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Existence and Stability of Almost Periodic Solutions of Nonlinear Damped Equations of a Suspended String

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Abstract. In this paper we shall show the existence and the stability of almost periodic solutions of the boundary value problem to a nonlinear suspended string equation with a linear damping term and an almost periodic weakly nonlinear forcing term. We treat both weak solutions and strong solutions. Also we show the existence of time global solutions of the initial boundary value problem to the equation.

1. Introduction

In recent researches of the behavior of suspended strings the existence of periodic oscillations of a nonlinear suspended string with finite length has been investigated under both the existence of damping (Hattori [4], Nagai [6] and Yamaguchi-Nagai-Matsukane [14]) and the nonexistence of damping (Yamaguchi [10, 11, 13]). On the other hand, for the existence of almost periodic and quasiperiodic oscillations of the suspended string only a few results have been obtained; it is shown in [10] that when the equation is *linear*, has a quasiperiodic forcing term and has *no damped term*, there exist infinitely many almost periodic and quasiperiodic oscillations of the suspended string. However there have been no works on the existence of almost periodic oscillations of the suspended string *when the equation is nonlinear*.

In this paper we shall be concerned with a suspended string to which *almost periodic nonlinear forces and a linear damping operate*. We show

(i) the existence of almost periodic oscillations ;

(ii) the exponential stability of the almost periodic oscillations.

Consider the following equation of the suspended string with a linear damping and weakly nonlinear forcing terms

$$\text{(SS)}\qquad \begin{aligned} &\partial_t^2 u(x,t) + L_m u(x,t) + \partial_t u(x,t) \\ &\quad = h(x,t) + \varepsilon f(x,t,u)\,, \quad (x,t) \in (0,a) \times R^1\,, \end{aligned}$$

where L_m is a second order differential operator whose principal part is degenerate at $x = 0$:

$$L_m = -\left(\frac{x}{m+1}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\right),$$

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m is a constant larger than -1, ε is a small parameter and $a > 0$ is a constant, the length of the string. For the derivation of the equation of the suspended string, see Koshlyakov-Gliner-Smirnov [5].

Throughout this paper except Section 4 we suppose that the forcing terms $h(x,t)$ and $f(x,t,u)$ are almost periodic in t uniformly with respect to other variables. See Section 2.3 for the definition of almost periodic functions.

In the first part of this paper we shall consider the following BVP (the boundary value problem) to Eq. (SS)

$$
\text{(P)} \qquad \begin{cases} \partial_t^2 u(x,t) + L_m u(x,t) + \partial_t u(x,t) \\ \qquad\qquad = h(x,t) + \varepsilon f(x,t,u)\,, \quad (x,t) \in (0,a) \times R^1\,, \\ u(a,t) = 0\,, \quad t \in R^1\,, \end{cases}
$$

and prove the existence of almost periodic solutions under suitable assumptions on m and h, f (see Section 3, Theorems 3.1 and 3.2). In the periodic cases where h and f are T-periodic ([4, 6, 11, 13, 14]) the time interval is taken as $[0,T]$ that is compact in R^1. Then in the suitable Lebesgue-type and Sobolev-type spaces with respect to the space-time variables x and t, the Fourier expansion method with respect to both x and t can be applied, and the periodic solutions of the linear BVP with $\varepsilon = 0$ are represented by the Fourier series in the spaces. From this fact, the existence of the periodic solutions of the linear BVP and the fundamental estimate of the inverse of the linear operator $\partial_t^2 + L_m + \partial_t$ are obtained. Then the contraction mapping principle, the Schauder fixed point theorem or the fixed point continuation method is applied to nonlinear BVP (P).

However in the almost periodic problems the time interval naturally is taken as $(-\infty, +\infty)$ that is not compact, whence different from the periodic problems, we are not able to apply the Fourier expansion method with respect to t. In this paper, instead of the Fourier expansion method in both x, t, we combine the Fourier expansion method with respect to x with the representation formula of the almost periodic solutions of the second order scalar ODE with almost periodic forcing term (Lemma 3.1). Using this method in some well-defined function spaces (see below and Section 2), we obtain the existence and uniqueness of an almost periodic solution of a linear BVP with $\varepsilon = 0$ and the basic energy estimates of the solution in suitable function spaces of almost periodic functions (see Section 3, Proposition 3.1). Then we apply the Picard iteration method to BVP (P), and show the existence of an almost periodic solution that is locally unique in the function space. We shall deal with the almost periodic solutions with weak regularity (Theorem 3.1) and strong or classical regularity (Theorem 3.2). In the former case we take $m \geq 0$ arbitrarily, while in the latter case we shall take $m = 0$ so that the generalized Sobolev-type inequality to the SS operator (see [10]) can be applied in order to obtain the regularity of the solutions.

In the second part we shall deal with the (local) exponential stability of the above almost periodic solutions of BVP (P). Let the almost periodic solution obtained in Section 3 be given, denoted by $u_0(x,t)$. Consider IBVP (initial boundary value problem) to the equation (SS)

$$
\text{(Q)}\qquad\begin{cases}\partial_t^2 u(x,t) + L_m u(x,t) + \partial_t u(x,t)\\ \qquad\qquad = h(x,t) + \varepsilon f(x,t,u)\,, \quad (x,t) \in (0,a) \times R^1\,,\\ u(a,t) = 0\,, \quad t \in R^1\,,\\ u(x,0) = \phi(x)\,, \quad \partial_t u(x,0) = \psi(x)\,, \quad x \in (0,a)\,.\end{cases}
$$

It turns out (Theorems 4.1–4.2) that IBVP (Q) has a unique time-global solution $u(x,t;\phi,\psi)$ in suitable function spaces for small ε. Then we show (Theorems 5.1–5.2) that any solution $u(x,t;\phi,\psi)$ starting from initial data (ϕ,ψ) in some neighborhood of the data $(u_0(x,0), \partial_t u_0(x,0))$ converges to $u_0(x,t)$ exponentially as t goes to $+\infty$. The rate of the exponential decay depends on the value of the max eigenvalue λ_j satisfying $4\lambda_j < 1$, where 1, the right hand side, means the damping constant. This statement will be proved by estimating the energy of the difference $u(x,t;\phi,\psi) - u_0(x,t)$ similar to the proofs of Theorems 4.1–4.2. In IBVP to nonlinear SS equations without a damping term Wongsawasdi-Yamaguchi [7, 8] and [13] proved the existence of both weak and classical time-global solutions. The solutions are stable; *i.e.,* the solutions in [7, 13] exist in the bounded stable set in $H_0^1(0,1;x^m)$ for all $t \in R^1$, and the solutions in [8] are bounded in $H^s(0,1;x^0)$ for all $t \in R^1$.

The operator L_m has the principal part that is degenerate at $x = 0$. In order to deal with this degeneracy, the Lebesgue and Sobolev type function spaces with power weight at $x = 0$ were introduced in [10], and the properties of the spaces and inequalities in such function spaces were studied (see [10, 11, 12, 13, 14]). In this paper, different from the above function spaces, we introduce function spaces whose elements have finite weighted norm at $x = 0$ in x-direction and uniformly bounded in $t \in R^1$ with its derivatives (see Section 2). We consider IBVP (Q) in such function spaces.

As we stated above, the time interval treated in this paper is not compact. Hence we define other necessary function spaces to deal with almost periodic solutions and functions, different from the periodic problems ([4, 14] and so on).

This paper is organized as follows. In Section 2 we introduce some necessary function spaces related to the SS operator. Also we define some function spaces of almost periodic functions, and state the properties of almost periodic functions, used in this paper. In Section 3 we consider the existence of almost periodic solutions of BVP (P). In Section 4 we show the existence of global solutions of the IBVP (Q). In Section 5 we prove the exponential stability of the almost periodic solutions.

2. Function Spaces, Eigenvalue Problem and Almost Periodic Functions

We introduce some function spaces with power weight at $x = 0$ (*cf.* [10, 11, 12, 13, 14]). We also state the definition and properties of almost periodic functions (*cf.* Amerio-Prouse [1] and Corduneanu [2]) in these function spaces used in later sections.

In this paper the constants such as $C_1, C_2, \ldots$ appeared *in the proofs of Theorems, Propositions and so on* will be suitably taken independent of x, t and u, if not specified.

2.1. Definitions of Function Spaces. Let O be any open set in R^n. Let Z_+ and R^1_+ be the set of nonnegative integers and the set of nonnegative real numbers, respectively. $L^2(O)$ and $H^s(O)$ are the usual Lebesgue and Sobolev spaces, respectively. Let m be any fixed nonnegative number.

Some function spaces below are defined in the above references, but we again write those spaces for readers.

By $L^2(0, a; x^m)$ we denote the Lebesgue-type space whose elements $f(x)$ are real-valued and measurable in $(0, a)$, and satisfy $x^{\frac{m}{2}} f \in L^2(0, a)$ with norm defined by

$$|f|_{L^2(0,a;x^m)} = \left(\int_0^a x^m f(x)^2 dx \right)^{\frac{1}{2}} .$$

The space is a Hilbert space with the inner product

$$(f, g)_{L^2(0,a;x^m)} = \int_0^a x^m f(x) g(x) dx .$$

Let $s, j \in Z_+$. We define the Sobolev-type Banach space $H^s(0, a; x^m)$ whose elements f and their weighted derivatives $x^{\frac{j}{2}} \partial_x^j f$ $(0 \le j \le s)$ belong to $L^2(0, a; x^m)$ with norm defined by

$$|f|_{H^s(0,a;x^m)} = \left(\sum_{j=0}^{s} |x^{\frac{j}{2}} \partial_x^j f(x)|^2_{L^2(0,a;x^m)} \right)^{\frac{1}{2}} .$$

Let $H^1_0(0, a; x^m)$ be a subspace of $H^1(0, a; x^m)$ whose elements f satisfy $f(a) = 0$. We also define a subspace $K^s(0, a; x^m)$ of $H^s(0, a; x^m)$ whose elements f satisfy $L^j_m f \in H^1_0(0, a; x^m)$ $(0 \le j \le [(s-1)/2])$. $K^s(0, a; x^m)$ is a Banach with the norm $|\cdot|_{H^s(0,a;x^m)}$. We set $K^0(0, a; x^m) = L^2(0, a; x^m)$. Clearly $K^1(0, a; x^m)$ and $K^2(0, a; x^m)$ are respectively identified with $H^1_0(0, a; x^m)$ and $H^2(0, a; x^m) \cap H^1_0(0, a; x^m)$.

Let I be an interval in R^1 and X be a Banach space with norm $|\cdot|_X$. We denote by $C^k(I; X)$ a function space whose elements $f(t)$ are k-times continuously differentiable in X. The norm is defined by

$$|f|_{C^k(I;X)} = \sup_{t \in R^1} \sum_{j=1}^{k} \left| \frac{d^j f}{dt^j}(t) \right|_X .$$

We write $C^0(I; X)$ as $C(I; X)$.

Let s and σ be positive integers with $s \geq \sigma$. Define the norm

$$|f(t)|_{s,\sigma,m} = \sum_{k=0}^{\sigma} \left| \frac{d^k f}{dt^k}(t) \right|_{H^{s-k}(0,a;x^m)}$$

for $f \in \bigcap_{k=0}^{\sigma} C^k(R^1; K^{s-k}(0,a;x^m))$. We denote by $F_m^{s,\sigma}$ the class of functions $f \in \bigcap_{k=0}^{\sigma} C^k(R^1; K^{s-k}(0,a;x^m))$ with $\sup_{t\in R^1} |f(t)|_{s,\sigma,m} < +\infty$. The norm of $F_m^{s,\sigma}$ is defined by

$$|f|_{F_m^{s,\sigma}} = \sup_{t \in R^1} |f(t)|_{s,\sigma,m} \,.$$

$F_m^{s,\sigma}$ is a Banach space with the norm $|\cdot|_{F_m^{s,\sigma}}$. We define $F_{m,+}^{s,\sigma}$ by replacing R^1 by R^1_+ in the definition of $F_m^{s,\sigma}$. The norm $|\cdot|_{F_{m,+}^{s,\sigma}}$ of $F_{m,+}^{s,\sigma}$ is defined in the same way as $|\cdot|_{F_m^{s,\sigma}}$.

2.2. Eigenvalue Problem for L_m. Consider the eigenvalue problem

$$\begin{cases} L_m\phi(x) = \lambda\phi(x)\,, & x \in (0,a)\,, \\ \phi(a) = 0\,. \end{cases} \tag{2.1}$$

It is known ([5]) that (2.1) has the eigenvalues λ_k and the corresponding eigenfunctions ϕ_k. Here λ_k and ϕ_k are given by

$$\lambda_k = \frac{\mu_k^2}{4(m+1)a}\,,$$

$$\phi_k(x) = \frac{x^{-\frac{m}{2}}}{a^{\frac{1}{2}} J_{m+1}(\mu_k)} J_m\left(\mu_k \sqrt{\frac{x}{a}}\right)$$

for $k = 1, 2, \ldots$, where J_m is the m-order Bessel function, and $\{\mu_k; k = 1, 2, \ldots\}$ is the set of all positive zero points of J_m with $\mu_1 < \mu_2 < \cdots$ and $\lim_{k\to+\infty} \mu_k = +\infty$. From this note that $\lim_{k\to+\infty} \lambda_k = +\infty$ holds. The sequence $\{\phi_k\}$ is the CONS in $L^2(0,a;x^m)$.

2.3. Almost Periodic Functions. In this subsection, we again write the definitions and the basic properties of almost periodic functions for readers (see also [1] and [2]). Furthermore we show necessary properties of almost periodic functions for later use and introduce function spaces of almost periodic functions, fundamental for the study of almost periodic solutions of this paper.

Let X be a Banach space. Let $f \in C(R^1; X)$. A function $f(t)$ is called X-almost periodic or almost periodic in X if for any $\mu > 0$ there exists a relatively dense set $AP_\mu(f) = \{\tau\}_\mu$ such that

$$|f(t+\tau) - f(t)|_X < \mu$$

for any $t \in R^1$ and any $\tau \in AP_\mu(f)$. Here we call $\{\tau\}_\mu$ *a relatively dense set* if for any $\mu > 0$ there exists a constant $l_\mu > 0$ such that any interval $(t, t + l_\mu)$, $t \in R^1$, contains at least one $\tau \in \{\tau\}_\mu$. Each $\tau \in AP_\mu(f)$ is called μ-almost period of f. We denote by APX the space of almost periodic functions in X with the uniform convergence norm $\sup_{t\in R^1} |\cdot|_X$.

Almost periodic function in R^1 is called *an almost periodic function in Bohr's sense* or *a numerical almost periodic function* that is the most basic almost periodic function.

The following lemma shows the relative compactness of APX (see [1, 2]).

LEMMA 2.1. *Let $f(t)$ be continuous in X. Then $f(t)$ belongs to APX if and only if for any real sequence $\{\alpha_j\}$ there exists a subsequence $\{\beta_j\}$ of $\{\alpha_j\}$ such that the sequence $\{f(t+\beta_j)\}$ converges uniformly in X.*

It is well-known ([1, 2]) that the sum and the difference of two X-almost periodic functions are X-almost periodic, and the product of numerical almost periodic function and X-almost periodic function is X-almost periodic.

LEMMA 2.2. *Let $\{f_n(t)\}$ be a sequence of almost periodic functions in X that converges uniformly to $f(t)$. Then $f(t)$ is almost periodic in X.*

REMARK 2.1. APX is a Banach space (see [1, 2]).

Let $f(t) \in APX$. Then there exists a sequence of real numbers $\{\lambda_k\}$ so that $f(t)$ is expanded into the Fourier series

$$f(t) \sim \sum_{k=1}^{\infty} f_k\, e^{i\lambda_k t}\,, \quad f_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t) e^{-i\lambda_k t} dt\,. \tag{2.2}$$

Here λ_k and $f_k \in X$ are called the characteristic exponents and the Fourier coefficients of $f(t)$, respectively. Conversely, if the Fourier series (2.2) converges uniformly in X, then it is almost periodic in X. Especially a finite trigonometric series $\sum_{k=1}^{m} g_k\, e^{i\mu_k t}$ is almost periodic, where $g_k \in X$ and $\mu_k \in R^1$.

Let $f(t,\lambda) \in C(R^1; X)$, where $\lambda \in \Lambda$ is a parameter. $f(t,\lambda)$ is called almost periodic in X uniformly with respect to $\lambda \in \Lambda$ if the relatively dense set is independent of λ.

LEMMA 2.3. *Let $f(t,\lambda)$ be almost periodic in X uniformly with respect to $\lambda \in \Lambda$, where Λ is a parameter domain in R^m, and differentiable with respect to (t,λ) up to order s. Assume that the derivatives $\partial_t^\alpha \partial_\lambda^\beta f(t,\lambda)$, $\alpha + |\beta| \le s$, are uniformly continuous in $R^1 \times \Lambda$, where $\beta = (\beta_1, \ldots, \beta_m)$, $|\beta| = |\beta_1| + \cdots + |\beta_m|$ and $\partial_\lambda^\beta = \partial_{\lambda_1}^{\beta_1} \cdots \partial_{\lambda_m}^{\beta_m}$. Then $\partial_t^\alpha \partial_\lambda^\beta f(t,\lambda)$ are almost periodic in X uniformly with respect to λ.*

The proof of this lemma is done in the similar way to [1].

LEMMA 2.4. *Let $f(t,\lambda)$ be almost periodic in X uniformly with respect to $\lambda \in \Lambda$, where Λ is a parameter domain in R^m. For any real sequence $\{\alpha_j\}$ there exists a subsequence $\{\beta_j\}$ of $\{\alpha_j\}$ independent of λ such that the sequence $\{f(t+\beta_j)\}$ converges uniformly in X.*

The proof is similar to that of Lemma 2.1 ([1]).

LEMMA 2.5. *Let $B_r(a)$ be a closed ball in X with radius r centered at $a \in X$. Let $F(t,u)$ be a mapping of $R^1 \times B_r(a)$ into X. Assume that $F(t,u)$ is uniformly continuous in $u \in B_r(a)$ uniformly in t and almost periodic in t uniformly with respect to $u \in B_r(a)$. Let $v(t)$ be an almost periodic function in X with $\{v(t)\}_{t\in R^1} \subset B_r(a)$. Then the composed function $F(t, v(t))$ is almost periodic in X.*

PROOF. We apply Lemma 2.1. Let $\{\alpha_j\}$ be any real sequence. Since $F(t,u)$ is almost periodic in t uniformly with respect to u, the sequence $\{F(t+\alpha_j, u)\}$ is relatively compact in X. Hence there exists a subsequence $\{\beta_j\}$ of $\{\alpha_j\}$ independent of u such that the subsequence $\{F(t+\beta_j, u)\}$ converges uniformly with respect to u. It means that for any $\varepsilon > 0$ there exists $n_1 = n_1(\varepsilon) \in N$ independent of u such that

$$|F(t+\beta_j, u) - F(t+\beta_k, u)|_X < \varepsilon \tag{2.3}$$

for any $j,\ k \geq n_1$ and any $u \in B_r(a)$. Since $F(t,u)$ is uniformly continuous in u uniformly with respect to $t \in R^1$, there exists $\delta > 0$ such that

$$|F(t,u) - F(t,w)|_X < \varepsilon \tag{2.4}$$

for $u,\ w \in B_r(a)$ with $|u-w|_X < \delta$ uniformly with respect to t. On the other hand, by the almost-periodicity of $v(t)$ we can choose a subsequence $\{\gamma_j\}$ of $\{\beta_j\}$ so that for a suitable $n_2 = n_2(\delta) \in N$ the following inequality

$$|v(t+\gamma_j) - v(t+\gamma_k)|_X < \delta \tag{2.5}$$

holds for $j,\ k \geq n_2$. Then we see

$$\begin{aligned}
&|F(t+\gamma_j, v(t+\gamma_j)) - F(t+\gamma_k, v(t+\gamma_k))|_X \\
&\quad \leq |F(t+\gamma_j, v(t+\gamma_j)) - F(t+\gamma_k, v(t+\gamma_j))|_X \\
&\qquad + |F(t+\gamma_k, v(t+\gamma_j)) - F(t+\gamma_k, v(t+\gamma_k))|_X
\end{aligned}$$

It follows from (2.3)–(2.5) that

$$|F(t+\gamma_j, v(t+\gamma_j)) - F(t+\gamma_k, v(t+\gamma_k))|_X < 2\varepsilon$$

for any $j,\ k \geq \max(n_1, n_2)$. Therefore $\{F(t+\gamma_j, v(t+\gamma_j))\}$ is a Cauchy sequence in X uniformly with respect to t. This means that the sequence $\{F(t+\alpha_j, v(t+\alpha_j))$ is relatively compact. Therefore we conclude from Lemma 2.1 that $F(t, v(t))$ is almost periodic in X. □

We introduce function spaces $APF_m^{s,\sigma}$ of almost periodic functions as follows : f belongs to $APF_m^{s,\sigma}$ if f belongs to $F_m^{s,\sigma}$ and each $d^k f(t)/dt^k$ is almost periodic in $K^{s-k}(0,a;x^m)$ for $0 \leq k \leq \sigma$.

Clearly the following proposition holds.

PROPOSITION 2.1. *$APF_m^{s,\sigma}$ is a Banach space with norm $|\cdot|_{F_m^{s,\sigma}}$.*

From the Sobolev lemma the differentiability of functions of $F_m^{s,\sigma}$, hence functions of $APF_m^{s,\sigma}$, is shown. Thus we obtain the following propositions. Note that $\sigma \le s$ is assumed (*cf.* Definition of $F_m^{s,\sigma}$).

PROPOSITION 2.2. *Let $\sigma \ge 1$. Then any $f \in F_m^{s,\sigma}$ belongs to $C(R^1; C^{s-1}(0,a))$.*

COROLLARY 2.1. *Let $s \ge 3$ and $\sigma \ge 2$. Then $f \in F_m^{s,\sigma}$ is of C^2 in $(0,a) \times R^1$.*

REMARK 2.2. It is necessary to state the relation between an almost periodic function in $APF_m^{s,\sigma}$ and a numerically almost periodic function that is the most basic and simple almost periodic function. See Assumption (B2) on the nonlinear function f. Clearly, if $f(x,t)$ is numerically almost periodic uniformly with respect to x, f belongs to $APL^r(0,a;x^m)$ for any $r \ge 0$ and $m \ge 0$. Similarly, if $f(x,t)$ with its derivatives $\partial_x^\alpha f(x,t)$ $(0 \le \alpha \le s)$ is numerically almost periodic uniformly with respect to x, then f belongs to $APH^s(0,a;x^m)$ for any $r \ge 0$ and $m \ge 0$. Conversely if f belongs to $APH^s(0,a;x^m)$ and m, s satisfy $s \ge m+2$, $f(x,t)$ and its derivatives $\partial_x^{[s-(m+2)]} f(x,t)$ are almost periodic uniformly with respect to x. This is shown by Proposition 4.2 in [11].

3. Almost Periodic Solutions of BVP (P)

In this section it is shown that BVP (P) has an almost periodic solution for small ε that is locally unique in suitable function space. To this end the Picard iteration method will be applied to BVP (P). It is necessary to show the existence of an almost periodic solution of a linear BVP and to derive the energy estimate of the solution in $F_m^{s+1,2}$.

From now on through this paper, we call a solution in $F_m^{1,1}$ the weak solution, and a solution in $F_m^{s+1,2} (s \ge 1)$ the strong solution. Also if the strong solution satisfies BVP or IBVP in the classical sense, *i.e.,* the solution is two-times differentiable with respect to (x,t), the solution is called the classical solution.

From now on we denote the operator $\partial_t^2 + L_m + \partial_t$ by A_m for brevity.

3.1. The Existence of Almost Periodic Solutions of Linear BVP. Consider the following linear BVP

$$\text{(LP)} \qquad \begin{cases} A_m\, w(x,t) = g(x,t)\,, & (x,t) \in (0,a) \times R^1\,, \\ w(a,t) = 0\,, & t \in R^1\,. \end{cases}$$

We assume the following condition on a forcing term g.

(A) $g(\cdot,t)$ is almost periodic in $K^s(0,a;x^m)$.

REMARK 3.1. Let $g \in C(R^1; K^s(0,a;x^m))$. Assume that $g(x,t)$ is numerically almost periodic uniformly with respect to x and the derivatives $\partial_x^j g(x,t)$ $(0 \le j \le s)$ are uniformly continuous in $[0,a] \times R^1$. Then it follows from Lemma 2.3 that g satisfies (A). See also Remark 2.2.

PROPOSITION 3.1. *Assume* (A). *Let* $s \geq 1$. *Then BVP* (LP) *has an almost periodic solution* w *unique in* $APF_m^{s+1,2}$ *satisfying*

$$|w|_{F_m^{s+1,2}} \leq C \sup_{t \in R^1} |g(\cdot, t)|_{H^s(0,a;x^m)} \,. \tag{3.1}$$

Moreover if $s = 0$, *the statement holds by replacing* $APF_m^{s+1,2}$ *and* $F_m^{s+1,2}$ *by* $APF_m^{1,1}$ *and* $F_m^{1,1}$, *respectively.*

To show this proposition we prepare the representation formula of almost periodic solution of the second order ODE.

LEMMA 3.1. *Let* λ_j *be any eigenvalue defined in Subsection* 2.2. *Consider the following scalar second order ODE*

$$y''(t) + \lambda_j y(t) + y'(t) = g(t) \,, \tag{3.2}$$

where $g(t)$ *is almost periodic in* R^1. *Then there exists a unique almost periodic solution* $y(t)$ *of* (3.2) *represented by*

$$y(t) = \frac{1}{2a_j} \int_0^\infty g(t-\tau)(e^{\nu_j^+ \tau} - e^{\nu_j^- \tau}) d\tau \,, \tag{3.3}$$

where $\nu_j^\pm$ *are the solutions of the corresponding characteristic equation of the second degree*

$$c^2 + c + \lambda_j = 0 \,,$$

given by

$$\nu_j^+ = -\frac{1}{2} + a_j \,, \quad \nu_j^- = -\frac{1}{2} - a_j \,, \quad a_j = \frac{(1-4\lambda_j)^{\frac{1}{2}}}{2} \,. \tag{3.4}$$

For the proof of this lemma, see [3, 9].

Let j_0 be the maximum of integers j so as to satisfy $1 - 4\lambda_j > 0$. If there exists no such number j_0, we take $j_0 = 0$. In what follows, we define a decay exponent by

$$\gamma = \begin{cases} 1/2 & (j_0 = 0) \\ 1/2 - a_1 & (j_0 \geq 1) \,. \end{cases} \tag{3.5}$$

Then it follows that

$$|e^{\nu_j^\pm t}| \leq e^{-\gamma t} \tag{3.6}$$

for any $j \in N$ and any $t \geq 0$.

PROOF OF PROPOSITION 3.1. We expand $g(x,t)$ into the Fourier series in $L^2(0, a; x^m)$

$$g(\cdot, t) = \sum_{j=1}^\infty g_j(t) \phi_j \,. \tag{3.7}$$

We look for the almost periodic solution $w(x,t)$ as the Fourier series in $L^2(0,a;x^m)$:

$$w(\cdot,t)=\sum_{j=1}^{\infty} w_j(t)\phi_j. \tag{3.8}$$

Substitute (3.7)–(3.8) into (LP) and compare the Fourier coefficients. Then we obtain the following system of second order ODEs

$$w_j''(t)+\lambda_j w_j(t)+w_j'(t)=g_j(t), \quad j=1,2,\dots . \tag{3.9}$$

Clearly $g_j(t)=(g(\cdot,t),\phi_j)_{L^2(0,a;x^m)}$ is numerically almost periodic. It follows from Lemma 3.1 that (3.9) has a unique almost periodic solution of the form

$$w_j(t)=\frac{1}{2a_j}\int_0^{\infty} g_j(t-\tau)(e^{\nu_j^+\tau}-e^{\nu_j^-\tau})\,d\tau\,, \tag{3.10}$$

where $\nu_j^{\pm}$ and a_j are defined by (3.4). Differentiating (3.10), we have

$$w_j'(t)=\frac{1}{2a_j}\int_0^{\infty} g_j(t-\tau)(\nu_j^+e^{\nu_j^+\tau}-\nu_j^-e^{\nu_j^-\tau})\,d\tau\,. \tag{3.11}$$

We show that (3.8) converges in $F_m^{s+1,2}$, whence w is the solution of BVP (LP). It is enough to show that the series $\sum_{j=1}^{\infty}(\lambda_j^{s+1}w_j(t)^2+\lambda_j^s w_j'(t)^2)$ converges uniformly with respect to $t\in R^1$. For, if the series converges, this is equivalent to $|w(t)|^2_{H^{s+1}(0,a;x^m)}+|\partial_t w(t)|^2_{H^s(0,a;x^m)}$ ([10]). We show that this quantity is estimated by $\sup_{t\in R^1}|g(\cdot,t)|^2_{H^s(0,a;x^m)}$. By this estimate it follows from Lemma 2.2 that $w(\cdot,t)$ and $\partial_t w(\cdot,t)$ are almost periodic in $K^{s+1}(0,a;x^m)$ and $K^s(0,a;x^m)$, respectively. The almost-periodicity of $\partial_t^2 w(\cdot,t)$ follows from (LP).

First we estimate $J\equiv\sum_{j=1}^{\infty}\lambda_j^{s+1}w_j(t)^2$. Using (3.10), (3.6) and $\lambda_j\le c\,|a_j|^2$, we calculate

$$\begin{aligned}J&\le\sum_{j=1}^{\infty}\lambda_j^{s+1}\left(\frac{1}{2|a_j|}\int_0^{\infty}|g_j(t-\tau)||e^{\nu_j^+\tau}-e^{\nu_j^-\tau}|d\tau\right)^2\\&\le C\sum_{j=1}^{\infty}\left(\lambda_j^{s/2}\int_0^{\infty}|g_j(t-\tau)|e^{-\gamma\tau}d\tau\right)^2.\end{aligned}$$

By applying the Minkowski inequality to the right hand side, we have

$$J\le C\left[\int_0^{\infty}\left\{\sum_{j=1}^{\infty}\left(\lambda_j^s\,|g_j(t-\tau)|^2e^{-2\gamma\tau}\right)\right\}^{\frac12}d\tau\right]^2\le C\sup_{t\in R^1}\sum_{j=1}^{\infty}\lambda_j^s\,g_j(t)^2\,.$$

Thus we obtain

$$\sum_{j=1}^{\infty}\lambda_j^{s+1}\,w_j(t)^2\le C\sup_{t\in R^1}\sum_{j=1}^{\infty}\lambda_j^s\,g_j(t)^2\,. \tag{3.12}$$

Note that the following inequalities holds ([10], Proposition 3.4)

$$\delta_1 \, |f|^2_{H^s(0,a;x^m)} \le \sum_{j=1}^{\infty} \lambda_j^s \, f_j^2 \le \delta_2 \, |f|^2_{H^s(0,a;x^m)} \tag{3.13}$$

for $f \in K^s(0,a;x^m)$, where $\delta_i > 0$ are constants and $f = \sum f_j \phi_j$. Conversely, if $f \in L^2(0,a;x^m)$ satisfies (3.13), f belongs to $K^s(0,a;x^m)$. Hence, from $g \in C(R^1; K^s(0,a;x^m))$ we obtain

$$|w(\cdot,t)|_{H^{s+1}(0,a;x^m)} \le C \sup_{t\in R^1} |g(\cdot,t)|_{H^s(0,a;x^m)} \, . \tag{3.14}$$

This means $w \in C(R^1; K^{s+1}(0,a;x^m))$.

In the similar way, using (3.11), we obtain

$$\sum_{j=1}^{\infty} \lambda_j^s \, w_j'(t)^2 \le C \sup_{t\in R^1} \sum_{j=1}^{\infty} \lambda_j^s \, g_j(t)^2 \, . \tag{3.15}$$

From this and (3.13) we see $\partial_t w \in C(R^1; K^s(0,a;x^m))$ and obtain the estimate

$$|\partial_t w(\cdot,t)|_{H^s(0,a;x^m)} \le C \sup_{t\in R^1} |g(\cdot,t)|_{H^s(0,a;x^m)} \, . \tag{3.16}$$

Using Eq. (3.9), we have

$$\sum_{j=1}^{\infty} \lambda_j^{s-1} w_j''(t)^2 \le C \sum_{j=1}^{\infty} (\lambda_j^{s+1} w_j(t)^2 + \lambda_j^{s-1} w_j'(t)^2 + \lambda_j^{s-1} g_j(t)^2) \, . \tag{3.17}$$

By (3.12), (3.15) and (3.13) we have $\partial_t^2 w \in C(R^1; K^{s-1}(0,a;x^m))$ and

$$|\partial_t^2 w(x,t)|_{H^{s-1}(0,a;x^m)} \le C \sup_{t\in R^1} |g(t)|_{H^s(0,a;x^m)} \, . \tag{3.18}$$

Therefore by (3.14), (3.16) and (3.18) we obtain $w \in F_m^{s+1,2}$ and the estimate (3.1).

Finally we show that the solution $w(\cdot,t)$ is almost periodic, more precisely w belongs to $APF_m^{s+1,2}$. Since $g(\cdot,t)$ is almost periodic in $K^s(0,a;x^m)$, it follows that for any $\varepsilon > 0$ there exists a relatively dense set $\{\tau\}_\varepsilon$ such that

$$|g(\cdot,t+\tau) - g(\cdot,t)|_{H^s(0,a;x^m)} < \varepsilon$$

for $t \in R^1$. Since $v(x,t) = w(\cdot,t+\tau) - w(\cdot,t)$ is a solution of BVP

$$\begin{cases} A_m v = g(x,t+\tau) - g(x,t) \, , & (x,t) \in (0,a) \times R^1 \, , \\ v(a,t) = 0 \, , \quad t \in R^1 \, , \end{cases}$$

it follows from (3.1) that

$$\sup_{t\in R^1} |v(\cdot,t)|_{s+1,2,m} \le C \sup_{t\in R^1} |g(\cdot,t+\tau) - g(\cdot,t)|_{H^s(0,a;x^m)} \le C\,\varepsilon \, .$$

This means $w \in APF_m^{s+1,2}$ with $C\varepsilon$-almost periods. The uniqueness is clear from (3.1).

From the above proof the case $s = 0$ is clear. □

3.2. The Existence of Almost Periodic Solutions of Nonlinear BVP. In this subsection we consider BVP (P)

$$\text{(P)} \qquad \begin{cases} A_m u(x,t) = h(x,t) + \varepsilon f(x,t,u)\,, & (x,t) \in (0,a) \times R^1\,, \\ u(a,t) = 0\,, \quad t \in R^1\,, \end{cases}$$

and show that under several conditions on h and f BVP (P) has locally unique almost periodic solutions in $APF_m^{1,1}$ (*weak solutions*) and $APF_m^{s+1,2}$ (*strong or classical solutions*).

3.2.1. The Existence of Almost Periodic Weak Solutions of BVP (P). First we show the existence of almost periodic *weak* solutions of BVP (P).

We assume the following conditions on h and f.

(A1) $h(\cdot,t)$ is almost periodic in $L^2(0,a;x^m)$.

(A2) (i) $f(x,t,\lambda)$ is continuous in $[0,a] \times R^1 \times R^1$. $f(x,t,\lambda)$ is locally Lipschitz continuous in $\lambda \in R^1$: For any $r > 0$ there exists a constant $\rho_0(r) > 0$ such that

$$|f(x,t,\lambda_1) - f(x,t,\lambda_2)| \le \rho_0(r)|\lambda_1 - \lambda_2| \tag{3.19}$$

for $\lambda_1,\ \lambda_2 \in (-r,r)$ and $(x,t) \in [0,a] \times R^1$.

(ii) $f(\cdot,t,\phi)$ is almost periodic in $L^2(0,a;x^m)$ uniformly with respect to any bounded function $\phi(x)$.

REMARK 3.2. Instead of (A1), assume that $h(x,t)$ is *numerically almost periodic* uniformly with respect to x (Remark 2.2). Then clearly h satisfies (A1). Also instead of (A2)(ii), assume that $f(x,t,\lambda)$ is *numerically almost periodic* uniformly with respect to $(x,\lambda) \in [0,a] \times R^1$. Then f satisfies (A2)(ii).

EXAMPLE 3.1. As typical examples of $f(x,t,\lambda)$ satisfying (A2), we can take

(1) $f(x,t,\lambda) = \alpha(x,t)\,|\lambda|^{\rho-1}\lambda\,,$

(2) $f(x,t,\lambda) = \alpha(x,t)\,|\lambda|^{\rho}\,,$

(3) $f(x,t,\lambda) = \alpha(x,t)\,\lambda^{\rho}$ for $\rho \in Z_+\,,$

(4) $f(x,t,\lambda) = \alpha(x,t)\,\sin\lambda\,,$

where $\alpha(x,t)$ is continuous in $(x,t) \in [0,a] \times R^1$ and numerically almost periodic in t uniformly with respect to $x \in [0,a]$ and $\rho \ge 1$ is a constant.

THEOREM 3.1. *Assume* (A1)–(A2). *Then there exists* $\varepsilon_0 > 0$ *such that BVP* (P) *has an almost periodic solution* $u \in APF_m^{1,1}$ *for any* ε, $|\varepsilon| \le \varepsilon_0$. *The solution u is locally unique in* $APF_m^{1,1}$.

PROOF. We prove this theorem by the Picard iteration method. We define a successive approximation sequence $\{u_n\}$ by the following scheme:

$$\begin{cases} A_m u_0 = h(x,t)\,, & (x,t) \in (0,a) \times R^1\,, \\ u_0(a,t) = 0\,, & t \in R^1\,, \end{cases} \tag{3.20}$$

and

$$\begin{cases} A_m u_{n+1} = h(x,t) + \varepsilon f(x,t,u_n)\,, & (x,t) \in (0,a) \times R^1\,, \\ u_{n+1}(a,t) = 0\,, & t \in R^1 \end{cases} \tag{3.21}$$

for $n = 0, 1, 2, \ldots$. We first show that the sequence $\{u_n\}$ is well-defined in $APF_m^{1,1}$. In fact, as (A1) holds, $u_0 \in APF_m^{1,1}$ by Proposition 3.1. Assuming that u_n belongs to $APF_m^{1,1}$, we show that u_{n+1} belongs to $APF_m^{1,1}$. We have only to show that $f(x,t,u_n)$ fulfills the condition (A) with $s = 0$ *i.e.,* the composed function $f(x,t,u_n)$ is almost periodic in $L^2(0,a;x^m)$. To this end first we show that $f(\cdot,t,u_n(\cdot,t))$ is continuous in $L^2(0,a;x^m)$. We calculate

$$\begin{aligned} &|f(\cdot,t,u_n(\cdot,t)) - f(\cdot,t_0,u_n(\cdot,t_0))|_{L^2(0,a;x^m)} \\ &\quad \le |f(\cdot,t,u_n(\cdot,t)) - f(\cdot,t,u_n(\cdot,t_0))|_{L^2(0,a;x^m)} \\ &\qquad + |f(\cdot,t,u_n(\cdot,t_0)) - f(\cdot,t_0,u_n(\cdot,t_0))|_{L^2(0,a;x^m)}\,. \end{aligned}$$

Since u_n belongs to $F_m^{1,1}$, $u_n(x,t)$ is bounded in $(0,a] \times R^1$ by the Sobolev-type inequality. By the local Lipschitzness of f with respect to λ and the continuity of $u_n(\cdot,t)$ in $L^2(0,a;x^m)$ the first term tends to 0 as $t \to t_0$. Since $f(x,t,\lambda)$ is continuous in $[0,a]\times R^1 \times R^1$, the second term in the right hand side tends to 0 as $t \to t_0$. Hence $f(\cdot,t,u_n(\cdot,t)) \in C(R^1; L^2(0,a;x^m))$. Also by the local Lipschitzness of f in (A2) $f(\cdot,t,\phi(\cdot))$ is a continuous mapping of $R^1 \times H_0^1(0,a;x^m)$ into $L^2(0.a;x^m)$. By (A2)(ii) $f(\cdot,t,\phi(\cdot))$ is almost periodic in $L^2(0,a;x^m)$ for $\phi \in H_0^1(0,a;x^m)$, since $H_0^1(0,a;x^m)$ is continuously embedded in $C([0,a])$. Hence by applying Lemma 2.5 to $f(\cdot,t,\phi)$, the composed function $f(\cdot,t,u_n(\cdot,t))$ is almost periodic in $L^2(0,a;x^m)$. Therefore $f(\cdot,t,u_n(\cdot,t))$ fulfills (A).

Let C be the same constant as in (3.1) in Proposition 3.1. We prove the following : For any $R > 2C \sup_{t\in R^1} |h(\cdot,t)|_{L^2(0,a;x^m)}$ there exists $\varepsilon_0 > 0$ such that for any ε, $|\varepsilon| \le \varepsilon_0$ the sequence $\{u_n\}$ satisfies

(i) $|u_n|_{F_m^{1,1}} \le R$ for $n = 0, 1, \ldots,$

(ii) $|u_{n+1} - u_n|_{F_m^{1,1}} \le \kappa\, |u_n - u_{n-1}|_{F_m^{1,1}}$ for $n = 1, 2, \ldots,$

where κ is a constant in $(0,1)$. We show (i) and (ii) by induction. First applying Proposition 3.1 to BVP (3.20), we obtain $u_0 \in F_m^{1,1}$ and $|u_0|_{F_m^{1,1}} \le R$. This shows that (i) holds for $n = 0$. Assume that $u_j \in F_m^{1,1}$ with $|u_j|_{F_m^{1,1}} \le R$ for $j = 1, 2, \ldots, n$. Then the sequence $\{u_n(x,t)\}$ is uniformly bounded in $(0,a) \times R^1$. In fact, applying the Sobolev-type inequality

([11]) to the middle term below, we have

$$|u_n(x,t)| \le \sup_{t\in R^1} |u_n(x,t)| \le C_1 \sup_{t\in R^1} |u_n(\cdot,t)|_{H^1(0,a;x^m)} \le C_1 |u_n|_{F_m^{1,1}} \le C_1 R\,.$$

From this $f(x,t,u_n)$ is uniformly bounded in $(x,t) \in (0,a) \times R^1$ by some constant $K > 0$ independent of n. Then again applying Proposition 3.1 with $s = 0$ to (3.21), we have

$$\begin{aligned} |u_{n+1}|_{F_m^{1,1}} &\le C\,(\sup_t |h(\cdot,t)|_{L^2(0,a;x^m)} + |\varepsilon| \sup_t |f(x,t,u_n)|_{L^2(0,a;x^m)}) \\ &\le \frac{R}{2} + |\varepsilon|\, C_2\, K\,, \end{aligned}$$

where C_2 depends only on a. Therefore by taking $\varepsilon_0 > 0$ so as to satisfy $\varepsilon_0 \le R/(2C_2\,K)$, u_{n+1} satisfies (i).

Next we show that (ii) holds. $v_n = u_{n+1} - u_n$ satisfies

$$\begin{cases} A_m v_n = \varepsilon(f(x,t,u_n) - f(x,t,u_{n-1}))\,, & (x,t) \in (0,a) \times R^1\,, \\ v_n(a,t) = 0\,, \quad t \in R^1\,. \end{cases}$$

Applying Proposition 3.1 to the above BVP and using (A2)(i), we have

$$\begin{aligned} |v_n|_{F_m^{1,1}} &\le C\,|\varepsilon| \sup_{t\in R^1} |f(x,t,u_n) - f(x,t,u_{n-1})|_{L^2(0,a;x^m)} \\ &\le C_3\,|\varepsilon| \sup_{t\in R^1} |v_{n-1}(\cdot,t)|_{L^2(0,a;x^m)} \\ &\le C_3\,|\varepsilon|\,|v_{n-1}|_{F_m^{1,1}}\,, \end{aligned}$$

where $C_3 = C\,\rho(C_1 R)$. Therefore again taking $\varepsilon_0 > 0$ such that $C_3\,|\varepsilon_0| = \kappa < 1$, we obtain

$$|u_{n+1} - u_n|_{F_m^{1,1}} \le \kappa\,|u_n - u_{n-1}|_{F_m^{1,1}} \tag{3.22}$$

for any $\varepsilon, |\varepsilon| \le \varepsilon_0$. Hence $\{u_n\}$ is the Cauchy sequence in $APF_m^{1,1}$. By the completeness of $APF_m^{1,1}$, $\{u_n\}$ converges to some $u \in APF_m^{1,1}$. Take $n \to +\infty$ in

$$u_{n+1} = A_m^{-1}(h + \varepsilon f(u_n))\,. \tag{3.23}$$

Then u is the solution of the integral equation $u = A_m^{-1}(h + \varepsilon f(u))$. This means that u is the solution of BVP (P) in $F_m^{1,1}$.

The local uniqueness of the solution is proved in the similar way to the above argument to show (3.22). □

From Theorem 3.1 and Remark 3.2 we obtain the following:

COROLLARY 3.1. *Assume that $h(x,t)$ is numerically almost periodic uniformly with respect to x and $f(x,t,\lambda)$ satisfies* (A2)(i) *and is numerically almost periodic uniformly with respect to (x,λ). Then the same conclusion as Theorem* 3.1 *holds. Moreover the solution is numerically almost periodic.*

3.2.2. The Existence of Almost Periodic Strong and Classical Solutions of BVP (P). In this part we show the existence of the almost periodic *strong* and *classical* solutions of BVP (P). More precisely, the almost periodic solutions belong to $APF_0^{s+1,2}$.

Throughout this part we assume that *m is equal to* 0 *and s is any fixed positive integer* ≥ 2.

We assume the following conditions on h and f.

(B1) $h(\cdot, t)$ is almost periodic in $K^s(0, a; x^0)$.

(B2) (i) $f(x, t, \lambda)$ is of C^s-class in $[0, a] \times R^1 \times R^1$, and its derivatives up to order s are uniformly continuous in $[0, a] \times R^1 \times I$, where I is any finite interval in R^1.

(ii) For $u \in K^{s+1}(0, a; x^0)$, $f(\cdot, t, u)$ belongs to $C(R^1; K^s(0, a; x^0))$.

(iii) $f(x, t, \lambda)$ is numerically almost periodic uniformly with respect to $(x, \lambda) \in [0, a] \times I$, where I is a finite interval in R^1.

LEMMA 3.2. *Assume* (B2) (i) (iii). *Then $f(\cdot, t, \lambda)$ is almost periodic in $H^s(0, a; x^0)$ uniformly with respect to λ.*

PROOF. As (B2) (i), (iii) holds, we apply Lemma 2.3 with $X = R^1$ to $f(x, t, \lambda)$ so that the derivatives $(\partial_x^\alpha f)(x, t, \lambda)$ $(\alpha \leq s)$ are numerically almost periodic uniformly with respect to (x, λ). Let $\{\alpha_j\}$ be any sequence in R^1. From the above fact and Remark 3.2 we have

$$\begin{aligned}
&|f(\cdot, t+\alpha_j, \lambda) - f(\cdot, t+\alpha_k, \lambda)|^2_{H^s(0,a;x^0)} \\
&\quad = \sum_{l=0}^{s} |x^{l/2}\{(\partial_x^l f)(\cdot, t+\alpha_j, \lambda) - (\partial_x^l f)(\cdot, t+\alpha_k, \lambda))\}|^2_{L^2(0,a;x^0)} \\
&\quad \leq C \sum_{l=0}^{s} \sup_{x,\lambda} |(\partial_x^l f)(x, t+\alpha_j, \lambda) - (\partial_x^l f)(x, t+\alpha_k, \lambda))|^2 .
\end{aligned}$$

Hence from Lemma 2.4 we can choose a subsequence $\{\beta_k\}$ of $\{\alpha_k\}$ independent of λ so that each term in the right hand side tends to 0 as $j, k \to \infty$. Therefore the conclusion holds from Lemma 2.1. □

Let $B_r(0)$ be a ball with radius r and center 0 in $K^s(0, a; x^0)$.

LEMMA 3.3. *Let $r > 0$ be any constant. Assume that* (B2) *holds. Then $f(\cdot, t, \phi(\cdot))$ is almost periodic in $K^s(0, a; x^0)$ uniformly with respect to $\phi \in B_r(0)$.*

PROOF. Let $\{\alpha_j\}$ be any real sequence. Then by Lemma 3.2 there exists a subsequence $\{\beta_j\}$ independent of λ such that the sequence $\{f(\cdot, t+\beta_j, \lambda)\}$ converges in $H^s(0, a; x^0)$ uniformly with respect to $\lambda \in R^1$. We calculate

$$\begin{aligned}
&|f(\cdot, t+\beta_j, \phi(\cdot)) - f(\cdot, t+\beta_k, \phi(\cdot))|^2_{H^s(0,a;x^0)} \\
&\quad = \sum_{l=0}^{s} |x^{l/2}\{\partial_x^l (f(\cdot, t+\beta_j, \phi) - f(\cdot, t+\beta_k, \phi)\}|^2_{L^2(0,a;x^0)} . \qquad (3.24)
\end{aligned}$$

By using the chain rule

$$\partial_x^l f(x,t,\phi) = \sum C_{\alpha\beta\gamma}\,(\partial_x^\beta \partial_\lambda^\gamma f)(x,t,\phi)(\partial_x^1\phi)^{\alpha_1}\cdots(\partial_x^l\phi)^{\alpha_l}\,, \tag{3.25}$$

where the summation is taken for $\alpha_1+\cdots+\alpha_l=\gamma$ and $\beta+\alpha_1+2\alpha_2+\cdots+l\alpha_l=l$, each term in the summation in (3.24) is estimated:

$$\begin{aligned}
&|x^{l/2}\{\partial_x^l(f(\cdot,t+\beta_j,\phi)-f(\cdot,t+\beta_k,\phi))\}|^2_{L^2(0,a;x^0)} \\
&= \Big|x^{l/2}\sum C_{\alpha\beta\gamma}\,\{(\partial_x^\beta\partial_\lambda^\gamma f)(x,t+\beta_j,\phi)-(\partial_x^\beta\partial_\lambda^\gamma f)(x,t+\beta_k,\phi)\} \\
&\quad\times(\partial_x^1\phi)^{\alpha_1}\cdots(\partial_x^l\phi)^{\alpha_l}\Big|^2_{L^2(0,a;x^0)} \\
&\le C\sum\sup_x|(\partial_x^\beta\partial_\lambda^\gamma f)(x,t+\beta_j,\phi)-(\partial_x^\beta\partial_\lambda^\gamma f)(x,t+\beta_k,\phi)|^2 \\
&\quad\times\Big|(x^{1/2}\partial_x^1\phi)^{\alpha_1}\cdots(x^{l/2}\partial_x^l\phi)^{\alpha_l}\Big|^2_{L^2(0,a;x^0)}\,.
\end{aligned}$$

The last term is estimated in the same way as in the proof of Proposition 4.2 in [14] :

$$\Big|(x^{1/2}\partial_x^1\phi)^{\alpha_1}\cdots(x^{l/2}\partial_x^l\phi)^{\alpha_l}\Big|^2_{L^2(0,a;x^0)} \le c\left(1+|\phi|^{2s}_{H^s(0,a;x^0)}\right).$$

Since $(\partial_x^\beta\partial_\lambda^\gamma f)(x,t,\phi)$ is numerically almost periodic in t uniformly with respect to (x,ϕ), we can choose a subsequence $\{r_k\}$ of $\{\beta_k\}$ such that $\{(\partial_x^\beta\partial_\lambda^\gamma f)(x,t+r_j,\phi)\}$ converges. Therefore it follows from (3.24) and the above calculations that $\{f(\cdot,t+r_j,\phi(\cdot))\}$ converges in $K^s(0,a;x^0)$ uniformly in $\phi\in B_r(0)$. Hence the conclusion follows. □

LEMMA 3.4. *Assume that* (B2) *holds. Then for* $u\in APF_0^{s+1,2}$ $f(\cdot,t,u(\cdot,t))$ *is almost periodic in* $K^s(0,a;x^0)$.

PROOF. We apply Lemma 2.5. Take $F(t,u)=f(\cdot,t,u)$ and $X=B_\infty(0)=K^s(0,a;x^0)$. From (B)(ii) F is a mapping of $R^1\times X$ into X. From the Moser-type inequality $f(\cdot,t,\phi)$ is Lipschitz continuous in $\phi\in B_r(0)$ uniformly with respect to t for any $r>0$. From Lemma 3.3 $f(\cdot,t,\phi)$ is almost periodic in $K^s(0,a;x^0)$ uniformly with respect to $\phi\in B_r(0)$. Let $u\in APF_0^{s+1,2}$. Then $u(\cdot,t)$ is continuous in $K^s(0,a;x^0)$, *i.e.,* $u\in C(R^1;K^s(0,a;x^0))$. Let $t_0\in R^1$ be fixed. By the triangle inequality we have

$$\begin{aligned}
&|f(\cdot,t,u(\cdot,t))-f(\cdot,t_0,u(\cdot,t_0))|_{H^s(0,a;x^0)} \\
&\quad\le|f(\cdot,t,u(\cdot,t))-f(\cdot,t,u(\cdot,t_0))|_{H^s(0,a;x^0)} \\
&\qquad+|f(\cdot,t,u(\cdot,t_0))-f(\cdot,t_0,u(\cdot,t_0))|_{H^s(0,a;x^0)}\,.
\end{aligned}$$

Since by (B2) (i) the Moser-type inequality ([14]) holds, the first term is estimated by $C\,|u(\cdot,t)-u(\cdot,t_0)|_{H^s(0,a;x^0)}$ that tends to 0 as $t\to t_0$. By (B2)(ii) the second term tends to 0 as $t\to t_0$. Hence $f(x,t,u(x,t))$ belongs to $C(R^1;K^s(0,a;x^0))$. By applying Lemma 2.5 to f, it follows that $f(\cdot,t,u(\cdot,t))$ is almost periodic in $K^s(0,a;x^0)$. Hence the conclusion holds. □

The following example of $f(x,t,\lambda)$ is a typical one that satisfies (B2).

EXAMPLE 3.2. We take

$$f(x,t,\lambda)=\alpha(x,t)\lambda^r\,,$$

where $\alpha(x,t)$ is of C^s-class in $[0,a]\times R^1$ and numerically almost periodic uniformly with respect to x, and r is any integer $\geq s$.

THEOREM 3.2. *Assume* (B1)–(B2). *Then there exists* $\bar{\varepsilon}_0>0$ *such that BVP* (P) *has an almost periodic solution* $u\in APF_0^{s+1,2}$ *for any* ε, $|\varepsilon|\leq\bar{\varepsilon}_0$. *The solution* u *is locally unique in* $APF_0^{s+1,2}$.

PROOF. To prove the theorem we again apply the Picard iteration method. We define the successive approximation sequence $\{u_n\}$ by (3.20)–(3.21).

In the proof we need to derive the higher order energy estimates of the solutions so that the differentiability of the solutions and so the almost-periodicity of the higher order derivatives of the solutions are obtained. For that main point is to estimate the higher order derivatives of the nonlinear term $f(x,t,u_n)$.

First we show by induction that $\{u_n\}$ is well defined in $APF_0^{s+1,2}$. By (B1) Proposition 3.1 assures $u_0\in APF_0^{s+1,2}$. Assume that u_n belongs to $APF_0^{s+1,2}$. We have only to show that $f(\cdot,t,u_n)$ is almost periodic in $K^s(0,a;x^0)$. Then we obtain $u_{n+1}\in APF_0^{s+1,2}$ by Proposition 3.1. By (B2) (ii) $f(\cdot,t,u_n)$ belongs to $C(R^1;K^s(0,a;x^0))$. By (B2) (i) (iii) it follows from Lemma 2.3 that $\partial_t^k\partial_\lambda^l f(x,t,\lambda)$, $k+l\leq s$ are numerically almost periodic uniformly with respect to $x,\ \lambda$. Hence $f(\cdot,t,\lambda)$ is almost periodic in $K^s(0,a;x^0)$ uniformly with respect to λ. It follows from Lemma 3.4 that $f(\cdot,t,u_n(\cdot,t))$ is almost periodic in $K^s(0,a;x^0)$.

By Proposition 3.1 we have

$$|u_{n+1}|_{F_0^{s+1,2}}\leq C\,|\varepsilon|\sup_{t\in R^1}|f(x,t,u_n)|_{H^s(0,a;x^0)}\,.$$

Let $|u_n|_{F_0^{s+1,2}}\leq\tilde{M}$. Then from Proposition 2.2 it follows that

$$|u_n(x,t)|\leq C\,|u_n|_{F_0^{s+1,2}}\leq C\tilde{M} \tag{3.26}$$

for any $(x,t)\in[0,a]\times R^1$. Then using the Moser-type inequality (see [14]), we see

$$\sup_{t\in R^1}|f(\cdot,t,u_n)|_{H^s(0,a;x^0)}\leq C_1(\sup_{t\in R^1}|u_n|^s_{H^s(0,a;x^0)}+1)\leq C_1(\tilde{M}^s+1)\,.$$

Hence we obtain

$$|u_{n+1}|_{F_0^{s+1,2}} \le C_2\,|\varepsilon|(\tilde{M}^s + 1).$$

Therefore taking $\varepsilon_1 > 0$ so as to satisfy $\varepsilon_1 \le \tilde{M}/\{C_2\,(\tilde{M}^s + 1)\}$, we obtain

$$|u_{n+1}|_{F_0^{s+1,2}} \le \tilde{M}$$

for any $\varepsilon, |\varepsilon| \le \varepsilon_1$.

We show the convergence of $\{u_n\}$ in $F_0^{s+1,2}$. Applying the Moser-type inequality to the difference of BVP (LP) for $n+1$ and n, and using Proposition 3.1, we have

$$\begin{aligned} &|u_{n+1} - u_n|_{F_0^{s+1,2}} \\ &\quad \le C_1\,|\varepsilon| \sup_{t \in R^1} |f(x,t,u_n(t)) - f(x,t,u_{n-1}(t))|_{H^s(0,a;x^0)} \\ &\quad \le C_2\,|\varepsilon| \sup_{t \in R^1} |u_n - u_{n-1}|_{H^s(0,a;x^0)}\,. \end{aligned}$$

Hence we have

$$|u_{n+1} - u_n|_{F_0^{s+1,2}} \le C_2\,|\varepsilon| |u_n - u_{n-1}|_{F_0^{s+1,2}}\,.$$

Therefore again taking $\varepsilon_1 > 0$ so as to satisfy $C_2\,|\varepsilon_1| \le \kappa$ for some constant $\kappa \in (0,1)$, we obtain

$$|u_{n+1} - u_n|_{F_0^{s+1,2}} \le \kappa\,|u_n - u_{n-1}|_{F_0^{s+1,2}}\,.$$

for any $\varepsilon, |\varepsilon| \le \varepsilon_1$. Hence $\{u_n\}$ is the Cauchy sequence in $F_0^{s+1,2}$, which leads to the convergence of $\{u_n\}$ to some $u \in F_0^{s+1,2}$ satisfying $|u|_{F_0^{s+1,2}} \le \tilde{M}$. u satisfies BVP (P) if $n \to \infty$ in (3.23).

From Lemma 2.2 we see the almost periodicity of u.

The local uniqueness is shown in the similar way to the above iteration convergent argument. □

4. Global Solutions of Nonlinear IBVP

Before showing the stability of almost periodic solutions obtained in the previous section, in this section we shall assure that IBVP (Q) has a unique global solution. We show the existence of time-global solutions of IBVP (LQ) and IBVP (Q) under more general conditions (see (C1)–(C2) and (D1)–(D3), respectively, below) than the conditions that the external forces h and f are almost periodic functions.

4.1. The Existence of a Solution of Linear IBVP. We consider a linear IBVP to a nonhomogeneous SS Eq.

$$\text{(LQ)} \qquad \begin{cases} A_m w(x,t) = g(x,t)\,, & (x,t) \in (0,a) \times R^1_+\,, \\ w(a,t) = 0\,, \quad t \in R^1_+\,, \\ w(x,0) = \phi(x)\,, \quad \partial_t w(x,0) = \psi(x)\,, \quad x \in (0,a)\,. \end{cases}$$

We assume the following conditions on a forcing term g and initial data $\phi,\ \psi$.

(C1) g belongs to $F^{s,0}_{m,+}$.

(C2) ϕ and ψ belong to $K^{s+1}(0,a;x^m)$ and $K^s(0,a;x^m)$, respectively.

The following proposition shows the existence and the uniqueness of a solution of the linear IBVP (LQ) and the energy estimate of the solutions and the higher order derivatives. The proposition plays an essential role in showing the exponential stability of the almost periodic solutions obtained in Section 3.

PROPOSITION 4.1. *Assume* (C1)–(C2). *Let* $s \geq 1$. *Then IBVP* (LQ) *has a unique global solution* w *in* $F^{s+1,2}_{m,+}$ *satisfying*

$$\text{(4.1)} \qquad \begin{aligned} |w(\cdot,t)|_{s+1,1,m} \leq C\Big\{ & e^{-\gamma t}\left(|\phi|_{H^{s+1}(0,a;x^m)} + |\psi|_{H^s(0,a;x^m)}\right) \\ & + \int_0^t e^{-\gamma\tau}|g(\cdot,t-\tau)|_{H^s(0,a;x^m)}d\tau\Big\} \end{aligned}$$

for $t \in R^1_+$, *where* γ *is the constant defined by* (3.5) *and* C *is independent of* $\phi,\ \psi,\ g$.

Moreover if $s = 0$, *the statement holds by replacing* $F^{s+1,2}_{m,+}$ *by* $F^{1,1}_{m,+}$, *respectively.*

It follows from the above proposition that for any initial data ϕ and ψ satisfying (C2) there exists a unique solution $u = B_m g \in F^{s+1,2}_{m,+}$ of IBVP (LQ) for $g \in F^{s,0}_{m,+}$, where $B_m = B_m(\phi,\psi)$ is a linear integral operator.

PROOF. We again use the Fourier method in the x-direction. Consider the Fourier series of g, ϕ and ψ

$$\text{(4.2)} \qquad g(\cdot,t) = \sum_{j=1}^{\infty} g_j(t)\,\phi_j\,, \quad \phi = \sum_{j=1}^{\infty} p_j\,\phi_j\,, \quad \psi = \sum_{j=1}^{\infty} q_j\,\phi_j\,.$$

From (C1)–(C2) the above Fourier series of $\phi,\ \psi$ and g converge in $K^{s+1}(0,a;x^m)$, $K^s(0,a;x^m)$ and $C(R^1;K^s(0,a;x^m))$, respectively. We expand $w(\cdot,t)$ into the formal Fourier series

$$\text{(4.3)} \qquad w(\cdot,t) = \sum_{j=1}^{\infty} w_j(t)\,\phi_j.$$

Substitute (4.2)–(4.3) into (LQ) and comparing the Fourier coefficients, we obtain

$$\begin{cases} w_j''(t) + \lambda_j w_j(t) + w_j'(t) = g_j(t)\,, \\ w_j(0) = p_j\,, \quad w_j'(0) = q_j\,, \quad j = 1, 2, \dots \end{cases} \tag{4.4}$$

Using the representation formula of the solutions of (4.4), we have

$$w_j(t) = b_j e^{\nu_j^- t} - c_j e^{\nu_j^+ t} + \frac{1}{2a_j}\int_0^t g_j(t-\tau)(e^{\nu_j^+\tau} - e^{\nu_j^-\tau})d\tau\,, \tag{4.5}$$

where a_j and $\nu_j^{\pm}$ are defined by (3.4) and

$$b_j = \frac{\nu_j^+ p_j - q_j}{2a_j}\,, \quad c_j = \frac{\nu_j^- p_j - q_j}{2a_j}\,.$$

Also we have

$$\begin{aligned} w_j'(t) &= \nu_j^- b_j e^{\nu_j^- t} - \nu_j^+ c_j e^{\nu_j^+ t} \\ &\quad + \frac{1}{2a_j}\int_0^t g_j(t-\tau)(\nu_j^+ e^{\nu_j^+\tau} - \nu_j^- e^{\nu_j^-\tau})d\tau\,. \end{aligned} \tag{4.6}$$

We estimate $\sum_{j=1}^{\infty}(\lambda_j^{s+1} w_j(t)^2 + \lambda_j^s w_j'(t)^2)$. This is done in the similar way in the proof of Proposition 3.1. We have, from (4.5)–(4.6)

$$\begin{aligned} \sum_{j=1}^{\infty}(\lambda_j^{s+1} w_j(t)^2 + \lambda_j^s w_j'(t)^2) &\le \sum_{j=1}^{\infty}\lambda_j^{s+1}\Bigg\{(|b_j|^2 + |c_j|^2)e^{-2\gamma t} \\ &\quad + \frac{1}{4|a_j|^2}\left|\int_0^t g_j(t-\tau)(e^{\nu_j^+\tau} - e^{\nu_j^-\tau})d\tau\right|^2\Bigg\} \end{aligned}$$

By the use of the inequality $\lambda_j(|b_j|^2 + |c_j|^2) \le c(\lambda_j|p_j|^2 + |q_j|^2)$ and the Minkowski inequality, it follows that

$$\begin{aligned} \sum_{j=1}^{\infty}(\lambda_j^{s+1} w_j(t)^2 + \lambda_j^s w_j'(t)^2) &\le c\,e^{-2\gamma t}\sum_{j=1}^{\infty}(\lambda_j^{s+1}|p_j|^2 + \lambda^s|q_j|^2) \\ &\quad + \int_0^t e^{-2\gamma\tau}\left\{\sum_{j=1}^{\infty}\lambda_j^s|g_j(t-\tau)|^2\right\}^{1/2} d\tau\,. \end{aligned}$$

Using the equivalence relation (3.13), we have

$$\begin{aligned} &|w(\cdot,t)|_{H^{s+1}(0,a;x^m)} + |\partial_t w(\cdot,t)|_{H^s(0,a;x^m)} \\ \le& C_1\left\{e^{-\gamma t}(|\phi|_{H^{s+1}(0,a;x^m)} + |\psi|_{H^s(0,a;x^m)}) + \int_0^t e^{-\gamma\tau}|g(\cdot,t-\tau)|_{H^s(0,a;x^m)}d\tau\right\}. \end{aligned}$$

This shows that the energy inequality (4.1) holds and w belongs to $F^{s+1,1}_{m,+}$. Using the equation in (LQ), we have

$$\sum_{j=1}^{\infty} \lambda_j^{s-1} w_j''(t)^2 \le C \sum_{j=1}^{\infty} \left\{ \lambda_j^{s+1} w_j(t)^2 + \lambda_j^{s-1} w_j'(t)^2 + \lambda_j^{s-1} g_j(t)^2 \right\} .$$

The right hand side converges uniformly in t. Hence by (3.13) we have $\partial_t^2 w \in C(R^1; K^{s-1}(0, a; x^m))$. Therefore w belongs to $F^{s+1,2}_{m,+}$. The uniqueness of the solution of IBVP (LQ) is clear. □

The following energy estimate will be used to show the existence of a unique global solution of the nonlinear IBVP (Q).

COROLLARY 4.1. *Under the same assumptions as Proposition* 4.1 *the solution* w *of IBVP* (LQ) *satisfies*

$$|w|_{F^{s+1,1}_{m,+}} \le C\left(|\phi|_{H^{s+1}(0,a;x^m)} + |\psi|_{H^s(0,a;x^m)} + \sup_{t \in R^1_+} |g(\cdot, t)|_{H^s(0,a;x^m)} \right) . \tag{4.7}$$

4.2. The Existence of a Global Solution of Nonlinear IBVP. We consider IBVP (Q)

$$\text{(Q)} \qquad \begin{cases} A_m u(x,t) = h(x,t) + \varepsilon f(x,t,u) , & (x,t) \in (0,a) \times R^1_+ , \\ u(a,t) = 0 , \quad t \in R^1_+ , \\ u(x,0) = \phi(x) , \quad \partial_t u(x,0) = \psi(x) , & x \in (0,a) . \end{cases}$$

4.2.1. Global Weak Solutions. First we deal with weak solutions of IBVP (Q). We assume the following conditions on h, f and ϕ, ψ.

(D1) h belongs to $C(R^1_+; L^2(0, a; x^m))$ with $\sup_{t \in R^1_+} |h(\cdot, t)|_{L^2(0,a;x^m)} < +\infty$.

(D2) (i) $f(x, t, \lambda)$ is continuous in $[0, a] \times R^1 \times R^1$.

(ii) $f(x, t, \lambda)$ is locally Lipschitz continuous in $\lambda \in R^1$ uniformly with respect to $(x, t) \in [0, a] \times R^1$: For any $r > 0$ there exists a constant $\rho_0(r) > 0$ such that

$$|f(x,t,\lambda_1) - f(x,t,\lambda_2)| \le \rho_0(r)|\lambda_1 - \lambda_2| \tag{4.8}$$

for $\lambda_1, \lambda_2 \in (-r, r)$ and $(x, t) \in [0, a] \times R^1$.

(iii) $f(x, t, \lambda)$ is locally bounded in $\lambda \in R^1$ uniformly with respect to $(x, t) \in [0, a] \times R^1$: For any $r > 0$ there exists a constant $\rho_1(r) > 0$ such that

$$|f(x,t,\lambda)| \le \rho_1(r) \tag{4.9}$$

for $\lambda \in (-r, r)$ and $(x, t) \in [0, a] \times R^1$.

(D3) ϕ and ψ belong to $K^1(0, a; x^m)$ and $L^2(0, a; x^m)$ respectively.

REMARK 4.1. The condition (A1) implies (D1), and the condition (A2) implies (D2).

EXAMPLE 4.1. The functions in Example 3.1; $f(x,t,\lambda) = \alpha(x,t)|\lambda|^{\rho-1}\lambda$, $f(x,t,\lambda) = \alpha(x,t)|\lambda|^{\rho}$, $f(x,t,\lambda) = \alpha(x,t)\lambda^{\rho}$ ($\rho \in Z_+$) and $f(x,t,\lambda) = \alpha(x,t)\sin\lambda$ satisfy (D2), where $\alpha(x,t)$ is continuous and bounded in $(x,t) \in [0,a] \times R^1_+$.

The following theorem shows the existence of a time-global weak solution of IBVP (Q).

THEOREM 4.1. *Assume* (D1)–(D3). *Let $R > 0$ be any constant. Take ϕ and ψ so as to satisfy $|\phi|_{H^1(0,a;x^m)} + |\psi|_{L^2(0,a;x^m)} \le R$. Then there exists $\varepsilon_1 > 0$ dependent on R such that IBVP* (Q) *for any ε, $|\varepsilon| \le \varepsilon_1$ has a global unique solution u in $F^{1,1}_{m,+}$.*

PROOF. Let $X = F^{1,1}_{m,+}$ and $Y = F^{0,0}_{m,+}$ for simplicity. Consider IBVP (LQ). By Proposition 4.1, for any initial data ϕ and ψ satisfying (D3) there exists a unique solution $u = B_m g \in X$ of IBVP (LQ) for $g \in Y$.

Now let us deal with IBVP (Q). We define a nonlinear operator $F_\varepsilon(u)$ by $B_m \circ (h(\cdot,t) + \varepsilon f(\cdot,t,u))$ for $u \in X$, where $\circ$ means the composition of operators. Set $r = |h|_Y$. We show that there exists $\varepsilon_1 > 0$ dependent on R and r such that F_ε for ε, $|\varepsilon| \le \varepsilon_1$ is a contracting mapping of a domain $B_{2C(R+r)} = \{u \in X;\, |u|_X \le 2C(R+r)\}$ into itself. Here C is the same constant as in (4.7) in Corollary 4.1. First we show that $f(x,t,u)$ belongs to Y for $u \in X$. Since u belongs to X, $u(x,t)$ is bounded in $[0,a] \times R^1$; $|u(x,t)| \le M$ for some $M > 0$. In fact, applying the Sobolev-type inequality [11] to the left hand side below, we have

$$|u(x,t)| \le M_0 |u(\cdot,t)|_{H^1(0,a;x^m)} \le M_0 \sup_{t\in R^1_+} |u(\cdot,t)|_{H^1(0,a;x^m)} \le M_0 |u|_X .$$

Hence by (D2) (iii) we have $|f(x,t,u)| \le \rho_1(M_0\,|u|_X)$. Therefore

$$|f(\cdot,t,u)|_Y \le C_1\,\rho_1(M_0\,|u|_X)$$

for $u \in X$. Let $|u|_X \le 2C(R+r)$. Then we have, by Corollary 4.1

$$|F_\varepsilon(u)|_X \le C\,\{R + r + |\varepsilon| C_1\,\rho_1(2M_0C(R+r))\} .$$

Take $\varepsilon_1 > 0$ so as to satisfy $\varepsilon_1 < (R+r)/\{C_1\,\rho_1(2M_0C(R+r))\}$. Then it follows that F_ε maps $B_{2C(R+r)}$ into $B_{2C(R+r)}$ for any ε with $|\varepsilon| < \varepsilon_1$.

Next we show that F_ε is a contraction mapping in X for any ε with $|\varepsilon| \le \varepsilon_1$ for a replaced constant ε_1. Then by the contraction mapping principle we obtain a unique fixed point u in $B_{2C(R+r)}$ that is the desired solution. Using Corollary 4.1 and (D2)(ii) in order, we have, for $u,\ v \in B_{2C(R+r)}$

$$\begin{aligned}|F_\varepsilon(u) - F_\varepsilon(v)|_X &\le C\,|\varepsilon| \sup_{t\in R^1_+} |f(\cdot,t,u) - f(\cdot,t,v)|_{L^2(0,a;x^m)} \\ &\le C\,|\varepsilon|\,\rho_0(2C(R+r)) \sup_{t\in R^1_+} |u(\cdot,t) - v(\cdot,t)|_{L^2(0,a;x^m)} \\ &\le C\,|\varepsilon|\,\rho_0(2C(R+r))\,|u-v|_X .\end{aligned}$$

Take further $\varepsilon_1 > 0$ so as to satisfy $\varepsilon_1 < 1/\{C\rho_0(2C(R+r))\}$. Then F_ε is contracting in $B_{2C(R+r)}$ for any ε with $|\varepsilon| \leq \varepsilon_1$. Therefore the conclusion follows. □

4.2.2. Global Strong and Classical Solutions. In this part we deal with the existence of strong and classical solutions of IBVP (Q). To this end some smoothness assumptions on the data are required. Also in this subsection we take $m = 0$ to obtain the regularity of the solutions, since we use the Sobolev-type inequality to derive the energy inequality. Through this subsection we assume $s \geq 1$.

(E1) h belongs to $F^{s,0}_{0,+}$.

(E2) (i) $f(x,t,\lambda)$ is of C^s-class in $[0,a] \times R^1_+ \times R^1$, and its derivatives up to order s are bounded in $[0,a] \times R^1_+ \times I$, where I is a finite interval in R^1.

(ii) For $u \in C(R^1_+; K^{s+1}(0,a;x^0)$ $f(\cdot,t,u)$ belongs to $C(R^1_+; K^s(0,a;x^0))$.

(E3) ϕ and ψ belong to $K^{s+1}(0,a;x^0)$ and $K^s(0,a;x^0)$, respectively.

REMARK 4.2. The condition (B2) implies the condition (E2).

The following theorem shows the existence of a global unique *strong solution* of IBVP (Q).

THEOREM 4.2. *Assume* (E1)–(E3). *Take* $m = 0$. *Let* $R > 0$ *be any constant. Take* ϕ *and* ψ *so as to satisfy* $|\phi|_{H^{s+1}(0,a;x^0)} + |\psi|_{H^s(0,a;x^0)} \leq R$. *Then there exists* $\bar{\varepsilon}_1 > 0$ *dependent on* R *such that IBVP* (Q) *for any* ε, $|\varepsilon| \leq \bar{\varepsilon}_1$ *has a global unique solution* u *in* $F^{s+1,2}_{0,+}$.

PROOF. The method of the proof is similar to the proof of Theorem 4.1 based on the contraction mapping principle. So we briefly describe the proof. In the proof the essential part is to use the higher order estimate of the nonlinear term to obtain the higher differentiability of the solutions.

Again take $F_\varepsilon(u) = B_0 \circ (h + \varepsilon f(\cdot,t,u))$. Let $X = F^{s+1,1}_{0,+}$, $Y = F^{s,0}_{0,+}$ and $B_c = \{u \in X;\, |u|_X \leq c\}$. Set $r = |h|_Y$. It follows from the Moser-type inequality ([14]) that $f(x,t,u)$ belongs to Y for $u \in X$. Let $u \in B_{2C(R+r)}$. Applying Corollary 4.1 and the Moser-type inequality ([14]) to F_ε in order, we estimate F_ε

$$\begin{aligned} |F_\varepsilon(u)|_X &\leq C\,(R + r + |\varepsilon| \sup_{t \in R^1_+} |f(\cdot,t,u)|_Y) \\ &\leq C\,\{R + r + c_1|\varepsilon|(1 + |u|^s_X)\} \\ &\leq C\,[R + r + c_1|\varepsilon|\{1 + (2C(R+r))^s\}]\,. \end{aligned}$$

By taking $\bar{\varepsilon}_1 > 0$ so as to satisfy $\bar{\varepsilon}_1 \leq (R+r)/[c_1\{1 + (2C(R+r))^s\}]$, $|F_\varepsilon|_X \leq 2C(R+r)$ holds for any ε with $|\varepsilon| \leq \bar{\varepsilon}_1$ *i.e.,* F_ε maps $B_{2C(R+r)}$ into $B_{2C(R+r)}$. Similarly applying Corollary 4.1 and the Moser-type inequality to the difference $F_\varepsilon(u) - F_\varepsilon(v)$, we show the Lipschitzness of F_ε in $B_{2C(R+r)}$ as follows: For $u,\ v \in B_{2C(R+r)}$

$$|F_\varepsilon(u) - F_\varepsilon(v)|_X \leq C\,|\varepsilon| \sup_t |f(\cdot,t,u) - f(\cdot,t,v)|_{H^s(0,a;x^0)}$$

$$\le C\,|\varepsilon|(|u|_X^{s-1} + |v|_X^{s-1})|u - v|_X$$
$$\le 2C\,|\varepsilon|\{2C(R + r)\}^{s-1}|u - v|_X\,.$$

Take again $\bar{\varepsilon}_1 > 0$ so as to satisfy $\bar{\varepsilon}_0 < 1/[2C\,\{2C(R + r)\}^{s-1}]$. Then F_ε is contracting in $B_{2C(R+r)}$ for ε with $|\varepsilon| \le \bar{\varepsilon}_1$. Hence by the contraction mapping principle F_ε has a unique fixed point u in $B_{2C(R+r)}$ that is the solution of IBVP (Q). Since $f(x, t, u)$ belongs to $C(R^1_+; K^s(0, a; x^0))$, we obtain $u \in F_0^{s+1,2}$. □

From Theorem 4.2 and Corollary 2.1 we obtain a *classical solution* of IBVP (Q) for $s \ge 2$.

COROLLARY 4.2. *Let $s \ge 2$. Then the solution u of IBVP* (Q) *in Theorem* 4.2 *is of C^2-class in $(x, t) \in (0, a] \times R^1$.*

5. Stability of Almost Periodic Solutions of BVP (P)

In this section we consider the stability of the almost periodic solutions obtained in Section 3. More precisely, we show that the solutions are locally exponentially stable. Main tool to show the stability is the energy estimate obtained in Proposition 4.1. We show the stability of both weak and strong almost periodic solutions of BVP (P).

From now on we always take ε such that the almost periodic solutions of BVP (P) in Theorems 3.1–3.2 exist respectively.

Let u_0 be the almost periodic solution of BVP (P) in Theorems 3.1–3.2. Let $\phi_0(x) = u_0(x, 0)$ and $\psi_0(x) = \partial_t u_0(x, 0)$. It follows from Theorems 3.1–3.2 that (ϕ_0, ψ_0) belong to $K^1(0, a; x^m) \times L^2(0, a; x^m)$ and $K^{s+1}(0, a; x^0) \times K^s(0, a; x^0)$, respectively according as u_0 belongs to $F^{1,1}_{m,+}$ and $F^{s+1,2}_{m,+}$.

We have the following result on the stability of the almost periodic weak solutions in Theorem 3.1.

THEOREM 5.1. *Assume* (A1)–(A2) *and* (D3). *Let u_0 be the almost periodic solution in Theorem* 3.1. *For any fixed constant $R > 0$, let ϕ and ψ satisfy*

$$|\phi - \phi_0|_{H^1(0,a;x^m)} + |\psi - \psi_0|_{L^2(0,a;x^m)} \le R\,.$$

Then there exists $\bar{\varepsilon} > 0$ dependent on R such that IBVP (Q) *for ε with $|\varepsilon| \le \bar{\varepsilon}$ has a unique solution u in $F^{1,1}_{m,+}$ satisfying the following asymptotic property*

$$\begin{aligned} &|u(\cdot, t) - u_0(\cdot, t)|_{H^1(0,a;x^m)} + |\partial_t(u(\cdot, t) - u_0(\cdot, t))|_{L^2(0,a;x^m)} \\ &\quad \le C\, e^{(-\gamma+\nu\,|\varepsilon|)t}, \quad t \in R^1_+\,, \end{aligned} \tag{5.1}$$

where $\nu > 0$ is a constant.

PROOF. Note that the conditions (A1)–(A2) and (D3) imply the conditions (D1)–(D3). Consider the following IBVP

$$
(5.2)\qquad \begin{cases} A_m v(x,t) = \varepsilon\,(f(x,t,v+u_0) - f(x,t,u_0))\,, & (x,t) \in (0,a)\times R^1_+\,,\\ v(x,0) = \phi(x) - \phi_0(x)\,, \quad \partial_t v(x,0) = \psi(x) - \psi_0(x)\,, & x \in (0,a)\,,\\ v(a,t) = 0\,, \quad t \in R^1_+\,. \end{cases}
$$

The nonlinear term $\varepsilon\,(f(x,t,v+u_0) - f(x,t,u_0))$ satisfies (D2), and the initial data satisfy (D3). Therefore by Theorem 4.1 there exists $\bar\varepsilon > 0$ dependent on R such that IBVP (5.2) for ε with $|\varepsilon| \le \bar\varepsilon$ has a solution v in $F^{1,1}_{m,+}$. Take $u = v + u_0$. Then u is a unique solution of IBVP (Q) in $F^{1,1}_{m,+}$. Also by Proposition 4.1 we have

$$
\begin{aligned}
&e^{\gamma t}(|\partial_t v(\cdot,t)|_{L^2(0,a;x^m)} + |v(\cdot,t)|_{H^1(0,a;x^m)})\\
&\quad \le C_1\left(L + |\varepsilon|\int_0^t e^{-\gamma\tau}|f(x,\tau,v+u_0) - f(x,\tau,u_0)|_{L^2(0,a;x^m)}d\tau\right).
\end{aligned}
$$

Here we set $L = |\phi - \phi_0|_{H^1(0,a;x^m)} + |\psi - \psi_0|_{L^2(0,a;x^m)}$. Using (A2) (ii), we see

$$
\begin{aligned}
&e^{\gamma t}(|\partial_t v(\cdot,t)|_{L^2(0,a;x^m)} + |v(\cdot,t)|_{H^1(0,a;x^m)})\\
&\quad \le C_1\left(L + c|\varepsilon|\int_0^t e^{\gamma\tau}|v(\cdot,\tau)|_{L^2(0,a;x^m)}d\tau\right).
\end{aligned}
$$

Set $\alpha(t) = e^{\gamma t}(|\partial_t v(\cdot,t)|_{L^2(0,a;x^m)} + |v(\cdot,t)|_{H^1(0,a;x^m)})$. Then we have the inequality

$$
\alpha(t) \le C_2(L + c|\varepsilon|\int_0^t \alpha(\tau)d\tau)\,.
$$

Therefore it follows from the Gronwall inequality that

$$
(5.3)\qquad \alpha(t) \le C_2 L e^{C_2 c|\varepsilon| t}\,.
$$

Hence we obtain (5.1) with $\nu = C_2 c$. □

Next we state the stability result of the strong and classical almost periodic solutions of BVP (P).

THEOREM 5.2. *Assume* (B1)–(B2) *and* (E3). *Let u_0 be the almost periodic solution in Theorem* 3.2. *Let $R > 0$ be any constant. Let ϕ and ψ satisfy*

$$
|\phi - \phi_0|_{H^{s+1}(0,a;x^0)} + |\psi - \psi_0|_{H^s(0,a;x^0)} \le R\,.
$$

Then there exists $\tilde\varepsilon > 0$ such that IBVP (Q) *for ε with $|\varepsilon| \le \tilde\varepsilon$ has a solution u in $F^{s+1,2}_{0,+}$ satisfying the following asymptotic property*

$$
\begin{aligned}
(5.4)\qquad &|u(\cdot,t) - u_0(\cdot,t)|_{H^{s+1}(0,a;x^0)} + |\partial_t(u(\cdot,t) - u_0(\cdot,t))|_{H^s(0,a;x^0)}\\
&\quad \le C\,e^{(-\gamma+\tilde\nu\,|\varepsilon|)t}\,, \quad t \in R^1_+\,,
\end{aligned}
$$

where $\tilde{\nu} > 0$ *is a constant.*

PROOF. The strategy of the proof is similar to that of the proof of Theorem 5.1 by using the energy estimate (Proposition 4.1). So we describe only the outline of the proof. In order to deal with the regularity of the solutions, we use the Moser-type inequality ([14]). The existence of the global solutions is shown, using Theorem 4.2.

Consider IBVP (5.2) and apply Proposition 4.1 and the Moser-type inequality to the IBVP. Then we have

$$\begin{aligned} & e^{\gamma t}(|\partial_t v(\cdot,t)|_{H^s(0,a;x^0)} + |v(\cdot,t)|_{H^{s+1}(0,a;x^0)}) \\ & \quad \le C_1\left(L_1 + |\varepsilon|\int_0^t e^{\gamma\tau}|f(x,\tau,u) - f(x,\tau,u_0)|_{H^s(0,a;x^0)}d\tau\right) \\ & \quad \le C_1\left(L_1 + c_1|\varepsilon|\int_0^t e^{\gamma\tau}|u(\cdot,\tau) - u_0(\cdot,\tau)|_{H^s(0,a;x^0)}d\tau\right). \end{aligned}$$

Here we set $L_1 = |\phi-\phi_0|_{H^{s+1}(0,a;x^0)} + |\psi-\psi_0|_{H^s(0,a;x^0)}$. Applying the Gronwall inequality to this inequality, we obtain (5.4). □

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