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Existence of Invariant Planes in a Complex Projective 3-Space under Discrete Projective Transformation Groups

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Abstract. Let Γ be a finitely generated discrete subgroup of PGL(4, **C**) acting on $\mathbf{P}^3$. Suppose that Γ leaves invariant a surface in $\mathbf{P}^3$. Then, except for a few cases, we can find a plane which is invariant by a finite index subgroup of Γ. The exceptional cases will be determined explicitly.

Introduction

Let Γ be a finitely generated discrete subgroup of PGL(4, **C**) acting on $\mathbf{P}^3$. By a *curve* (resp. *surface*), we shall mean an irreducible compact complex space of dimension one (resp. two). We say that a set M is (Γ-)*semi-invariant*, if we can find a finite index subgroup Γ_0 of Γ such that $g(M) = M$ for any $g \in \Gamma_0$. If $\Gamma_0 = \Gamma$, we say M is (Γ-) *invariant*.

In this note, we shall prove the following.

THEOREM A. *Suppose that Γ leaves invariant a curve C and a surface S such that $C \subset S$. Then, there are Γ-semi-invariant planes, except for the following two cases.*

I. *By a suitable system of homogeneous coordinates* $[z_0 : z_1 : z_2 : z_3]$ *on* $\mathbf{P}^3$, *C and S are given by*

$$C = \{z_0 = z_1 = 0\}, \quad S = \{z_0 z_3 - z_1 z_2 = 0\},$$

and, every element $\sigma \in \Gamma$ is represented by a matrix of the form

$$\begin{pmatrix} A_\sigma & 0 \\ c_\sigma A_\sigma & A_\sigma \end{pmatrix}, \quad c_\sigma \in \mathbf{C}.$$

Further, there are no Γ-invariant surfaces other than S.

II. *C is a twisted cubic curve, and S is the tangential surface of C.*

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Here the *tangential surface* of a twisted cubic curve C is the union of the tangent lines to C.

THEOREM B. *Assume that Γ leaves invariant a surface S in $\mathbf{P}^3$. If Γ admits no semi-invariant planes, then S is either a non-singular quadric, the tangential surface of a twisted cubic curve, or a cone over a non-singular conic.*

As an application of our results, we shall give an explicit geometric construction of finitely generated discrete subgroups in PGL(4, $\mathbf{C}$) without semi-invariant surfaces (Section 6).

In section 1, in the first half, we collect together some terms and well-known elementary facts from classical Kleinian group theory and, in the second half, make preparations for the proof of Theorem A. Here we see that there are two cases of C to be considered, the line case and the space rational curve case. The line case will be considered in section 2 and the space rational case in section 3. In section 4, we shall prove Theorems A, B and their corollaries. In section 5, we shall give examples in each of the exceptional cases. In section 6, we shall construct finitely generated discrete subgroups in PGL(4, $\mathbf{C}$) without semi-invariant surfaces. In section 7, we introduce the Eichler cohomology and prove a lemma which is used in constructing an example in section 5.

1. Preliminaries

1.1. Elementary facts from Kleinian group theory. The contents of this subsection will be used in sections 3, 5 and 7. Here we collect some terms and well-known facts from the classical Kleinian group theory.

A subgroup G of a Lie group $\mathcal{G}$ is said to be a *discrete subgroup* if G is a discrete subset of $\mathcal{G}$. This is equivalent to the fact that the identity element $1 \in \mathcal{G}$ has an open neighborhood U such that $U \cap G = \{1\}$.

Let G be a discrete subgroup of PGL(2.$\mathbf{C}$). Then G acts on $\mathbf{P}^1$ naturally. A point $z \in \mathbf{P}^1$ is called a properly discontinuous point, if z has a neighborhood U such that the set

$$\{g \in G : g(U) \cap U \neq \emptyset\}$$

is finite. The set of all properly discontinuous points on $\mathbf{P}^1$ is called *the discontinuity region* of G, which we indicate by $\Omega(G)$. Obviously $\Omega(G)$ is an open set, but may happen to be empty. We call the group G a *Kleinian group* [1], if $\Omega(G)$ is not empty. It can be proved that the action of G on $\Omega(G)$ is properly discontinuous. The complement

$$\Lambda(G) = \mathbf{P}^1 \setminus \Omega(G)$$

[1] We adopt here the classical definition of the Kleinian groups. In [MT], any discrete subgroups of PGL(2, $\mathbf{C}$) is said to be Kleinian, see [MT, Theorem 1.19]

of the discontinuity region is called the *limit set* of G. If the cardinality of $\Lambda(G)$ is less than 3, G is said to be *elementary* ([MT, Definition p.41]). The following fact follows from [MT, Theorem 2.4].

PROPOSITION 1. *If G admits a non-empty G-invariant finite set, then G is elementary.*

We use also the following

PROPOSITION 2 (Proposition 2.2 in [MT]). *If G is elementary, then G contains an abelian subgroup of finite index.*

Any element in PSL(2, **C**) has a matrix representative conjugate to one of the following:

$$(1)\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (2)\ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\ |\lambda| = 1, \quad (3)\ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\ |\lambda| \neq 0, 1\,.$$

The element conjugate to (1) is said to be *parabolic*, the one conjugate to (2) is said to be *elliptic*, and the one conjugate to (3) is said to be *loxodromic*.

PROPOSITION 3 (Lemma 2.3 in [MT]). *If G is non-elementary, then G contains loxodromic elements.*

1.2. Two Cases. In this subsection, we shall make preparation for the proof of Theorem A. The conditions we impose on Γ are the following.

A1. Γ is a finitely generated discrete infinite subgroup of PSL(4, **C**).
A2. Γ leaves invariant a curve C in $\mathbf{P}^3$.
A3. Γ admits no semi-invariant planes.

PROPOSITION 4. *If Γ contains a solvable subgroup of finite index, then Γ admits a semi-invariant plane.*

PROOF. Suppose that $\Gamma_0 \subset \Gamma$ is a solvable subgroup of finite index. Let p : SL (4, **C**) $\to$ PSL(4, **C**) be the natural projection. Then $\tilde{\Gamma}_0 = p^{-1}(\Gamma_0)$ is also solvable. Hence $\tilde{\Gamma}_0$ is conjugate in SL(4, **C**) to a subgroup of the triangular subgroup of SL(4, **C**) by the matrix theory. Therefore Γ_0 admits an invariant plane. ∎

PROPOSITION 5. *We can assume that C is a line, or a space rational curve.*

PROOF. Suppose that C is not a line. If C is a plane curve, then the plane is Γ-invariant. This contradicts the assumption A3. Therefore C is a space curve. Then the restriction

$$r : \Gamma \to \mathrm{Aut}(C)$$

is injective. Hence Aut(C) is an infinite group. Therefore C is either a (possibly singular) rational curve, or a non-singular elliptic curve. If C is non-singular elliptic, then Aut(C) contains a finite index abelian subgroup. Hence so is Γ. Then Γ admits a semi-invariant plane by Proposition 4. This also contradicts the condition A3. Hence C is a space rational curve. ∎

In view of Proposition 5, we consider the line case in section 2, and the space rational curve case in section 3 separately.

2. Line case

In this section, in addition to the three conditions (A1), (A2) and (A3) on Γ, we assume that

$$C \text{ is a line contained in a } \Gamma\text{-invariant surface } S.$$

Then we have the following.

THEOREM 1. *S is a non-singular quadric.*

To prove the theorem, we determine the defining equation of S. Let $[z_0 : z_1 : z_2 : z_3]$ be a system of homogeneous coordinates on $\mathbf{P}^3$ such that C is given by

$$C = \{z_0 = z_1 = 0\}.$$

Let $F = F(z_0, z_1, z_2, z_3) = 0$ be the defining equation of S. Put $m = \deg F$. Since $C \subset S$, F is written as

(1)

$$F(z_0, z_1, z_2, z_3) = \sum_{k=1}^{m} F_k(z_0, z_1, z_2, z_3), \quad F_k(z_0, z_1, z_2, z_3) = \sum_{i+j=k} a_{ij}^{(k)}(z_2, z_3) z_0^i z_1^j,$$

where the $a_{i,j}^{(k)} = a_{i,j}^{(k)}(z_2, z_3)$ are homogeneous polynomials of degree $m-k$. Put

$$n = \min\{k : F_k \neq 0\}.$$

Obviously, $n \geq 1$.

Let $\mu : M \to \mathbf{P}^3$ be the blowing-up of $\mathbf{P}^3$ centered with C. The exceptional set $E = \mu^{-1}(C)$ is biholomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Let $([u_0 : u_1], [v_0 : v_1])$ be a system of coordinates on E. Here $[u_0 : u_1]$ and $[v_0 : v_1]$ are the homogeneous coordinates on $\mathbf{P}^1$ such that $\mu|E : E \to C$ is given by

$$\mu([u_0 : u_1], [v_0 : v_1]) = [0 : 0 : v_0 : v_1] \in \mathbf{P}^3.$$

Let $\tilde{S}$ be the proper transform of S by μ and $\tilde{C} = \tilde{S} \cap E$. Then $\tilde{C}$ is given by

$$F_n(u_0, u_1, v_0, v_1) = \sum_{i+j=n} a_{ij}^{(n)}(v_0, v_1) u_0^i u_1^j. \tag{2}$$

Consider $\tilde{C}$ with its reduced structure. If $\tilde{C}$ has singular points, put

$$\Gamma_1 = \{\sigma \in \Gamma : \sigma \text{ fixes each singular point of } \tilde{C}\}.$$

Then Γ_1 is a subgroup of Γ with finite index. Let $([p_0 : p_1], [q_0 : q_1]) \in \tilde{C}$ be one of the singular points. Then Γ_1 leaves the plane $p_1 z_0 - p_0 z_1 = 0$ invariant. If $\tilde{C}$ has no singular points, then $\tilde{C}$ would be a finite disjoint union of non-singular curves in E. Note that μ maps every connected component of $\tilde{C}$ onto $C \simeq \mathbf{P}^1$.

Suppose that $\tilde{C}$ is non-singular, i.e., that $\tilde{C}$ consists of mutually disjoint non-singular curves. If $\tilde{C}$ has a component, say $\tilde{C}_1$, such that $\mu|\tilde{C}_1 : \tilde{C}_1 \to C$ is of $\deg \mu|\tilde{C}_1 \geq 2$. Then $\mu|\tilde{C}_1$ would have branch points. Let B be the set of branch points of $\mu|\tilde{C}$, and put

$$\Gamma_2 = \{\sigma \in \Gamma : \sigma \text{ fixes each point of } B\}\,.$$

Since B is a finite set, Γ_2 is a subgroup of Γ with finite index. Take any point $([p_0 : p_1], [q_0 : q_1]) \in B$. Then the plane $p_1 z_0 - p_0 z_1 = 0$ would be Γ_2-invariant.

Suppose that $\tilde{C}$ is non-singular, and that μ maps every component of $\tilde{C}$ bijectively onto C. Suppose further that $\tilde{C}$ has distinct components $\tilde{C}_1$ and $\tilde{C}_2$. Since $\tilde{C}_1$ and $\tilde{C}_2$ do not intersect each other, we see that both are of the form

$$\tilde{C}_j = \{[p_0^j : p_1^j]\} \times \mathbf{P}^1 \subset E, \quad j = 1, 2 \tag{3}$$

for some $[p_0^j : p_1^j] \in \mathbf{P}^1$. Put

$$\Gamma_3 = \{\sigma \in \Gamma : \sigma(\tilde{C}_1) = \tilde{C}_1\}\,.$$

Then Γ_3 is a subgroup of Γ with finite index and the plane $p_1^1 z_0 - p_0^1 z_1 = 0$ would be Γ_3-invariant.

Thus it remains to consider the case where $\tilde{C}$ consists of a unique non-singular curve, and where μ maps $\tilde{C}$ bijectively onto C. In this case, $\tilde{C}$ is defined by a homogeneous polynomial $G(u_0, u_1; v_0, v_1)$ of the four variables of the form

$$G(u_0, u_1; v_0, v_1) = g_1(v_0, v_1)u_0 - g_0(v_0, u_1)u_1\,, \tag{4}$$

where $d = \deg g_0 = \deg g_1$. Define a holomorphic map

$$\varphi : \mathbf{P}^1 \to \mathbf{P}^1$$

by

$$\varphi([v_0 : v_1]) = [g_0(v_0, v_1) : g_1(v_0, v_1)]\,.$$

Note that every $\sigma \in \Gamma$ can be written as

$$\begin{pmatrix} A_\sigma & 0 \\ C_\sigma & D_\sigma \end{pmatrix},$$

where $A_\sigma, C_\sigma, D_\sigma$ are $(2, 2)$-matrices with $\det A_\sigma \cdot \det D_\sigma \neq 0$. The action of σ on E is given by

$$[u, v] \mapsto [A_\sigma u, D_\sigma v]\,,$$

where $u = (u_0, u_1)$, $v = (v_0, v_1)$. Therefore we have the commutative diagram

$$\begin{array}{ccc} \mathbf{P}^1 & \xrightarrow{\varphi} & \mathbf{P}^1 \\ D_\sigma \downarrow & & \downarrow A_\sigma \\ \mathbf{P}^1 & \xrightarrow{\varphi} & \mathbf{P}^1 \end{array}$$

for any $\sigma \in \Gamma$.

If φ is a constant map, then put $[p_0 : p_1] = \varphi(\mathbf{P}^1)$. Then the plane $p_1 z_0 - p_0 z_1 = 0$ is left invariant by Γ.

Suppose that φ is not constant. First we consider the case $\deg \varphi = d \geq 2$. In this case, the set

$$R = \{v \in \mathbf{P}^1 : d\varphi(v) = 0\}\,.$$

is a non-empty finite set. Put

$$\Gamma_4 = \{\sigma \in \Gamma : A_\sigma \text{ fixes every point of} R\}\,.$$

Then, Γ_4 is a subgroup of Γ with finite index. This implies that for a suitable homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ on $\mathbf{P}^3$, the matrices A_σ and D_σ are lower triangular for all $\sigma \in \Gamma_4$. Therefore $z_0 = 0$ is left invariant by Γ_4.

Lastly, we consider the case $\deg \varphi = d = 1$. Since $\varphi \in \mathrm{PSL}(2, \mathbf{C})$ in this case, replacing coordinates z_2, z_3 of $[z_0 : z_1 : z_2 : z_3]$ suitably, we can assume that

$$A_\sigma = D_\sigma$$

holds for every $\sigma \in \Gamma$. Then G can be written as

$$G(u_0, u_1; v_0, v_1) = v_1 u_0 - v_0 u_1\,. \tag{5}$$

Recall that we have been studying $\tilde{C}$ with its reduced structure. By (4) and (2), F_n in (1) is of the form

$$F_n(z_0, z_1, z_2, z_3) = (z_0 z_3 - z_1 z_2)^n$$

where $m = 2n$. Thus we have shown the following.

LEMMA 2.1. *We can choose a system of homogeneous coordinates* $[z_0 : z_1 : z_2 : z_3]$ *on* $\mathbf{P}^3$ *such that*

$$C = \{z_0 = z_1 = 0\}\,,$$

$$F_n(z_0, z_1, z_2, z_3) = (z_0 z_3 - z_1 z_2)^n, \quad m = 2n\,, \tag{6}$$

and that every element $\sigma \in \Gamma$ *is represented by a matrix of the form*

$$\begin{pmatrix} A_\sigma & 0 \\ C_\sigma & A_\sigma \end{pmatrix}. \tag{7}$$

Before going into the process of determining F, we shall check the conditions that can be imposed on Γ. Let

$$\rho : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$$

be the homomorphism defined by $\rho(\sigma) = A_\sigma$. Since Γ is finitely generated, replacing Γ with its subgroup of finite index, we can assume that $\rho(\Gamma)$ is torsion free. This implies, in particular, that $\rho(\Gamma)$ contains *no* elements represented by matrices of the following forms

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \quad \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \tag{8}$$

where $b, c \in \mathbf{C}^*$, and ε is a root of unity other than ± 1.

LEMMA 2.2. *If every element of $\rho(\Gamma)$ is parabolic, then there is a Γ-invariant plane.*

PROOF. By taking a suitable conjugate of Γ, we can assume that $\rho(\Gamma)$ contains an element represented by $J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Now take any element $\sigma \neq 1$ in $\rho(\Gamma)$, which is represented by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C})$. Since AJ, $A^{-1}J$ are parabolic or identities, their traces are ± 2. Hence, we obtain $|a + d + b| = |a + d - b| = |a + d| = 2$. This implies $b = 0$. Therefore every element of $\rho(\Gamma)$ is represented by a lower triangular matrix. Hence $z_0 = 0$ is a Γ-invariant plane. ∎

By Lemma 2.2, we can assume that Γ contains an element σ represented by

$$\begin{pmatrix} A_\sigma & 0 \\ C_\sigma & A_\sigma \end{pmatrix}, \quad A_\sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \tag{9}$$

where $\alpha \in \mathbf{C}^*$ is not a root of unity. Put

$$C_\sigma = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix}.$$

Introduce a new system of homogeneous coordinates $[z'_0 : z'_1 : z'_2 : z'_3]$ on $\mathbf{P}^3$ by

$$\begin{cases} z_0 = z'_0 \\ z_1 = z'_1 \\ z_2 = z'_2 + \dfrac{\gamma_{01}}{\alpha^{-1} - \alpha} z'_1 \\ z_3 = z'_3 + \dfrac{\gamma_{10}}{\alpha - \alpha^{-1}} z'_0. \end{cases} \tag{10}$$

Then σ can be represented by the matrix

$$\begin{pmatrix} A_\sigma & 0 \\ C_\sigma & A_\sigma \end{pmatrix}, \quad A_\sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad C_\sigma = \begin{pmatrix} \gamma_{00} & 0 \\ 0 & \gamma_{11} \end{pmatrix}. \tag{11}$$

From now on in this section, we fix the homogeneous coordinates above on $\mathbf{P}^3$, by which the element σ is represented by the matrix (11). Note that the equation (6) remains unchanged. Here we sum up the conditions imposed on Γ.

(i) Γ has no elements of finite order,

(ii) Γ consists of elements satisfying (7),

(iii) Γ contains an element of the form (11),

(iv) $\rho(\Gamma)$ has no elements of finite order. Thus $\rho(\Gamma)$ contains no elements represented by matrices of (8),

(v) $\rho(\Gamma)$ contains an element of the form

$$A = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}, \quad |\alpha_{01}| + |\alpha_{10}| \neq 0\,.$$

Recall that conditions (i) and (iv) are fulfilled, if Γ is replaced with its subgroup of finite index. Condition (iii) is a consequence of Lemma 2.2. We impose the condition (v) on Γ, since otherwise Γ will have trivially a Γ-invariant plane. By the conditions (iv) and (v), every $A \in \rho(\Gamma)$ with $|\alpha_{01}| + |\alpha_{01}| \neq 0$ satisfies either $\alpha_{01}\alpha_{11} \neq 0$ or $\alpha_{00}\alpha_{10} \neq 0$.

Now we are going into the process of determining F. We write F in the form (1). By (6), we know that F is written as

$$F = F_n + F_{n+1} + \cdots + F_m, \quad m = \deg F = 2n \tag{12}$$

where

$$F_n = (z_0z_3 - z_1z_2)^n\,.$$

In general, for a polynomial

$$G = \sum_{i_0,i_1,i_2,i_3} g_{i_0i_1i_2i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}\,,$$

we indicate by

$$[G]_k = \sum_{i_0+i_1=k} g_{i_0i_1i_2i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}$$

the partial sum of terms with $i_0 + i_1 = k$. Thus $F_k = [F]_k$. We put

$$F_k = \sum_{i_0+i_1=k} a_{i_0i_1i_2i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}\,.$$

SUBLEMMA 2.1. *Under the conditions* (i)–(iv) *on* Γ, F *is of the form*

$$F(z_0, z_1, z_2, z_3) = \sum_{k=0}^{n} F_{n+k}(z_0, z_1, z_2, z_3),$$

where

$$F_{n+k}(z_0, z_1, z_2, z_3) = (z_0 z_1)^k \sum_{i_2+i_3=n-k} a_{i_2 i_3}(z_0 z_3)^{i_3}(z_1 z_2)^{i_2}.$$

PROOF. Let $\sigma \in \Gamma$ be an element of (iii), i.e., of the form (11). We already know that F_n is a polynomial of $z_0 z_3$ and $z_1 z_2$. From $\sigma^* F = F$, it follows that

$$\left[\sigma^* F\right]_{n+1} = F_{n+1}. \tag{13}$$

On the other hand, we have

$$\left[\sigma^* F\right]_{n+1} = F_{n+1}(\alpha z_0, \alpha^{-1} z_1, \alpha z_2, \alpha^{-1} z_3) + \left[\sigma^* F_n\right]_{n+1}, \tag{14}$$

where

$$\left[\sigma^* F_n\right]_{n+1} = n\left(\alpha\gamma_{11} - \alpha^{-1}\gamma_{00}\right) z_0 z_1 (z_0 z_3 - z_1 z_2)^{n-1}.$$

Hence, by (13), we have

$$\begin{aligned} F_{n+1}(z_0, z_1, z_2, z_3) = {} & F_{n+1}(\alpha z_0, \alpha^{-1} z_1, \alpha z_2, \alpha^{-1} z_3) \\ & + n\left(\alpha\gamma_{11} - \alpha^{-1}\gamma_{00}\right) z_0 z_1 (z_0 z_3 - z_1 z_2)^{n-1} \end{aligned} \tag{15}$$

Comparing the terms $z_0 z_1 (z_0 z_3)^k (z_1 z_2)^{n-k-1}$ in (15), we have

$$F_{n+1}(z_0, z_1, z_2, z_3) = F_{n+1}(\alpha z_0, \alpha^{-1} z_1, \alpha z_2, \alpha^{-1} z_3) \tag{16}$$

and

$$\alpha\gamma_{11} - \alpha^{-1}\gamma_{00} = 0. \tag{17}$$

Let

$$F_{n+1} = \sum_{i_0+i_1=n+1,\ i_2+i_3=n-1} a_{i_0 i_1 i_2 i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}.$$

By (16), we see that non-zero terms $a_{i_0 i_1 i_2 i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}$ satisfy $i_0 + i_2 = i_1 + i_3$. Therefore we have $i_0 = i_3 + 1$ and $i_1 = i_2 + 1$, and

$$a_{i_0 i_1 i_2 i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3} = a_{i_0 i_1 i_2 i_3}(z_0 z_1)(z_0 z_3)^{i_3}(z_1 z_2)^{i_2}.$$

Hence we have

$$F_{n+1} = z_0 z_1 \sum_{i_2+i_3=n-1} a_{i_2 i_3}(z_0 z_3)^{i_3}(z_1 z_2)^{i_2}. \tag{18}$$

Further, (17) implies

$$\sigma^*(F_n) = F_n\,. \tag{19}$$

By (18), we have

$$\begin{aligned}\sigma^*(F_{n+1}) &= F_{n+1}(\alpha z_0, \alpha^{-1} z_1, \alpha z_2 + \gamma_{00} z_0, \alpha^{-1} z_3 + \gamma_{11} z_1)\\ &= z_0 z_1 \sum_{i_2+i_3=n-1} a_{i_2 i_3}(z_0 z_3 + \alpha\gamma_{11} z_0 z_1)^{i_3}(z_1 z_2 + \alpha^{-1}\gamma_{00} z_0 z_1)^{i_2}.\end{aligned}$$

Therefore

(20) $\sigma^*(F_{n+1})$ is a polynomial of $z_0 z_1$, $z_0 z_3$ and $z_1 z_2$.

Next, we consider F_{n+2}. Using (19), we have

$$\begin{aligned}F_{n+2} &= \left[\sigma^* F\right]_{n+2}\\ &= \left[\sigma^*(F_n) + \sigma^*(F_{n+1}) + \sigma^*(F_{n+2})\right]_{n+2}\\ &= \left[\sigma^*(F_{n+1})\right]_{n+2} + \left[\sigma^*(F_{n+2})\right]_{n+2}\,.\end{aligned} \tag{21}$$

Since

$$\left[\sigma^*(F_{n+2})\right]_{n+2} = F_{n+2}(\alpha z_0, \alpha^{-1} z_1, \alpha z_2, \alpha^{-1} z_3)\,.$$

it follows from from (21) that

$$F_{n+2}(z_0, z_1, z_2, z_3) - F_{n+2}(\alpha z_0, \alpha^{-1} z_1, \alpha z_2, \alpha^{-1} z_3) = \left[\sigma^*(F_{n+1})\right]_{n+2} \tag{22}$$

By (20), $\sigma^*(F_{n+1})$ is a polynomial of $z_0 z_1$, $z_0 z_3$ and $z_1 z_2$. On the other hand, the left-hand side of (22) contains no terms of monomials of $z_0 z_1$, $z_0 z_3$ and $z_1 z_2$, since these terms remain invariant by the action $(z_0, z_1, z_2, z_3) \mapsto (\alpha z_0, \alpha^{-1} z_1, \alpha z_2, \alpha^{-1} z_3)$. Hence we have

$$\begin{aligned}&F_{n+2}(z_0, z_1, z_2, z_3) = F_{n+2}(\alpha z_0, \alpha^{-1} z_1, \alpha z_2, \alpha^{-1} z_3),\\ &\left[\sigma^*(F_{n+1})\right]_{n+2} = 0\,.\end{aligned} \tag{23}$$

Let

$$F_{n+2} = \sum_{i_0+i_1=n+2,\ i_2+i_3=n-2} a_{i_0 i_1 i_2 i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}\,.$$

By (23), we see that non-zero terms $a_{i_0 i_1 i_2 i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}$ satisfy $i_0 + i_2 = i_1 + i_3$. Therefore we have $i_0 = i_3 + 2$ and $i_1 = i_2 + 2$, and

$$a_{i_0 i_1 i_2 i_3} z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3} = a_{i_0 i_1 i_2 i_3}(z_0 z_1)^2 (z_0 z_3)^{i_3} (z_1 z_2)^{i_2}\,.$$

Hence we have

$$F_{n+2} = (z_0 z_1)^2 \sum_{i_2+i_3=n-2} a_{i_2 i_3}(z_0 z_3)^{i_3}(z_1 z_2)^{i_2}\,. \tag{24}$$

From this, it follows also that

(25) $\sigma^*(F_{n+2})$ is a polynomial of z_0z_1, z_0z_3 and z_1z_2 .

By (24), (20) and (25), we can proceed to the next induction step F_{n+3}. Thus the sublemma is proved inductively. ∎

REMARK 1. By (17), σ of (iii) is of the form

$$(26)\qquad \begin{pmatrix} A_\sigma & 0 \\ c_\sigma A_\sigma & A_\sigma \end{pmatrix}, \quad c_\sigma \in \mathbf{C}\,.$$

SUBLEMMA 2.2. *Under the conditions* (i)–(v) *on Γ, F is of the form*

$$F = \sum_{k=0}^{n} a_k (z_0z_1)^k (z_0z_3 - z_1z_2)^{n-k}, \quad a_0 = 1\,.$$

PROOF. Put $y = z_0z_3 - z_1z_2$ for short. By Sublemma 2.1, it is enough to show that each F_{n+k} is divisible by y^{n-k}. Since $F_n = y^n$, we are done for $k = 0$. Now we shall prove the lemma by induction on $k \geq 0$. Suppose that F_{n+j} is determined for $j \leq k$ and consider the case $k+1$. Let

$$F = \sum_{j=0}^{k} a_j (z_0z_1)^j y^{n-j} + \sum_{j=k+1}^{n} F_{n+j}\,,$$

where $a_j \in \mathbf{C}$ for some k with $0 \leq j \leq k < n$. Choose $\tau \in \Gamma$ of (v). Put

$$(27)\quad \begin{pmatrix} A_\tau & 0 \\ C_\tau & A_\tau \end{pmatrix}, \quad A_\tau = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}, \quad C_\tau = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix}, \quad |\alpha_{01}| + |\alpha_{10}| \neq 0\,.$$

Put also

$$\begin{aligned} w_0 &= \alpha_{00}z_0 + \alpha_{01}z_1, \\ w_1 &= \alpha_{10}z_0 + \alpha_{11}z_1, \\ v_0 &= \gamma_{00}z_0 + \gamma_{01}z_1, \quad & w_2 &= \alpha_{00}z_2 + \alpha_{01}z_3, \\ v_1 &= \gamma_{10}z_0 + \gamma_{11}z_1, \quad & w_3 &= \alpha_{10}z_2 + \alpha_{11}z_3\,. \end{aligned}$$

We put

$$\Delta_\tau = w_0v_1 - w_1v_0\,.$$

Note that

$$(28)\qquad y = z_0z_3 - z_1z_2 = w_0w_3 - w_1w_2\,,$$

and that

$$\tau^* y = y + \Delta_\tau\,.$$

From $\tau^* F = F$, it follows that

$$F_{n+k+1} = \left[\tau^* \left(\sum_{j=0}^{k} a_j (z_0 z_1)^j y^{n-j} + F_{n+k+1} \right) \right]_{n+k+1}$$

$$= y^{n-k-1} \sum_{j=0}^{k} {}_{n-j}C_{n-k-1} a_j (w_0 w_1)^j \Delta_\tau^{k+1-j} + \left[\tau^*(F_{n+k+1}) \right]_{n+k+1}$$

$$= y^{n-k-1} \sum_{j=0}^{k} {}_{n-j}C_{n-k-1} a_j (w_0 w_1)^j \Delta_\tau^{k+1-j} + F_{n+k+1}(w_0, w_1, w_2, w_3). \tag{29}$$

In view of Sublemma 2.1, we can write F_{n+k+1} as

$$F_{n+k+1} = (z_0 z_1)^{k+1} \sum_{i_2+i_3=n-k-1} a_{i_2 i_3} (z_0 z_3)^{i_3} (z_1 z_2)^{i_2} .$$

Suppose that $n - k - 1 > 0$. Then, letting $z_0 = z_2 = 0$ in (29), we have

$$\begin{aligned} 0 &= F_{n+k+1}(\alpha_{01} z_1, \alpha_{11} z_1, \alpha_{01} z_3, \alpha_{11} z_3) \\ &= (\alpha_{01} z_1 \alpha_{11} z_1)^{k+1} \sum_{i_2+i_3=n-k-1} a_{i_2 i_3} (\alpha_{01} z_1 \alpha_{11} z_3)^{i_3} (\alpha_{11} z_1 \alpha_{01} z_3)^{i_2} \\ &= (\alpha_{01} \alpha_{11})^n z_1^{n+k+1} z_3^{n-k-1} \sum_{i_2+i_3=n-k-1} a_{i_2 i_3} . \end{aligned} \tag{30}$$

Similarly, letting $z_1 = z_3 = 0$ in (29), we have

$$\begin{aligned} 0 &= F_{n+k+1}(\alpha_{00} z_0, \alpha_{10} z_0, \alpha_{00} z_2, \alpha_{10} z_2) \\ &= (\alpha_{00} z_0 \alpha_{10} z_0)^{k+1} \sum_{i_2+i_3=n-k-1} a_{i_2 i_3} (\alpha_{00} z_0 \alpha_{10} z_2)^{i_3} (\alpha_{10} z_0 \alpha_{00} z_2)^{i_2} \\ &= (\alpha_{00} \alpha_{10})^n z_0^{n+k+1} z_2^{n-k-1} \sum_{i_2+i_3=n-k-1} a_{i_2 i_3} . \end{aligned} \tag{31}$$

Recall that, by the conditions (iv) and (v), we have either $\alpha_{01}\alpha_{11} \neq 0$ or $\alpha_{00}\alpha_{10} \neq 0$. Hence by (30), (31), we have

$$\sum_{i_2+i_3=n-k-1} a_{i_2 i_3} = 0 . \tag{32}$$

This condition (32) implies that F_{n+k+1} is divisible by y. Define $G_1(z_0, z_1, z_2, z_3)$ by

$$F_{n+k+1}(z_0, z_1, z_2, z_3) = y G_1(z_0, z_1, z_2, z_3) ,$$

where G_1 is of the form

$$G_1 = (z_0 z_1)^{k+1} \sum_{i_2+i_3=n-k-2} a_{i_2 i_3} (z_0 z_3)^{i_3} (z_1 z_2)^{i_2} . \tag{33}$$

Further, by (29) and (28), we have

$$G_1(z_0, z_1, z_2, z_3) = y^{n-k-2}\tilde{\Delta}_\tau + G_1(w_0, w_1, w_2, w_3)\,, \tag{34}$$

where

$$\tilde{\Delta}_\tau = \sum_{j=0}^{k} {}_{n-j}C_{n-k-1}a_j(w_0w_1)^j \Delta_\tau^{k+1-j}\,.$$

Therefore we can repeat the argument from (30) while $n-k-1 > 0$. This implies that F_{n+k+1} is divisible by y^{n-k-1}. This proves the sublemma. ∎

REMARK 2. Until now, we have not used the assumption that S is irreducible.

PROPOSITION 6. *Assume that the conditions* (i)–(iv) *on Γ are fulfilled with respect to a system* $[z_0 : z_1 : z_2 : z_3]$ *of homogeneous coordinates on* $\mathbf{P}^3$, *where*

$$\ell = \{[z_0 : z_1 : z_2 : z_3] : z_0 = z_1 = 0\}\,.$$

If S is a Γ-invariant surface, then the defining equation F of S can be given by

$$F = (z_0z_3 - z_1z_2 + tz_0z_1)^n \tag{35}$$

for some $t \in \mathbf{C}$. Further, each element $\sigma \in \Gamma$ is represented by a matrix of the form

$$\begin{pmatrix} A_\sigma & 0 \\ c_\sigma A_\sigma + A_\sigma T - TA_\sigma & A_\sigma \end{pmatrix}, \quad T = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad c_\sigma \in \mathbf{C} \tag{36}$$

PROOF. By Sublemma 2.2, any irreducible factor of F is of the form

$$z_0z_3 - z_1z_2 + tz_0z_1, \quad t \in \mathbf{C}\,. \tag{37}$$

Since S is irreducible, F is of the form (35). Each $\sigma \in \Gamma$ is represented by a matrix of the form $\begin{pmatrix} A_\sigma & 0 \\ C_\sigma & A_\sigma \end{pmatrix}$. To determine C_σ, we define

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

Since $\sigma \in \Gamma$ leaves the equation (37) invariant, we have the relation of parings $\langle\ ,\ \rangle$ of 2-vectors

$$\langle A_\sigma z, J(TA_\sigma z + C_\sigma z + A_\sigma z')\rangle = \langle z, J(Tz + z')\rangle\,,$$

where $z = {}^t(z_0, z_1)$ and $z' = {}^t(z_2, z_3)$. By the relation ${}^tA_\sigma JA_\sigma = J$, we have $\langle z, Jz'\rangle = \langle A_\sigma z, JA_\sigma z'\rangle$. Hence we obtain

$$\langle z, \left({}^tA_\sigma JTA_\sigma + {}^tA_\sigma JC_\sigma - JT\right) z\rangle = 0\,.$$

This implies

$$ {}^t A_\sigma J T A_\sigma + {}^t A_\sigma J C_\sigma - J T = c_\sigma J $$

for some $c_\sigma \in \mathbf{C}$. Hence we have

$$ C_\sigma = c_\sigma A_\sigma + A_\sigma T - T A_\sigma . $$

Thus we have the proposition. ∎

Theorem 1 is clear by the proposition above. ∎

COROLLARY 2.1. *Let S_1 and S_2 be surfaces which are invariant by a finitely generated discrete subgroup $\Gamma \subset \mathrm{PGL}(4, \mathbf{C})$. Suppose that $S_1 \cap S_2$ contains a line and that Γ admits no semi-invariant planes. Then $S_1 = S_2$.*

PROOF. By Proposition 6, both S_1 and S_2 are quadrics defined respectively by

$$ S_1 : F_1 = z_0 z_3 - z_1 z_2 + t_1 z_0 z_1, \quad t_1 \in \mathbf{C} $$
$$ S_2 : F_2 = z_0 z_3 - z_1 z_2 + t_2 z_0 z_1, \quad t_2 \in \mathbf{C}. $$

For $\sigma \in \Gamma$, we have $\sigma^* F_j = (\det A_\sigma) F_j$. Therefore the polynomial $F_1 - F_2 = (t_1 - t_2) z_0 z_1$ is also Γ-invariant. Since Γ admits no semi-invariant plane, we have $t_1 = t_2$. Hence $S_1 = S_2$. ∎

3. Space rational curve case

In this section, in addition to the three conditions (A1), (A2) and (A3) on Γ, we assume that

C is a space rational curve not contained in any plane.

Let

$$ \varphi : \mathbf{P}^1 \to C \subset \mathbf{P}^3 \tag{38} $$
$$ \varphi([z : w]) = [v_0([z : w]) : v_1([z : w]) : v_2([z : w]) : v_3([z : w])] \tag{39} $$

be the normalization of C, where $n = \deg v_j$, $0 \le j \le 3$. Let $\varphi^{(k)}$ be the k-th associated curve of φ (see [GH, p.263]) :

$$ \varphi^{(0)} = \varphi, \quad \varphi^{(1)} : \mathbf{P}^1 \to \mathrm{Gr}(4, 2), \quad \varphi^{(2)} : \mathbf{P}^1 \to \mathbf{P}^{3\vee} . $$

Since C is Γ-invariant, every $\sigma \in \Gamma$ induces a holomorphic automorphism $\tilde{\sigma} \in \mathrm{Aut}(\mathbf{P}^1)$ which makes the diagram

$$ \begin{array}{ccc} \mathbf{P}^1 & \xrightarrow{\varphi} & \mathbf{P}^3 \\ \tilde{\sigma} \downarrow & & \downarrow \sigma \\ \mathbf{P}^1 & \xrightarrow{\varphi} & \mathbf{P}^3 \end{array} $$

commutative. Let $\tilde{\Gamma}$ denote the subgroup of $\mathrm{Aut}(\mathbf{P}^1)$ generated by $\tilde{\sigma}$ for $\sigma \in \Gamma$. Since Γ is a discrete group of $\mathrm{PSL}(4, \mathbf{C})$, $\tilde{\Gamma}$ is also discrete in $\mathrm{PSL}(2, \mathbf{C})$. If $\tilde{\Gamma}$ is an elementary group, then $\tilde{\Gamma}$ contains an abelian subgroup of finite index. Hence Γ admits a semi-invariant plane by Proposition 4. Therefore we can assume that $\tilde{\Gamma}$ contains no elementary subgroup of finite index. Consider the sets

$$\Sigma^{(k)} = \{x \in \mathbf{P}^1 : \operatorname{rank} d\varphi^{(k)}(x) = 0\}, \quad k = 0, 1, 2$$

and

$$\Sigma = \bigcup_{k=0}^{2} \Sigma^{(k)} .$$

PROPOSITION 7. *If $\Sigma \neq \emptyset$, then there is a semi-invariant plane.*

PROOF. Note that Σ is a $\tilde{\Gamma}$-invariant finite set. Hence $\tilde{\Gamma}$ is elementary by Proposition 1. This contradicts our assumption. ∎

It is well-known that $\Sigma = \emptyset$ if and only if C is a twisted cubic. Suppose that C is given by

$$\varphi : \mathbf{P}^1 \ni [1 : t] \to [1 : t : t^2 : t^3] \in \mathbf{P}^3 . \tag{40}$$

By Proposition 3, we can take a loxodromic element $\tilde{\sigma}$ in $\tilde{\Gamma}$. Choosing the coordinates on $\mathbf{P}^1$ and $\mathbf{P}^3$ suitably, $\tilde{\sigma} \in \tilde{\Gamma}$ is given by

$$\tilde{\sigma}([1 : t]) = [1 : \alpha t] \tag{41}$$

where $0 < |\alpha| < 1$. Let S be the tangential surface of C. Note that the tangential surface is irreducible, Γ-invariant and containing C.

PROPOSITION 8. *There is no Γ-invariant surface containing C other than S.*

PROOF. By an easy calculation, the tangential surface S is of degree 4 given by

$$F(z_0, z_1, z_2, z_3) = z_0^2 z_3^2 - 6z_0z_1z_2z_3 - 3z_1^2z_2^2 + 4z_0z_2^3 + 4z_1^3z_3 = 0 .$$

Hence, we see that S contains the lines

$$\ell_{01} = \{z_0 = z_1 = 0\} \quad \text{and} \quad \ell_{23} = \{z_2 = z_3 = 0\} . \tag{42}$$

Suppose that there is a Γ-invariant surface S_1 other than S. Let $F_1(z_0, z_1, z_2, z_3)$ be the defining equation of S_1. Put $m = \deg F_1$. Then by (41), F_1 satisfies

$$\begin{aligned} F_1(tz_0, tz_1, tz_2, tz_3) &= t^m F_1(z_0, z_1, z_2, z_3), \\ F_1(z_0, tz_1, t^2z_2, t^3z_3) &= t^N F_1(z_0, z_1, z_2, z_3), \end{aligned}$$

for any $t \in \mathbf{C}$, and some fixed natural number N. If $N > m$, then F_1 contains no terms with $z_0^{i_0} z_1^{i_1}$ with $m = i_0 + i_1$. Hence $\ell_{23} \subset S_1$. If $N < 2m$, then F_1 contains no terms with $z_2^{i_2} z_3^{i_3}$

with $m = i_2 + i_3$. Hence $\ell_{01} \subset S_1$. Therefore S_1 contains either ℓ_{01} or ℓ_{23}. Hence by (42), $S \cap S_1$ contains either ℓ_{01} or ℓ_{23}. Since both S and S_1 are Γ-invariant, there is a subgroup Γ_1 of Γ which leaves invariant the line in $S \cap S_1$. Thus we are in the case of section 2. Since S is not a non-singular quadric, we have a contradiction. ∎

4. Proof of main results

Let Γ be a finitely generated discrete subgroup of PGL(4, $\mathbf{C}$) acting on $\mathbf{P}^3$.

PROOF OF THEOREM A. For the first case, it is enough to use $[z_0 : z_1 : z_2 : z_3 + tz_1]$ as a new system of homogeneous coordinates on $\mathbf{P}^3$ in Proposition 6. The uniqueness follows from Corollary 2.1. The second case is obvious by Proposition 8. ∎

PROOF OF THEOREM B. Suppose that the singular locus of S is a finite set points. Then Γ contains a finite index subgroup Γ_1 which fixes a singular point p. Then the tangent cone S_0 at p of S is Γ_1-invariant. If $S_0 \neq S$, then $S_0 \cap S$ is a finite union of curves. Let $C_0 \subset S_0 \cap S$ be a curve passing through p. Then the tangent cone of C_0 at p contains a line L, which is contained also in S_0. Obviously, L is Γ_1-semi-invariant. Thus the pair L and S_0 satisfies the condition of section 2. Hence by Theorem 1, S_0 is a non-singular quadric. This contradicts the fact that p is a singular point of S. Hence $S = S_0$, i.e., S is a cone over a non-singular plane curve C. If C is not rational, then Γ_1 contains a solvable subgroup of finite index. Hence there are semi-invariant planes by Proposition 4. If C is a line, then S is the plane left invariant by Γ_1. Thus C is a non-singular conic. If the singular locus of S contains a curve, then we can apply Theorem A and obtain that S is either a quadric or the tangential surface of a twisted cubic curve. Suppose that S is non-singular. Put $n = \deg S$. By assumption, we have $n \geq 2$. If $n = 2$, then there is nothing to prove. It is well-known that every non-singular surface of degree 3 contains exactly 27 lines. Hence each one of these lines is semi-invariant. Thus we are in the case Theorem 1 and we see that this case doesn't occur. Suppose that $n \geq 4$. Note that the group

$$G = \{g \in \mathrm{PSL}(4, \mathbf{C}) : g(S) = S\}$$

is a closed algebraic subgroup of PSL(4, $\mathbf{C}$). It is easy to see that, if $n \geq 4$, S does not admit non-zero tangent vector fields. Hence we see that G is a finite group. This implies Γ itself is a finite group, a contradiction. ∎

COROLLARY 1. *If Γ leaves invariant distinct two surfaces in $\mathbf{P}^3$, then there is a semi-invariant plane.*

PROOF. The two surfaces intersect in curves, and Γ contains is a finite index subgroup which leaves invariant each irreducible component of the intersection. Hence the corollary follows from Theorem A. ∎

As an easy consequence, we have the following, which was proved in [K1] by a rather long case-by-case checking.

COROLLARY 2 ([K1, Lemma 5.9]). *If there is a non-trivial rational function on* $\mathbf{P}^3$ *which is invariant under the action of a finitely generated discrete subgroup* Γ *of* $\mathrm{PGL}(4, \mathbf{C})$, *then* Γ *contains a subgroup* Γ_0 *of finite index such that* Γ_0 *leaves invariant a 2-dimensional projective plane in* $\mathbf{P}^3$.

COROLLARY 3. *Let* C *be a* Γ*-invariant curve in* $\mathbf{P}^3$. *If* Γ *admits no semi-invariant planes, then* C *is a line or a twisted cubic curve.*

PROOF. Suppose that C is not a line. Then, C is a space rational curve by Proposition 5. Then, by Propositions 7, we conclude that C is a twisted cubic curve. ∎

5. Examples

Let Γ be a finitely generated discrete subgroup of $\mathrm{PGL}(4, \mathbf{C})$ acting on $\mathbf{P}^3$. Suppose that a surface $S \subset \mathbf{P}^3$ is Γ-invariant, but that Γ admits no semi-invariant planes. Then, by Theorem B, S is either a non-singular quadric, tangential surface of a twisted cubic curve, or a cone over a non-singular conic. In this section, we shall give an example for each case.

5.1. Non-singular quadric. Let G be a finitely generated non-elementary Kleinian group, and $c \in \mathrm{Hom}(G, \mathbf{C})$ any element. Define the group Γ by

$$\Gamma = \left\{ \begin{pmatrix} A & 0 \\ c(A)A & A \end{pmatrix} : A \in G \right\},$$

It is easy to see that Γ is a finitely generated discrete subgroup of $\mathrm{PGL}(4, \mathbf{C})$ and that the line

$$z_0 = z_1 = 0$$

and the surface

$$z_0 z_3 - z_1 z_2 = 0$$

are Γ-invariant.

LEMMA 5.1. *Γ admits no semi-invariant plane.*

PROOF. Consider the Segre map

$$s : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3,$$

where

$$s([u_0 : u_1], [v_0 : v_1]) = [u_0 v_0 : u_0 v_1 : u_1 v_0 : u_1 v_1] .$$

Then s defines a group homomorphism

$$\tilde{s} : \mathrm{PSL}(2, \mathbf{C}) \times \mathrm{PSL}(2, \mathbf{C}) \to \mathrm{PGL}(4, \mathbf{C}) .$$

Obviously, we have

$$\tilde{s}\,((J_c, A)) = \begin{pmatrix} A & 0 \\ cA & A \end{pmatrix}, \quad \text{where } J_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

Suppose that a finite index subgroup Γ_1 leaves a plane

$$p_0 z_0 + p_1 z_1 + p_2 z_2 + p_3 z_3 = 0$$

invariant. Put

$$p = (p_0, p_1), \ q = (p_2, p_3), \ u = {}^t(u_0, u_1), \ \text{and} \ v = {}^t(v_0, v_1).$$

Then, for $h = (J_c, A) \in \tilde{s}^{-1}(\Gamma_1)$, by the invariance of the plane we have that

$$u_0 pAv + (cu_0 + u_1) qAv = \mu(h)(u_0 pv + u_1 qv)$$

for some $\mu(h) \in \mathbf{C}^*$. Thus we have

$$\begin{aligned} pAv + cqAv &= \mu(h) pv, \\ cqAv &= \mu(h) qv, \end{aligned}$$

and hence

$$pA + cqA = \mu(h) p, \tag{43}$$

$$cqA = \mu(h) q. \tag{44}$$

The equality (44) implies $q = 0$, since otherwise G would be an elementary group. Then again, the equality (43) implies that G is elementary, a contradiction. ∎

5.2. Tangential surface to a twisted cubic curve. Let G be a non-elementary discrete subgroup of PGL(2, **C**). Note that any element in PGL(2, **C**) defines a linear transformation $H^0(\mathbf{P}^1, \mathcal{O}(3)) \simeq \mathbf{C}^4$. Hence G defines a discrete subgroup $\Gamma \subset$ PGL(4, **C**), which gives an example of this case. Indeed, if Γ admits a semi-invariant plane, then the intersection of the plane with the twisted cubic curve would be a semi-invariant set. Thus a finite index subgroup of G would have a fixed point. This implies that G is elementary, a contradiction.

5.3. Cone over a non-singular conic. Let $\{1, z, z^2\}$ be a basis of $H^0(\mathbf{P}^1, \mathcal{O}(2))$ ($\simeq \Pi_2$), and

$$\Phi : \mathbf{P}^1 \to \mathbf{P}^3$$

be the embedding defined by $\Phi([1 : z]) = [0 : 1 : z : z^2]$. Let G be a non-elementary Kleinian group with $A_2(\Omega(G), G) \neq 0$, and consider the Eichler cohomology $H^1(G, \Pi_2)$ (see Appendix for the definition). Let $e = \{e(\gamma)\}_{\gamma \in G} \in Z^1(G, \Pi_2)$ be a cocycle which is not zero in $H^1(G_0, \Pi_2)$ for any subgroup $G_0 \subset G$ of finite index. We can choose such e by Lemma 7.1 in Appendix. For $\gamma \in G$, we write

$$e(\gamma) = e_1(\gamma) + e_2(\gamma) z + e_3(\gamma) z^2,$$

where $e_j(\gamma) \in \mathbf{C}$, and

$$\gamma(z) = \frac{a(\gamma)z + b(\gamma)}{c(\gamma)z + d(\gamma)}, \quad \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix} \in \mathrm{SL}(2, \mathbf{C}) .$$

Define a matrix by

$$\rho_e(\gamma) = \begin{pmatrix} 1 & e_1(\gamma) & e_2(\gamma) & e_3(\gamma) \\ 0 & d(\gamma)^2 & 2c(\gamma)d(\gamma) & c(\gamma)^2 \\ 0 & b(\gamma)d(\gamma) & a(\gamma)d(\gamma) + b(\gamma)c(\gamma) & a(\gamma)c(\gamma) \\ 0 & b(\gamma)^2 & 2a(\gamma)b(\gamma) & a(\gamma)^2 \end{pmatrix} .$$

Then the map

$$\rho_e : G \to \mathrm{PGL}(4, \mathbf{C})$$

will be a group homomorphism. The image group $\rho_e(G)$ is an automorphism of the cone over the conic $\Phi(\mathbf{P}^1)$ with the vertex $[1 : 0 : 0 : 0]$.

PROPOSITION 9. *$\rho_e(G)$ admits no semi-invariant planes.*

PROOF. Suppose contrary that there is a finite index subgroup G_0 of G such that $\rho_e(G_0)$ admits an invariant plane. Then we see that e represents zero in $H^1(G_0, \Pi_2)$ by an easy calculation. This contradicts the choice of e. ∎

Proposition 9 shows that $\rho_e(G)$ gives an example of Γ in this case.

REMARK 3. Fix any integer $q > 2$. Let S be a cone over a non-singular rational curve embedded in $\mathbf{P}^{2q-1}$ by the complete linear system $|\mathcal{O}(2q - 2)|$ on $\mathbf{P}^1$. Let Γ be any finitely generated Kleinian group. Then, for each cocycle in the Eichler cohomology $H^1(\Gamma, \Pi_{2q-2})$, we can construct by the same method as above, a subgroups of $\mathrm{PGL}(2q, \mathbf{C})$ which leaves S invariant.

6. Discrete subgroups without invariant surfaces

In this section, we shall construct by a geometric method an example of finitely generated discrete subgroups in $\mathrm{PGL}(4, \mathbf{C})$ without semi-invariant surfaces.

A domain $\Omega \subset \mathbf{P}^3$ is said to be *large*, if Ω contains a line. Every holomorphic automorphism of a large domain extends to an element of $\mathrm{PGL}(4, \mathbf{C})$ ([K1, Lemma 3.1]). Let Ω be a large domain and Γ a properly discontinuous group of holomorphic automorphisms of Ω acting freely on Ω. Suppose that the quotient manifold $X = \Omega/\Gamma$ is compact. It is not difficult to see that the quotient manifold $X = \Omega/\Gamma$ also contains an open subdomain which is biholomorphic to a tubular neighborhood of a line in $\mathbf{P}^3$.

When we are given two such compact quotients $X_1 = \Omega_1/\Gamma_1, X_2 = \Omega_2/\Gamma_2$ of large domains, we can connect complex analytically these two to obtain another compact quotient

$X_3 = \Omega_3/\Gamma_3$ of a large domain. The manifold X_3 is called a *connected sum*[2] of X_1 and X_2 and denoted by $X_3 = X_1\#X_2$. The connected sum is a kind of complex analytic surgery analogous to the classical Klein combination. We describe the connected sum more explicitly in a special case below.

In connected sums, $\mathbf{P}^3$ behaves like an unit, i.e., $X \simeq X\#\mathbf{P}^3 \simeq \mathbf{P}^3\#X$ holds. Conversely, only $\mathbf{P}^3$ is the unit. In deed, if $X \simeq X\#X_1$, then $\pi_1(X)$ is isomorphic to the free product $\pi_1(X) * \pi_1(X_1)$ by van Kampen's theorem. On the other hand, $\pi_1(X) \simeq \pi_1(X) * \pi_1(X_1)$ implies $\pi_1(X_1) = \{1\}$ by Grushko's theorem. Hence we have $X_1 \simeq \mathbf{P}^3$, since a simply connected compact 3-manifold with a projective structure is $\mathbf{P}^3$. The connected sum $X_1\#X_2$ is said to be *trivial* if one of X_j's is $\mathbf{P}^3$. A compact quotient manifold $X = \Omega/\Gamma$ is said to be *prime*, if X is *not* biholomorphic to any non-trivial connected sum $X_1\#X_2$.

In studying compact quotients of large domains, the existence of invariant planes is sometimes crucial. To construct our example, we make use of the following fact.

THEOREM 2 ([K1, Theorem 5.1]). *Let $X = \Omega/\Gamma$ be a compact quotient of a large domain. If Γ admits a semi-invariant plane, then X is prime.*

If the compact quotient $X = \Omega/\Gamma$ admits a non-constant meromorphic function, then Γ admits a semi-invariant plane by Corollary 2 (or [K1, Lemma 5.9]). Hence X is prime by the above theorem. By a recent result on extension of holomorphic maps and by Corollary 2, we can refine the argument of [K1] to prove that, *if $X = \Omega/\Gamma$ admits a non-constant meromorphic function, then X is biholomorphic to either* $\mathbf{P}^3$*, a Blanchard manifold, or an L-Hopf manifold* (see [K2]). Here, a *Blanchard manifold* is a compact complex 3-manifolds whose universal covering is the complement of a line in $\mathbf{P}^3$. An *L-Hopf manifold* is a compact complex 3-manifolds whose universal covering is the complement of two skew lines in $\mathbf{P}^3$.

Now we shall construct an examples of *finitely generated discrete subgroups of* PGL(4, **C**) *without semi-invariant surfaces*, using connected sums together with the results obtained in previous sections.

Let Γ_1 be the infinite cyclic subgroup in PGL(4, **C**) generated by

$$\alpha : [z_0 : z_1 : z_2 : z_3] \mapsto [\alpha_0 z_0 : \alpha_1 z_1 : \alpha_2 z_2 : \alpha_3 z_3],$$

where α_j, $j = 0, 1, 2, 3$, are non-zero constants satisfying the following two conditions:

1. The inequality $|\alpha_0| \le |\alpha_1| < |\alpha_2| \le |\alpha_3|$ holds.
2. The equality $\alpha_i\alpha_j = \alpha_k\alpha_\ell$ holds if and only if the two sets $\{i, j\}$ and $\{k, \ell\}$ coincide.

For example, we can define $\alpha_0 = 1,\ \alpha_1 = c,\ \alpha_2 = c^3,\ \alpha_3 = c^7$ for a constant $c \in \mathbf{C}^*$ with $|c| > 1$, or $\alpha_0 = 1,\ \alpha_1 = e^{\sqrt{2}i},\ \alpha_2 = 2,\ \alpha_3 = 2e^{\sqrt{3}i}$, etc. By the second condition above, we have easily the following

LEMMA 6.1. *No (irreducible) surfaces of degree 2 in $\mathbf{P}^3$ are Γ_1-semi-invariant.*

[2] See [K1] for the details. The connected sum here was called *the connecting operation* there.

Let $X_1 = \Omega_1/\Gamma_1$ be the L-Hopf manifold defined by $\Omega_1 = \mathbf{P}^3 \setminus (\{z_0 = z_1 = 0\} \cup \{z_2 = z_3 = 0\})$ and $\Gamma_1 = \langle \alpha \rangle$.

Let Γ_2 be the rank 4 free abelian subgroup in PGL(4, $\mathbf{C}$) defined by

$$\Gamma_2 = \left\{ g_j = \begin{pmatrix} I & A_j \\ 0 & I \end{pmatrix} : j = 1, 2, 3, 4 \right\},$$

where I is the identity matrix of size 2, and

$$A_1 = I, \quad A_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $X_2 = \Omega_2/\Gamma_2$ be the Blanchard manifold defined by $\Omega_2 = \mathbf{P}^3 \setminus \{z_2 = z_3 = 0\}$ and $\Gamma_2 = \langle g_1, g_2, g_3, g_4 \rangle$.

Now consider the connected sum $X = X_1 \# X_2$. The construction of X is as follows. Consider the open subdomain

$$U_\varepsilon = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbf{P}^3 : |z_0|^2 + |z_1|^2 < \varepsilon^2 \left(|z_2|^2 + |z_3|^2 \right) \right\},$$

where $\varepsilon > 0$. Put $U = U_1$. Note that, for any $\varepsilon > 0$, U_ε is biholomorphic to U, and that any tubular neighborhood of a line in $\mathbf{P}^3$ has a neighborhood biholomorphic to U. For a subset $M \subset \mathbf{P}^3$, we indicate by Int M, $[M]$ and ∂M, the set of interior points, the closure, and the set of boundary points of M, respectively.

First we define an open embedding $j_1 : U_2 \to X_1$. Note that $\alpha([U]) \subset U$. We put

$$Z = [U] \setminus \alpha(U),$$

which is a compact subset contained in Ω_1. The boundary ∂Z has two connected components

$$\Sigma_1 = \partial[U] \quad \text{and} \quad \Sigma_2 = \alpha(\Sigma_1).$$

The manifold X_1 is obtained by identifying Σ_1 and Σ_2 by α. We can find a line ℓ_1 in Int(Z). For example, the line defined by

$$z_0 = \mu z_2, \quad z_1 = \mu z_3, \quad \text{with} \quad \max \left\{ \left| \frac{\alpha_0}{\alpha_2} \right|, \left| \frac{\alpha_1}{\alpha_3} \right| \right\} < |\mu| < 1$$

is contained in Int(Z). Let $W_1 \subset$ Int(Z) be a tubular neighborhood of ℓ_1 which is biholomorphic to U_2. We define the open embedding $j_1 : U_2 \to X_1$ by the composition of a biholomorphic map $\tilde{j}_1 : U_2 \to W_1$ and the canonical projection $\Omega_1 \to X_1$.

Next we define an open embedding $j_2 : U_2 \to X_2$. Let

$$p : \Omega_2 \to \mathbf{C}^2$$

be the projection defined by

$$[z_0 : z_1 : z_2 : z_3] \mapsto (x_1, x_2),$$

where

$$x_1 = \frac{z_0\bar{z}_2 + \bar{z}_1 z_3}{|z_2|^2 + |z_3|^2}, \quad x_2 = \frac{z_1\bar{z}_2 - \bar{z}_0 z_3}{|z_2|^2 + |z_3|^2}.$$

Then p defines a trivial C^∞ S^2-bundle over $\mathbf{C}^2$. Consider the translations τ_j of $\mathbf{C}^2$ defined by

$$\tau_1 : (x_1, x_2) \to (x_1 + 1, x_2), \quad \tau_2 : (x_1, x_2) \to (x_1 + i, x_2)$$

$$\tau_3 : (x_1, x_2) \to (x_1, x_2 + 1), \quad \tau_4 : (x_1, x_2) \to (x_1, x_2 + i).$$

Then we have

$$p \circ g_j = \tau_j \circ p, \quad j = 1, 2, 3, 4. \tag{45}$$

Hence the fundamental domain F of Γ_2 is given by

$$F = p^{-1}\left(\left\{|\operatorname{Re} x_1| \le \frac{1}{2},\ |\operatorname{Im} x_1| \le \frac{1}{2},\ |\operatorname{Re} x_2| \le \frac{1}{2},\ |\operatorname{Im} x_2| \le \frac{1}{2}\right\}\right).$$

Fix any $0 < r < \frac{1}{2}$ and put

$$W_2 = p^{-1}(\{|x_1|^2 + |x_2|^2 < r^2\}).$$

Then, in view of (45), $W_2 \cap g(W_2) = \emptyset$ for any $g \in \Gamma_2 \setminus \{1\}$. Therefore there is a subdomain in X_2 which is biholomorphic to W_2. Note that W_2 is biholomorphic to $U_r (\simeq U_2)$. Define the open embedding $j_2 : U_2 \to X_2$ by the composition of a biholomorphic map $\tilde{j}_2 : U_2 \to W_2$ and the canonical projection $\Omega_2 \to X_2$.

Consider the domain

$$N(2) = U_2 \setminus [U_{\frac{1}{2}}]$$

in $\mathbf{P}^3$. Define $\sigma \in \mathrm{PGL}(4, \mathbf{C})$ by

$$\sigma([z_0 : z_1 : z_2 : z_3]) = [z_2 : z_3 : z_0 : z_1].$$

Then σ is an involution of $N(2)$. We define the connected sum $X = X_1 \# X_2$ by

$$X = \left(X_1 \setminus j_1([U_{\frac{1}{2}})]\right) \bigcup \left(X_2 \setminus j_2([U_{\frac{1}{2}})]\right),$$

where $x_1 \in j_1(N(2))$ and $x_2 \in j_2(N(2))$ are identified if and only if

$$x_2 = j_2 \circ \sigma \circ j_1^{-1}(x_1).$$

Note that $\tilde{j}_2 \circ \sigma \circ \tilde{j}_1^{-1}$ extends to an element $\tau \in \mathrm{PGL}(4, \mathbf{C})$. By this construction, X becomes also a compact quotient of a large domain $\Omega \subset \mathbf{P}^3$ by a subgroup $\Gamma \subset \mathrm{PGL}(4, \mathbf{C})$. More explicitly, Γ is generated by $\tau \circ \alpha \circ \tau^{-1}$ and Γ_2 in $\mathrm{PGL}(4, \mathbf{C})$. The fundamental domain of Γ

is given as a compact subset of Ω_2 by

$$\left(F \setminus \tilde{j}_2\left(U_{\frac{1}{2}}\right)\right) \bigcup \tau\left(Z \setminus \tilde{j}_1\left(U_{\frac{1}{2}}\right)\right)$$

which is the fundamental domain F of Γ_2 with disjoint tubular neighborhoods of 2 skew lines in F deleted. Thus Γ *is a finitely generated discrete subgroup* of PGL(4, **C**), and is isomorphic to the free product of Γ_1 and Γ_2 by van Kampen's theorem.

PROPOSITION 10. *Γ do not admit any semi-invariant surface.*

PROOF. Suppose contrary that there is a Γ-semi-invariant surface S. Then there is a subgroup $\Gamma' \subset \Gamma$ of finite index such that S is Γ'-invariant. Since X is not prime, S is not a plane by Theorem 2. Since a finite index subgroup $\Gamma_1' \subset \Gamma_1$ is contained in Γ', we see that S is neither a surface of degree 2 by Lemma 6.1. Hence S is the tangential surface of a twisted cubic curve by Theorem B.

Let C denote the twisted cubic curve. Since C is the singular locus of S, C is Γ'-invariant. Hence Γ' defines a subgroup $G(\simeq \Gamma')$ of PGL(2, **C**) which induces Γ' by the holomorphic map (40). Note that Γ' contains a finite index subgroup of Γ_2, which is a free abelian group of rank $= 4$. Hence, so does G. But free abelian subgroups of rank 4 cannot be discrete in PGL(2, **C**). Hence G is not discrete in PGL(2, **C**). Consequently, Γ' is not discrete in PGL(4, **C**). This is a contradiction. ∎

REMARK 4. There is a small neighborhood V of $\tau \in$ PGL(4, **C**) such that, for any $\gamma \in V$, the group $\Gamma_\gamma := (\gamma\Gamma_1\gamma^{-1}) * \Gamma_2$ is a finitely generated discrete subgroup of PGL(4, **C**) that admit no Γ_γ-semi-invariant surfaces.

REMARK 5. Using the manifold X constructed above and any other compact quotient manifold Y of a large domain, we can form the connected sum $X\#Y$ that provides an example for which Proposition 10 holds.

7. Appendix : Eichler cohomology

We recall some facts related to the Eichler cohomology. Fix a finitely generated non-elementary Kleinian group G. Let Π_2 be the vector space of polynomials in z of degree at most 2. Then G acts from the right on Π_2 by the rule

$$P\gamma = P(\gamma(z))(c_\gamma z + d_\gamma)^2\,,$$

where

$$\gamma(z) = \frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma} \in G\,.$$

A map $e : G \to \Pi_2$ is called a *cocycle* if

$$e(\gamma_1 \circ \gamma_2) = e(\gamma_1)\gamma_2 + e(\gamma_2)\,.$$

For a fixed $Q \in \Pi_2$, its *coboundary* δQ is a cocycle defined by

$$\delta Q(\gamma) = Q\gamma - Q, \quad \gamma \in G\,.$$

Let $Z^1(G, \Pi)$ be the vector space spanned by the cocycles and $B^1(G, \Pi)$ its vector subspace spanned by the coboundaries. Then the *Eichler cohomology* is the vector space defined by

$$H^1(G, \Pi_2) = Z^1(G, \Pi_2)/B^1(G, \Pi_2)\,.$$

By [B, Lemma 1], we see that

$$\dim H^1(G, \Pi_2) \leq 3(N-1)\,,$$

where N is a number of generators of G. The equality holds if G is a free group of N generators.

Put $\Omega = \Omega(G)$ and $\Lambda = \mathbf{P}^1 \setminus \Omega$. By Ahlfors' finiteness theorem, there are finite number of compact curves $C_1, \dots, C_r$ such that

$$\Omega/G = C_1^* \cup \cdots \cup C_r^*\,,$$

where C_j^* is C_j with a finite number of points $\{p_1, \dots, p_{s_j}\}$ deleted. Let D_j be the divisor on C_j defined by $p_1 + \cdots + p_{s_j}$. Each C_j^* is given by Ω_j/G_j, where Ω_j is a connected component of Ω and G_j is the stabilizer subgroup of G. Let K_{C_j} be the canonical line bundle of C_j. Lifting up an element $\omega \in \oplus_{j=1}^r H^0(C_j, 2K_{C_j} + D_j)$ to Ω, we obtain a holomorphic quadratic 1-form $\tilde{\omega} = \phi(z)dz^{\otimes 2}$ on Ω satisfying

$$\gamma^*\tilde{\omega} = \tilde{\omega}, \quad \gamma \in G\,,$$

which is called a *cusp form* of weight (-4). In other words, a cusp form $\tilde{\omega}$ of weight (-4) is a G-invariant holomorphic quadratic 1-form on Ω whose norm

$$\|\tilde{\omega}\| := \sup_{z\in\Omega} \lambda_\Omega^{-2}(z)|\phi(z)|$$

is finite, where $\lambda_\Omega(z)|dz|$ is a Poincaré metric on Ω. Let $A_2(\Omega, G)$ denote the vector space of the cusp forms. By L. Bers [B, Theorem 3], there is a canonical injective antilinear map

$$\beta : A_2(\Omega, G) \to H^1(G, \Pi_2)\,.$$

Now let G_0 be a subgroup of G with finite index. In general, for a discrete group $G \subset \mathrm{PGL}(2, \mathbf{C})$ and its any subgroup G_0 of finite index, the equality $\Omega(G) = \Omega(G_0)$ holds ([MT, Proposition 2.30]). Hence we have the covering projection $\pi : \Omega/G_0 \to \Omega/G$. Therefore we have the commutative diagram

$$\begin{array}{ccc} A_2(\Omega, G_0) & \xrightarrow{\beta_0} & H^1(G_0, \Pi_2) \\ \uparrow i & & \uparrow r \\ A_2(\Omega, G) & \xrightarrow{\beta} & H^1(G, \Pi_2)\,, \end{array}$$

where i is the inclusion and r is the restriction.

LEMMA 7.1. *Suppose that* $\tilde{\omega} \in A_2(\Omega, G)$ *is an element such that* $\beta(\tilde{\omega}) \neq 0$. *Then* $r \circ \beta(\tilde{\omega}) \neq 0$ *for any subgroup* $G_0 \subset G$ *of finite index.*

PROOF. In the diagram above, i is obviously injective, and so is β by the theorem of Bers cited above. Hence we have $r \circ \beta(\tilde{\omega}) = \beta_0 \circ i(\tilde{\omega}) \neq 0$. ∎

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