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The Adams Inequality on Weighted Morrey Spaces

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Abstract. We introduce new weight classes, and extend the Adams inequality to weighted Morrey spaces. We also investigate the boundedness of the modified fractional integral operator from weighted Morrey spaces to Lipschitz or BMO spaces.

1. Introduction

Hardy, Littlewood, and Sobolev proved the boundedness of the fractional integral operator I_α from L^p to L^q on Euclidean spaces, where $1/q = 1/p - \alpha/n,\ 1 < p < n/\alpha$. Peetre [8] studied the boundedness of I_α on classical Morrey spaces. Adams [1] showed the boundedness of I_α on classical Morrey spaces whose result improved the Spanne and Peetre inequality (see Chiarenza and Frasca [2]). In this paper, we will extend the Adams result to weighted Morrey spaces. Komori and Shirai [4] proved the Spanne and Peetre inequality on weighted Morrey spaces only for subcritical indices. We will introduce new weight classes and prove the Adams inequality on weighted Morrey spaces. Furthermore we consider the boundedness of the modified fractional integral operator from weighted Morrey spaces to Lipschitz or BMO spaces.

This article is organized as follows. In Section 2 we define the ordinary fractional integral operator, the weighted Morrey space, new weight classes, and recall some known results. In Section 3 we will state the main result. In Section 4 we give some lemmas. In Section 5 we prove these main results.

The following notation is used: Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. For a set $E \subset \mathbf{R}^n$ we denote the Lebesgue measure of E by $|E|$. We denote the characteristic function of E by χ_E. We write a ball of radius R centered at x_0 by $B = B(x_0, R) := \{x; |x - x_0| < R\}$ and $aB := \{x; |x - x_0| < aR\}$, for any $a > 0$. We call a nonnegative locally integrable function w on $\mathbf{R}^n$ a weight function. Also we write $w(E) = \int_E w(x)dx$. For $1 < p < \infty$, p' is the conjugate index if satisfies $1/p + 1/p' = 1$. The letter C shall always denote a positive

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constant which is independent of essential parameters and not necessarily the same at each occurrence.

2. Definitions and known results

We define two fractional integral operators and weighted Morrey, Lipschitz and BMO spaces.

DEFINITION 2.1 (Fracional integral operators). Let $0 < \alpha < n$. One define fractional integral operator I_α and modified fractional integral operator $\tilde{I}_\alpha$:

$$I_\alpha f(x) := \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy ,$$

$$\tilde{I}_\alpha f(x) := \int_{\mathbf{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \chi_{\{|y|\geq 1\}}(y) \right) f(y) dy .$$

Next we define weighted Morrey spaces.

DEFINITION 2.2 (Weighted Morrey spaces). Let $1 \leq p < \infty$ and $0 \leq \lambda < 1$. Suppose that u and v are weight functions on $\mathbf{R}^n$. We define weighted Morrey space $L^{p,\lambda}(u,v)$:

$$L^{p,\lambda}(u,v) := \left\{ f \in L^1_{loc}(\mathbf{R}^n) : \|f\|_{L^{p,\lambda}(u,v)} < \infty \right\} ,$$

where

$$\|f\|_{L^{p,\lambda}(u,v)} := \sup_{B\subset\mathbf{R}^n, B:ball} \left(\frac{1}{v(B)^\lambda} \int_B |f(x)|^p u(x) dx \right)^{\frac{1}{p}} .$$

REMARK 1. When $u = 1$ and $v = 1$, weighted Morrey spaces are classical Morrey spaces.

DEFINITION 2.3 (Lipschitz and BMO spaces). Let $0 \leq \varepsilon < 1$.

$$Lip_\varepsilon(\mathbf{R}^n) := \left\{ f \in L^1_{loc}(\mathbf{R}^n) : \|f\|_{Lip_\varepsilon(\mathbf{R}^n)} < \infty \right\} ,$$

where

$$\|f\|_{Lip_\varepsilon(\mathbf{R}^n)} := \sup_{B\subset\mathbf{R}^n, B:ball} \inf_{c\in\mathbf{C}} \frac{1}{|B|^{1+\frac{\varepsilon}{n}}} \int_B |f(x) - c| dx .$$

We denote $BMO(\mathbf{R}^n) := Lip_0(\mathbf{R}^n)$.

Hardy, Littlewood and Sobolev(cf. Lu, Ding and Yan [5]) proved the boundedness of I_α from L^p to L^q where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Adams [1] proved the boundedness on Morrey spaces.

THEOREM A (Adams). *Let $0 < \alpha < n$, $0 \leq \lambda < 1$, $1 < p < \frac{n(1-\lambda)}{\alpha}$. If $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{q,\lambda}(\mathbf{R}^n)} \leq C \|f\|_{L^{p,\lambda}(\mathbf{R}^n)} . \tag{1}$$

When $p \geq \frac{n}{\alpha}(1-\lambda)$, Spanne and Peetre(cf. Peetre [8]) proved the following.

THEOREM B (Spanne and Peetre). *Let* $0 < \alpha < n$ *and* $0 \leq \lambda < 1$. *If* $0 \leq \varepsilon = n\left(\frac{\alpha}{n} - \frac{1-\lambda}{p}\right) < 1$, *then there exists a constant* $C > 0$ *such that*

$$\|\tilde{I}_\alpha f\|_{Lip_\varepsilon(\mathbf{R}^n)} \leq C\|f\|_{L^{p,\lambda}(\mathbf{R}^n)}.$$

Let w be a non-negative function. We will recall the Muckenhoupt class.

DEFINITION 2.4. Let $1 < p < \infty$ and $0 < \lambda < 1$. We say $w \in A_p(\mathbf{R}^n)$ if

$$\sup_{B\subset\mathbf{R}^n, B:ball}\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}} < \infty.$$

We say $w \in A_1(\mathbf{R}^n)$ if there exists a constant $C > 0$ such that for every ball $B \subset \mathbf{R}^n$

$$\frac{1}{|B|}\int_B w(x)dx \leq C \operatorname*{ess.inf}_{x\in B} w(x).$$

We put $A_\infty(\mathbf{R}^n) := \bigcup_{p\geq 1} A_p(\mathbf{R}^n)$. We say $\tilde{A}_{p,\lambda}(\mathbf{R}^n)$ if

$$\sup_{B\subset\mathbf{R}^n, B:ball}\left(\frac{1}{|B|}\int_B w(x)dx\right)^{\frac{\lambda}{p}}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda p'}{p}}dx\right)^{\frac{1}{p'}} < \infty.$$

REMARK 2. $w \in \tilde{A}_{p,\lambda}(\mathbf{R}^n)$ if and only if $w \in A_{1+\frac{p-1}{\lambda}}(\mathbf{R}^n)$.

DEFINITION 2.5. Let $1 < p, q < \infty$. We say $w \in A_{p,q}(\mathbf{R}^n)$ if

$$\sup_{B\subset\mathbf{R}^n, B:ball}\left(\frac{1}{|B|}\int_B w(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|B|}\int_B w(x)^{-p'}dx\right)^{\frac{1}{p'}} < \infty.$$

We say $w \in A_{p,\infty}(\mathbf{R}^n)$ if

$$\sup_{B\subset\mathbf{R}^n, B:ball} \operatorname*{ess.sup}_{x\in B} w(x)\left(\frac{1}{|B|}\int_B w(x)^{-p'}dx\right)^{\frac{1}{p'}} < \infty.$$

REMARK 3. $w \in A_{p,\infty}(\mathbf{R}^n)$ if and only if $w^{-p'} \in A_1(\mathbf{R}^n)$.

Muckenhoupt and Wheeden [7] proved the weighted boundedness of I_α.

THEOREM C (Muckenhoupt and Wheeden). *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $w \in A_{p,q}(\mathbf{R}^n)$, *then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{L^q(w^q)} \leq C\|f\|_{L^p(w^p)},$$

that is,

$$\left(\int_{\mathbf{R}^n}|I_\alpha f(x)|^q w(x)^q dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbf{R}^n}|f(x)|^p w(x)^p dx\right)^{\frac{1}{p}}.$$

3. Main results

Our results are the following. Note that our results are true for $\lambda = 0$. We only treat the weighted case of $0 < \lambda < 1$, because the unweighted case of $\lambda = 0$ follows from the weighted case (see Remark 4 (1) below). In the following we assume that $1 < p < \infty$.

THEOREM 1. *Let* $0 < \alpha < n$, $0 < \lambda < 1 - \frac{\alpha}{n}$ *and* $1 < p < \frac{n}{\alpha}(1-\lambda)$. *Assume that*

$$w \in \tilde{A}_{p,\lambda}(\mathbf{R}^n)\,. \tag{2}$$

Then there exists a constant $C > 0$ *such that*

$$\|I_\alpha f\|_{L^{q,\lambda}(w^\lambda,w)} \le C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)}\ ,$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}\,.$$

REMARK 4. (1) The condition $0 < \lambda < 1 - \frac{\alpha}{n}$ is necessary for $\frac{n}{\alpha}(1-\lambda) > 1$. When $w = 1$ in Theorem 1, we obtain Theorem A. When λ tends to 0, the result corresponds to the Hardy, Littlewood and Sobolev theorem.

(2) We can obtain the Spanne and Peetre inequality on weighted Morrey spaces by Theorem 1(see Chiarenza and Frasca [2]). We omit the details.

When $p \ge \frac{n}{\alpha}(1-\lambda)$, we obtain the following. Let

$$\varepsilon = \alpha - \frac{n(1-\lambda)}{p}\,. \tag{3}$$

THEOREM 2. *Let* $0 < \alpha < n$, $0 \le \lambda < 1$, $\frac{n}{\alpha}(1-\lambda) \le p < \frac{n}{\alpha}$ *and* $0 \le \varepsilon < 1$. *Assume that*

$$w^{\frac{\lambda}{p}} \in A_{p,q}(\mathbf{R}^n) \quad \textit{where} \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\,. \tag{4}$$

Then there exists a constant $C > 0$ *such that*

$$\left\|\tilde{I}_\alpha f\right\|_{Lip_\varepsilon(\mathbf{R}^n)} \le C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)}\ .$$

Especially

$$\left\|\tilde{I}_\alpha f\right\|_{BMO(\mathbf{R}^n)} \le C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)} \quad \textit{when} \quad p = \frac{n}{\alpha}(1-\lambda)\,.$$

REMARK 5. In Theorem 1, the condition (2) is stronger than (4), since $\frac{\lambda q}{p} < 1$. However (2) can not be replaced with (4). Because the weight w which satisfies (4) may not be locally integrable when $\frac{\lambda q}{p} < 1$.

Note that (2) and (4) are mutually equivalent when $p = \frac{n}{\alpha}(1-\lambda)$.

THEOREM 3. *Let* $0 < \alpha < n$, $0 < \lambda < 1$, $p \geq \frac{n}{\alpha}$ *and* $0 \leq \varepsilon < 1$. *Assume that*

$$w^{\frac{\lambda}{p}} \in A_{\frac{n}{\alpha},\infty}(\mathbf{R}^n)\,. \tag{5}$$

Then there exists a constant $C > 0$ *such that*

$$\left\|\tilde{I}_\alpha f\right\|_{Lip_\varepsilon(\mathbf{R}^n)} \leq C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)}\ .$$

REMARK 6. The $A_{\frac{n}{\alpha},\infty}(\mathbf{R}^n)$ condition can be regarded as limiting case of the $A_{p,q}(\mathbf{R}^n)$ condition when p tends to n/α.

4. Some lemmas

The following two lemmas are important to prove Theorem 3.

LEMMA 4.1. *Let* $0 < \alpha < n$, $0 < \lambda < 1$ *and* $\frac{n}{\alpha} \leq p < \infty$. *If* $w^{\frac{\lambda}{p}} \in A_{\frac{n}{\alpha},\infty}(\mathbf{R}^n)$ *then*

$$w \in \tilde{A}_{p,\lambda}(\mathbf{R}^n)\,.$$

PROOF. Since $p \geq \frac{n}{\alpha}$, by Hölder's inequality and definition of $A_{\frac{n}{\alpha},\infty}(\mathbf{R}^n)$, we have for every ball $B \subset \mathbf{R}^n$,

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B w(x)dx\right)^{\frac{\lambda}{p}}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda p'}{p}}dx\right)^{\frac{1}{p'}}\\
&\quad\leq \left(\frac{1}{|B|}\int_B w(x)dx\right)^{\frac{\lambda}{p}}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda}{p}(\frac{n}{\alpha})'}dx\right)^{\frac{1}{(\frac{n}{\alpha})'}}\\
&\quad\leq \operatorname*{ess.\,sup}_{x\in B} w(x)^{\frac{\lambda}{p}}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda}{p}(\frac{n}{\alpha})'}dx\right)^{\frac{1}{(\frac{n}{\alpha})'}}\\
&\quad\leq C\,.
\end{aligned}$$

□

LEMMA 4.2. *Let* $0 < \alpha < n$, $0 < \lambda < 1$ *and* $\frac{n}{\alpha} \leq p < \infty$. *If* $w^{\frac{\lambda}{p}} \in A_{\frac{n}{\alpha},\infty}(\mathbf{R}^n)$ *then there exist* $p_0 < \frac{n}{\alpha}$ *and a constant* $C > 0$ *such that for every ball* $B \subset \mathbf{R}^n$,

$$w^{\frac{\lambda}{p}} \in A_{p_0,q_0}(\mathbf{R}^n)\,, \tag{6}$$

$$\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda q_0'}{p}}dx\right)^{\frac{1}{q_0'}} \leq C\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda p'}{p}}dx\right)^{\frac{1}{p'}}\ , \tag{7}$$

where $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$.

PROOF. Note that if p_0 is near $\frac{n}{\alpha}$ then q_0 is large, that is, q_0' is near 1 and $\frac{n-\alpha}{n} \cdot p_0'$ is near 1. First we prove (6). Since $w^{-\frac{\lambda}{p}(\frac{n}{\alpha})'} \in A_1(\mathbf{R}^n)$, by the reverse Hölder inequality (cf. Lu, Ding and Yan [5]) we can take p_0 sufficiently near $\frac{n}{\alpha}$ such that

$$\begin{aligned}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda p_0'}{p}}dx\right)^{\frac{1}{p_0'}} &= \left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda}{p}(\frac{n}{\alpha})'\cdot\frac{n-\alpha}{n}\cdot p_0'}dx\right)^{\frac{n}{n-\alpha}\cdot\frac{1}{p_0'}\cdot\frac{n-\alpha}{n}} \\ &\le C\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda}{p}\cdot(\frac{n}{\alpha})'}dx\right)^{\frac{n-\alpha}{n}} \\ &= C\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda}{p}\cdot(\frac{n}{\alpha})'}dx\right)^{\frac{1}{(\frac{n}{\alpha})'}} .\end{aligned}$$

On the other hand, we have

$$\left(\frac{1}{|B|}\int_B w(x)^{\frac{\lambda q_0}{p}}dx\right)^{\frac{1}{q_0}} \le \operatorname*{ess.sup}_{x\in B} w(x)^{\frac{\lambda}{p}} .$$

Therefore we obtain

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda p_0'}{p}}dx\right)^{\frac{1}{p_0'}}\left(\frac{1}{|B|}\int_B w(x)^{\frac{\lambda q_0}{p}}dx\right)^{\frac{1}{q_0}} \\ &\le C\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda}{p}\cdot(\frac{n}{\alpha})'}dx\right)^{\frac{1}{(\frac{n}{\alpha})'}} \operatorname*{ess.sup}_{x\in B} w(x)^{\frac{\lambda}{p}} \\ &\le C .\end{aligned}$$

Next we prove (7). Since $w^{-\frac{\lambda}{p}(\frac{n}{\alpha})'} \in A_1(\mathbf{R}^n)$, we have $w^{-\frac{\lambda}{p}} \in A_1(\mathbf{R}^n)$. By the reverse Hölder inequality we can take p_0 slightly less than $\frac{n}{\alpha}$ so that

$$\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda}{p}q_0'}dx\right)^{\frac{1}{q_0'}} \le \frac{C}{|B|}\int_B w(x)^{-\frac{\lambda}{p}}dx \le C\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda p'}{p}}dx\right)^{\frac{1}{p'}} .$$

Thus, Lemma 4.2 is proved. □

The following two lemmas are due to Adams [1]. Let M_μ be the fractional maximal function

$$M_\mu f(x) := \sup_{B\ni x}\frac{1}{|B|^\mu}\int_B |f(y)|\,dy\,, \quad 0 \le \mu \le 1 .$$

LEMMA 4.3. *Let* $0 < \alpha < n$, $1 \le p < \frac{n(1-\lambda)}{\alpha}$ *and* $0 \le \lambda < 1$. *Then there exists a constant* $C > 0$ *such that*

$$|I_\alpha f(x)| \le C\big(M_{1+\frac{\lambda-1}{p}}f(x)\big)^{\frac{\alpha}{n}\cdot\frac{p}{1-\lambda}} M_1 f(x)^{1-\frac{\alpha}{n}\cdot\frac{p}{1-\lambda}} . \tag{8}$$

LEMMA 4.4. *Suppose that* $0 < \alpha < n$, $x_0 \in \mathbf{R}^n$, $r > 0$, $B = B(x_0, r)$ *and* $x \in B$. *Then the following inequality holds*:

$$|I_\alpha f_\infty(x)| \le \int_r^\infty \rho^{\alpha-n} \left(\int_{B(x,\rho)} |f(y)|\, dy \right) \frac{d\rho}{\rho}. \tag{9}$$

where $f_\infty(x) := f(x)\chi_{(2B)^c}(x)$.

We derive the following pointwise inequality from Hölder's inequality. This approach has the advantage that can be also used in the weighted context:

LEMMA 4.5. *If* $w \in \tilde{A}_{p,\lambda}(\mathbf{R}^n)$, *then*

$$M_{1+\frac{\lambda-1}{p}} f(x) \le C \, \|f\|_{L^{p,\lambda}(w^\lambda, w)} .$$

PROOF. Let $x \in \mathbf{R}^n$ fix. for every $B \ni x$, we have the following:

$$\begin{aligned}
\frac{1}{|B|^{1+\frac{\lambda-1}{p}}} \int_B |f(y)|\, dy &= \frac{1}{|B|^{1+\frac{\lambda-1}{p}}} \int_B |f(y)|\, w(y)^{\frac{\lambda}{p}} w(y)^{-\frac{\lambda}{p}} dx \\
&\le \frac{1}{|B|^{1+\frac{\lambda-1}{p}}} \left(\int_B |f(y)|^p \, w(y)^\lambda dy \right)^{\frac{1}{p}} \cdot \left(\int_B w(y)^{-\frac{\lambda p'}{p}} dy \right)^{\frac{1}{p'}} \\
&= \frac{1}{|B|^{1+\frac{\lambda-1}{p}}} \left(\frac{1}{w(B)^\lambda} \int_B |f(y)|^p \, w(y)^\lambda dy \right)^{\frac{1}{p}} \cdot w(B)^{\frac{\lambda}{p}} \cdot w^{-\frac{\lambda p'}{p}}(B)^{\frac{1}{p'}} \\
&\le \|f\|_{L^{p,\lambda}(w^\lambda, w)} \left(\frac{1}{|B|} \int_B w(y) dy \right)^{\frac{\lambda}{p}} \cdot \left(\frac{1}{|B|} \int_B w(y)^{-\frac{\lambda p'}{p}} dy \right)^{\frac{1}{p'}} \\
&\le C \, \|f\|_{L^{p,\lambda}(w^\lambda, w)} .
\end{aligned}$$

□

5. Proof of main theorems

We will prove Theorem 1 by using two pointwise inequalities obtained in the previous section.

PROOF OF THEOREM 1. For every $x_0 \in \mathbf{R}^n$ and $r > 0$, let $B = B(x_0, r)$ and

$$f(x) = f(x)\chi_{2B}(x) + f(x)\chi_{(2B)^c}(x) = f_0(x) + f_\infty(x) .$$

First we estimate $I_\alpha f_0$. Because of Lemmas 4.3 and 4.5 we obtain

$$|I_\alpha f_0(x)| \le C \, \|f\|_{L^{p,\lambda}(w^\lambda, w)}^{1-\frac{p}{q}} \cdot M_1 f_0(x)^{\frac{p}{q}} .$$

On the other hand, since $w \in \tilde{A}_{p,\lambda}(\mathbf{R}^n)$, by Hölder's inequality we have $w^\lambda \in A_p(\mathbf{R}^n)$. By the weighted L^p boundedness of M_1(cf. Lu, Ding and Yan [5]) and doubling condition on

$\tilde{A}_{p,\lambda}(\mathbf{R}^n)$ we obtain

$$\begin{aligned}
&\int_B |I_\alpha f_0(x)|^q \, w(x)^\lambda dx \\
&\quad \le C \, \|f\|^{q-p}_{L^{p,\lambda}(w^\lambda,w)} \int_B M_1 f_0(x)^p w(x)^\lambda dx \\
&\quad \le C \, \|f\|^{q-p}_{L^{p,\lambda}(w^\lambda,w)} \cdot \left(\frac{1}{w(2B)^\lambda} \int_{2B} |f(x)|^p \, w(x)^\lambda dx \right) \cdot w(2B)^\lambda \\
&\quad \le C \, \|f\|^{q}_{L^{p,\lambda}(w^\lambda,w)} \cdot w(B)^\lambda .
\end{aligned}$$

Thus we have

$$\|I_\alpha f_0\|_{L^{q,\lambda}(w^\lambda,w)} \le C \, \|f\|_{L^{p,\lambda}(w^\lambda,w)} \, .$$

Next we estimate $|I_\alpha f_\infty(x)|$. By Lemma 4.4, we have for $x \in B$

$$|I_\alpha f_\infty(x)| \le \int_r^\infty \rho^{\alpha-n} \left(\int_{B(x,\rho)} |f(y)| \, dy \right) \frac{d\rho}{\rho} \, .$$

Moreover by the same method as the proof of Lemma 4.5, we have

$$\int_{B(x,\rho)} |f(y)| \, dy \le C \, \|f\|_{L^{p,\lambda}(w^\lambda,w)} \, \rho^{n(\frac{\lambda}{p}+\frac{1}{p'})} \, .$$

Hence we obtain

$$\begin{aligned}
|I_\alpha f_\infty(x)| &\le C \, \|f\|_{L^{p,\lambda}(w^\lambda,w)} \int_r^\infty \rho^{\alpha-n+n(\frac{\lambda}{p}+\frac{1}{p'})} \frac{d\rho}{\rho} \\
&\le C \, \|f\|_{L^{p,\lambda}(w^\lambda,w)} \, r^{\alpha+\frac{n\lambda}{p}-\frac{n}{p}} \\
&\le C \, |B|^{\frac{\lambda-1}{q}} \, \|f\|_{L^{p,\lambda}(w^\lambda,w)} \, .
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\left(\frac{1}{w(B)^\lambda} \int_B |I_\alpha f_\infty(x)|^q \, w(x)^\lambda dx \right)^{\frac{1}{q}} \\
&\quad \le C \, |B|^{\frac{\lambda-1}{q}} \cdot w^\lambda(B)^{\frac{1}{q}} \cdot w(B)^{-\frac{\lambda}{q}} \, \|f\|_{L^{p,\lambda}(w^\lambda,w)} \\
&\quad \le C \, \|f\|_{L^{p,\lambda}(w^\lambda,w)} \, .
\end{aligned}$$

That is, we obtain the desired result. □

Theorems 2 and 3 are obtained from the following proposition.

PROPOSITION 1. *Let* $0 < \alpha < n$, $0 \le \lambda < 1$ *and* $\frac{n}{\alpha}(1-\lambda) \le p$. *Assume that*

$$w \in \tilde{A}_{p,\lambda}(\mathbf{R}^n) \, , \tag{10}$$

and there exists p_0 *such that* $1 < p_0 < \frac{n}{\alpha}$, $p_0 \le p$ *and*

$$w^{\frac{\lambda}{p}} \in A_{p_0,q_0}(\mathbf{R}^n)\,, \tag{11}$$

$$\sup_{B\subset\mathbf{R}^n,\,B:ball}\left(\frac{1}{|B|}\int_B w(x)dx\right)^{\frac{\lambda}{p}}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda q_0'}{p}}dx\right)^{\frac{1}{q_0'}} < \infty\,, \tag{12}$$

where $\frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n}$. *Then there exists a constant* $C > 0$ *such that*

$$\left\|\tilde{I}_\alpha f\right\|_{Lip_\varepsilon(\mathbf{R}^n)} \le C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)}\,,$$

where $0 \le \varepsilon = \alpha - \frac{n(1-\lambda)}{p} < 1$.

Assuming this proposition temporarily, we shall prove Theorems 2 and 3. First we prove Theorem 2.

PROOF OF THEOREM 2. Under the assumption of Theorem 2, we can take $p_0 = p$ in Proposition 1. In fact, because $w^{\frac{\lambda}{p}} \in A_{p,q}(\mathbf{R}^n)$ and $\frac{\lambda q}{p} > 1$, we have $w \in \tilde{A}_{p,\lambda}(\mathbf{R}^n)$. Since $p'/q' > 1$, we have

$$\sup_{B\subset\mathbf{R}^n,\,B:ball}\left(\frac{1}{|B|}\int_B w(x)dx\right)^{\frac{\lambda}{p}}\left(\frac{1}{|B|}\int_B w(x)^{-\frac{\lambda}{p}q'}dx\right)^{\frac{1}{q'}} < \infty\,,$$

by the Hölder inequality. Therefore w satisfies the conditions in Proposition 1. □

Next we prove Theorem 3.

PROOF OF THEOREM 3. We can apply Lemma 4.1 and Lemma 4.2 by the assumptions of Theorem 3. Hence by Lemma 4.1 w satisfies (10), and by Lemma 4.2, we can find p_0 which satisfies (6) and (7). Therefore w satisfies (11). By (7) and (10), w satisfies (12). Therefore by using Proposition 1, we obtain the desired result. □

Finally we prove Proposition 1.

PROOF OF PROPOSITION 1. For every $x_0 \in \mathbf{R}^n$ and $r > 0$, let $B = B(x_0, r)$ and

$$f(x) = f(x)\chi_{2B}(x) + f(x)\chi_{(2B)^c}(x) = f_0(x) + f_\infty(x)\,,$$

$$c_0 := -\int_{|y|\ge 1}\frac{f_0(y)}{|y|^{n-\alpha}}dy\,,$$

$$c_1 := \tilde{I}_\alpha(f_\infty)(x_0)\,,$$

$$c := c_0 + c_1\,.$$

We obtain the pointwise inequality:

$$\left|\tilde{I}_\alpha f(x) - c\right| \le |I_\alpha f_0(x)| + \int_{\mathbf{R}^n}\left|\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}}\right||f_\infty(y)|\,dy = I + II\,.$$

First we estimate I. Note that by the condition (10) w satisfies the doubling condition. Since $w^{\frac{\lambda}{p}} \in A_{p_0,q_0}(\mathbf{R}^n)$, by Theorem C and (12), we have

$$\begin{aligned}
\int_B I\,dx &= \int_B |I_\alpha f_0(x)|\,dx \\
&\le \left(\int_B |I_\alpha f_0(x)|^{q_0}\, w(x)^{\frac{\lambda}{p}q_0} dx\right)^{\frac{1}{q_0}} \left(\int_B w(x)^{-\frac{\lambda}{p}q_0'} dx\right)^{\frac{1}{q_0'}} \\
&\le C\left(\int_{2B} |f(x)|^{p_0}\, w(x)^{\frac{\lambda}{p}p_0} dx\right)^{\frac{1}{p_0}} \left(\int_B w(x)^{-\frac{\lambda q_0'}{p}} dx\right)^{\frac{1}{q_0'}} \\
&\le C\left(\int_{2B} |f(x)|^{p}\, w(x)^{\lambda} dx\right)^{\frac{1}{p}} w^{-\frac{\lambda q_0'}{p}}(B)^{\frac{1}{q_0'}} \cdot |B|^{\frac{1}{p_0}-\frac{1}{p}} \\
&\le C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)} \cdot w(B)^{\frac{\lambda}{p}} \cdot w^{-\frac{\lambda q_0'}{p}}(B)^{\frac{1}{q_0'}} \cdot |B|^{\frac{1}{p_0}-\frac{1}{p}} \\
&\le C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)}\, |B|^{1+\frac{\varepsilon}{n}}\ .
\end{aligned}$$

Next we estimate II. Let $x \in B$. By the condition (10) we have

$$\begin{aligned}
II &\le C\sum_{k=1}^{\infty} \int_{2^k r \le |x_0-y| < 2^{k+1} r} \frac{|x-x_0|}{|x_0-y|^{n-\alpha+1}}\,|f(y)|\,dy \\
&\le C\sum_{k=1}^{\infty} \frac{r}{(2^k r)^{n-\alpha+1}} \int_{2^{k+1}B} |f(y)|\,dy \\
&\le C\sum_{k=1}^{\infty} \frac{r}{(2^k r)^{n-\alpha+1}} \left(\int_{2^{k+1}B} |f(y)|^p\, w(y)^\lambda dy\right)^{\frac{1}{p}} \cdot \left(\int_{2^{k+1}B} w(y)^{-\frac{\lambda p'}{p}} dy\right)^{\frac{1}{p'}} \\
&= C\sum_{k=1}^{\infty} \frac{r}{(2^k r)^{n-\alpha+1}} \left(\frac{1}{w(2^{k+1}B)^\lambda}\int_{2^{k+1}B} |f(y)|^p\, w(y)^\lambda dy\right)^{\frac{1}{p}} \\
&\quad \times w(2^{k+1}B)^{\frac{\lambda}{p}} \cdot w^{-\frac{\lambda p'}{p}}(2^{k+1}B)^{\frac{1}{p'}} \\
&\le C\sum_{k=0}^{\infty} \frac{r}{(2^{k+1} r)^{n-\alpha+1}} \|f\|_{L^{p,\lambda}(w^\lambda,w)} \cdot \left|2^{k+1}B\right|^{\frac{\lambda}{p}+\frac{1}{p'}} \\
&\quad \times \left(\frac{1}{\left|2^{k+1}B\right|}\int_{2^{k+1}B} w(y)dy\right)^{\frac{\lambda}{p}} \cdot \left(\frac{1}{\left|2^{k+1}B\right|}\int_{2^{k+1}B} w(y)^{-\frac{\lambda p'}{p}} dy\right)^{\frac{1}{p'}} \\
&\le C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)}\, |B|^{\frac{\alpha}{n}-1+\frac{\lambda}{p}+1-\frac{1}{p}} \sum_{k=1}^{\infty} 2^{-kn\left(1-\frac{\alpha}{n}+\frac{1}{n}-\frac{\lambda}{p}-1+\frac{1}{p}\right)} \\
&\le C\,|B|^{\frac{\varepsilon}{n}}\, \|f\|_{L^{p,\lambda}(w^\lambda,w)}\ ,
\end{aligned}$$

since $-\frac{\alpha}{n}+\frac{1}{n}-\frac{\lambda}{p}+\frac{1}{p}>0$. Thus we have

$$\begin{aligned}\left\|\tilde{I}_\alpha f\right\|_{Lip_\varepsilon(\mathbf{R}^n)} &\leq C \sup_{B\subset\mathbf{R}^n, B:ball} \frac{1}{|B|^{1+\frac{\varepsilon}{n}}}\int_B \left|\tilde{I}_\alpha f(x)-c\right| dx \\ &\leq C \sup_{B\subset\mathbf{R}^n, B:ball} \frac{1}{|B|^{1+\frac{\varepsilon}{n}}}\left(\int_B I dx+\int_B II dx\right) \\ &\leq C\,\|f\|_{L^{p,\lambda}(w^\lambda,w)}\,.\end{aligned}$$

This is the desired result. □

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