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Some Relationships between the Geometry of the Tangent Bundle and the Geometry of the Riemannian Base Manifold

Guillermo HENRY and Guillermo KEILHAUER

Universidad de Buenos Aires

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Abstract. We compute the curvature tensor of the tangent bundle of a Riemannian manifold endowed with a natural metric and we get some relationships between the geometry of the base manifold and the geometry of the tangent bundle.

1. Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Let $\pi : TM \longrightarrow M$ and $P : O(M) \longrightarrow M$ be the tangent and the orthonormal bundle over M respectively. In this paper we deal with a certain class of Riemannian metrics on TM. A metric G belongs to this class if the canonical projection $\pi : (TM, G) \longrightarrow (M, g)$ is a Riemannian submersion, the horizontal distribution induced by the Levi-Civita connection of (M, g) is orthogonal to the vertical distribution, and G is the image by a natural operator of order two of the metric g. The Sasaki metric and the Cheeger-Gromoll metric are well known examples of this class of metrics, and they were extensively studied by Kowalski [7], Aso [2], Sekizawa [11], Musso and Tricerri [9], Gudmundsson and Kappos [4] among others. The notion of *natural tensor* on the tangent bundle of a Riemannian manifold as a tensor that is the image by a natural operator of order two of the base manifold metric was introduced and characterized by Kowalski and Sekizawa in [8]. In [3], Calvo and the second author showed that for a given Riemannian manifold (M, g), any $(0, 2)$ tensor field on TM admits a global matrix representation. Using this one to one relationship, they defined and characterized, without making use of the theory of differential invariants, what they also called a *natural tensor*. In the symmetric case this concept coincides with the one defined by Kowalski and Sekizawa. In [5], the first author gives a new approach to the concept of naturality, introducing the notion of *s-space* and λ-*naturality*. This approach avoids jets and natural operator theory and generalizes the one given in [3] and [8].

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In section 2, we introduce natural metrics on TM by means of [3]. For any $q \in M$, let M_q be the tangent space of M at q. Let $\psi : N := O(M) \times \mathbf{R}^n \longrightarrow TM$ be the projection defined by

$$\psi(q, u, \xi) = \sum_{i=1}^{n} \xi^i u_i \tag{1}$$

where $u = (u_1, \ldots, u_n)$ is an orthonormal basis for M_q and $\xi = (\xi^1, \ldots, \xi^n) \in \mathbf{R}^n$. It is well known (see [9]), that for a fixed Riemannian metric G on TM a suitable Riemannian metric G^* on N can be defined such that $\psi : (N, G^*) \longrightarrow (TM, G)$ is a Riemannian submersion. Based on this fact and the O'Neill formula, in Section 3, we compute the curvature tensor of (TM, G), when G is a natural metric. As an application, we get in Section 4 some relationships between the geometry of TM and the geometry of M. In [1] Abbassi and Sarih studied some relationships between the geometry of TM and the geometry of M, when TM is endowed with a $g - natural\ metric$. For example (Theorem 0.1) states that if (TM, G) is flat, then (M, g) is flat. Since in this paper we deal with a subclass of $g - natural\ metrics$ we get Corollary 4.2 as a converse of this theorem. Throughout, all geometric objects are assumed to be differentiable, i.e. C^∞.

2. Preliminaries

Let ∇ be the Levi-Civita connection of g and $K : TTM \longrightarrow TM$ be the connection map induced by ∇. For any $q \in M$ and $v \in M_q$, let $\pi_{*_v} : (TM)_v \longrightarrow M_q$ be the differential map of π at v, and $K_v : (TM)_v \longrightarrow M_q$ be the restriction of K to $(TM)_v$.

Since the linear map $\pi_{*_v} \times K_v : (TM)_v \longrightarrow M_q \times M_q$ defined by $(\pi_{*_v} \times K_v)(b) = (\pi_{*_v}(b), K_v(b))$ is an isomorphism that maps the horizontal subspace $(TM)_v^h = \ker K_v$ onto $M_q \times \{0_q\}$ and the vertical subspace $(TM)_v^v = \ker \pi_{*_v}$ onto $\{0_q\} \times M_q$, where 0_q denotes the zero vector, we define differentiable mappings $e_i, e_{n+i} : N = O(M) \times \mathbf{R}^n \longrightarrow TTM$ for $i = 1, \ldots, n$ and $v = \psi(q, u, \xi)$ by

$$\begin{aligned} e_i(q, u, \xi) &= (\pi_{*_v} \times K_v)^{-1}(u_i, 0_q)\,, \\ e_{n+i}(q, u, \xi) &= (\pi_{*_v} \times K_v)^{-1}(0_q, u_i)\,. \end{aligned} \tag{2}$$

The action of the orthonormal group $O(n)$ of $\mathbf{R}^{n \times n}$ on N is given by the family of maps $R_a : N \longrightarrow N$, $a \in O(n)$, $R_a(q, u, \xi) = (q, u.a, \xi.a)$ where $u.a = (\sum_{i=1}^n a_1^i u_i, \ldots, \sum_{i=1}^n a_n^i u_i)$ and $\xi.a = (\sum_{i=1}^n a_1^i \xi^i, \ldots, \sum_{i=1}^n a_n^i \xi^i)$. Clearly, $\psi \circ R_a = \psi$. It follows from (2) that

$$e_j(R_a(p, u, \xi)) = \sum_{i=1}^{n} e_i(p, u, \xi) a_j^i \quad \text{for } j = 1, \ldots, n$$

and

$$e_{n+j}(R_a(p,u,\xi)) = \sum_{i=1}^{n} e_{n+i}(p,u,\xi)a^i_j \quad \text{for } j = 1,\dots,n\,.$$

For any $(0,2)$ tensor field T on TM we define the differentiable function ${}^gT : N \longrightarrow \mathbf{R}^{2n\times 2n}$ as follows: If $(q,u,\xi) \in N$ and $v = \psi(q,u,\xi)$, let ${}^gT(q,u,\xi)$ be the matrix of the bilinear form $T_v : (TM)_v \times (TM)_v \longrightarrow \mathbf{R}$ induced by T on $(TM)_v$ with respect to the basis $\{e_1(q,u,\xi),\dots,e_{2n}(q,u,\xi)\}$. One sees easily that ${}^gT$ satisfies the following invariance property:

$${}^gT \circ R_a = (L(a))^t.{}^gT.L(a) \tag{3}$$

where $L : O(n) \longrightarrow \mathbf{R}^{2n\times 2n}$ is the map defined by

$$L(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Moreover, there is a one to one correspondence between the $(0,2)$ tensor fields on TM and differentiable maps ${}^gT$ satisfying (3).

A tensor field T on TM will be call *natural with respect to* g if ${}^gT$ depends only on the parameter ξ, (see [3]). In the sense of [5], the collection $\lambda = (N, \psi, O(n), \tilde{R}, \{e_i\})$ is a s-space over TM, with base change morphism L; and the natural tensors with respect to g are the λ-*natural tensors with respect to* TM.

Writing ${}^gT$ in the block form ${}^gT = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$, where $A_i : N \longrightarrow \mathbf{R}^{n\times n}$, it follows from Lemma 3.1 of [3] that T is natural if there exist differentiable functions $\alpha_i, \beta_i : [0,+\infty) \longrightarrow \mathbf{R}$ $(i = 1,2,3,4)$, such that

$$A_i(p,u,\xi) = \alpha_i(|\xi|^2)Id_{n\times n} + \beta_i(|\xi|^2)\xi^t.\xi$$

where $|\xi|$ denotes the norm of ξ induced by the canonical inner product of $\mathbf{R}^n$. In that case T is said to be a $g-natural\ metric$ if in addition T is a Riemannian metric.

It is easy to check that a (0,2)- tensor field T on TM is a $g-natural\ metric$ if and only if T is natural, $A_2 = A_4$, $\alpha_3(t) > 0$, $\alpha_1(t).\alpha_3(t) - \alpha_2^2(t) > 0$, $\phi_3(t) > 0$ and $\phi_1(t)\phi_3(t) - \phi_2^2(t) > 0$ for all $t \geq 0$; where $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$ for $i = 1,2,3$.

In this paper we will call G a natural metric on TM if:

1. G is a Riemannian metric such that $\pi : (TM, G) \longrightarrow (M, g)$ is a Riemannian submersion.
2. For $v \in TM$, the subspaces $(TM)^v_v$ and $(TM)^h_v$ are orthogonal.
3. G is natural with respect to g.

It follows that G is a natural metric on TM if

$$ {}^{g}G(p,u,\xi) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & \alpha(|\xi|^2).Id_{n\times n} + \beta(|\xi|^2)(\xi)^t.\xi \end{pmatrix} \tag{4} $$

where $\alpha, \beta : [0,+\infty) \longrightarrow \mathbf{R}$ are differentiable functions satisfying $\alpha(t) > 0$, and $\alpha(t) + t\beta(t) > 0$ for all $t \geq 0$.

REMARK 2.1. The Sasaki metric G_s corresponds to the case $\alpha = 1$, $\beta = 0$; and the Cheeger-Gromoll metric G_{ch} to the case $\alpha(t) = \beta(t) = \frac{1}{1+t}$.

3. Curvature equations

In this section we compute the curvature tensor of TM endowed with a natural metric. Since this computation involves well known objects defined on N, we shall begin to describe them briefly using the connection map.

3.1. Canonical constructions on N. Let θ^i, ω^i_j be the canonical 1-forms on $O(M)$, which in terms of the connection map are defined as follows:

$$ \theta^i(q,u)(b) = g_q(P_{*(q,u)}(b), u_i) \tag{5} $$

and

$$ \omega^i_j(q,u)(b) = g_q(K((\pi_j)_{*(q,u)}(b)), u_i) \tag{6} $$

where $\pi_j : O(M) \longrightarrow TM$ is the j^{th} projection, i.e. $\pi_j(q,u) = u_j$ and $1 \leq i, j \leq n$.

From now on, let θ^i, ω^i_j, $d\xi^i$ be the pull backs of the canonical 1-forms on $O(M)$ and the usual 1-forms on $\mathbf{R}^n$ by the projections $P_1 : N \longrightarrow O(M)$ and $P_2 : N \longrightarrow \mathbf{R}^n$ respectively.

For any $z \in N$ let us denote by $V_z = \ker \psi_{*_z}$ and $H_z := \{b \in N_z : \omega^i_j(z)(b) = 0, 1 \leq i < j \leq n\}$ the vertical and the horizontal subspace of N_z respectively. By letting (see [9])

$$ \theta^{n+i} = d\xi^i + \sum_{j=1}^{n} \xi^j.\omega^i_j \tag{7} $$

we get that for any $z \in N$, $\{\theta^1(z), \ldots, \theta^{2n}(z), \{\omega^i_j(z)\}\}$ is a basis for N^*_z and $V_z := \{b \in N_z : \theta^l(z)(b) = 0$ for $1 \leq l \leq 2n\}$.

Let $H_1, \ldots, H_{2n}, \{V^l_m\}_{1\leq l<m\leq n}$ be the dual frame of $\{\theta^1, \ldots, \theta^{2n}, \{\omega^i_j\}\}$. These vector fields were constructed as follow: If $z = (q,u,\xi)$, let c_i be the geodesic that satisfies $c_i(0) = q$ and $\dot{c}_i(0) = u_i$. Let $E^i_1, \ldots, E^i_n$ be the parallel vector fields along c_i such that $E^i_l(0) = u_l$. If we define $\gamma_i(t) = (c_i(t), E^i_1(t), \ldots, E^i_n(t), \xi)$, then

$$ H_i(z) = \dot{\gamma}_i(0) \tag{8} $$

and

$$H_{n+i}(z) = (i_{(q,u)})_{*_\xi}\left(\frac{\partial}{\partial \xi^i}|_\xi\right) \tag{9}$$

for $1 \leq i \leq n$, where $i_{(q,u)} : \mathbf{R}^n \longrightarrow N$ is the inclusion map given by $i_{(q,u)}(\xi) = (q, u, \xi)$.

Let $\sigma_z : O(n) \longrightarrow N$ be the map defined by $\sigma_z(a) = R_a(z) = z.a$. Since $V_z = \ker(\psi_{*_z}) = (\sigma_z)_{*Id}(\mathfrak{o}(n))$, where $\mathfrak{o}(n)$ is the space of skew symmetric matrices of $\mathbf{R}^{n\times n}$, let

$$V^l_m(z) = (\sigma_z)_{*id}(A^l_m) \tag{10}$$

where $[A^l_m]^l_m = 1$, $[A^l_m]^m_l = -1$ and $[A^l_m]^i_j = 0$ otherwise. Hence,

$$\psi_{*_z}(V^l_m(z)) = 0\,. \tag{11}$$

An easy check shows that

$$\psi_{*_z}(H_i(z)) = e_i(z) \tag{12}$$

and

$$\psi_{*_z}(H_{n+i}(z)) = e_{n+i}(z)\,. \tag{13}$$

Let $\omega = \sum_{1\leq i<j\leq n} \omega^i_j \otimes \omega^i_j$, if G is a Riemannian metric on TM then

$$G^* = \psi^*(G) + \omega \tag{14}$$

is also a Riemannian metric on N. It follows easily that $V_z \perp_{G^*} H_z$ and $\psi_{*_z} : H_z \longrightarrow (TM)_{\psi(z)}$ is an isometry, therefore $\psi : (N, G^*) \longrightarrow (TM, G)$ is a Riemannian submersion. We shall use this fact to compute the curvature tensor of (TM, G) when G is a natural metric.

REMARK 3.1. Let X be a vector field on TM, the horizontal lift of X is a vector field X^h on N such that $X^h(z) \in H_z$ and $\psi_{*_z}(X^h(z)) = X(\psi(z))$. If $X(\psi(z)) = \sum_{i=1}^{2n} x^i(z)e_i(z)$, from (11), (12) and (13) it follows that $X^h(z) = \sum_{i=1}^{2n} x^i(z)H_i(z)$.

PROPOSITION 3.2. *For $1 \leq i, j, l, m \leq n$ let $R_{ijlm} : N \longrightarrow \mathbf{R}$ be the maps defined by $R_{ijlm}(q, u, \xi) = g(R(u_i, u_j)u_l, u_m)$, where R is the curvature tensor of (M, g). The Lie bracket on vertical and horizontal vector fields on N satisfies:*

a) $[H_i, H_j] = \sum_{l,m=1}^n R_{ijlm}\xi^m H_{n+l} + \frac{1}{2}\sum_{l,m=1}^n R_{ijlm} V^l_m$.
b) $[H_i, H_{n+j}] = 0$.
c) $[H_i, V^l_m] = \delta_{il} H_m - \delta_{im} H_l$.
d) $[H_{n+i}, H_{n+j}] = 0$.
e) $[H_{n+i}, V^l_m] = \delta_{il} H_{n+m} - \delta_{im} H_{n+l}$.
f) $[V^i_j, V^l_m] = \delta_{il} V^m_j + \delta_{jl} V^i_m + \delta_{im} V^j_l + \delta_{jm} V^l_i$.
g) *If $f : N \longrightarrow \mathbf{R}$ is a function that depends only on the parameter ξ, then $H_i(f) = 0$ and $V^i_j(f) = \xi^i H_{n+j}(f) - \xi^j H_{n+i}(f)$.*

h) *If* X *and* Y *are tangent vector fields on* TM *and* $v = \psi(q,u,\xi)$ *then* $[X^h, Y^h]^v|_{(q,u,\xi)} = \sum_{1\le l<m\le n} g_q(R(\pi_*(X(v)), \pi_*(Y(v)))u_l, u_m)V^l_m(q,u,\xi)$.

The proof is straightforward and follows by taking local coordinates in M and the induced one in TM and evaluating the forms θ^i, θ^{n+i}, ω^i_j on the fields $[H_r, H_s]$, $[H_r, V^l_m]$ and $[V^l_m, V^{l'}_{m'}]$ for $1\le r,s\le 2n$, $1\le l<m\le n$ and $1\le l'<m'\le n$.

3.2. The main result. From now on, let $\bar{R}$ and R^* be the curvature tensors of (TM, G) and (N, G^*) respectively. For simplicity we denote by $\langle\ ,\ \rangle$ the metrics G and G^*. Since $\psi : (N, G^*) \longrightarrow (TM, G)$ is a Riemannian submersion, by the O'Neill formula (see [10]) we have that

$$\begin{aligned}\langle \bar{R}(X,Y)Z, W\rangle \circ \psi = {}& \langle R^*(X^h, Y^h)Z^h, W^h\rangle + \frac{1}{4}\langle [Y^h, Z^h]^v, [X^h, W^h]^v\rangle \\ & - \frac{1}{4}\langle [X^h, Z^h]^v, [Y^h, W^h]^v\rangle - \frac{1}{2}\langle [Z^h, W^h]^v, [X^h, Y^h]^v\rangle\,. \end{aligned} \tag{15}$$

If $Y^h(z) = \sum_{j=1}^{2n} y^j(z)H_j(z)$, $Z^h(z) = \sum_{k=1}^{2n} z^k(z)H_k(z)$ and $W^h(z) = \sum_{l=1}^{2n} w^l(z)H_l(z)$, then the first term of the right side of equality (15) is

$$\langle R^*(X^h, Y^h)Z^h, W^h\rangle = \sum_{ijkl=1}^{2n} x^i y^j z^k w^l \langle R^*(H_i, H_j)H_k, H_l\rangle\,.$$

On the other hand, if $v = \psi(q,u,\xi)$, it follows from Proposition 3.2 (part h) that

$$\begin{aligned}&\langle [X^h, Y^h]^v, [Z^h, W^h]^v\rangle|_{(q,u,\xi)} \\ &= \frac{1}{2}\sum_{r,s=1}^{n} \langle R(\pi_*(X(v)), \pi_*(Y(v)))u_r, u_s\rangle . \langle R(\pi_*(Z(v)), \pi_*(W(v)))u_r, u_s\rangle\,. \end{aligned} \tag{16}$$

REMARK 3.3. In order to compute $\langle \bar{R}(X(v), Y(v))Z(v), W(v)\rangle$ it is sufficient to evaluate the right side of (15) on points of N of the form $z = (q, u, t, 0, \ldots, 0)$ such that $v = \psi(z)$, where $t = |v|$, and where $|v|$ is the norm induced by the metric g.

Let $f : [0, +\infty) \longrightarrow \mathbf{R}$ be a differentiable map. From now on, let us denote by $\dot{f}(t)$ the derivative of f at t.

THEOREM 3.4. *Let* G *be a natural metric on* TM. *Let* α *and* β *be the functions that characterizes* G. *If* $1\le i,j,k,l\le n$ *and* $z = (q,u,t,0,\ldots,0)$ *we have that*

a) $\langle R^*(H_i(z), H_j(z))H_k(z), H_l(z))\rangle$

$$= t^2\alpha(t^2).\sum_{r=1}^{n}\left\{\frac{1}{2}R_{ijr1}(z)R_{klr1}(z) + \frac{1}{4}R_{ilr1}(z)R_{kjr1}(z) + \frac{1}{4}R_{jlr1}(z)R_{ikr1}(z)\right\}$$

$$+ \sum_{1\le r<s\le n} \left\{ \frac{1}{2} R_{ijr1}(z) R_{klrs}(z) + \frac{1}{4} R_{ilr1}(z) R_{kjrs}(z) + \frac{1}{4} R_{jlr1}(z) R_{ikrs}(z) \right\}$$

$$+ R_{ijkl}(z) .$$

b) *Let* $\varepsilon_{ijkl} = \delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik}$, *then*

b.1) *If no index is equal to one*, *then*

$$\langle R^*(H_{n+i}(z), H_{n+j}(z)) H_{n+k}(z), H_{n+l}(z)\rangle = \varepsilon_{ijkl} F(t^2)$$

where $F : [0, +\infty) \longrightarrow \mathbf{R}$ *is defined by*

$$F(t) = \frac{\alpha(t)\beta(t) - t(\dot\alpha(t))^2 - 2\alpha(t)\dot\alpha(t)}{\alpha(t) + t\beta(t)} . \tag{17}$$

b.2) *If some index equals one*, *for example* $l = 1$, *then*

$$\langle R^*(H_{n+i}(z), H_{n+j}(z)) H_{n+k}(z), H_{n+1}(z)\rangle = \varepsilon_{ijk1} H(t^2)$$

where $H : [0, +\infty) \longrightarrow \mathbf{R}$ *is defined by*

$$H(t) = \phi(t) \frac{\partial}{\partial t} \ln(\alpha\Delta)|_t - 2\dot\phi(t) \tag{18}$$

and $\phi(t) = \alpha(t) + t\dot\alpha(t)$, $\Delta(t) = \alpha(t) + t\beta(t)$.

c) $\langle R^*(H_i(z), H_{n+j}(z)) H_{n+k}(z), H_{n+l}(z)\rangle = 0$.

d) $\langle R^*(H_{n+i}(z), H_{n+j}(z)) H_k(z), H_l(z)\rangle$

$$= \frac{1}{2}(2\alpha(t^2) + (\delta_{i1} + \delta_{j1})\beta(t^2)t^2) R_{ijkl}(z) + \frac{1}{2}\delta_{i1}(\beta(t^2) - 2\dot\alpha(t^2)) t^2 R_{klj1}(z)$$

$$+ \frac{1}{2}\delta_{j1}(2\dot\alpha(t^2) - \beta(t^2)) t^2 R_{kli1}(z)$$

$$+ \frac{(\alpha(t^2))^2 t^2}{4} \sum_{r=1}^{n} \{R_{krj1}(z) R_{rli1}(z) - R_{kri1}(z) R_{rlj1}(z)\} .$$

e) $\langle R^*(H_i(z), H_{n+j}(z)) H_k(z), H_{n+l}(z)\rangle$

$$= \frac{1}{2}\alpha(t^2) R_{kilj}(z) + \frac{(\alpha(t^2))^2 t^2}{4} \sum_{r=1}^{n} R_{krj1}(z) R_{ril1}(z)$$

$$+ \frac{t^2}{2}(\delta_{j1} + \delta_{l1}) \dot\alpha(t^2) (R_{kil1}(z) - R_{kij1}(z)) .$$

f) $\langle R^*(H_i(z), H_j(z)) H_{n+k}(z), H_l(z))\rangle$

$$= \frac{\alpha(t^2)t}{2} \{\langle \nabla_D R(E^i_j(s), E^l_j(s)) E^k_j(s)|_{s=0}, u_1\rangle - \langle \nabla_D R(E^j_i(s), E^l_i(s)) E^k_i(s)|_{s=0}, u_1\rangle\} .$$

The proof follows from the Koszul formula and Proposition 3.2 and it involves a lot of calculation. For more details we refer the reader to [6] pages 132–151.

THEOREM 3.5. *The curvature tensor $\bar{R}$ evaluated on $e_i(z)$, $e_{n+i}(z)$ satisfies*:

a) $\langle \bar{R}(e_i(z), e_j(z))e_k(z), e_l(z)\rangle$

$$= t^2\alpha(t^2)\sum_{r=1}^{n}\left\{\frac{1}{2}R_{ijr1}(z)R_{klr1}(z) + \frac{1}{4}R_{ilr1}(z)R_{kjr1}(z) + \frac{1}{4}R_{jlr1}(z)R_{ikr1}(z)\right\} + R_{ijkl}(z)\,.$$

b) b.1) *If no index is equal to one*, *then*

$$\langle \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z)\rangle = \varepsilon_{ijkl}.F(t^2)\,. \tag{19}$$

b.2) *If some index equals one*, *for example $l = 1$*, *then*

$$\langle \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+k}(z), e_{n+1}(z)\rangle = \varepsilon_{ijk1}.H(t^2)\,. \tag{20}$$

c) $\langle \bar{R}(e_i(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z)\rangle = 0.$

d) $\langle \bar{R}(e_{n+i}(z), e_{n+j}(z))e_k(z), e_l(z)\rangle$

$$= \frac{1}{2}(2\alpha(t^2) + (\delta_{i1} + \delta_{j1})\beta(t^2)t^2)R_{ijkl}(z) + \frac{1}{2}\delta_{i1}(\beta(t^2) - 2\dot{\alpha}(t^2))t^2R_{klj1}(z) + \frac{1}{2}\delta_{j1}(2\dot{\alpha}(t^2) - \beta(t^2))t^2R_{kli1}(z) + \frac{(\alpha(t^2))^2t^2}{4}\sum_{r=1}^{n}\{R_{krj1}(z)R_{rli1}(z) - R_{kri1}(z)R_{rlj1}(z)\}\,.$$

e) $\langle \bar{R}(e_i(z), e_{n+j}(z))e_k(z), e_{n+l}(z)\rangle$

$$= \frac{1}{2}\alpha(t^2)R_{kilj}(z) + \frac{(\alpha(t^2))^2t^2}{4}\sum_{r=1}^{n}R_{krj1}(z)R_{ril1}(z) + \frac{t^2}{2}(\delta_{j1} + \delta_{l1})\dot{\alpha}(t^2)(R_{kil1}(z) - R_{kij1}(z))\,.$$

f) $\langle \bar{R}(e_i(z), e_j(z))e_{n+k}(z), e_l(z))\rangle$

$$= \frac{\alpha(t^2)t}{2}\{\langle \nabla_D R(E^i_j(s), E^l_j(s))E^k_j(s)|_{s=0}, u_1\rangle - \langle \nabla_D R(E^j_i(s), E^l_i(s))E^k_i(s)|_{s=0}, u_1\rangle\}\,.$$

PROOF. The proof is straightforward and follows form Theorem 3.4 and equality (15). □

The functions F and H satisfy the following proposition:

PROPOSITION 3.6. *Let* $\alpha, \beta : [0, +\infty) \longrightarrow \mathbf{R}$ *be differentiable functions such that* $\alpha(t) > 0$ *and* $\alpha(t) + t\beta(t) > 0$ *for all* $t \geq 0$. *If* F *is the zero function, then*:

i) $\beta(t) = \frac{t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)}{\alpha(t)}$.
ii) $\alpha(t)(\alpha(t) + t\beta(t)) = (t\dot{\alpha}(t) + \alpha(t))^2$.
iii) $\alpha(t) + t\dot{\alpha}(t) > 0$.
iv) $H(t) = 0$ *for all* $t \geq 0$.

PROOF. Assertion i) follows from equality (17) and ii) is a consequence of i). Equality ii) shows that $\alpha(t) + t\dot{\alpha}(t) \neq 0$ for all $t \geq 0$, and since $\alpha(0) + 0.\dot{\alpha}(0) = \alpha(0) > 0$, then we get iii). Equality ii) says that $\alpha.\Delta = \phi^2$, and assertion iii) says that $\phi > 0$. Therefore, from equality (18) we get that $H = 0$. □

COROLLARY 3.7. *Let* $\alpha, \beta : [0, +\infty) \longrightarrow \mathbf{R}$ *be differentiable functions such that* $\alpha(t) > 0, \alpha(t) + t\dot{\alpha}(t) > 0$ *and* $\alpha(t) + t\beta(t) > 0$ *if* $t \geq 0$. *If* H *is the zero function, then it is also* F.

PROOF. Since $\phi > 0$ and $H = 0$, the equality (18) implies that $\ln(\alpha\Delta) = \ln(\phi^2) + C$ for some constant C. In particular $2\ln(\alpha(0)) = 2\ln(\alpha(0)) + C$, hence $C = 0$. Since $\alpha.\Delta = \phi^2$, we obtain that $F = 0$. □

4. Geometric consequences of curvature equations

In this section the Riemannian metric G on TM is assumed natural. Throughout this paper, G is characterized by the functions α and β. As in Remark 3.3, if $v \in TM$, let $z = (q, u, t, 0, \ldots, 0) \in N$ such that $\psi(z) = v$ and $t = |v|$. From Theorem 3.5 and Proposition 3.6 we get immediately

COROLLARY 4.1 (Theorem 0.1, [1]). *If* (TM, G) *is flat then* (M, g) *is flat.*

PROOF. This follows from part a) of Theorem 3.5 by setting $t = 0$. □

COROLLARY 4.2. *If* $\dim M \geq 3$, (TM, G) *is flat if and only if* (M, g) *is flat and*

$$\beta(t) = \frac{t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t)}{\alpha(t)}$$

PROOF. Assume that (TM, G) is flat. From Theorem 3.5 part b.1) and $1 < i < j \leq n$ we have that

$$\langle \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+i}(z), e_{n+j}(z)\rangle = -F(t^2)$$

Therefore $F = 0$, and the desired equality on β follows from Proposition 3.6 part i).

Assuming that (M, g) is flat and $\beta(t) = (t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t))/\alpha(t)$, we only need to show that

$$\langle \bar{R}(e_{n+i}(z), e_{n+j}(z))e_{n+k}(z), e_{n+l}(z)\rangle = 0 \tag{21}$$

for $1 \leq i, j, k, l \leq 2n$. The other cases also satisfies (21) because $R = 0$. Equality on β implies that $F = 0$, therefore by Proposition 3.6 part iv) we have that $H = 0$, and equality (21) is satisfied. □

We get also the following result:

COROLLARY 4.3. *If* $\dim M = 2$, *(TM, G) is flat if and only if (M, g) is flat and* $H = 0$.

REMARK 4.4. Let $\alpha(t) > 0$ be a differentiable function that satisfies $t\dot{\alpha}(t) + \alpha(t) > 0$ for all $t \geq 0$ and define $\beta(t) = (t(\dot{\alpha}(t))^2 + 2\alpha(t)\dot{\alpha}(t))/\alpha(t)$. If we consider the natural metric G induced by α and β, then (TM, G) is flat if (M, g) is flat.

REMARK 4.5. The above Corollaries generalize the well known fact that (TM, G_s) is flat if and only if (M, g) if flat (Kowalski [7], Aso [2]). This fact follows from the Corollaries by taking $\alpha = 1$ and $\beta = 0$.

We will denote by K and $\bar{K}$ the sectional curvatures of (M, g) and (TM, G) respectively.

THEOREM 4.6. *Let $v \in TM$ and $z = (q, u, t, 0, \ldots, 0) \in N$ such that $\psi(z) = v(t = |v|)$. We have the following expression for the sectional curvature of (TM, G) :*

a) *For* $1 \leq i, j \leq n$:

$$\bar{K}(e_i(z), e_j(z)) = K(u_i, u_j) - \frac{3}{4}\alpha(t^2)|R(u_i, u_j)v|^2 .$$

b) b.1) *If* $2 \leq i, j \leq n$ *and* $i \neq j$

$$\bar{K}(e_{n+i}(z), e_{n+j}(z)) = \frac{F(t^2)}{(\alpha(t^2))^2} .$$

b.2) *If* $2 \leq i \leq n$

$$\bar{K}(e_{n+1}(z), e_{n+j}(z)) = \frac{H(t^2)}{\alpha(t^2)(\alpha(t^2) + t^2\beta(t^2))} .$$

c) *For* $1 \leq i, j \leq n$:

$$\bar{K}(e_i(z), e_{n+j}(z)) = \frac{\alpha(t^2)}{4}|R(u_j, v)u_i|^2 .$$

In particular $\bar{K}(e_i, e_{n+1}) = 0$ if $1 \leq i \leq n$ because $v = tu_1$.

PROOF. From equality (4) we get that $\{e_1(z), \ldots, e_{2n}(z)\}$ is an orthogonal basis for $(TM)_v$ such that $\langle e_i(z), e_j(z)\rangle = \delta_{ij}$ if $1 \leq i, j \leq n$, $\langle e_{n+1}(z), e_{n+1}(z)\rangle = \alpha(t^2) + t^2\beta(t^2)$ and $\langle e_{n+i}(z), e_{n+i}(z)\rangle = \alpha(t^2)$ if $2 \leq i \leq n$. Let $1 \leq i, j \leq n$, $i \neq j$. By setting $k = j$ and

$l = i$ in equation a) of Theorem 3.5 we have that

$$\bar{K}(e_i(z), e_j(z)) = -\langle \bar{R}(e_i(z), e_j(z))e_j(z), e_i(z)\rangle = R_{ijji}(z) - \frac{3}{4}t^2\alpha(t^2)\sum_{r=1}^{n} R^2_{ij1r}(z)\,.$$

Since $K(u_i, u_j) = R_{ijji}(z)$ and $v = tu_1$, we can write

$$\bar{K}(e_i(z), e_j(z)) = K(u_i, u_j) - \frac{3}{4}\alpha(t^2)|R(u_i, u_j)v|^2\,.$$

Part b) follows directly from equations b.1) and b.2) of Theorem 3.5.

Since $|e_i(z)| = 1$ and $\langle e_i(z), e_{n+j}(z)\rangle = 0$ for $1 \le i, j \le n$, from Theorem 3.5 equation e), we see that

$$\begin{aligned}\bar{K}(e_i(z), e_{n+j}(z)) &= -\frac{(\alpha(|v|^2))^2|v|^2}{4(\alpha(|v|^2) + \delta_{j1}\beta(|v|^2)|v|^2)}\sum_{r=1}^{n} R_{irj1}(z)R_{rij1}(z)\\ &= \frac{\alpha(|v|^2)}{4}\sum_{r=1}^{n}[g(R(u_j, u_1|v|)u_i, u_r)]^2 = \frac{\alpha(|v|^2)}{4}|R(u_j, v)u_i|^2\,. \quad \square\end{aligned}$$

COROLLARY 4.7.

i) *(TM, G) is never a manifold with negative sectional curvature.*
ii) *If $\bar{K}$ is constant, then (TM, G) and (M, g) are flat.*
iii) *If $\bar{K}$ is bounded and $\lim_{t\to+\infty} t\alpha(t) = +\infty$, then (M, g) is flat.*
iv) *If $c \le \bar{K} \le C$ (possibly $c = -\infty$ and $C = +\infty$), then $c \le K \le C$.*

PROOF. Assertions i), ii) and iii) follow from Theorem 4.6 part c). Let $q \in M$ and $u = (u_1, \ldots, u_n)$ be an orthonormal basis for M_q. Then, if we consider $z = (q, u, 0, \ldots, 0)$ and $v = 0_q$, from Theorem 4.6 part a) we have that $\bar{K}(e_i(z), e_j(z)) = K(u_i, u_j)$ and part iv) holds. Also ii) follows from Theorem 3.5) part a) taking $t = 0$. $\square$

COROLLARY 4.8. *Let (M, g) be a manifold of constant sectional curvature K_0 and TM endowed with a natural metric G, then we have for $z = (q, u, t, 0, \ldots, 0)$ and $\psi(z) = v$ that*

a) $\bar{K}(e_i(z), e_j(z)) = K_0 - \frac{3}{4}(K_0)^2\alpha(|v|^2)(\delta_{i1} + \delta_{j1})|v|^2$ *with $i \ne j$.*
b) $\bar{K}(e_i(z), e_{n+j}(z)) = \frac{\alpha(|v|^2)}{4}K_0|v|^2(\delta_{ij} + \delta_{i1})$.

The vertical case $\bar{K}(e_{n+i}, e_{n+j})$ is as Theorem 4.6 *part* b).

From Theorem 4.6 we get the following result

COROLLARY 4.9. *Let G_1 and G_2 be two natural metrics on TM such that are characterized by the functions $\{\alpha_i\}_{i=1,2}$ and $\{\beta_i\}_{i=1,2}$ respectively. If $\bar{K}_1(u)(V, W) = \bar{K}_2(u)(V, W)$ for all $u \in TM$ and $V, W \in (TM)_u$ and (M, g) is not flat, then $\alpha_1 = \alpha_2$.*

REMARK 4.10. Let $G_{+\exp}$ and $G_{-\exp}$ be the natural metrics on TM defined by

$$ {}^{g}G_{+\exp}(q,u,\xi) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & A^{+}(\xi) \end{pmatrix} \quad \text{and} \quad {}^{g}G_{-\exp}(q,u,\xi) = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & A^{-}(\xi) \end{pmatrix} $$

where $A^{+}(\xi) = e^{|\xi|^2}(Id_{n\times n} + \xi^t.\xi)$ and $A^{-}(\xi) = e^{-|\xi|^2}(Id_{n\times n} + \xi^t.\xi)$. We call $G_{+\exp}$ and $G_{-\exp}$ the positive and negative exponential metrics.

It is known ([11]) that TM endowed with the Cheeger-Gromoll metric is never a manifold of constant sectional curvature. Theorem 4.6 applied to $G_{+\exp}$ and $G_{-\exp}$ shows that these metrics satisfy the same property.

4.1. Ricci tensor and scalar curvature. Let $Ricc$ and $\bar{R}icc$ be the Ricci tensor of (M,g) and (TM,G) respectively. We will denote by S and $\bar{S}$ the scalar curvatures of (M,g) and (TM,G).

THEOREM 4.11. *For $1 \leq i,j \leq n$ and $z=(q,u,t,0,\dots,0)$ we have the following expressions for $\bar{R}icc$:*

a) $\bar{R}icc(e_i(z),e_j(z)) = -\frac{\alpha(t^2)t^2}{2} \sum_{1\leq r,l\leq n} R_{irl1}(z)R_{jrl1}(z) + Ricc(u_i,u_j).$

b) $$\bar{R}icc(e_i(z),e_{n+j}(z)) = -\frac{\alpha(t^2)t^2}{2} \sum_{1\leq r\leq n} \{\langle \nabla_D R(E_r^i,E_r^r)E_r^j|_{s=0},u_1\rangle - \langle \nabla_D R(E_i^r,E_i^r)E_i^j|_{s=0},u_1\rangle\}.$$

c) c.1) *If $2\leq i\leq n$, then*

$$\bar{R}icc(e_{n+i}(z),e_{n+i}(z)) = \frac{t^2\alpha(t^2)}{4}\sum_{1\leq r,l\leq n} R^2_{rli1}(z) + \frac{(n-2)}{\alpha(t^2)}F(t^2) + \frac{1}{\alpha(t^2)+t^2\beta(t^2)}H(t^2)\,.$$

c.2) *If $2\leq i,j\leq n$ and $i\neq j$, then*

$$\bar{R}icc(e_{n+i}(z),e_{n+j}(z)) = \frac{t^2\alpha(t^2)}{4}\sum_{1\leq r,l\leq n} R_{rli1}(z)R_{rlj1}(z).$$

c.3) *If $1\leq j\leq n$, then*

$$\bar{R}icc(e_{n+1}(z),e_{n+j}(z)) = \frac{(n-1)}{\alpha(t^2)}H(t^2)\delta_{j1}.$$

PROOF. Let $\bar{e}_1(z),\dots,\bar{e}_{2n}(z)$ be the orthonormal basis for $(TM)_v$ induced by the orthogonal basis $e_1(z),\dots,e_{2n}(z)$, where $\psi(z)=v$. For $X,Y\in(TM)_v$ we have that

$$\bar{R}icc(X,Y) = \sum_{l=1}^{2n}\langle \bar{R}(X,\bar{e}_l(z))\bar{e}_l(z),Y\rangle.$$

Equalities a), b) and c) follow directly from Theorem 3.5 and the fact that $\langle e_{n+1}(z), e_{n+1}(z)\rangle = \alpha(t^2) + t^2\beta(t^2)$ and $\langle e_{n+i}(z), e_{n+i}(z)\rangle = \alpha(t^2)$ if $2 \leq i \leq n$. □

In [1], it is shown in the general g-Riemannian natural case that if (TM, G) is Einstein then (M, g) is Einstein. In our situation we have

COROLLARY 4.12. *If (TM, G) is Einstein, then (M, g) and (TM, G) are flats.*

PROOF. Let c be a constant such that $\bar{R}icc = cG$. In order to prove that $R = 0$, it is enough to show that for any $q \in M$ and any orthonormal basis $u = \{u_1, \dots, u_n\}$ for M_q the following equalities are satisfied

$$\langle R(u_i, u_r)u_l, u_1\rangle = 0 \tag{22}$$

for $1 \leq i, r, l \leq n$. Let $v \in M_q$, $v \neq 0$ and $z = (q, u, t, 0, \dots, 0) \in N$ such that $\psi(z) = tu_1 = v$. Since $G(e_i(z), e_j(z)) = \delta_{ij}$ if $1 \leq i, j \leq n$, from Theorem 4.11 part a) we have that

$$c\delta_{ij} = -\frac{\alpha(t^2)t^2}{2} \sum_{1\leq r,l\leq n} R_{irl1}(z)R_{jrl1}(z) + Ricc(u_i, u_j). \tag{23}$$

Taking $t = 0$, we get that $Ricc(u_i, u_j) = c\delta_{ij}$. Replacing these values for $i = j$ in (23) we obtain that

$$0 = -\frac{\alpha(t^2)t^2}{2} \sum_{1\leq r,l\leq n} (\langle R(u_i, u_r)u_l, u_1\rangle)^2$$

for $t \geq 0$ and equality (22) is satisfied. Since $Ricc = c.g$ and $R = 0$, it follows that $\bar{R}icc = 0$. Using that (TM, G) is Ricci flat and $R = 0$, from Theorem 4.11 parts c.1) and c.3) one gets that $H = F = 0$. From Theorem 3.5 we have that $\bar{R} = 0$. □

REMARK 4.13. It is easy to see from Theorem 4.11 that if (M, g) is not flat or if not exists a constant k such that $H(t) = k\alpha(t)$ and $(n-2)[\alpha(t) + t\beta(t)]F(t) = \alpha(t)k[(n-2)\alpha(t) + (n-1)t\beta(t)]$, then $\bar{R}icc$ is not a $\lambda - natural\ tensor$ (see [5]).

COROLLARY 4.14. *Let $v \in TM$ and $z = (\pi(v), u_1, \dots, u_n, t, 0, \dots, 0) \in N$ such that $v = u_1 t$. The scalar curvature of (TM, G) at v is given by*

$$\bar{S}(v) = S(\pi(v)) - \frac{t^2\alpha(t^2)}{4} \sum_{irl=1}^{n} R^2_{irl1}(z) + \frac{2(n-1)}{\alpha(t^2)(\alpha(t^2) + \beta(t^2)t^2)} H(t^2) + \frac{(n-1)(n-2)}{(\alpha(t^2))^2} F(t^2)\,.$$

PROOF. Since $\{\bar{e}_1(z), \dots, \bar{e}_{2n}(z)\}$ is an orthonormal basis for $(TM)_v$ and the scalar curvature $\bar{S}(v) = \sum_{l=1}^{2n} Ricc(\bar{e}_l(z), \bar{e}_l(z))$, the expression for $\bar{S}$ follows straightforward from Theorem 4.11. □

REMARK 4.15. Corollary 4.14 applied to $G_{+\exp}$ and $G_{-\exp}$ reads:

$$S_{+\exp}(v) = S(\pi(v)) - (n-1)e^{-|v|^2}\frac{[2+(n-2)(1+|v|^2)]}{(1+|v|^2)} - \frac{e^{|v|^2}}{4}\sum_{i,j=1}^{n}|R(u_i,u_j)v|^2$$

and

$$S_{-\exp}(v) = S(\pi(v)) + \frac{(n-1)e^{|v|^2}}{1+|v|^2}\left[(n-2)(3-|v|^2) + \frac{6+2|v|^2}{1+|v|^2}\right] - \frac{e^{-|v|^2}}{4}\sum_{i,j=1}^{n}|R(u_i,u_j)v|^2 .$$

PROPOSITION 4.16. *If (M,g) is a manifold of constant sectional curvature K_0, then*

$$S_{+\exp}(v) = (n-1)\left\{K_0\left(n - \frac{K_0}{2}|v|^2 e^{|v|^2}\right) - e^{-|v|^2}\frac{[2+(n-2)(1+|v|^2)]}{(1+|v|^2)}\right\}.$$

and

$$S_{-\exp}(v) = (n-1)\left\{K_0\left(n - \frac{K_0}{2}|v|^2 e^{-|v|^2}\right) + \frac{e^{|v|^2}}{1+|v|^2}\left[(n-2)(3-|v|^2) + \frac{6+2|v|^2}{1+|v|^2}\right]\right\}.$$

COROLLARY 4.17. *Let (M,g) be a flat manifold, then we have that*:

a) $S_{+\exp} < 0$.
b) *If* $\dim M = 2$, *then* $S_{-\exp} > 0$.
c) *If* $\dim \geq 3$, $S_{-\exp}(v) > 0$ *if and only if* $0 \leq |v|^2 < \frac{(n-1)+\sqrt{4(n-2)n+1}}{n-2}$.
d) *If* $\dim \geq 3$, $S_{-\exp}(v) = 0$ *if and only if* $|v|^2 = \frac{(n-1)+\sqrt{4(n-2)n+1}}{n-2}$.

PROOF. It follows from Proposition 4.16. □

REMARK 4.18. In [1], it is shown (Theorem 0.3) that if G is a g-natural metric on TM and (TM,G) has constant scalar curvature, then (M,g) has constant scalar curvature. In our case, this property follows immediately from Corollary 4.14, taking $t=0$. We can see that if (TM,G) has constant scalar curvature $\bar{S}$ and $F=0$, then (TM,G) is flat. If $F=0$ by Proposition 3.6, $H=0$, and by Corollary 4.14 it follows that $R=0$. Finally, from Theorem 3.5 we get that (TM,G) is flat.

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Present Addresses:
GUILLERMO HENRY
DEPARTAMENTO DE MATEMÁTICA, FCEYN,
UNIVERSIDAD DE BUENOS AIRES,
CIUDAD UNIVERSITARIA, PABELLÓN I, BUENOS AIRES, C1428EHA, ARGENTINA.
e-mail: ghenry@dm.uba.ar

GUILLERMO KEILHAUER
DEPARTAMENTO DE MATEMÁTICA, FCEYN,
UNIVERSIDAD DE BUENOS AIRES,
CIUDAD UNIVERSITARIA, PABELLÓN I, BUENOS AIRES, C1428EHA, ARGENTINA.
e-mail: wkeilh@dm.uba.ar