this paper we consider surjective multiplicatively range-preserving maps between topological algebras $C(X)$ and $C(Y)$ for realcompact spaces $X$ and $Y$ (not necessarily compact or locally compact) and show that such preserving maps are weighted composition operators which induce homeomorphisms between $X$ and $Y$. A similar characterization will be given for multiplicatively range-preserving maps defined between (topological) algebras of continuous functions on a metric space $X$ satisfying the Lipschitz condition of order $\alpha$, for some $0 < \alpha \leq 1$, on each compact subset of $X$.

For two algebras (or spaces of functions) $A$ and $B$ a map $T : A \to B$ is called *separating* if $fg = 0$ implies $TfTg = 0$ for all $f, g \in A$ and *biseparating* if $T$ is bijective and $T^{-1}$ is separating as well. Clearly algebra homomorphisms and multiplicatively range-preserving maps are separating. Weighted composition operators on algebras of functions are important typical examples of linear separating maps. On the other hand, if $X$ and $Y$ are compact Hausdorff spaces, then any continuous linear separating map $T$ from $C(X)$ onto $C(Y)$ is a weighted composition operator of the form $(Tf)(y) = h(y)f(\varphi(y))$, $y \in Y$ and $f \in A$, where $h$ is a continuous complex-valued function on $Y$ and $\varphi : Y \to X$ is continuous, in particular, if $T$ is bijective, then $\varphi$ is a homeomorphism [12]. This result has been extended to regular Banach function algebras satisfying Ditkin's condition in [6]. The study of separating maps between various Banach algebras has attracted a considerable interest in recent years. For example, separating maps between $C^*$-algebras have been considered in [4] and biseparating maps between (vector-valued) Lipschitz functions have been studied in [2] and [15]. A more general situation would be to consider the separating maps between not necessarily normable algebras of functions. For an integer $m \geq 0$, linear separating functionals on $C^m(\Omega)$, the algebra of all $m$-times continuously differentiable complex-valued functions on an open subset $\Omega$ of $\mathbf{R}^n$, $n \in \mathbf{N}$, were studied in [16]. Moreover, additive biseparating maps from $C(X)$ onto $C(Y)$, for completely regular spaces $X$ and $Y$, were discussed in [1] and [3]. In fact, in the latter case such maps induce homeomorphisms between the real compactifications of $X$ and $Y$ [3].

There are also some results related to the automatic continuity of linear separating maps. For example, any bijective linear separating map between regular Banach function algebras satisfying Ditkin's condition is automatically continuous [6]. For more results on automatic continuity of such maps see [3, 12, 16].

In the second part of the paper we give a description of a (not necessarily bijective) linear separating map from a certain subalgebra $A$ of continuous functions on a Hausdorff space $X$ into $C(Y)$ for some Hausdorff space $Y$. The result can be applied for the case where $X$ is a normal space, $A = C(X)$ and for the case where $X$ is a metric space and $A$ is the algebra of all complex-valued functions on $X$ satisfying the Lipschitz condition of order $\alpha$, $0 < \alpha \leq 1$, on each compact subset of $X$.

## 2. Preliminaries

By a *topological algebra* we mean a complex algebra $A$ with a (Hausdorff) vector space topology making the multiplication of $A$ jointly continuous. A topological algebra whose topology can be defined by a family of submultiplicative seminorms is called a *locally multiplicatively convex algebra* (*an lmc-algebra*). A *Fréchet algebra* is an lmc-algebra $A$ whose topology is generated by a sequence $(p_n)$ of submultiplicative seminorms such that the metric induced by $(p_n)$ is complete. The set of all continuous complex-valued homomorphisms on a Fréchet algebra $A$ will be denoted by $M_A$. We always endow $M_A$ with the Gelfand topology. A unital commutative Fréchet algebra $A$ is said to be *regular* if for each closed subset $F$ of $M_A$ and a point $\varphi \in M_A \backslash F$, there exists an element $a \in A$ such that $\hat{a}(\varphi) = 1$ and $\hat{a} = 0$ on $F$, where $\hat{a}$ is the Gelfand transform of $a \in A$. We refer the reader to [7] for some classical results on Fréchet algebras.

For an arbitrary Hausdorff space $X$ we denote the algebra of all continuous complex-valued functions on $X$ by $C(X)$ and the subalgebra of $C(X)$ consisting of all bounded functions, respectively compact support functions by $C_b(X)$, respectively $C_c(X)$. For a point $x \in X$ we denote the evaluation homomorphism on $C(X)$ at this point by $\delta_x$ and for an element $f \in C_b(X)$ we denote the supremum norm of $f$ on $X$ by $\|f\|_X$.

Let $X$ be a locally compact Hausdorff space. We denote the algebra of all continuous complex-valued functions on $X$ vanishing at infinity by $C_0(X)$. A subalgebra $A$ of $C_0(X)$ is called a *function algebra* on $X$ if $A$ separates strongly the points of $X$, i.e. for each $x, z \in X$ with $x \neq z$, there exists $f \in A$ with $f(x) \neq f(z)$ and for each $x \in X$, there exists $f \in A$ with $f(x) \neq 0$. We follow [5] for the definition of a *Banach function algebra*, that is, a function algebra on $X$ which is a Banach algebra with respect to a norm. A *uniform algebra* on $X$ is a function algebra which is a closed subalgebra of $(C_0(X), \|.\|_X)$. A Banach function algebra

$A$ on a locally compact Hausdorff space $X$ is called *natural* if the maximal ideal space $M_A$ of $A$ coincides with $X$, via the evaluation homomorphisms. We note that in this case by [5, Proposition 4.1.2] we have $X \cong M_A$ through the map $x \mapsto \delta_x$.

When $X$ is compact, all Banach function algebras on $X$ are assumed to contain the constant functions.

A Banach function algebra $A$ on a locally compact Hausdorff space $X$ is said to satisfy *Ditkin's condition* if for each $\varphi \in M_A \cup \{0\}$ and $a \in A$ with $\hat{a}(\varphi) = 0$ there exists a sequence $\{a_n\}$ in $A$ such that each $\widehat{a_n}$ has compact support and vanishes on a neighborhood of $\varphi$ and, furthermore, $\|a_n a - a\| \to 0$.

For metric spaces $(X, d_1)$ and $(Y, d_2)$ and $\alpha \in (0, 1]$ we call a map $f : X \to Y$ a Lipschitz function of order $\alpha$ if $L_{X,\alpha}(f) = \sup\left\{\frac{d_2(f(x), f(y))}{d_1^{\alpha}(x,y)} : x, y \in X, x \neq y\right\}$ is finite. For $\alpha = 1$ such functions are referred to as Lipschitz functions.

If $(K, d)$ is a compact metric space and $0 < \alpha \leq 1$, then the algebra $\mathrm{Lip}(K, \alpha)$ of all complex-valued Lipschitz functions of order $\alpha$ on $K$ is a natural Banach function algebra on $K$ with respect to the following Lipschitz norm

$$\|f\| = \|f\|_K + L_{K,\alpha}(f) \quad (f \in \mathrm{Lip}(K, \alpha)).$$

Moreover, for each closed subset $F$ of $K$ and open neighborhood $U$ of $F$ there exists a function $f \in \mathrm{Lip}(K, \alpha)$, such that $0 \leq f \leq 1$, $f|_F = 1$ and $f|_{K \setminus U} = 0$. In particular, $\mathrm{Lip}(K, \alpha)$ is a regular Banach function algebra.

Let $(X, d)$ be an arbitrary metric space and let $0 < \alpha \leq 1$. We define $\mathrm{Lip}_c(X, \alpha)$ as the algebra of all complex-valued functions on $X$ which satisfy the Lipschitz condition of order $\alpha$ on each compact subset of $X$, i.e. for each compact subset $K$ of $X$, $f|_K \in \mathrm{Lip}(K, \alpha)$. Since each metric space $X$ is a $k$-space, in the sense that a subset $U$ of $X$ is open whenever $U \cap K$ is open in $K$ for every compact subset of $X$, it follows that all functions in $\mathrm{Lip}_c(X, \alpha)$ are necessarily continuous on $X$. It is easy to verify that $\mathrm{Lip}_c(X, \alpha)$ is a complete lmc-algebra under the topology defined by the family $(p_K)$ of seminorms, where $K$ ranges over all compact subsets of $X$ and for each $f \in \mathrm{Lip}_c(X, \alpha)$, $p_K(f)$ is the Lipschitz norm of $f|_K$ in the Banach algebra $\mathrm{Lip}(K, \alpha)$.

For a compact metric space $K$ and an arbitrary metric space $X$ we write $\mathrm{Lip}(K)$ and $\mathrm{Lip}_c(X)$, respectively, for $\mathrm{Lip}(K, \alpha)$ and $\mathrm{Lip}_c(X, \alpha)$ whenever $\alpha = 1$.

## 3. Multiplicatively Range-preserving Maps on $C(X)$ and $\mathrm{Lip}_c(X, \alpha)$

In this section we give a description of multiplicatively range-preserving maps from $C(X)$ onto $C(Y)$ for realcompact spaces $X$ and $Y$ which are not necessarily compact or locally compact, and from $\mathrm{Lip}_c(X, \alpha)$ onto $\mathrm{Lip}_c(Y, \beta)$ for metric spaces $X$ and $Y$ and $\alpha, \beta \in (0, 1]$. We show that in both cases such maps are essentially weighted composition operators which induce homeomorphisms between $X$ and $Y$.

Before stating the results we prove the following simple lemma which concludes that multiplicatively range-preserving maps between certain subalgebras of continuous functions are increasing in modulus of functions in both directions (Corollary 3.3).

LEMMA 3.1. *Let $X$ be a Hausdorff space and let $A$ be a subalgebra of $C(X)$ with the property that for each $x \in X$ and each neighborhood $V$ of $x$, there exists a bounded function $f \in A$ such that $|f(x)| = 1 = \|f\|_X$ and $f = 0$ on $X \setminus V$. Then for $f, g \in A$, $|f| \leqslant |g|$ if and only if for every $c \geqslant 0$ and $h \in A$, $|gh| \leqslant c$ implies $|fh| \leqslant c$.*

PROOF. The "only if" part is trivial. For the converse assume, on the contrary, that there exists $x_0 \in X$ such that $|f(x_0)| > |g(x_0)|$. Set $\gamma = \frac{1}{2}(|f(x_0)| + |g(x_0)|)$, then $|g(x_0)| < \gamma < |f(x_0)|$ and hence there exists a neighborhood $V$ of $x_0$ such that $|g(x)| < \gamma$ on $V$. Now, by the hypothesis, we can find a bounded function $h \in A$ such that $|h(x_0)| = 1 = \|h\|_X$ and $h = 0$ on $X \setminus V$. Thus $|gh| \leq \gamma$ on $X$ while $|fh(x_0)| > \gamma$. □

We can easily deduce the following corollaries from the above lemma. It should be noted that similar results are known for uniform algebras on compact Hausdorff spaces (see Corollary 1 and Lemma 7 in [17]):

COROLLARY 3.2. *Let $X$ and $A$ be as in the preceding lemma and let $f, g \in A$. If $(fh)(X) = (gh)(X)$ for every $h \in A$, then $|f| = |g|$.*

COROLLARY 3.3. *Assume that $A$ and $B$ are subalgebras of continuous functions on Hausdorff spaces $X$ and $Y$, respectively, having the property stated in Lemma 3.1. If $T : A \to B$ is a surjective multiplicatively range-preserving map, then for $f, g \in A$, $|f| \leq |g|$ if and only if $|Tf| \leq |Tg|$.*

THEOREM 3.4. *Let $X$ and $Y$ be realcompact spaces and let $T : C(X) \to C(Y)$ be a surjective multiplicatively range-preserving map. Then there exists a homeomorphism $\varphi$ from $Y$ onto $X$ such that*

$$(Tf)(y) = (T1)(y)f(\varphi(y)) \quad (f \in C(X), y \in Y).$$

PROOF. Since, by assumption, $T$ is multiplicatively range-preserving, $(T1)(Y) \subseteq \{-1, 1\}$ and the restriction $\widetilde{T}$ of $T$ to $C_b(X)$ maps $C_b(X)$ onto $C_b(Y)$. Now the density of $X$ and $Y$ in their Stone-Čech compactifications $\beta X$ and $\beta Y$ implies easily that $\widetilde{T}$ is a multiplicatively spectrum-preserving map from $C(\beta X)$ onto $C(\beta Y)$. Thus $(\widetilde{T}1)(\beta Y) \subseteq \{-1, 1\}$ and by [9, Theorem 3.2] there exists a homeomorphism $\varphi : \beta Y \to \beta X$ such that

$$(Tf)(y) = (\widetilde{T}f)(y) = (T1)(y)f(\varphi(y)) \quad (f \in C_b(X), y \in \beta Y) \tag{1}$$

We first show that $\varphi(Y) \subseteq X$. Let $y \in Y$ and assume, on the contrary, that $\varphi(y) \in \beta X \setminus X$. Then by [21, P. 81] there is a function $f_0 \in C_b(X)$ with $f_0(\varphi(y)) = 1$ and $|f_0| < 1$ on $X$, which is impossible, since $|f_0|(X) = |Tf_0|(Y) = |f_0 \circ \varphi|(Y)$, by (1). This concludes that $\varphi(Y) \subseteq X$. It is now simple to observe that $\varphi$ is a homeomorphism from $Y$ onto $X$.

We now claim that (1) holds for every $f \in C(X)$ and $y \in Y$. Since $(T1)(Y) \subseteq \{1, -1\}$ and $f \mapsto T1\,Tf$ defines a multiplicatively range-preserving map from $C(X)$ onto $C(Y)$, we can assume, without loss of generality, that $T(1) = 1$. So rewriting (1) we have

$$(Tf)(y) = f(\varphi(y)) \quad (f \in C_b(X), y \in \beta Y) \tag{2}$$

Let $y \in Y$, $f \in C(X)$ and take $a = f(\varphi(y))$ and $b = (Tf)(y)$. We shall show that $a = b$. If $a = 0$, then for an arbitrary $\varepsilon > 0$ set $V = \{x \in X : |f(x)| < \varepsilon\}$. Then $V$ is a neighborhood of $\varphi(y)$. Since $X$ is completely regular, we can take $g \in C_b(X)$ with $g(\varphi(y)) = 1 = \|g\|_X$ and $g = 0$ on $X \setminus V$. Obviously $fg \in C_b(X)$ and $\|fg\|_X < \varepsilon$, hence $TfTg \in C_b(Y)$ with $\|TfTg\|_Y < \varepsilon$. Since $(Tg)(y) = g(\varphi(y)) = 1$, by (2), we get

$$|(Tf)(y)| = |(Tf)(y)\,(Tg)(y)| \leq \|Tf\,Tg\|_Y < \varepsilon\,,$$

which implies that $b = (Tf)(y) = 0 = a$ as $\varepsilon > 0$ was arbitrary. So we may assume that $a \neq 0$. Set $x = \varphi(y)$ and let $V$ be an arbitrary neighborhood of $x$ in $X$. Choose $g \in C_b(X)$ with $g(x) = 1 = \|g\|_X$ and $g = 0$ on $X \setminus V$. Then there exists $h \in C_b(X)$ such that $Th = \min(|Tf|, |(Tf)(y)|)$. Since $(Tg)(y) = 1$, by (2), it follows that

$$|(Tf)(y)| = |(Tf)(y)\,(Tg)(y)| = |(Th)(y)\,(Tg)(y)| \leq \|Th\,Tg\|_Y = \|hg\|_X\,.$$

Hence there exists a point $x_0 \in V$ such that $|(Tf)(y)| \leq |hg(x_0)| \leq |h(x_0)|$. Since $V$ is an arbitrary neighborhood of $x$ and $h$ is continuous, we conclude that $|(Tf)(y)| \leq |h(x)|$. On the other hand, $|Th| \leq |Tf|$ on $Y$ and so by Corollary 3.3, $|h| \leq |f|$ on $X$. Hence $|(Tf)(y)| \leq |h(x)| \leq |f(x)| = |f(\varphi(y))|$, that is $|b| \leq |a|$. A similar argument implies the other inequality, therefore $|a| = |b|$. Now consider the closed subsets $F_0 = \{z \in X : |f(z) - a| \geq |a|/2\}$ and

$$F_n = \left\{z \in X : \frac{|a|}{2^{n+1}} \leq |f(z) - a| \leq \frac{|a|}{2^n}\right\} \quad (n \in \mathbf{N})$$

of $X$ (the idea of considering such subsets comes from [8]). Then for each $i \geq 0$ there exists a positive function $u_i$ in $C_b(X)$ such that $u_i(x) = 1 = \|u_i\|_X$ and $u_i = 0$ on $F_i$. Clearly the series $u_0 \sum_{i=1}^{\infty} \frac{u_i}{2^i}$ converges uniformly on $X$ to a function $u \in C_b(X)$. Obviously for every $z \in F_0$, $fu(z) = 0$. If $z \in F_n$, for some $n \geq 1$, then a simple calculation shows that $|fu(z)| < |a|$ and if $z \in X \setminus \bigcup_{n=0}^{\infty} F_n$, then $f(z) = a$. Therefore, for every $z \in X$ either $fu(z) = a$ or $|fu(z)| < |a|$. Consequently, $(fu)(X) \subseteq \{\lambda \in \mathbf{C} : |\lambda| < |a|\} \cup \{a\}$. Now since $(Tu)(y) = 1$, by (2), it follows that

$$b = (Tf)(y) = (Tf)(y)\,(Tu)(y) \in (fu)(X)\,,$$

and consequently $b = a$, as $|b| = |a|$. □

In the next theorem we prove the above result for multiplicatively range-preserving maps from $\mathrm{Lip}_c(X, \alpha)$ onto $\mathrm{Lip}_c(Y, \beta)$, where $X$ and $Y$ are metric spaces and $\alpha, \beta \in (0, 1]$. The argument in this case is different from the above and comes essentially from [18, Theorem 5].

Before stating the theorem we prove the following proposition which will be used in the proof of Theorem 3.6.

PROPOSITION 3.5. *Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces and $\alpha,\ \beta \in (0, 1]$. Let $T : \mathrm{Lip}_c(X, \alpha) \to \mathrm{Lip}_c(Y, \beta)$ be a weighted composition operator of the form $(Tf)(y) = h(y)f(\varphi(y))$, $f \in \mathrm{Lip}_c(X, \alpha)$, $y \in Y$, where h is a non-vanishing continuous complex-valued function on Y and $\varphi$ is a continuous function from Y into X. Then $\varphi$ satisfies the Lipschitz condition of order $\beta$ on each compact subset of Y.*

PROOF. We first note that since $h = T1 \in \mathrm{Lip}_c(Y, \beta)$ is non-vanishing, $\frac{1}{h} \in \mathrm{Lip}_c(Y, \beta)$. Hence for each $f \in \mathrm{Lip}_c(X, \alpha)$, $f \circ \varphi \in \mathrm{Lip}_c(Y, \beta)$. Let $K$ be a compact subset of $Y$, $y_0$ be a fixed point of $K$ and $H = \varphi(K)$. Since $\mathrm{Lip}(H) \subseteq \mathrm{Lip}(H, \alpha)$ and each function $f \in \mathrm{Lip}(H, \alpha)$ can be extended to a bounded function $\widetilde{f}$ on $X$ satisfying the Lipschitz condition of order $\alpha$ on the whole $X$, we can define a linear map $T_K : \mathrm{Lip}(H) \to \mathrm{Lip}(K, \beta)$ by $T_K(f) = (\widetilde{f} \circ \varphi)|_K$. Clearly $T_K$ is well-defined and continuous by the Closed Graph theorem. Let $t = \|T_K\|$, then for every $f \in \mathrm{Lip}(H)$, $\|T_K(f)\| \le t\|f\|$. Since for every pairs $y_1,\ y_2$ of points of $K$, the function $f$ defined on $H$ by

$$f(x) = d_1(x, \varphi(y_1)) - d_1(\varphi(y_1), \varphi(y_0)) \quad (x \in H)$$

is an element of $\mathrm{Lip}(H)$ with $f(\varphi(y_0)) = 0$ and $L_{H,1}(f) \le 1$ and, furthermore, $L_{H,1}(f) > 0$ whenever $\varphi(y_1) \ne \varphi(y_2)$ it follows easily that

$$d_1(\varphi(y_1), \varphi(y_2)) = \sup\left\{\frac{|f(\varphi(y_1)) - f(\varphi(y_2))|}{L_{H,1}(f)} : f \in \mathrm{Lip}(H), L_{H,1}(f) \ne 0, f(\varphi(y_0)) = 0\right\}.$$

There is another norm $\|f\|' = \max(L_{H,1}(f), |f(\varphi(y_0))|)$, $f \in \mathrm{Lip(H)}$, on $\mathrm{Lip}(H)$ which is equivalent to the norm of $\mathrm{Lip}(H)$, and so there exists a positive scalar $s$ such that $\|f\| \le s\|f\|'$ for all $f \in \mathrm{Lip}(H)$. Hence

$$d_1(\varphi(y_1), \varphi(y_2)) \le \sup\left\{ts\frac{|g(y_1) - g(y_2)|}{L_{K,\beta}(g)} : g \in T_K(\mathrm{Lip}(H)),\ L_{K,\beta}(g) \ne 0,\ g(y_0) = 0\right\}$$
$$\le t s d_2^{\beta}(y_1, y_2)\,.$$

Therefore, $\sup_{y_1,y_2 \in K} \frac{d_1(\varphi(y_1),\varphi(y_2))}{d_2^{\beta}(y_1,y_2)} \le ts$, that is, $\varphi$ satisfies the Lipschitz condition of order $\beta$ on $K$. □

We note that for each metric space $(X, d)$ and $0 < \alpha \le 1$, $\mathrm{Lip}_c(X, \alpha)$ satisfies the hypothesis of Lemma 3.1. Indeed, for each $x_0 \in X$ and any neighborhood $V$ of $x_0$, $h(x) = 1 - \max\left(0, 1 - \frac{d^\alpha(x, X\backslash V)}{d^\alpha(x_0, X\backslash V)}\right)$, $x \in X$, defines an element of $\mathrm{Lip}_c(X, \alpha)$ with $h(x_0) = 1 = \|h\|_X$ and $h = 0$ on $X\backslash V$.

THEOREM 3.6. *Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces, let $\alpha,\ \beta \in (0, 1]$ and let $T : \mathrm{Lip}_c(X, \alpha) \to \mathrm{Lip}_c(Y, \beta)$ be a surjective multiplicatively range-preserving map. Then there is a homeomorphism $\varphi$ from Y onto X such that $\varphi$ (respectively $\varphi^{-1}$) is a Lipschitz function* *of order $\beta$ (respectively $\alpha$) on each compact subset and for each $y \in Y$ and $f \in Lip_c(X, \alpha)$,*

$$(Tf)(y) = (T1)(y)\, f(\varphi(y))\,.$$

PROOF. Since $(T1)^2 = 1$ and the map $\widetilde{T} : \mathrm{Lip}_c(X, \alpha) \to \mathrm{Lip}_c(Y, \beta)$ defined by $\widetilde{T}(f) = T1\, Tf$ is a surjective multiplicatively range-preserving map, we can assume, without loss of generality, that $T(1) = 1$.

We follow the same argument as in the proof of [18, Theorem 5] to show that $T$ is injective and homogeneous.

To prove the injectivity of $T$, suppose that $f, g \in \mathrm{Lip}_c(X, \alpha)$ such that $Tf = Tg$. Then for every $h \in \mathrm{Lip}_c(X, \alpha)$, $(fh)(X) = (TfTh)(Y) = (TgTh)(Y) = (gh)(X)$ and hence $|f| = |g|$, by Corollary 3.2. If there exists an $x_0 \in X$ such that $f(x_0) \neq g(x_0)$, then we can find $r > 0$ such that $|f(x_0) - g(x_0)| > r$. Let $V$ be an open neighborhood of $x_0$ such that $|f(x) - f(x_0)| < r$ holds on $V$ and let $h(x) = 1 - \max\big(0, 1 - \frac{d_1^{\alpha}(x, X\backslash V)}{d_1^{\alpha}(x_0, X\backslash V)}\big)$, $x \in X$. Then $h$ is an element of $\mathrm{Lip}_c(X, \alpha)$ and we observe that $(fh)(X)$ is contained in the product $[0, 1]D$, where $D = \{z \in \mathbf{C} : |z - f(x_0)| < r\}$ while $g(x_0)h(x_0) \notin [0, 1]D$ since $|g(x_0)| = |f(x_0)|$. This contradiction shows that $T$ is injective.

The above argument shows, in particular, that two functions $f, g \in \mathrm{Lip}_c(X, \alpha)$ are equal if and only if $(fh)(X) = (gh)(X)$, for every $0 \leq h \in \mathrm{Lip}_c(X, \alpha)$. Similar identification holds for two elements in $\mathrm{Lip}_c(Y, \beta)$.

Now let $f \in \mathrm{Lip}_c(X, \alpha)$ and $\lambda \in \mathbf{C}$. Then for every $h \in \mathrm{Lip}_c(X, \alpha)$, $((\lambda Tf)Th)(Y) = \lambda(TfTh)(Y) = \lambda(fh)(X) = ((\lambda f)h)(X) = (T(\lambda f)Th)(Y)$ and hence $T(\lambda f) = \lambda Tf$, by the above identification, i.e., $T$ is homogeneous.

To any point $y$ in $Y$, there exists a function $h_y$ in $\mathrm{Lip}_c(Y, \beta)$ such that $0 \leq h_y \leq 1$, $h_y(y) = 1$ and $h_y(z) < 1$ for all $z \neq y$; for example, $h_y(z) = \max(0, 1 - d_2^{\beta}(z, y))$, $z \in Y$, satisfies the requirements. We claim that for each $y \in Y$ and each element $h_y$ in $\mathrm{Lip}_c(Y, \beta)$ with the above mentioned properties, $T^{-1}(h_y)$ takes the value 1 exactly at one point in $X$, independent of the choice of $h_y$. We first note that since $T^{-1}(h_y)(X) = h_y(Y)$, the range of $T^{-1}(h_y)$ contains the value 1. Suppose now that there exist two different points $x_1, x_2 \in X$ such that $T^{-1}(h_y)(x_1) = T^{-1}(h_y)(x_2) = 1$. Let $V_1$ and $V_2$ be disjoint neighborhoods of $x_1$ and $x_2$ in $X$. Then, as before, we can find elements $f_1, f_2$ in $\mathrm{Lip}_c(X, \alpha)$ such that $0 \leq f_i \leq 1$, $f_i(x_i) = 1$ and $f_i = 0$ on $X\backslash V_i$, $i = 1, 2$. Clearly $f_1 f_2 = 0$ and replacing $f_i$ by $\min(f_i, T^{-1}(h_y))$, for $i = 1, 2$, we can assume that $f_1 + f_2 \leq T^{-1}(h_y)$. Let $g_i = Tf_i$, $i = 1, 2$, then $g_i \leq h_y$ by Corollary 3.3. Since $T$ is multiplicatively range-preserving, $g_1 g_2 = 0$ and therefore $g_1 + g_2 \leq h_y$. Since $g_i(Y) = (Tf_i)(Y) = f_i(X)$, there exists $y_i \in Y$ such that $g_i(y_i) = 1$. Then $y_1 \neq y_2$ by $g_1 g_2 = 0$, and consequently $h_y(y_1) = 1 = h_y(y_2)$, a contradiction. We now claim that for each $y \in Y$ the unique point in $X$ at which $T^{-1}(h_y)$ attains its maximum value 1 is independent of the choice of $h_y$. Let $h_y$ and $h'_y$ be two elements in $\mathrm{Lip}_c(Y, \beta)$ having the above mentioned properties. Since $\min(h_y, h'_y)$ also has the same properties, it follows that $\min(T^{-1}(h_y), T^{-1}(h'_y))$ attains its maximum value at a unique point

$x \in X$. Hence $T^{-1}(h_y)$ and $T^{-1}(h'_y)$ take their maximum value at the same point, as desired. So we can define a function $\varphi : Y \to X$ such that for each $y \in Y$, $\varphi(y)$ is the unique point in $X$ at which $T^{-1}(h_y)$ attains its maximum module, where $h_y$ is an arbitrary function in $\mathrm{Lip}_c(Y, \beta)$ such that $0 \le h_y \le 1$, $h_y(y) = 1$ and $h_y(z) < 1$ for all $z \ne y$.

Now we show that for any $0 \le f \in \mathrm{Lip}_c(X, \alpha)$,

$$Tf(y) = f(\varphi(y)) \quad (y \in Y) \tag{3}$$

Let $f \in \mathrm{Lip}_c(X, \alpha)$ be a non-negative function and let $y \in Y$. Set $a = f(\varphi(y))$ and $b = (Tf)(y)$. Considering the function $h_y$ as above we take $g_y = h_y$ if $b = 0$ and $g_y = \min\left(h_y, \frac{Tf}{b}\right)$ if $b \ne 0$. Hence $bg_y \le Tf$ and so by Corollary 3.3, $bT^{-1}(g_y) = T^{-1}(bg_y) \le f$. Since $0 \le g_y \le h_y$, $T^{-1}(g_y)(\varphi(y)) = 1$, and consequently $b \le a$. Similar argument shows that $a \le b$ and this establishes (3).

We now show that $\varphi$ is continuous. Let $y_0 \in Y$ and let $V$ be a neighborhood of $\varphi(y_0)$ in $X$. As before, we can find a function $f_{\varphi(y_0)} \in \mathrm{Lip}_c(X, \alpha)$ such that $0 \le f_{\varphi(y_0)} \le 1$, $f_{\varphi(y_0)}(\varphi(y_0)) = 1$ and $f_{\varphi(y_0)} = 0$ on $X \setminus V$. Hence $W = \{y \in Y : (Tf_{\varphi(y_0)})(y) > 1/2\}$ is a neighborhood of $y_0$ with $\varphi(W) \subseteq V$, by (3), that is, $\varphi$ is continuous. Since our conditions are symmetric with respect to $T$ and $T^{-1}$, there exists a continuous map $\psi$ from $X$ into $Y$ with the same properties as $\varphi$. Thus $f(x) = (Tf)(\psi(x))$ for all $x \in X$ and $0 \le f \in \mathrm{Lip}_c(X, \alpha)$. Therefore for all $x \in X$ and $0 \le f \in \mathrm{Lip}_c(X, \alpha)$, $f(x) = f(\varphi(\psi(x)))$. Similarly $g(y) = g(\psi(\varphi(y)))$ for all $y \in Y$ and $0 \le g \in \mathrm{Lip}_c(Y, \beta)$. Hence $\psi$ is the inverse of $\varphi$, i.e., $\varphi$ is a homeomorphism.

Now if $f$ is an arbitrary element of $\mathrm{Lip}_c(X, \alpha)$, then for each non-negative function $h \in \mathrm{Lip}_c(X, \alpha)$,

$$(Tf \cdot (h \circ \varphi))(Y) = (TfTh)(Y) = (fh)(X) = (f \circ \varphi)(h \circ \varphi)(Y),$$

which concludes that $Tf = f \circ \varphi$, by the identification stated earlier.

Finally it follows, from the previous proposition, that $\varphi$ and $\varphi^{-1}$ satisfy the Lipschitz condition of order $\beta$ and $\alpha$, respectively, on the compact subsets of $Y$ and $X$, respectively. □

## 4. Linear Separating Maps on $C(X)$ and $\mathrm{Lip}_c(X, \alpha)$

As it was mentioned before, linear separating maps $T : A \to B$ between regular Banach function algebras $A$ and $B$, where $A$ satisfies the Ditkin's condition, were discussed by Font in [6]. The same proofs can be applied for the case where $A$ is a Banach function algebra satisfying the following ($\star$)-property

> ($\star$) *There exists a scalar $c$ such that for each compact subset $K$ of $X$ and each open neighborhood $U$ of $K$ there exists a function $f \in A$ with $\|f\|_X \le c$, $f = 1$ on $K$ and $f = 0$ on $X \setminus U$,*

which is slightly stronger than the regularity of $A$. However, in the next lemma we give an elementary and short proof for this case and then, using this result we give a description of linear separating maps between (not necessarily normable) algebras of type $C(X)$ and $\mathrm{Lip}_c(X,\alpha)$ for a normal, respectively, metric space $X$.

LEMMA 4.1. *Let $A$ be a Banach function algebra on a locally compact Hausdorff space $X$ having the above $(\star)$-property. Then the following statements hold.*

(i) *Any $\|\cdot\|_X$-continuous linear separating functional on $A$ is a scalar multiple of an evaluation homomorphism.*

(ii) *Let $T : A \to C(Y)$, where $Y$ is an arbitrary Hausdorff space, be a linear separating map and let $Y_c$ be the set of all points $y \in Y$ such that $\delta_y \circ T$ is nonzero and $\|\cdot\|_X$-continuous where $\delta_y$ is the evaluation functional at $y$. Then there exist continuous functions $h : Y_c \to \mathbf{C}$ and $\varphi : Y_c \to X$ such that*

$$(Tf)(y) = h(y)f(\varphi(y)) \quad (f \in A, y \in Y_c)\,.$$

PROOF. (i) Let $\psi$ be a $\|\cdot\|_X$-continuous linear separating functional on $A$. Extending $\psi$ to a continuous linear functional on $C_0(X)$ we can correspond a regular Borel measure $\mu$ to $\psi$ such that $\psi(f) = \int_X f d\mu$ for all $f \in A$. We shall show that the support of $\mu$ is a singleton. Let $F$ and $G$ be arbitrary disjoint compact subsets of $X$ and let $U$ and $V$ be disjoint compact neighborhoods of $F$ and $G$, respectively. By regularity of $\mu$ there exist decreasing sequences $\{U_n\}$ and $\{V_n\}$ of open neighborhoods of $F$ and $G$, respectively, such that for each $n$, $U_n \subseteq U$ and $|\mu|(U_n\backslash F) \le 2^{-n}$, similarly $V_n \subseteq V$ and $|\mu|(V_n\backslash G) \le 2^{-n}$. By the hypothesis, there exist sequences $\{f_n\}$ and $\{g_n\}$ in $A$ such that $\|f_n\|_X \le c$, $f_n|_F = 1$ and $f_n = 0$ outside $U_n$ and similarly $\|g_n\|_X \le c$, $g_n|_G = 1$ and $g_n = 0$ outside $V_n$, for some constant $c$. Obviously, $\psi(f_n) = \int_X f_n d\mu \to \mu(F)$ and $\psi(g_n) = \int_X g_n d\mu \to \mu(G)$. Since for each $n$, $f_n g_n = 0$ it follows that $\psi(f_n)\psi(g_n) = 0$, $n \in \mathbf{N}$, that is $\mu(F)\mu(G) = 0$. Since this result is valid for all pairs of disjoint compact subsets of $X$, the regularity of $\mu$ implies that if $F$ and $G$ are disjoint compact subsets of $X$, then $|\mu|(F)|\mu|(G) = 0$. Now assume to the contrary that there exist two distinct points $x_1$ and $x_2$ in the support of $\mu$ and choose compact neighborhoods $U$ and $V$ of $x_1$ and $x_2$, respectively whose closures $\overline{U}$ and $\overline{V}$ are disjoint. Then by the above argument either $|\mu|(\overline{U}) = 0$ or $|\mu|(\overline{V}) = 0$, which is impossible. Therefore, $\mu$ has one point in its support, as desired.

(ii) Let $Y_c = \{y \in Y : \delta_y \circ T$ is nonzero and $\|\cdot\|_X -$ continuous$\}$. Then by (i), for each $y \in Y_c$ there exist a nonzero scalar $h(y)$ and an element $\varphi(y)$ in $X$ such that $(\delta_y \circ T)(f) = h(y)f(\varphi(y))$ holds for all $f \in A$. We note that for each $y \in Y_c$ the scalar $h(y)$ and the element $\varphi(y)$ with the above property are uniquely determined. Indeed, if $h'(y) \in \mathbf{C}$ and $\varphi'(y) \in X$ satisfy the same condition, then by property $(\star)$ we can find a function $f \in A$ such that $f(\varphi(y)) = 1 = f(\varphi'(y))$. Hence $h(y) = (Tf)(y) = h'(y)$ which concludes easily that $\varphi(y) = \varphi'(y)$. Thus $\varphi : Y_c \to X$ and $h : Y_c \to \mathbf{C}$ are well-defined.

We shall show that the functions $\varphi$ and $h$ obtained in this way are continuous. Let $y_0 \in Y_c$ and let $U$ be an open neighborhood of $\varphi(y_0)$ in $X$. Consider, by property $(\star)$, a function $f \in A$ whose cozero set $\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$ is contained in $U$ and $f(\varphi(y_0)) \neq 0$. Clearly $(Tf)(y_0) \neq 0$ and $\mathrm{coz}(Tf) \cap Y_c$ is an open neighborhood of $y_0$ in $Y_c$

such that $\varphi(\mathrm{coz}(Tf) \cap Y_c) \subseteq \mathrm{coz}(f) \subseteq U$, that is $\varphi$ is continuous. It is now simple to observe that $h$ is continuous as well. □

REMARK. It is easy to see that the set $Y_c$ defined in the above theorem is the largest subset of $Y$ for which there exist continuous functions $h : Y_c \to \mathbf{C}$ and $\varphi : Y_c \to X$ such that $(Tf)(y) = h(y) f(\varphi(y))$ for all $y \in Y_c$ and $f \in A$.

Let $X$ be a Hausdorff space and for each compact subset $K$ of $X$ let $(A_K, p_K)$ be a Banach function algebra on $K$ such that for all pairs of compact subsets $K$ and $K'$ of $X$ with $K' \supseteq K$, we have $A_{K'}|_K = A_K$ and $p_K(f|_K) \le p_{K'}(f)$ for all $f \in A_{K'}$. Then the algebra $A = \{f \in C(X) : f|_K \in A_K,$ for each compact subset $K\}$ is an lmc-algebra with respect to the family $(\tilde{p}_K)_K$ of seminorms defined by $\tilde{p}_K(f) = p_K(f|_K)$, $f \in A$, where $K$ ranges over all compact subsets of $X$. For simplicity we use the same notation $p_K$ instead of $\tilde{p}_K$. The algebra $C(X)$, for an arbitrary Hausdorff space $X$, endowed with the compact-open topology and the algebra $\mathrm{Lip}_c(X, \alpha)$, for a metric space $X$ and $0 < \alpha \le 1$, with the topology defined earlier, are examples of lmc-algebras which can be expressed in this way.

In the next theorem, we give a description of linear separating maps defined either on $C(X)$, for a normal space $X$ or on $\mathrm{Lip}_c(X, \alpha)$, for a metric space $X$ and $0 < \alpha \le 1$. However the main part of the proof (except the continuity of $\varphi$) is valid for all lmc-algebras $A$ defined as above whenever for each compact subset $K$ of $X$, $A|_K = A_K$, $A_K$ is regular, closed under conjugation and $\mathrm{Re}(A_K)$ is closed under maximum.

THEOREM 4.2. *Let $X$ be a normal space, $Y$ be a Hausdorff space and $A = C(X)$. If $T : A \to C(Y)$ is a linear separating map, then there exists a continuous map $\varphi : Y_c \to X$, where $Y_c$ consists of all points $y \in Y$ such that $\delta_y \circ T$ is nonzero and continuous with respect to the compact-open topology, such that $(Tf)(y) = (T1)(y) f(\varphi(y))$, for all $y \in Y_c$ and $f \in A$. The same conclusion holds when $A = \mathrm{Lip}_c(X, \alpha)$, for a metric space $X$ and $0 < \alpha \le 1$.*

PROOF. We prove both cases simultaneously. Let $X$ be either a normal space or a metric space and let $0 < \alpha \le 1$. For the first case we set $A = C(X)$ and $A_K = C(K)$, for each compact subset $K$ of $X$, and for the second case we set $A = \mathrm{Lip}_c(X, \alpha)$ and $A_K = \mathrm{Lip}(K, \alpha)$ for each compact subset $K$ of $X$. Then clearly for each compact subset $K$ of $X$, $A|_K = A_K$.

We first show that each linear separating functional $\psi$ on $A$ which is continuous with respect to the compact-open topology is a scalar multiple of an evaluation homomorphism at some point of $X$. Since $\psi$ is continuous

$$|\psi(f)| \le c\|f\|_K \quad (f \in A) \tag{4}$$

holds for some constant $c$ and a compact subset $K$ of $X$. The fact that in both cases $A|_K = A_K$ together with (4) imply that the linear functional $\psi_K$ defined on $A_K$ by $\psi_K(f|_K) = \psi(f)$, $f \in A$, is well-defined and $\|\cdot\|_K$-continuous. We claim that $\psi_K$ is separating as well. Let $f, g \in A_K$ with $fg = 0$. Since $A_K$ is conjugate closed and $\mathrm{Re}(A_K)$ is closed under maximum, $f$ and $g$ can be decomposed as $f = f_1 - f_2 + i(f_3 - f_4)$ and $g = g_1 - g_2 + i(g_3 - g_4)$

where $f_i, g_i, i = 1, \ldots, 4$, are positive functions in $A_K$ with $f_1 f_2 = f_3 f_4 = 0$ and $g_1 g_2 = g_3 g_4 = 0$. It is easy to see that $f_i g_j = 0$, $i, j = 1, \ldots, 4$. So without loss of generality we can assume that $f$ and $g$ are positive functions in $A_K$. We can now choose a real function $\tilde{h}$ in $A$ such that $\tilde{h}|_K = f - g$. Then it is easy to see that the functions $\tilde{f} = \max(\tilde{h}, 0)$ and $\tilde{g} = \max(-\tilde{h}, 0)$ as elements of $A$ are extensions of $f$ and $g$, respectively such that $\tilde{f}\tilde{g} = 0$. Hence $\psi_K(f)\psi_K(g) = \psi(\tilde{f})\psi(\tilde{g}) = 0$, i.e., $\psi_K$ is separating. Now since $A_K$ is a regular Banach function algebra on $K$ which is closed under conjugation and $\mathrm{Re}(A_K)$ is closed under maximum, it follows that $A_K$ has ($\star$)-property and so by the preceding lemma $\psi_K$ is a scalar multiple of an evaluation homomorphism on $A|_K = A_K$, i.e., $\psi_K = \alpha\delta_{x_K}$, for some scalar $\alpha$ and $x_K \in K$, that is $\psi(f) = \alpha\delta_{x_K}(f)$ for all $f \in A$.

We now pass to the general case. Let $T : A \to C(Y)$ be a linear separating map and let $Y_c$ be the set of all points $y \in Y$ such that $\delta_y \circ T$ is a nonzero linear functional continuous with respect to the compact-open topology. By the above argument for each $y \in Y_c$, there exist a nonzero scalar $h(y)$ and an element $\varphi(y)$ in $X$ with $(\delta_y \circ T)(f) = h(y)f(\varphi(y))$ for all $f \in A$.

Continuity of the function $h : Y_c \to \mathbf{C}$ is obvious, since $h$ is, indeed, the restriction of $T1$ to $Y_c$. As in the proof of Lemma 4.1(ii) we can show that the function $\varphi : Y_c \to X$ is also continuous. □

REMARK. One can apply the proof of Proposition 3.5 to show that for metric spaces $X$ and $Y$ and $\alpha, \beta \in (0, 1]$ and for any linear separating map $T$ from $\mathrm{Lip}_c(X, \alpha)$ into $\mathrm{Lip}_c(Y, \beta)$, the map $\varphi$ given in the previous theorem satisfies the Lipschitz condition of order $\beta$ on each compact subset of $Y_c$.

It should be noted that, in general, a separating map need not be continuous. Indeed in [12] it was shown that for any infinite compact Hausdorff space $X$, there is a discontinuous linear separating functional on $C(X)$. In the following we extend Proposition 5 in [16] concerning the existence of a discontinuous linear separating functional $\varphi$ on $C_b(\Omega)$, where $\Omega$ is an open subset of $\mathbb{R}^n$, $n \in \mathbf{N}$, such that $\varphi(1) = 1$ and $\varphi = 0$ on $C_c(\Omega)$ to certain subalgebras of $C_b(X)$, where $X$ is a locally compact $\sigma$-compact Hausdorff space.

Let $X$ be a locally compact $\sigma$-compact space which is not compact. For each compact subset $K$ of $X$ let $(A_K, p_K)$ be a natural Banach function algebra on $K$ such that the family $\{(A_K, p_K) : K \subseteq X \text{ is compact}\}$ satisfies the requirements stated before Theorem 4.2, i.e., for any pair $K$ and $K'$ of compact subsets of $X$ with $K' \supseteq K$, $A_{K'}|_K = A_K$ and $p_K(f|_K) \leq p_{K'}(f)$, $f \in A_{K'}$. Let $A = \{f \in C(X) : f|_K \in A_K, \text{ for each compact subset } K\}$. Since $X$ is locally compact and $\sigma$-compact there is a sequence $\{K_n\}$ of compact subsets of $X$ such that $X = \bigcup K_n$ and for each $n \in \mathbf{N}$, $K_n$ is contained in the interior $\mathrm{int}(K_{n+1})$ of $K_{n+1}$. Hence each compact subset of $X$ is contained in some $K_n$ and so in this case $A = \{f \in C(X) : f|_{K_n} \in A_{K_n}, n \in \mathbf{N}\}$. In particular, $A$ is a Fréchet algebra under the topology defined earlier.

THEOREM 4.3. *Let $X$ be a locally compact $\sigma$-compact space which is not compact and let $A$ be as above. Assume, in addition, that for each compact subset $K$ of $X$ and open* *neighborhood $U$ of $K$ there exists a function $f \in A$ such that $f|_K = 1$, $f = 0$ on $X \backslash U$. Then there exists a discontinuous linear separating functional $\varphi$ on $A_b = A \cap C_b(X)$ (endowed with the relative topology inherited from $A$) such that $\varphi(1) = 1$ and $\varphi = 0$ on $A_c = A \cap C_c(X)$.*

PROOF. Let the sequence $\{K_n\}$ of compact subsets of $X$ be chosen as above. We first establish the theorem for the case where $A = C(X)$, equipped with the compact-open topology. Let $x \in \beta\mathbf{N}\backslash\mathbf{N}$ and consider the subspace $V = \{a \in l^\infty : x \notin \text{supp}(\hat{a})\}$ of $l^\infty$. Then clearly $e = (1, 1, \dots) \notin V$ and $u = \left(1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^{n-1}}, \dots\right) \notin V$. Since $X$ is not compact we can choose a sequence $\{x_n\}$ in $X$ such that $x_n \in K_n \backslash K_{n-1}$, for all $n \geq 2$. Clearly the set $\{x_n : n \in \mathbf{N}\}$, where $x_1$ is an arbitrary element of $K_1$, has no limit point. Now choose a sequence $\{f_n\}$ in $C_0(X)$ with $0 \leq f_n \leq 1/2^n$, $f_n|_{K_n} = 1/2^n$ and $f_n = 0$ outside $\text{int}(K_{n+1})\backslash\{x_{n+1}\}$. Set $f_0 = \sum_{n=1}^{\infty} f_n$. Then $f_0 \in C_0(X)$ and $(f_0(x_1), f_0(x_2), \dots) = u$. Since $\{x_n : n \in \mathbf{N}\}$ is contained in $\text{supp}(f_0)$, it follows that $\text{supp}(f_0)$ is not compact. Let $\varphi$ be a linear functional on $l^\infty$ such that $\varphi|_V = 0$, $\varphi(e) = 1$ and $\varphi(u) \neq 0$ and let $S$ be the linear map from $C_b(X)$ into $l^\infty$ defined by $S(f) = (f(x_1), f(x_2), \dots)$, $f \in C_b(X)$. We claim that for all $f \in C_c(X)$, $S(f) \in V$. For suppose that $f \in C_c(X)$, then $\text{supp}(f) \subseteq K_{N_0}$ for some $N_0$. Hence $f(x_n) = 0$, for all $n \geq N_0 + 1$, that is the set $\{n \in \mathbf{N} : f(x_n) \neq 0\}$ is finite. Therefore, $\text{supp}(\widehat{Sf})$ is a finite subset of $\mathbf{N}$ and hence $x \notin \text{supp}(\widehat{Sf})$, i.e., $Sf \in V$. We can easily verify that the linear functional $T = \varphi \circ S$ is a discontinuous linear separating functional on $C_b(X)$ (with respect to the compact-open topology) which has the desired properties.

Now consider the general case. Using the hypotheses on $A$, one can verify easily that $A_c = A \cap C_c(X)$ is dense in $A$. Therefore, the restriction of the linear functional $T$ obtained in the first part of the proof to $A$ is already discontinuous with respect to the relative topology inherited from $A$, as desired. □

COROLLARY 4.4. *Let $X$ be a locally compact $\sigma$-compact space which is not compact. If either $A = C(X)$ or $A = \text{Lip}_c(X, \alpha)$, $0 < \alpha \leq 1$, when $X$ is, in addition, a metric space, then there exists a discontinuous linear separating functional on $A_b = A \cap C_b(X)$ (with respect to the topology of $A$) which vanishes on each element of $A$ with compact support.*

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