TOKYO J. MATH.
VOL. 34, NO. 1, 2011

# Diffeomorphism Classes of Real Bott Manifolds

Admi NAZRA

*Andalas University*

(Communicated by Y. Matsumoto)

**Abstract.** A real Bott manifold is obtained as the orbit space of the $n$-torus $T^n$ by a free action of an elementary abelian 2-group $(\mathbb{Z}_2)^n$. This paper deals with the classification of 5-dimensional real Bott manifolds and studies certain types of $n$-dimensional real Bott manifolds ($n \geq 6$).

## Introduction

A real Bott tower is described as a sequence of $\mathbb{RP}^1$-bundles of height $n$ which is the real restriction to a Bott tower introduced in [1]. The total space of such a sequence is called a *real Bott manifold*. From the viewpoint of group actions, an $n$-dimensional real Bott manifold is the quotient of the $n$-dimensional torus $T^n = S^1 \times \cdots \times S^1$ by the product $(\mathbb{Z}_2)^n$ of cyclic group of order 2. A *Bott matrix* $A$ of size $n$ is an upper triangular matrix whose diagonal entries are 1 and the other entries are either 1 or 0. By the definition, the number of distinct Bott matrices of size $n$ is $2^{\frac{1}{2}(n^2-n)}$. The free action of $(\mathbb{Z}_2)^n$ on $T^n$ can be expressed by each row of the Bott matrix $A$ whose orbit space $M(A) = T^n/(\mathbb{Z}_2)^n$ is the real Bott manifold. It is easy to see that $M(A)$ is a compact euclidean space form (Riemannian flat manifold). Then we can apply the Bieberbach theorem [7] to classify real Bott manifolds. Using this theorem, the classification of real Bott manifolds up to dimension 4 has been obtained in [5], [2].

In [3] we have proved that every $n$-dimensional real Bott manifold $M(A)$ admits an injective Seifert fibred structure which has the form $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$, that is there is a $k$-torus action on $M(A)$ whose quotient space is an $(n-k)$-dimensional real Bott orbifold $M(B)/(\mathbb{Z}_2)^s$ by some $(\mathbb{Z}_2)^s$-action $(1 \leq s \leq k)$. Moreover we have proved the smooth rigidity that two real Bott manifolds $M(A_i)$ $i = 1, 2$ are diffeomorphic if and only if the corresponding actions $((\mathbb{Z}_2)^{s_i}, M(B_i))$ are equivariantly diffeomorphic. When the low dimensional real Bott manifolds with $(\mathbb{Z}_2)^s$-actions are classified, we can determine the diffeomorphism classes of higher dimensional ones by the above rigidity. We have classified real Bott manifolds up to dimension 4 (see [6]).

Received December 8, 2009; revised April 2, 2010
2000 *Mathematics Subject Classification*: 53C24 (Primary), 57S25 (Secondary)
*Key words and phrases*: Bott tower, Bieberbach group, flat riemannian manifold, real Bott manifold, Seifert fibration, diffeomorphism

The main purpose of this paper is to determine: $(a)$ Diffeomorphism classes of 5-dimensional real Bott manifolds from the classifications of $2, 3, 4$-dimensional ones with $(\mathbb{Z}_2)^s$-actions ($s = 1, 2$), $(b)$ Classification of certain type of $n$-dimensional real Bott manifolds $M(A)$.

We have obtained the following to $(a)$ (compare Theorem 6).

*Theorem A.* There are 54 diffeomorphism classes of 5-dimensional real Bott manifolds.

Since each $n$-dimensional real Bott manifold has the form $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$, it is shown that the diffeomorphism class of $M(A)$ is determined by the equivariant diffeomorphism class of the action $((\mathbb{Z}_2)^s, M(B))$ $(1 \leq s \leq k)$ as above. We prove that if $k = 1$, there are 12 nonequivariant diffeomorphism classes of 4-dimensional real Bott manifolds with $\mathbb{Z}_2$-actions. Then they create 29 diffeomorphism classes of such 5-dimensional real Bott manifolds. When $k = 2$, there are 4 nonequivariant diffeomorphism classes of 3-dimensional real Bott manifolds with $(\mathbb{Z}_2)^s$-actions ($s = 1, 2$). Then from these, there are 19 diffeomorphism classes of the 5-dimensional real Bott manifolds. When $k = 3$, there are 2 nonequivariant diffeomorphism classes of 2-dimensional real Bott manifolds with $(\mathbb{Z}_2)^s$-actions ($s = 1, 2$). Then there are 4 diffeomorphism classes of the 5-dimensional real Bott manifolds. When $k = 4$, the 1-dimensional real Bott manifold is $S^1$ with conjugate action of $\mathbb{Z}_2$, there exists only one such a 5-dimensional real Bott manifold. Finally if $k = 5$, the 5-dimensional real Bott manifold is $T^5$. As a consequence, the total number of 5-dimensional diffeomorphism classes is 54. The details of the proof is in section 3.

It is far to determine the number of diffeomorphism classes of $n$-dimensional real Bott manifolds for $n \geq 6$. However, we shall solve the special types of higher dimensional real Bott manifolds.

*Theorem B.* Let $T^{n-2}$ be the maximal torus action on an $n$-dimensional real Bott manifold $(n \geq 4)$. Then the diffeomorphism classes of such real Bott manifolds consists of 4.

See Theorem 7 for the proof.

Theorem A and Theorem B can also be obtained by a different method, see [4].

*Proposition C.* The following hold.

(i) The diffeomorphism class is unique for the real Bott manifold of the form $M(A) = T^k \underset{\mathbb{Z}_2}{\times} T^{n-k}$ for any $k$ $(1 \leq k \leq n-1)$. In particular, if $k = n$ then $M(A) = T^n$.

(ii) Let $M(A)$ be a real Bott manifold which fibers $S^1$ over the real Bott manifold $M(B)$ for which $M(B)$ is either $T^k \times_{(\mathbb{Z}_2)^s} \mathrm{K}$ or $T^k \times_{(\mathbb{Z}_2)^s} T^2$ $(k \geq 2)$. Here K is a Klein bottle. Then the number of diffeomorphism classes of such $M(A)$ is 3.

(iii) Let $M(A)$ be a real Bott manifold which fibers $S^1$ over the real Bott manifold $M(B)$ where $M(B) = S^1 \times_{\mathbb{Z}_2} T^k$ $(k \geq 2)$, then the number of diffeomorphism classes of such $M(A)$ is $[\frac{k}{2}] + 1$. Here $[x]$ is the integer part of $x$.

We prove these results in Section 4 (see Proposition 5, Corollary 2, Corollary 3, Theorem 8 respectively).

A special kind of Bott matrices is introduced in Section 1. We consider such a class of

Bott matrices in (4.27).

*Theorem D.* Let $M(A) = S^1 \times_{\mathbb{Z}_2} M(B)$ be an $n$-dimensional real Bott manifold. Suppose that $B$ is either one of the list in (4.27). Then $M(B)$ are diffeomorphic to each other and the number of diffeomorphism classes of such real Bott manifolds $M(A)$ above is $(k+1)2^{n-k-3}$ $(k \geq 2, n-k \geq 3)$.

See Theorem 9 for the proof.

## 1. Preliminaries

**1.1. Seifert fiber space.** Each $i$-th row of a Bott matrix $A$ defines a $\mathbb{Z}_2$-action on $T^n$ by $g_i(z_1, \ldots, z_n) = (z_1, \ldots, z_{i-1}, -z_i, \tilde{z}_{i+1}, \ldots, \tilde{z}_n)$, $(i = 1, \ldots, n)$ where $(i,i)$-(diagonal) entry 1 acts as $z_i \to -z_i$ while $\tilde{z}_j$ is either $z_j$ or $\bar{z}_j$ depending on whether $(i,j)$-entry $(i < j)$ is 0 or 1 respectively. Note that $\bar{z}$ is the conjugate of the complex number $z \in S^1$. It is always trivial; $z_j \to z_j$ whenever $j < i$. Here $(z_1, \ldots, z_n)$ are the standard coordinates of the $n$-dimensional torus $T^n$ whose universal covering is the $n$-dimensional euclidean space $\mathbb{R}^n$. The projection $p\colon \mathbb{R}^n \to T^n$ is denoted by

$$p(x_1, \ldots, x_n) = (exp(2\pi \mathbf{i} x_1), \ldots, exp(2\pi \mathbf{i} x_n)) = (z_1, \ldots, z_n)\,.$$

Those $\langle g_1, \ldots, g_n \rangle$ constitute the generators of $(\mathbb{Z}_2)^n$. It is easy to see that $(\mathbb{Z}_2)^n$ acts freely on $T^n$ such that the orbit space $M(A) = T^n/(\mathbb{Z}_2)^n$ is a smooth compact manifold. In this way, given a Bott matrix $A$ of size $n$, we obtain a free action of $(\mathbb{Z}_2)^n$ on $T^n$.

Let $\pi(A) = \langle \tilde{g}_1, \ldots, \tilde{g}_n \rangle$ be the lift of $(\mathbb{Z}_2)^n = \langle g_1, \ldots, g_n \rangle$ to $\mathbb{R}^n$. Then we get

$$\tilde{g}_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \frac{1}{2} + x_i, \tilde{x}_{i+1}, \ldots, \tilde{x}_n)$$

where $\tilde{x}_j$ is either $x_j$ or $-x_j$. It is easy to see that $\pi(A)$ acts properly discontinuously and freely on $\mathbb{R}^n$ as euclidean motions. Note that $\pi(A)$ is a Bieberbach group which is a discrete uniform subgroup of the euclidean group $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ (cf. [7]). It follows that $M(A) = T^n/(\mathbb{Z}_2)^n = \mathbb{R}^n/\pi(A)$.

Now let us recall moves **I**, **II** and **III** [3] which are applied to a Bott matrix $A$ of size $n$ under which the diffeomorphism class of $M(A)$ does not change.

**I**. If the $j$-th column has all 0-entries except for the $(j,j)$-entry 1 for some $j > 1$, then interchange the $j$-th column and the $(j-1)$-th column. Next, interchange the $j$-th row and the $(j-1)$-th row.

This move **I** is interpreted in terms of the coordinates $z_j$'s of $T^n$ and the generators $g_j$'s of $(\mathbb{Z}_2)^n$ as follows: $z_j \to z'_{j-1}$, $z_{j-1} \to z'_j$, $g_j \to g'_{j-1}$, $g_{j-1} \to g'_j$. It is easy to see that the resulting matrix $A'$ under move **I** is again a Bott matrix such that $M(A)$ is diffeomorphic to $M(A')$.

We perform move **I** iteratively to get a Bott matrix $A'$

$$A' = \left(\begin{array}{c|c} I_k & C \\ \hline 0 & B \end{array}\right) \quad B = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \tag{1.1}$$

where $I_k$ is a maximal block of identity matrix of size $k$, the entries of the $*$ are either 1 or 0, $B$ is a Bott matrix of size $(n-k)$ which represents a real Bott manifold $M(B) = T^{n-k}/(\mathbb{Z}_2)^{n-k}$. Since $I_k$ is a maximal block of identity matrix, each $k + j$ $(j = 1, \ldots, n-k)$-th column of $A'$ has at least two non zero elements.

Associated with $A'$, the $(\mathbb{Z}_2)^n$-action splits into $(\mathbb{Z}_2)^k \times (\mathbb{Z}_2)^{n-k}$ and $T^n$ splits into $T^k \times T^{n-k}$. Hence

$$M(A) = T^n/(\mathbb{Z}_2)^n \cong \frac{T^k \times T^{n-k}}{(\mathbb{Z}_2)^k \times (\mathbb{Z}_2)^{n-k}} = T^k \underset{(\mathbb{Z}_2)^k}{\times} M(B) = M(A')\,. \tag{1.2}$$

Note that the above $(\mathbb{Z}_2)^k$-action of (1.2) is not necessarily effective on $M(B)$ but we can reduce it to the effective $(\mathbb{Z}_2)^s$-action on $M(B)$ for some $s$ $(1 \le s \le k)$. In order to do so, we have two more moves.

**II**. If there is an $m$-th row $(1 \le m \le k)$ whose entries in C are all zero, then divide $T^k \times M(B)$ by the corresponding $\mathbb{Z}_2$-action. For example, suppose $M(A_1) = T^2 \times_{(\mathbb{Z}_2)^2} M(B)$ with

$$A_1 = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

By move **II**, $M(A_1) \cong T^2 \times_{\mathbb{Z}_2} M(B)$.

**III**. If the $p$-th row and $\ell$-th row $(1 \le p < \ell \le k)$ have the common entries in $C$, then compose the $\mathbb{Z}_2$-action of $p$-th row with $l$-th row and divide $T^k \times M(B)$ by this $\mathbb{Z}_2$-action. For example, suppose $M(A_2) = T^2 \times_{(\mathbb{Z}_2)^2} M(B)$ with

$$A_2 = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

By move **III**, $M(A_2) \cong T^2 \times_{\mathbb{Z}_2} M(B) \cong M(A_1)$.

By an iteration of **II, III**, the quotient is again diffeomorphic to $T^k \times_{(\mathbb{Z}_2)^k} M(B)$ but eventually the $(\mathbb{Z}_2)^k$-action is reduced to the effective $(\mathbb{Z}_2)^s$-action on $T^k \times M(B)$. Therefore $A'$ reduces to

$$A'' = \left(\begin{array}{c|c|c} I_{k-s} & 0 & 0 \\ \hline 0 & I_s & * \\ \hline 0 & 0 & B \end{array}\right) \tag{1.3}$$

in which $M(A') = T^k \times_{(\mathbb{Z}_2)^k} M(B) = \frac{T^{k-s} \times T^s \times M(B)}{(\mathbb{Z}_2)^{k-s} \times (\mathbb{Z}_2)^s} = M(A'')$. Since $(\mathbb{Z}_2)^{k-s}$ acts trivially on $T^s \times M(B)$, we have $M(A'') \cong T^k \times_{(\mathbb{Z}_2)^s} M(B)$.

From now on, we write $M(A)$ instead of $M(A'')$.

REMARK 1. Since $(\mathbb{Z}_2)^s$ acts trivially on $T^{k-s}$,

$$\begin{aligned} M(A) &\cong T^k \underset{(\mathbb{Z}_2)^s}{\times} M(B) = T^{k-s} \times T^s \underset{(\mathbb{Z}_2)^s}{\times} M(B) \\ &\cong (S^1)^{k-s} \times T^s \underset{(\mathbb{Z}_2)^s}{\times} M(B) = (S^1)^{k-s} \times M(B') \end{aligned}$$

where $M(B') = T^s \times_{(\mathbb{Z}_2)^s} M(B)$. That is, for $s < k$, a real Bott manifold $M(A)$ is the product of $(S^1)^{k-s}$ and an $(n-k+s)$-dimensional real Bott manifold $M(B')$. In particular, if $M(A) = T^{n-1} \times_{\mathbb{Z}_2} S^1$ then it is diffeomorphic to $(S^1)^{n-2} \times$ Klein bottle.

REMARK 2. From the submatrix $*$ of (1.3), the group $(\mathbb{Z}_2)^s = \langle g_{k-s+1}, \dots, g_k \rangle$ acts on $T^k \times M(B)$ by

$$\begin{aligned} &g_i(z_1, \dots, z_{k-s+1}, \dots, z_k, [z_{k+1}, \dots, z_n]) \\ &= (z_1, \dots, z_{k-s+1}, \dots, -z_i, \dots, z_k, [\tilde{z}_{k+1}, \dots, \tilde{z}_n]) \end{aligned} \tag{1.4}$$

where $\tilde{z} = \bar{z}$ or $z$. So there induces an action of $(\mathbb{Z}_2)^s$ on $M(B)$ by

$$g_i([z_{k+1}, \dots, z_n]) = [\tilde{z}_{k+1}, \dots, \tilde{z}_n]\,. \tag{1.5}$$

Moreover in [3],

THEOREM 1 (Structure). *Given a real Bott manifold $M(A)$, there exists a maximal $T^k$-action ($k \geq 1$) such that $M(A) = T^k \times_{(\mathbb{Z}_2)^s} M(B)$ is an injective Seifert fiber space over the $(n-k)$-dimensional real Bott orbifold $M(B)/(\mathbb{Z}_2)^s$;*

$$T^k \to M(A) \to M(B)/(\mathbb{Z}_2)^s\,. \tag{1.6}$$

*There is a central extension of the fundamental group $\pi(A)$ of $M(A)$:*

$$1 \to \mathbb{Z}^k \to \pi(A) \to Q_B \to 1 \tag{1.7}$$

*such that*

(i) *$\mathbb{Z}^k$ is the maximal central free abelian subgroup*

(ii) *The induced group $Q_B$ is the semidirect product $\pi(B) \rtimes (\mathbb{Z}_2)^s$ for which $\mathbb{R}^{n-k}/\pi(B) = M(B)$.*

See [3] for the proof.

By this theorem, a real Bott manifold $M(A)$ which admits a maximal $T^k$-action ($k \geq 1$) can be created from an $(n-k)$-dimensional real Bott manifold $M(B)$ by a $(\mathbb{Z}_2)^s$-action, and the corresponding Bott matrix $A$ has the form as in (1.3) above.

**1.2. Affine maps between real Bott manifolds.** Next, we can apply the following theorem to check whether two real Bott manifolds are diffeomorphic.

THEOREM 2 (Rigidity). *Let $M(A_1)$, $M(A_2)$ be $n$-dimensional real Bott manifolds and $1 \to \mathbb{Z}^{k_i} \to \pi(A_i) \to Q_{B_i} \to 1$ be the associated group extensions* ($i = 1, 2$). *Then the following are equivalent*:

(i) *$\pi(A_1)$ is isomorphic to $\pi(A_2)$.*

(ii) *There exists an isomorphism of $Q_{B_1} = \pi(B_1) \rtimes (\mathbb{Z}_2)^{s_1}$ onto $Q_{B_2} = \pi(B_2) \rtimes (\mathbb{Z}_2)^{s_2}$ preserving $\pi(B_1)$ and $\pi(B_2)$.*

(iii) *The action $((\mathbb{Z}_2)^{s_1}, M(B_1))$ is equivariantly diffeomorphic to the action $((\mathbb{Z}_2)^{s_2}, M(B_2))$.*

See [3] for the proof. Here Bott matrices $A_1$ and $A_2$ are created from $B_1$ and $B_2$ respectively.

Note that two real Bott manifolds $M(A_1)$ and $M(A_2)$ are diffeomorphic if and only if $\pi(A_1)$ is isomorphic to $\pi(A_2)$ by the Bieberbach theorem [7]. Moreover by Theorems 1 and 2 we have,

REMARK 3. Let real Bott manifolds $M(A_i) = T^{k_i} \times_{(\mathbb{Z}_2)^{s_i}} M(B_i)$ $(i = 1, 2)$. If $M(A_1)$ and $M(A_2)$ are diffeomorphic then the following hold.

(i) $k_1 = k_2$.

(ii) $M(B_1)$ and $M(B_2)$ are diffeomorphic.

(iii) $s_1 = s_2$.

Therefore two real Bott manifolds which admit different maximal $T^k$-action are not diffeomorphic. If they have the same maximal $T^k$-action, then the quotients $((\mathbb{Z}_2)^{s_i}, M(B_i))$ are compared. If $M(B_1)$ is not diffeomorphic to $M(B_2)$ or $s_1 \neq s_2$, then $M(A_1)$ and $M(A_2)$ are not diffeomorphic. So our task is to distinguish the $(\mathbb{Z}_2)^{s_i}$-action on $M(B_i)$ when it is the case that $s_1 = s_2 = s$ and $M(B_1)$ is diffeomorphic to $M(B_2)$.

**1.3. Type of fixed point set.** Note that from (1.5), the action of $(\mathbb{Z}_2)^s$ on $M(B)$ is defined by $\alpha[(z_1, \dots, z_{n-k})] = [\alpha(z_1, \dots, z_{n-k})] = [(\tilde{z}_1, \dots, \tilde{z}_{n-k})]$ for $\alpha \in (\mathbb{Z}_2)^s$ and $\tilde{z} = z$ or $\bar{z}$. Since $M(B) = T^{n-k}/(\mathbb{Z}_2)^{n-k}$, the action $\langle\alpha\rangle$ lifts to a linear (affine) action on $T^{n-k}$ naturally: $\alpha(z_1, \dots, z_{n-k}) = (\tilde{z}_1, \dots, \tilde{z}_{n-k})$. Then the fixed point set is characterized by the equation: $(\tilde{z}_1, \dots, \tilde{z}_{n-k}) = g(z_1, \dots, z_{n-k})$ for some $g \in (\mathbb{Z}_2)^{n-k}$. It is also an affine subspace of $T^{n-k}$. So the fixed point sets of $(\mathbb{Z}_2)^s$ are affine subspaces in $M(B)$.

Let $B$ be the Bott matrix as in (1.1). By a repetition of move **I**, $B$ has the form

$$B = \begin{pmatrix} I_2 & C_{23} & \dots & \dots & C_{2\ell} \\ & I_3 & C_{34} & \dots & C_{3\ell} \\ & & \ddots & & \dots \\ & 0 & & I_{\ell-1} & C_{(\ell-1)\ell} \\ & & & & I_\ell \end{pmatrix} \tag{1.8}$$

where rank$B$=$n-k$=rank$I_2$+…+rank$I_\ell$ and $I_i$ $(i=2,\dots,\ell)$ is the identity matrix, $C_{jt}$ $(j=2,\dots,\ell-1, t=3,\dots,\ell)$ is a $p_j\times q_t$ matrix ($p_j$=rank$I_j$, $q_t$=rank$I_t$).

Note that by the Bieberbach theorem (cf. [7]), if $f$ is an isomorphism of $\pi(A_1)$ onto $\pi(A_2)$, then there exists an affine element $g=(h,H)\in \mathrm{A}(n)=\mathbb{R}^n\rtimes \mathrm{GL}(n,\mathbb{R})$ such that

$$f(r)=grg^{-1}\ (\forall r\in\pi(A_1))\,. \tag{1.9}$$

Recall that if $M(A_1)$ is diffeomorphic to $M(A_2)$ then $M(B_1)$ is diffeomorphic to $M(B_2)$. This implies that $B_1$ and $B_2$ have the form as in (1.8).

Using (1.9) and according to the form of $B$ in (1.8) we obtain that

$$g=\left(\begin{pmatrix}\mathbf{h}_1\\ \mathbf{h}_2\\ \vdots\\ \mathbf{h}_\ell\end{pmatrix},\begin{pmatrix}H_1 & & & \\ & H_2 & & 0\\ 0 & & \ddots & \\ & & & H_\ell\end{pmatrix}\right) \tag{1.10}$$

where $\mathbf{h}_i$ is an $s_i\times 1$ ($s_i$=rank $I_i$) column matrix ($\mathbf{h}_1$ is a $k\times 1$ column matrix), $H_i\in\mathrm{GL}(s_i,\mathbb{R})$ $(i=2,\dots,\ell)$, $H_1\in\mathrm{GL}(k,\mathbb{R})$ (see Remark 3.2 [3]).

Let $\bar f\colon Q_{B_1}\to Q_{B_2}$ be the induced isomorphism from $f$ (cf. Theorem 2). Now the affine equivalence $\bar g\colon\mathbb{R}^{n-k}\to\mathbb{R}^{n-k}$ has the form

$$\bar g=\left(\begin{pmatrix}\mathbf{h}_2\\ \vdots\\ \mathbf{h}_\ell\end{pmatrix},\begin{pmatrix}H_2 & & 0\\ & \ddots & \\ 0 & & H_\ell\end{pmatrix}\right) \tag{1.11}$$

which is equivariant with respect to $\bar f$. The pair $(\bar f,\bar g)$ induces an equivariant affine diffeomorphism $(\hat f,\hat g)\colon((\mathbb{Z}_2)^s, M(B_1))\to((\mathbb{Z}_2)^s, M(B_2))$.

Let rank$H_i=b_i$ $(i=2,\dots,\ell)$. (Note that $b_2+\cdots+b_\ell=n-k$.) Since $M(B_1)=T^{n-k}/(\mathbb{Z}_2)^{n-k}$, $\bar g$ induces an affine map $\tilde g$ of $T^{n-k}$. Put

$$X_{b_2}=\begin{pmatrix}x_1\\ \vdots\\ x_{b_2}\end{pmatrix},\dots,X_{b_\ell}=\begin{pmatrix}x_{b_{\ell'}+1}\\ \vdots\\ x_{b_{\ell'}+b_\ell}\end{pmatrix},\ w_{b_i}=p(X_{b_i})\in T^{b_i}\ (i=2,\dots,\ell)\,,$$

$$b_{\ell'}=b_2+\cdots+b_{\ell-1}\,.$$

Since $\tilde g p=p\bar g$, $\tilde g({}^t w_{b_2},\dots,{}^t w_{b_\ell})=({}^t w'_{b_2},\dots,{}^t w'_{b_\ell})$ where $w'_{b_i}=p(\mathbf{h}_i+H_iX_{b_i})\in T^{b_i}$. That is, $\tilde g$ preserves each $T^{b_i}$ of $T^{n-k}=T^{b_2}\times\cdots\times T^{b_\ell}$, so does $\hat g$ on

$$M(B_1)=\{[z_1,\dots,z_{b_2};z_{b_2+1},\dots,z_{b_2+b_3};\dots\dots;z_{b_{\ell'}+1},\dots,z_{b_{\ell'}+b_\ell}]\}\,.$$

We say that $\hat g$ *preserves the type* $(b_2,\dots,b_\ell)$ of $M(B_1)$. As $\hat g$ is $\hat f$-equivariant, it also preserves the type corresponding to the fixed point sets between $((\mathbb{Z}_2)^s, M(B_1))$ and $((\mathbb{Z}_2)^s, M(B_2))$.

PROPOSITION 1. *The $(\mathbb{Z}_2)^s$-action on $M(B)$ is distinguished by the number of components and types of each positive dimensional fixed point subsets.*

See [3] for the proof.

DEFINITION 1. We say that two Bott matrices $A$ and $A'$ are *equivalent* (denoted by $A \sim A'$) if $M(A)$ and $M(A')$ are diffeomorphic.

## 2. Examples

We shall give some real Bott manifolds in order to determine diffeomorphism classes of 5-dimensional ones. We introduce the following Bott matrices created from $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

$$A_1 = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & & B & \end{array}\right), \quad A_2 = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B & \end{array}\right),$$

$$A_3 = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & & B & \end{array}\right), \quad A_4 = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B & \end{array}\right).$$

Then we obtain the 5-dimensional real Bott manifolds $M(A_i)$ for which the $(\mathbb{Z}_2)^2$-action on $M(B)$ is given by the first two rows of $A_i$ $(i = 1, 2, 3, 4)$. We prove that there are two distinct diffeomorphism classes among $M(A_i)$ $(i = 1, 2, 3, 4)$.

**a)** $M(A_1)$ is diffeomorphic to $M(A_2)$. For this, the $(\mathbb{Z}_2)^2$-actions on $M(B)$ corresponding to $A_1$ and $A_2$ are given as follows:

(i) $g_1([z_3, z_4, z_5]) = [\bar{z}_3, \bar{z}_4, z_5] = [g_3(\bar{z}_3, \bar{z}_4, z_5)] = [-\bar{z}_3, z_4, z_5]$,
$g_2([z_3, z_4, z_5]) = [z_3, \bar{z}_4, z_5]$,

(ii) $h_1([z_3, z_4, z_5]) = [z_3, \bar{z}_4, z_5]$, $h_2([z_3, z_4, z_5]) = [\bar{z}_3, z_4, z_5]$.

There is an equivariant diffeomorphism $\varphi\colon ((\mathbb{Z}_2)^2, M(B)) \to ((\mathbb{Z}_2)^2, M(B))$ defined by $\varphi([z_3, z_4, z_5]) = ([\mathbf{i}z_3, z_4, z_5])$ such that $\varphi g_1 = h_2\varphi$ and $\varphi g_2 = h_1\varphi$. Hence the result follows from Theorem 2.

**b)** $M(A_2)$ is not diffeomorphic to $M(A_3)$. If $M(A_2)$ and $M(A_3)$ are diffeomorphic, by Theorem 2 there is an equivariant diffeomorphism $\varphi\colon ((\mathbb{Z}_2)^2, M(B)) \to ((\mathbb{Z}_2)^2, M(B))$. Let $\bar{\varphi} : \mathbb{R}^3 \to \mathbb{R}^3$ be the lift of $\varphi$. According to the form of $B$, the affine element $\bar{\varphi}$ has the form

$$\bar{\varphi} = \left( \begin{pmatrix} a_2 \\ a_3 \\ a_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \tag{2.1}$$

for some $a_i \in \mathbb{R}$ ($i = 2, 3, 4$) (see (1.11)). Since $M(B) = T^3/(\mathbb{Z}_2)^3$, $\bar{\varphi}$ induces an affine map of $T^3$. By the formula of (2.1), it preserves each $S^1$ of $T^3 = S^1 \times S^1 \times S^1$, so does $\varphi$ on $M(B)$. Since $\varphi$ is equivariant, it also preserves the type $(1, 1, 1)$ of the fixed point sets of $((\mathbb{Z}_2)^2, M(B))$. That is, if $[z_3, z_4, z_5]$ is a fixed point set of $((\mathbb{Z}_2)^2, M(B))$, then $\varphi$ preserves each coordinate $z_i$ ($i = 3, 4, 5$) (i.e., $\varphi[z_3, z_4, z_5] = [exp(2\pi \mathbf{i} a_2)z_3, exp(2\pi \mathbf{i} a_3)z_4, exp(2\pi \mathbf{i} a_4)z_5]$).

The fixed point sets of $((\mathbb{Z}_2)^2, M(B))$ corresponding to $A_2$ and $A_3$ are as follows:

(i) 3 components $T^2 = \{[z_3, 1, z_5], [1, z_4, z_5], [\mathbf{i}, z_4, z_5]\}$,
4 components $S^1 = \{[z_3, \mathbf{i}, 1], [\mathbf{i}, 1, z_5], [z_3, \mathbf{i}, \mathbf{i}], [1, 1, z_5]\}$,
4 points $\{[\mathbf{i}, \mathbf{i}, 1], [\mathbf{i}, \mathbf{i}, \mathbf{i}], [1, \mathbf{i}, 1], [1, \mathbf{i}, \mathbf{i}]\}$,

(ii) 3 components $T^2 = \{[z_3, z_4, \mathbf{i}], [1, z_4, z_5], [z_3, z_4, 1]\}$,
4 components $S^1 = \{[1, z_4, 1], [\mathbf{i}, 1, z_5], [1, z_4, \mathbf{i}], [\mathbf{i}, \mathbf{i}, z_5]\}$,
and 4 points $\{[\mathbf{i}, \mathbf{i}, 1], [\mathbf{i}, \mathbf{i}, \mathbf{i}], [\mathbf{i}, 1, 1], [\mathbf{i}, 1, \mathbf{i}]\}$.

We see that the number of components of fixed point sets of $((\mathbb{Z}_2)^2, M(B))$ corresponding to $A_2$ and $A_3$ is the same. Since the type of fixed point set is preserved, $\varphi$ maps $T^2 = \{[z_3, 1, z_5]\}$, $z_3, z_5 \in S^1$ ((i) in $A_2$) onto the fixed point set $T^2 = \{[w_3, exp(2\pi \mathbf{i} a_3), w_5]\}$ ($w_3, w_5 \in S^1$) of $A_3$. However there is no type of such fixed point set in (ii) of $A_3$. Therefore by Proposition 1, $M(A_2)$ and $M(A_3)$ are not diffeomorphic.

**c)** $M(A_3)$ is diffeomorphic to $M(A_4)$. In this case, the $(\mathbb{Z}_2)^2$-actions on $M(B)$ corresponding to $A_3$ and $A_4$ are given as follows:

(i) $g_1([z_3, z_4, z_5]) = [\bar{z}_3, z_4, z_5]$, $g_2([z_3, z_4, z_5]) = [z_3, z_4, \bar{z}_5]$,

(ii) $h_1([z_3, z_4, z_5]) = [\bar{z}_3, z_4, \bar{z}_5]$, $h_2([z_3, z_4, z_5]) = [\bar{z}_3, z_4, z_5]$.

We change the generator $h_1$ by $h_1'$: $h_1'([z_3, z_4, z_5]) = h_1 h_2[z_3, z_4, z_5] = [z_3, z_4, \bar{z}_5]$. Define an equivariant diffeomorphism $\varphi\colon ((\mathbb{Z}_2)^2, M(B)) \to ((\mathbb{Z}_2)^2, M(B))$ to be $\varphi([z_3, z_4, z_5]) = ([z_3, z_4, z_5])$ such that $\varphi g_1 = h_2 \varphi$ and $\varphi g_2 = h_1' \varphi$. Hence $M(A_3)$ is diffeomorphic to $M(A_4)$ by Theorem 2.

## 3. Five-Dimensional Real Bott manifolds

Before giving the classification of 5-dimensional real Bott manifolds, we recall the classification of 2, 3, 4-dimensional ones as stated in [5], [6], [2].

THEOREM 3. *The diffeomorphism classes of* 2*-dimensional real Bott manifolds consist of two. The corresponding Bott matrices are as follows.*

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

THEOREM 4. *The diffeomorphism classes of* 3*-dimensional real Bott manifolds consist of four. The corresponding Bott matrices are classified into four equivalence classes as follows*:

a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$

b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

d) $I_3$.

THEOREM 5. *The diffeomorphism classes of* 4*-dimensional real Bott manifolds consist of twelve. The corresponding Bott matrices are classified into twelve equivalence classes as follows*:

i) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

ii) $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

iii) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

iv) $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

v) $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

vi) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

vii) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

viii) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

ix) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$

$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

x) $$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

xi) $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

xii) $I_4$.

Using the classification results of Theorem 3, 4, 5, we shall classify 5-dimensional real Bott manifolds.

**3.1. $S^1$-actions with 4-dimensional quotients.** The Bott matrices of $M(A)$ admitting $S^1$-actions have the following form

$$\left(\begin{array}{c|cccc} 1 & 1 & a_{13} & a_{14} & a_{15} \\ \hline 0 & & & B & \end{array}\right)$$

where $a_{13}, a_{14}, a_{15} \in \{0, 1\}$. In this case $M(B)$ corresponds to the Bott matrices $B$ in Theorem 5. Taking the first Bott matrix from i) as $B_1$, we consider the following Bott matrices.

$$A_1 = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 0 \\ \hline 0 & & & B_1 & \end{array}\right) A_2 = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 1 & 0 \\ \hline 0 & & & B_1 & \end{array}\right)$$
$$A_3 = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 1 & 1 \\ \hline 0 & & & B_1 & \end{array}\right).$$

Then the fixed point sets of the $\mathbb{Z}_2$-actions on $M(B_1)$ corresponding to $A_1$, $A_2$ and $A_3$ respectively are as follows: (1) $T^3$, $T^2$, 4 points, (2) 2 components $T^2$, 4 components $S^1$, (3) $T^2$, 4 components $S^1$, 4 points. (1), (2) and (3) have the different fixed point sets each other so each $A_i$ $(i = 1, 2, 3)$ is not equivalent by Proposition 1.

From the first Bott matrix in ii), say $B_2$, we created the following Bott matrices.

$$A_4 = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 0 & & & B_2 & \end{array}\right) A_5 = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 0 \\ \hline 0 & & & B_2 & \end{array}\right)$$
$$A_6 = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 0 \\ \hline 0 & & & B_2 & \end{array}\right).$$

The fixed point sets of the $\mathbb{Z}_2$-actions on $M(B_2)$ corresponding to $A_4$, $A_5$ and $A_6$ are obtained as: (1) 3 components $T^2$, 4 points, (2) $T^3$, 4 components $S^1$, (3) 8 components $S^1$. In view of the fixed points, similarly $A_i$ $(i = 4, 5, 6)$ are not equivalent to each other.

From the first Bott matrix in iii), say $B_3$, we created the following Bott matrices.

$$A_7 = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 0 \\ \hline 0 & & B_3 & & \end{array}\right) A_8 = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 1 & 0 \\ \hline 0 & & B_3 & & \end{array}\right)$$
$$A_9 = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 1 \\ \hline 0 & & B_3 & & \end{array}\right) A_{10} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 1 & 1 \\ \hline 0 & & B_3 & & \end{array}\right).$$

The fixed point sets of the $\mathbb{Z}_2$-actions on $M(B_3)$ corresponding to $A_7$, $A_8$, $A_9$ and $A_{10}$ are as follows:

(1) $T^3, T^2, S^1$, 2 points,

(2) 2 components $T^2 = \{[1, z_3, 1, z_5], [\mathbf{i}, \mathbf{i}, z_4, z_5]\}$, 3 components $S^1 = \{[\mathbf{i}, 1, 1, z_5], [1, z_3, \mathbf{i}, 1], [1, z_3, \mathbf{i}, \mathbf{i}]\}$, 2 points=$\{[\mathbf{i}, 1, \mathbf{i}, 1], [\mathbf{i}, 1, \mathbf{i}, \mathbf{i}]\}$,

(3) 2 components $T^2 = \{[1, z_3, z_4, 1], [1, z_3, z_4, \mathbf{i}]\}$, 3 components $S^1 = \{[\mathbf{i}, \mathbf{i}, \mathbf{i}, z_5], [\mathbf{i}, 1, z_4, 1], [\mathbf{i}, 1, z_4, \mathbf{i}]\}$, 2 points=$\{[\mathbf{i}, \mathbf{i}, 1, 1], [\mathbf{i}, \mathbf{i}, 1, \mathbf{i}]\}$,

(4) $T^2$, 5 components $S^1$, 2 points.

Note that the fixed point sets of (2) and (3) coincide, but the type of them are different. (Compare **b)** in Section 2 for the type (1, 1, 1, 1).) Hence $A_8$ and $A_9$ are not equivalent. As the fixed point sets (1), (4) and (2) (or (3)) are all different, each $A_i$ $(i = 7, 8, 9, 10)$ is not equivalent.

From the first Bott matrix in iv), say $B_4$, we create the following Bott matrices.

$$A_{11} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 0 \\ \hline 0 & & B_4 & & \end{array}\right) A_{12} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 1 & 0 \\ \hline 0 & & B_4 & & \end{array}\right)$$
$$A_{13} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 1 \\ \hline 0 & & B_4 & & \end{array}\right) A_{14} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 1 & 1 \\ \hline 0 & & B_4 & & \end{array}\right).$$

The fixed point sets of the $\mathbb{Z}_2$-actions on $M(B_4)$ corresponding to $A_{11}$, $A_{12}$, $A_{13}$ and $A_{14}$ are as follows: (1) $T^3$, 3 components $S^1$, 2 points, (2) $T^2$, 5 components $S^1$, 2 points, (3) 3 components $T^2$, $S^1$, 2 points, (4) 2 components $T^2$, 3 components $S^1$, 2 points. By Proposition 1, $A_i$ $(i = 11, 12, 13, 14)$ are not equivalent to each other.

The Bott matrices $A_i$ $(i = 15, 16, 17, 18)$ below are created from the first Bott matrix in v), say $B_5$, while $A_{19}$ is created from the second Bott matrix in v), say $B_6$.

$$A_{15} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 0 \\ \hline 0 & & B_5 & & \end{array}\right) A_{16} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 1 & 0 \\ \hline 0 & & B_5 & & \end{array}\right)$$
$$A_{17} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 1 \\ \hline 0 & & B_5 & & \end{array}\right) A_{18} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 1 & 1 \\ \hline 0 & & B_5 & & \end{array}\right)$$

$$A_{19} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 0 & & B_6 & & \end{array}\right).$$

The fixed point sets of the $\mathbb{Z}_2$-actions on $M(B_i)$ ($i = 5, 6$) corresponding to $A_{15}$, $A_{16}$, $A_{17}$, $A_{18}$ and $A_{19}$ are as follows: (1) $T^3$, 2 components $S^1$, 4 points, (2) 3 components $T^2$, 2 components $S^1$, (3) 2 components $T^2$, 2 components $S^1$, 4 point, (4) $T^2$, 6 components $S^1$, (5) 2 components $T^2$, 4 components $S^1$. By Proposition 1, $A_i$ ($i = 15, 16, 17, 18, 19$) are not equivalent to each other.

The following Bott matrices are created from the Bott matrix vi), say $B_7$.

$$A_{20} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & 0 & 0 \\ \hline 0 & & B_7 & & \end{array}\right) A_{21} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 0 & & B_7 & & \end{array}\right).$$

The fixed point sets of the $\mathbb{Z}_2$-actions on $M(B_7)$ corresponding to $A_{20}$ and $A_{21}$ are as follows: (1) $T^3$, 8 points, (2) 2 components $T^2$, 4 components $S^1$. By Proposition 1, $A_{20}$ and $A_{21}$ are not equivalent.

The Bott matrix $A_{22}$ is created from the first Bott matrix in vii), say $B_8$.

$$A_{22} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 0 & & B_8 & & \end{array}\right).$$

The following Bott matrices are created from the first Bott matrix in viii), say $B_9$.

$$A_{23} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 0 & & B_9 & & \end{array}\right) A_{24} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 0 \\ \hline 0 & & B_9 & & \end{array}\right).$$

The fixed point sets of the $\mathbb{Z}_2$-actions on $M(B_9)$ corresponding to $A_{23}$ and $A_{24}$ are as follows: (1) 2 components $T^2$, 8 points, (2) 8 components $S^1$. By Proposition 1, $A_{23}$ is not equivalent to $A_{24}$.

The following Bott matrices are created from the first Bott matrix in ix), say $B_{10}$.

$$A_{25} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 0 & & B_{10} & & \end{array}\right) A_{26} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 1 \\ \hline 0 & & B_{10} & & \end{array}\right).$$

The fixed point sets of the $\mathbb{Z}_2$-actions on $M(B_{10})$ corresponding to $A_{25}$ and $A_{26}$ are as follows: (1) 2 components $T^2$, 2 components $S^1$, 4 points, (2) 6 components $S^1$, 4 points. By Proposition 1, $A_{25}$ and $A_{26}$ are not equivalent.

The Bott matrix $A_{27}$ (resp. $A_{28}$) below is created from the first Bott matrix in x), say $B_{11}$, (resp. xi), say $B_{12}$).

$$A_{27} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 0 & & B_{11} & & \end{array}\right) A_{28} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 0 \\ \hline 0 & & B_{12} & & \end{array}\right).$$

Finally from $I_4$, we get $A_{29} = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 \\ \hline 0 & & I_4 & & \end{array}\right)$.

Since each Bott matrix $B$ of i) to xii) is not equivalent to each other, the resulting Bott matrix $A$ is not equivalent. Totally, 29 Bott matrices $A_i$ $(i = 1, \dots, 29)$ are not equivalent to each other. When we take the second Bott matrix $B'$ from i), the resulting Bott matrix $A'$ gives an action $(\mathbb{Z}_2, M(B'))$. We can check that $(\mathbb{Z}_2, M(B'))$ is equivariantly diffeomorphic to one of the actions $(\mathbb{Z}_2, M(B))$ corresponding to $A_1$, $A_2$, $A_3$ by the ad hoc argument. (Compare Section 2 for the argument to find an equivariant diffeomorphism.) Once there exists such an equivariant diffeomorphism, $A'$ is equivalent to one of $A_1$, $A_2$, $A_3$ by Theorem 2. Similarly, if $A'$ is another Bott matrix created from the first Bott matrix in i), we can check that the corresponding $(\mathbb{Z}_2, M(B))$ is equivariantly diffeomorphic to one of the actions $(\mathbb{Z}_2, M(B))$ corresponding to $A_1$, $A_2$, $A_3$. (Note that the total number of Bott matrices created from the first Bott matrix in i) is 8.) This argument works not only for the case i) but also for the cases from ii) to xii). As a consequence the Bott matrix $A'$ created from Bott matrices from ii) to xii) is equivalent to one of $A_i$'s $(i = 4, \dots, 29)$. In summary, we obtain the following but the proof is omitted because of a tedious argument.

LEMMA 1. *A Bott matrix created from any one of Bott matrices of Theorem* 5 *is equivalent to one of the Bott matrices* $A_i$ $(i = 1, \dots, 29)$ *above.*

PROPOSITION 2. *There are* 29 *diffeomorphism classes of the case* $S^1$*-actions with* 4*-dimensional quotients.*

**3.2. $T^2$-actions with 3-dimensional quotients.** The Bott matrices of $M(A)$ admitting $T^2$-actions have the following form

$$\left(\begin{array}{c|c} I_2 & * \\ \hline 0 & \mathrm{B} \end{array}\right).$$

The following Bott matrices are created from the first Bott matrix $B$ of a), say $B_{13}$, in Theorem 4.

$$A_{30} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B_{13} & \end{array}\right) A_{31} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & & B_{13} & \end{array}\right)$$

$$A_{32} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B_{13} & \end{array}\right) A_{33} = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & & B_{13} & \end{array}\right)$$

$$A_{34} = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & & B_{13} & \end{array}\right) A_{35} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B_{13} & \end{array}\right).$$

The fixed point sets of the $(\mathbb{Z}_2)^s$-actions $(s = 1, 2)$ on $M(B_{13})$ corresponding to $A_{30}$, $A_{31}$, $A_{32}$, $A_{33}$, $A_{34}$ and $A_{35}$ are as follows: (1) $T^2$, $S^1$, 2 points, (2) 3 components $S^1$, 2 points, (3) 3 components $T^2$, 4 components $S^1$, 4 points, (4) $T^2$, 8 components $S^1$, 4 points, (5) 3 components $T^2$, 4 components $S^1$, 4 points, (6) 2 components $T^2$, 6 components $S^1$, 4 points. Compared (3) with (5), we see from **b)** in Section 2 that $A_{32}$ is not equivalent to $A_{34}$ . By Proposition 1, Bott matrices $A_i$ ($i = 30, 31$) (resp. $A_j$ ($j = 32, 33, 34, 35$)) are not equivalent to each other. Moreover, by Remark 3, Bott matrices $A_i$ ($i = 30, 31$) are not equivalent to $A_j$ ($j = 32, 33, 34, 35$) because the $(\mathbb{Z}_2)^2$-action corresponding to $A_j$ ($j = 32, 33, 34, 35$) cannot be reduced to a $\mathbb{Z}_2$-action.

The following Bott matrices are created from the Bott matrix b), say $B_{14}$, in Theorem 4.

$$A_{36} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B_{14} & \end{array}\right) A_{37} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & & B_{14} & \end{array}\right)$$

$$A_{38} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B_{14} & \end{array}\right) A_{39} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B_{14} & \end{array}\right)$$

$$A_{40} = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & & B_{14} & \end{array}\right).$$

The fixed point sets of the $(\mathbb{Z}_2)^s$-actions ($s = 1, 2$) on $M(B_{14})$ corresponding to $A_{36}$, $A_{37}$, $A_{38}$, $A_{39}$ and $A_{40}$ are as follows: (1) $T^2$, 4 points, (2) 4 components $S^1$, (3) 2 components $T^2$, 4 components $S^1$, 8 points, (4) 3 components $T^2$, 4 components $S^1$, 4 points, (5) 12 components $S^1$. By Remark 3, Bott matrices $A_i$ ($i = 36, 37$) are not equivalent to $A_j$ ($j = 38, 39, 40$) because the $(\mathbb{Z}_2)^2$-action corresponding to $A_j$ ($j = 38, 39, 40$) cannot be reduced to a $\mathbb{Z}_2$-action. On the other hand, by Proposition 1, Bott matrices $A_i$ ($i = 36, 37$) (resp. $A_j$ ($j = 38, 39, 40$)) are not equivalent to each other.

The Bott matrices $A_i$ ($i = 41, 42, 43, 44$) below are created from the first Bott matrix in c), say $B_{15}$, of Theorem 4 while $A_{45}$ is created from the second Bott matrix in c), say $B_{16}$.

$$A_{41} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & & B_{15} & \end{array}\right) A_{42} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B_{15} & \end{array}\right)$$

$$A_{43} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & & B_{15} & \end{array}\right) A_{44} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & & B_{15} & \end{array}\right)$$

$$A_{45} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & B_{16} & \end{array}\right).$$

The fixed point sets of the $(\mathbb{Z}_2)^2$-actions on $M(B_i)$ ($i = 15, 16$) corresponding to $A_{42}$, $A_{43}$, $A_{44}$ and $A_{45}$ are as follows: (1) 3 components $T^2$, 4 components $S^1$, 4 points, (2) $T^2$, 8 components $S^1$, 4 points, (3) 2 components $T^2$, 4 components $S^1$, 8 points, (4) 2 components $T^2$, 6 components $S^1$, 4 points. By Remark 3, $A_{41}$ is not equivalent to $A_i$ ($i = 42, 43, 44, 45$) because the $(\mathbb{Z}_2)^2$-action corresponding to $A_i$ ($i = 42, 43, 44, 45$) cannot be reduced to a $\mathbb{Z}_2$-action. Then by Proposition 1, Bott matrices $A_i$ ($i = 42, 43, 44, 45$) are not equivalent to each other.

The following Bott matrices are created from $I_3$.

$$A_{46} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & & I_3 & \end{array}\right) A_{47} = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & & I_3 & \end{array}\right)$$

$$A_{48} = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & & I_3 & \end{array}\right).$$

The fixed point sets of the $(\mathbb{Z}_2)^2$-actions on $M(B)$ corresponding to $A_{47}$ and $A_{48}$ are as follows: (1) 2 components $T^2$, 4 components $S^1$, 8 points, (2) 12 components $S^1$. By Remark 3, $A_{46}$ is not equivalent to $A_i$ ($i = 47, 48$), and by Proposition 1, $A_{47}$ is not equivalent to $A_{48}$.

Since each Bott matrix B of a) to d) is not equivalent to each other, the resulting Bott matrix $A$ is not equivalent. Totally, 19 Bott matrices $A_i$ ($i = 30, \ldots, 48$) are not equivalent to each other.

When we take the second Bott matrix $B'$ from a) of Theorem 4, the resulting Bott matrix $A'$ gives an action $((\mathbb{Z}_2)^s, M(B'))$ ($s = 1, 2$). We can check that $((\mathbb{Z}_2)^s, M(B'))$ is equivariantly diffeomorphic to one of the actions $((\mathbb{Z}_2)^s, M(B))$ corresponding to $A_i$ ($i = 30, \ldots, 35$) by the ad hoc argument. (Compare Section 2 for the argument to find an equivariant diffeomorphism.) Once there exists such an equivariant diffeomorphism, $A'$ is equivalent to one of $A_i$'s ($i = 30, \ldots, 35$) by Theorem 2. Similarly, if $A'$ is another Bott matrix created from the first Bott matrix in a) of Theorem 4, we can check that the corresponding $((\mathbb{Z}_2)^s, M(B))$ ($s = 1, 2$) is equivariantly diffeomorphic to one of the actions $((\mathbb{Z}_2)^s, M(B))$ corresponding to $A_i$ ($i = 30, \ldots, 35$). (Note that the total number of Bott matrices created from the first Bott matrix in a) is 76.) This argument also works for the case b), c) and d). As a consequence the Bott matrix $A'$ created from Bott matrices in b), c) and d) is equivalent to one of $A_i$'s ($i = 36, \ldots, 48$). In summary, we obtain the following.

LEMMA 2. *A Bott matrix created from any one of Bott matrices of Theorem* 4 *is equivalent to one of the Bott matrices* $A_i$ $(i = 30, \ldots, 48)$ *above.*

PROPOSITION 3. *There are* 19 *diffeomorphism classes of the case* $T^2$*-actions with* 3*-dimensional quotients.*

**3.3.** $T^3$**-actions with** 2**-dimensional quotients.** The Bott matrices of $M(A)$ admitting $T^3$-actions have the following form

$$\left(\begin{array}{c|c} I_3 & * \\ \hline 0 & \mathrm{B} \end{array}\right).$$

In this case a Bott matrix $B$ is either $A_1$ or $A_2$ in Theorem 3. The Bott matrices $A_{49}$ and $A_{50}$ (resp. $A_{51}$ and $A_{52}$) below are created from $A_1$ (resp. $A_2$).

$$A_{49} = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) A_{50} = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$A_{51} = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) A_{52} = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Since $A_1$ and $A_2$ in Theorem 3 are not equivalent, none of $A_{49}$ and $A_{50}$ is equivalent to $A_{51}$ or $A_{52}$. Then $A_{49}$ (resp. $A_{51}$) is not equivalent to $A_{50}$ (resp. $A_{52}$), because $(\mathbb{Z}_2)^2$-action on $M(B)$ corresponding to $A_{50}$ (or $A_{52}$) cannot be reduced to a $\mathbb{Z}_2$-action. If $A'$ is another Bott matrix created from $A_1$ in Theorem 3, we can check that the corresponding $((\mathbb{Z}_2)^s, M(A_1))$ $(s = 1, 2)$ is equivariantly diffeomorphic to one of the actions $((\mathbb{Z}_2)^s, M(A_1))$ corresponding to $A_{49}$ and $A_{50}$ by the ad hoc argument. Once there exists such an equivariant diffeomorphism, $A'$ is equivalent to $A_{49}$ or $A_{50}$ by Theorem 2. This argument works also for the case $A_2$ in Theorem 3. As a consequence another Bott matrix $A'$ created from $A_2$ is equivalent to $A_{51}$ or $A_{52}$. Thus we obtain the following.

LEMMA 3. *A Bott matrix created from any one of Bott matrices in Theorem* 3 *is equivalent to one of the Bott matrices* $A_i$ $(i = 49, 50, 51, 52)$ *above.*

PROPOSITION 4. *There are* 4 *diffeomorphism classes of the case* $T^3$*-actions with* 2*-dimensional quotients.*

**3.4. $T^4$-actions with one-dimensional quotients.** The Bott matrices of $M(A)$ admitting $T^4$-actions have the following form

$$\left(\begin{array}{cccc|c} & I_4 & & & * \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

In this case $M(B) = M(1) = S^1$. It is easy to check by using moves **II** and **III**, it consists of just one diffeomorphism class, where the corresponding Bott matrix is

$$A_{53} = \left(\begin{array}{cccc|c} & I_4 & & & 0 \\ & & & & 1 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Obviously the corresponding Bott matrix of size 5 of a real Bott manifold admitting $T^5$-action is the identity matrix of rank 5. Combined with Propositions 2, 3, 4 and the case of

$T^4$-actions above we get the following theorem.

THEOREM 6. *The diffeomorphism classes of* 5*-dimensional real Bott manifolds consist of* 54.

## 4. Classification of $n$-dimensional Real Bott Manifolds

In this section we shall prove some results regarding the classification of certain types of $n$-dimensional real Bott manifolds.

THEOREM 7. *The number of diffeomorphism classes of $n$-dimensional real Bott manifolds* ($n \geq 4$) *which admit the maximal $T^{n-2}$-actions* ($i.e.\ s = 1, 2$ ) *is* 4:

$$M(A) = T^{(n-2)} \underset{(\mathbb{Z}_2)^s}{\times} M(B)\,.$$

PROOF. Since there are two diffeomorphism classes of 2-dimensional real Bott manifolds $M(B)$ (see Theorem 3), the real Bott manifolds $M(A)$ created from $M(B)$ correspond to the following Bott matrices

$$A_1 = \left(\begin{array}{c|c} I_{n-2} & * \\ \hline 0 & B_1 \end{array}\right), \quad A_2 = \left(\begin{array}{c|c} I_{n-2} & * \\ \hline 0 & B_2 \end{array}\right)$$

where $B_1 = I_2$, $B_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Let us consider $A_1$. If the entries in each row of $*$ are the same then by moves **II** or **III**, $A_1$ is equivalent to

$$\left(\begin{array}{c|cc} I_{n-2} & \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} \\ \hline 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right). \tag{4.1}$$

Otherwise by moves **II**, **III** or the equivariant diffeomorphism $\varphi$: $((\mathbb{Z}_2)^2,\ M(B_1)) \rightarrow ((\mathbb{Z}_2)^2, M(B_1))$ defined by $\varphi[z_{n-1}, z_n] = [z_{n-1}, z_n]$, $A_1$ is equivalent to

$$\left(\begin{array}{c|cc} I_{n-2} & \begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right). \tag{4.2}$$

However (4.1) is not equivalent to (4.2) because the $(\mathbb{Z}_2)^2$-action on $M(B_1)$ corresponding to (4.2) cannot be reduced to a $\mathbb{Z}_2$-action on it.

Let us consider $A_2$. If the entries in each row of $*$ are the same then $A_2$ is equivalent to

$$\left(\begin{array}{c|cc} I_{n-2} & 0 & 0 \\ & 1 & 1 \\ \hline & 1 & 1 \\ 0 & 0 & 1 \end{array}\right), \tag{4.3}$$

or if the entries in the second column of $*$ are all zero then $A_2$ is equivalent to

$$\left(\begin{array}{c|cc} I_{n-2} & 0 & 0 \\ & 1 & 0 \\ \hline & 1 & 1 \\ 0 & 0 & 1 \end{array}\right) \tag{4.4}$$

by moves **II** or **III**. However (4.3) and (4.4) are equivalent by the equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_2)) \to (\mathbb{Z}_2, M(B_2))$ defined by $\varphi([z_{n-1}, z_n]) = [\mathbf{i}z_{n-1}, z_n]$. Otherwise $A_2$ is equivalent to

$$\left(\begin{array}{c|cc} & 0 & 0 \\ I_{n-2} & 1 & 0 \\ & 0 & 1 \\ \hline & 1 & 1 \\ 0 & 0 & 1 \end{array}\right) \tag{4.5}$$

by moves **II**, **III** or the equivariant diffeomorphism $\varphi\colon ((\mathbb{Z}_2)^2,\ M(B_2)) \to ((\mathbb{Z}_2)^2,\ M(B_2))$ defined by $\varphi[z_{n-1}, z_n] = [z_{n-1}, z_n]$. Moreover (4.4) and (4.5) are not equivalent because the $(\mathbb{Z}_2)^2$-action on $M(B_2)$ corresponding to (4.5) cannot be reduced to a $\mathbb{Z}_2$-action on it.

Therefore there are 4 equivalence classes of the Bott matrices corresponding to $M(A)$. $\square$

PROPOSITION 5. *The diffeomorphism class is unique for the real Bott manifold of the form $M(A) = T^k \underset{\mathbb{Z}_2}{\times} T^{n-k}$ for any $k$ $(1 \leq k \leq n-1)$. In particular, if $k = n$ then $M(A) = T^n$.*

PROOF. Since $M(A)$ admits the maximal $T^k$-action and $A$ is created from $I_{n-k}$, there is only one Bott matrix $A$, namely

$$A = \left(\begin{array}{c|c} I_k & 0 \\ & 1 \quad \dots \quad 1 \\ \hline 0 & I_{n-k} \end{array}\right). \tag{4.6}$$

Clearly, if $k = n$ then $A = I_n$ the identity matrix of rank $n$. $\square$

Obviously, by Theorem 7 and this proposition, there are 6 diffeomorphism classes of $n$-dimensional real Bott manifolds $M(A)$ $(n \geq 4)$ admitting the maximal $T^k$-action for $k = n-2, n-1, n$.

COROLLARY 1. *If* $M(A) = S^1 \times_{\mathbb{Z}_2} M(B)$ *where* $M(B) = T^k \times_{\mathbb{Z}_2} S^1$, *then for any* $k \geq 1$ *there is only one diffeomorphism class.*

PROOF. Since $M(B) = T^k \times_{\mathbb{Z}_2} S^1$,

$$B = \begin{pmatrix} & I_k & & * \\ & & & 1 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

The Bott matrices $A$ created from $B$ are

$$\left(\begin{array}{c|cccc} 1 & 1 & \dots & 1 & 1 \\ \hline 0 & & B & & \end{array}\right) \text{and} \left(\begin{array}{c|cccc} 1 & 1 & \dots & 1 & 0 \\ \hline 0 & & B & & \end{array}\right)$$

which are equivalent by the equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B)) \to (\mathbb{Z}_2, M(B))$ defined by $\varphi([z_2, \dots, z_{k+1}, z_{k+2}]) = [z_2, \dots, \mathbf{i}z_{k+1}, z_{k+2}]$. □

COROLLARY 2. *Let* $M(A)$ *be a real Bott manifold which fibers* $S^1$ *over the real Bott manifold* $M(B)$ *for which* $M(B)$ *is* $T^k \times_{(\mathbb{Z}_2)^s} \mathrm{K}$ $(k \geq 2)$. *Here* K *is a Klein bottle. Then the number of diffeomorphism classes of such* $M(A)$ *is* 3.

PROOF. Since $M(B) = T^k \times_{(\mathbb{Z}_2)^s} \mathrm{K}$ $(s = 1, 2)$, there are 2 distinct diffeomorphism classes of $M(B)$ corresponding to Bott matrices in (4.4) and (4.5)) (say $B_1$ and $B_2$ respectively). The Bott matrices of size $(k+3)$ created from $B_1$ with the $\mathbb{Z}_2$-actions are as follows

$$\left(\begin{array}{c|ccccc} 1 & 1 & \dots & 1 & 0 & 0 \\ \hline 0 & & & B_1 & & \end{array}\right), \left(\begin{array}{c|ccccc} 1 & 1 & \dots & 1 & 1 & 0 \\ \hline 0 & & & B_1 & & \end{array}\right) \tag{4.7}$$

$$\left(\begin{array}{c|ccccc} 1 & 1 & \dots & 1 & 0 & 1 \\ \hline 0 & & & B_1 & & \end{array}\right), \left(\begin{array}{c|ccccc} 1 & 1 & \dots & 1 & 1 & 1 \\ \hline 0 & & & B_1 & & \end{array}\right). \tag{4.8}$$

The Bott matrices in (4.7) (resp. (4.8)) are equivalent by the equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_1)) \to (\mathbb{Z}_2, M(B_1))$ defined by $\varphi([z_2, \dots, z_{k+1}, z_{k+2}, z_{k+3}]) = [z_2, \dots, \mathbf{i}z_{k+1}, z_{k+2}, z_{k+3}]$.

On the other hand, Bott matrices in (4.7) are not equivalent to (4.8) because the maximal fixed point sets of $(\mathbb{Z}_2, M(B_1))$ corresponding to the Bott matrices in (4.7) and (4.8) are $T^2$ and $S^1$, respectively.

It is easy to see that each Bott matrix created from $B_2$ is equivalent to

$$\left(\begin{array}{c|ccccc} 1 & 1 & \dots & 1 & 0 & 0 \\ \hline 0 & & & B_2 & & \end{array}\right) \tag{4.9}$$

by $\varphi\colon (\mathbb{Z}_2, M(B_2)) \to (\mathbb{Z}_2, M(B_2))$ which is defined by one of the following

$$\varphi([z_2, \dots, z_k, z_{k+1}, z_{k+2}, z_{k+3}]) = [z_2, \dots, \mathbf{i}z_k, z_{k+1}, z_{k+2}, z_{k+3}],$$

$$\varphi([z_2, \dots, z_k, z_{k+1}, z_{k+2}, z_{k+3}]) = [z_2, z_3, \dots, z_k, \mathbf{i}z_{k+1}, z_{k+2}, z_{k+3}],$$

$$\varphi([z_2, \dots, z_k, z_{k+1}, z_{k+2}, z_{k+3}]) = [z_2, z_3, \dots, \mathbf{i}z_k, \mathbf{i}z_{k+1}, z_{k+2}, z_{k+3}].$$

Obviously, the Bott matrix (4.9) is not equivalent to the Bott matrices in (4.7) and (4.8) because $B_1$ is not equivalent to $B_2$. Therefore there are 3 equivalence classes of the Bott matrices corresponding to $M(A)$. □

COROLLARY 3. *Let $M(A)$ be a real Bott manifold which fibers $S^1$ over the real Bott manifold $M(B)$ for which $M(B)$ is $T^k \times_{(\mathbb{Z}_2)^s} T^2$ $(k \geq 2)$. Then the number of diffeomorphism classes of such $M(A)$ is 3.*

PROOF. Since $M(B) = T^k \times_{(\mathbb{Z}_2)^s} T^2$ $(s = 1, 2)$, there are 2 distinct diffeomorphism classes of $M(B)$ which correspond to the Bott matrices in (4.1) and (4.2). Using a similar observation as in the proof of Corollary 2, one can prove that there are 3 equivalence classes of the Bott matrices corresponding to $M(A)$. □

Now if we create Bott matrices from (4.6) (for $k = 1$) with $\mathbb{Z}_2$-actions then we will get the classification of the corresponding real Bott manifolds as follows.

THEOREM 8. *Let $M(A)$ be a real Bott manifold which fibers $S^1$ over the real Bott manifold $M(B)$ where $M(B) = S^1 \times_{\mathbb{Z}_2} T^k$ $(k \geq 2)$, then the diffeomorphism classes of such $M(A)$ is $[\frac{k}{2}] + 1$. Here $[x]$ is the integer part of $x$.*

PROOF. Since $M(B) = S^1 \times_{\mathbb{Z}_2} T^k$,

$$B = \left(\begin{array}{c|ccc} 1 & 1 & \dots & 1 \\ \hline 0 & & I_k & \end{array}\right). \tag{4.10}$$

The Bott matrices $A_i$ of size $(k + 2)$ created from (4.10) with $\mathbb{Z}_2$-actions are as follows

$$A_i = \left(\begin{array}{c|cc} 1 & 1 & * \\ \hline 0 & & B \end{array}\right), \quad (i = 1, \dots, 2^k)\,. \tag{4.11}$$

We apply the different $\mathbb{Z}_2$-actions on $M(B)$ such that the Bott matrices $A_i$ are as follows:

$$\begin{aligned} A_1 &= \left(\begin{array}{c|cccc} 1 & \{1\}_2 & \{0\}_3 & \dots & \{0\}_{2+k} \\ \hline 0 & & B & & \end{array}\right), \\ A_2 &= \left(\begin{array}{c|ccccc} 1 & \{1\}_2 & \{1\}_3 & \{0\}_4 & \dots & \{0\}_{2+k} \\ \hline 0 & & & B & & \end{array}\right), \\ &\vdots \\ A_{[\frac{k}{2}]+1} &= \left(\begin{array}{c|cccccc} 1 & \{1\}_2 & \dots & \{1\}_{1+([\frac{k}{2}]+1)} & \{0\}_{2+([\frac{k}{2}]+1)} & \dots & \{0\}_{2+k} \\ \hline 0 & & & B & & & \end{array}\right). \end{aligned} \tag{4.12}$$

Here $\{y\}_i$ means $y$ in the $i$-th spot. It is easy to check that the maximal fixed point sets of $(\mathbb{Z}_2, M(B))$ corresponding to $A_i$ $(i = 1, 2, \dots, [\frac{k}{2}] + 1)$ are $T^k$, $T^{k-1}$, …, $T^{k-[\frac{k}{2}]}$, respectively. Hence they are not equivalent to each other.

On the other hand, for $[\frac{k}{2}]+1 < l \leq (k+1)$, Bott matrix

$$\left(\begin{array}{c|cccccc} 1 & \{1\}_2 & \dots & \{1\}_{1+l} & \{0\}_{2+l} & \dots & \{0\}_{2+k} \\ \hline 0 & & & & B & & \end{array}\right) \tag{4.13}$$

is equivalent to one of the Bott matrices in (4.12). To show this, consider the $g_1$-action corresponding to (4.13):

$$\begin{aligned} g_1([z_2, \dots, z_{1+l}, z_{l+2}, \dots, z_{k+2}]) &= [\bar{z}_2, \dots, \bar{z}_{l+1}, \overbrace{z_{l+2}, \dots, z_{k+2}}^{(k+1)-l}] \\ &= [g_2(\bar{z}_2, \dots, \bar{z}_{l+1}, z_{l+2}, \dots, z_{k+2})] \\ &= [-\bar{z}_2, z_3, \dots, z_{l+1}, \bar{z}_{l+2}, \dots, \bar{z}_{k+2}]\,. \end{aligned}$$

Since $(k+1)-l < k-[\frac{k}{2}] \leq [\frac{k}{2}]+1$, there is an equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B)) \to (\mathbb{Z}_2, M(B))$ defined by $\varphi([z_2, \dots, z_{l+1}, z_{l+2}, \dots, z_{k+2}]) = [\mathbf{i}z_2, z_{l+2}, \dots, z_{k+2}, z_3, \dots, z_{l+1}]$ such that $\varphi g_1 = h_1\varphi$ for some $h_1$-action corresponding to one of the Bott matrices in (4.12).

The other Bott matrices $A_i$ $(i \neq 1, \dots, [\frac{k}{2}]+1)$ may have the form

$$A' = \left(\begin{array}{c|cccc} 1 & 1 & \hat{1} & \dots & \hat{1} \\ \hline 0 & & & B & \end{array}\right), \tag{4.14}$$

where $\hat{1} \in \{0,1\}$. We prove that $A'$ is equivalent to one of the Bott matrices in (4.12). Suppose that the number of entries $\hat{1}$'s for $\hat{1}=1$ is $t$. Applying move **I** on $A'$ such that the entries $\hat{1}$'s for $\hat{1}=1$ are placed in series, we get a new Bott matrix

$$A'' = \left(\begin{array}{c|ccccccc} 1 & \{1\}_2 & \{1\}_3 & \dots & \{1\}_{2+t} & \{0\}_{3+t} & \dots & \{0\}_{2+k} \\ \hline 0 & & & & B & & & \end{array}\right),$$

which is still equivalent to $A'$. Obviously, $A'' = A_i$ for some $i = 1, \dots, [\frac{k}{2}]+1$ if $0 \leq t \leq [\frac{k}{2}]$, or $A''$ is the same as (4.13) if $t > [\frac{k}{2}]$. Hence $A'$ is equivalent to one of the Bott matrices in (4.12). This completes the proof of the theorem. $\square$

From now on, we use the notation $(\mathbb{Z}_2, M(B_j))_i$ which means that the $\mathbb{Z}_2$-action on $M(B_j)$ corresponds to a Bott matrix $A_{ij}$.

Next we state three lemmas which will be used to prove Theorem D in Introduction.

LEMMA 4. *Let $M(A) = S^1 \times_{\mathbb{Z}_2} M(B_1)$ be an $n$-dimensional real Bott manifold with*

$$B_1 = \begin{pmatrix} 1 & 1 & 1 & \dots & \dots & \dots & 1 \\ & 1 & 1 & \dots & \dots & \dots & 1 \\ & & \ddots & & & & \vdots \\ & & & 1 & 1 & \dots & 1 \\ & 0 & & & & I_k & \end{pmatrix}, \tag{4.15}$$

*$k \geq 2$, and $n-k \geq 3$. Such real Bott manifolds $M(A)$ corresponding to Bott matrices derived from $A_{j'1}$ $(j = 1, \ldots, k, (k+1))$ in* (4.16) *are not diffeomorphic to each other.* (*That is, there are $k(2^{n-k-3}-1)+(2^{n-k-3}-(n-k-2))$ nonequivalent Bott matrices derived from* (4.16).)

$$A_{j'1} = \left(\begin{array}{c|cccccccccc} 1 & 1 & 0 & \{\hat{1}\}_4 & \ldots & \{\hat{1}\}_{n-k} & \{1\}_{n-k+1} & \ldots & \{1\}_{n-k+(j-1)} & \{0\} & \ldots & \{0\}_n \\ \hline 0 & \multicolumn{11}{c}{B_1} \end{array}\right) \tag{4.16}$$

*for $j = 1, \ldots, k, (k+1)$, where $\hat{1}$ is either* 0 *or* 1,

$$\begin{gathered} (\{\hat{1}\}_4, \ldots, \{\hat{1}\}_{n-k}) \neq (0, \ldots, 0) \\ (\textit{resp.}\ (\{\hat{1}\}_4, \ldots, \{\hat{1}\}_{n-k}) \neq (\overbrace{0, \ldots, 0}^{l}, 1, \ldots, 1),\ l = 0, 1, \ldots, n-k-3) \end{gathered} \tag{4.17}$$

*for Bott matrix $A_{j'1}$ $(j = 1, \ldots, k)$ (resp. $A_{(k+1)'1}$).*

PROOF. Recall that, if $M(A_{m1})$ is diffeomorphic to $M(A_{q1})$ (i.e., $A_{m1}$ is equivalent to $A_{q1}$, $m \neq q$), by Theorem 2, there is an equivariant diffeomorphism

$$\begin{aligned} &(\Phi, \varphi)\colon (\mathbb{Z}_2 = \langle\alpha\rangle, M(B_1))_m \to (\mathbb{Z}_2 = \langle\beta\rangle, M(B_1))_q, \text{ such that} \\ &\varphi(\alpha[z_2, \ldots, z_n]) = \Phi(\alpha)\varphi[z_2, \ldots, z_n] = \beta\varphi[z_2, \ldots, z_n]\,. \end{aligned}$$

Let $\bar{\varphi}\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the lift of $\varphi$. According to the form of $B_1$, the affine element $\bar{\varphi}$ has the form

$$\bar{\varphi} = \left(\left(\frac{\mathbf{a}}{\mathbf{b}}\right), \left(\begin{array}{c|c} I_{n-k-1} & 0 \\ \hline 0 & D \end{array}\right)\right) \tag{4.18}$$

where $D$ is a nonsingular submatrix of rank $k$, ${}^t\mathbf{a} = (a_2, \ldots, a_{n-k})$ and ${}^t\mathbf{b} = (b_{n-k+1}, \ldots, b_n)$ (compare (1.11)). Since $M(B_1) = T^{n-1}/(\mathbb{Z}_2)^{n-1}$, $\bar{\varphi}$ induces an affine map $\tilde{\varphi}$ of $T^{n-1}$.

Put $X = \begin{pmatrix} x_{n-k+1} \\ \vdots \\ x_n \end{pmatrix}$. Since $\tilde{\varphi}p = p\bar{\varphi}$,

$$\begin{aligned} \tilde{\varphi}(z_2, \ldots, z_{n-k}, z_{n-k+1}, \ldots, z_n) &= (\ell_2 z_2, \ldots, \ell_{n-k} z_{n-k}, p\,{}^t(\mathbf{b} + DX)) \\ &= (\ell_2 z_2, \ldots, \ell_{n-k} z_{n-k}, c_{n-k+1} w_{n-k+1}, \ldots, c_n w_n) \end{aligned} \tag{4.19}$$

where $\ell_j = exp(2\pi\mathbf{i}a_j)$ $(j = 2, \ldots, n-k)$, $c_s = exp(2\pi\mathbf{i}b_s)$ $(s = n-k+1, \ldots, n)$, $(w_{n-k+1}, \ldots, w_n) = p\,{}^t(DX)$.

On the other hand, since $M(B_1) = T^{n-1}/(\mathbb{Z}_2)^{n-1}$, the action $\langle\alpha\rangle$ lifts to an action on $T^{n-1}$ such that we have the commutative diagram

$$
\begin{tikzcd}
(\mathbb{Z}_2)^{n-1} \arrow[d] & (\mathbb{Z}_2)^{n-1} \arrow[d] \\
(\alpha, T^{n-1}) \arrow[r, "\tilde{\varphi}"] \arrow[d, "Pr"'] & (g\beta, T^{n-1}) \arrow[d, "Pr"'] \\
(\alpha, M(B_1))_m \arrow[r, "\varphi"] & (\beta, M(B_1))_q
\end{tikzcd}
$$

for some $g \in (\mathbb{Z}_2)^{n-1} = \langle g_2, \dots, g_n \rangle$. This means that

$$
\begin{aligned}
Pr(\tilde{\varphi}(\alpha(z_2, \dots, z_n))) &= \varphi(Pr(\alpha(z_2, \dots, z_n))) = \varphi(\alpha(Pr(z_2, \dots, z_n))) \\
&= \Phi(\alpha)\varphi(Pr(z_2, \dots, z_n)) = \beta\varphi(Pr(z_2, \dots, z_n)) \\
&= \beta Pr(\tilde{\varphi}(z_2, \dots, z_n)) = Pr(\beta\tilde{\varphi}(z_2, \dots, z_n)),
\end{aligned}
$$

$$
\text{(i.e., } \tilde{\varphi}(\alpha(z_2, \dots, z_n)) = g\beta\tilde{\varphi}(z_2, \dots, z_n).\text{)} \tag{4.20}
$$

(Note that $g_i$ $(i = 2, \dots, n)$ corresponds to the $i$-th row of $A_{m1}$ and $A_{q1}$.) This implies that $\tilde{\varphi}$ maps the fixed point set of $(\alpha, T^{n-1})$ to that of $(g\beta, T^{n-1})$ diffeomorphically. From the commutative diagram, we also have, for $g \in (\mathbb{Z}_2)^{n-1}$,

$$
\begin{aligned}
Pr(\tilde{\varphi}(g(z_2, \dots, z_n))) &= \varphi(Pr(g(z_2, \dots, z_n))) = \varphi(Pr(z_2, \dots, z_n)) \\
&= Pr(\tilde{\varphi}(z_2, \dots, z_n))\,.
\end{aligned}
$$

Hence there is an element $h \in (\mathbb{Z}_2)^{n-1}$ such that

$$
\tilde{\varphi}(g(z_2, \dots, z_n)) = h\tilde{\varphi}(z_2, \dots, z_n),\ \ \tilde{\varphi}g\tilde{\varphi}^{-1} = h\,. \tag{4.21}
$$

Here the action $(\alpha, M(B_1))_{j'}$ (similarly for $(\beta, M(B_1))_{j'}$) corresponding to $A_{j'1}$ is written as

$$
\alpha(z_2, \dots, z_n) = (\bar{z}_2, z_3, \overset{\alpha}{\widehat{z_4}}, \dots, \overset{\alpha}{\widehat{z_{n-k}}}, \bar{z}_{n-k+1}, \dots, \bar{z}_{n-k+j-1}, z_{n-k+j}, \dots, z_n)
$$

where $\overset{\alpha}{\widehat{z_j}} (\in \{z_j, \bar{z}_j\})$ means an $\alpha$-action on $z_j$. Note that $\overset{\alpha}{\widehat{z_j}}$ is either $z_j$ or $\bar{z}_j$ depending on whether $\hat{1}$ is 0 or 1 respectively.

To show that Bott matrices derived from $A_{j'1}$ $(j = 1, \dots, k, (k+1))$ in (4.16) are not equivalent to each other, we shall prove the following claims.

CLAIM 1. Bott matrices $A_{j'1}$ $(j = 1, \dots, k, (k+1))$ are not equivalent to each other.

Suppose that $(\alpha, M(B_1))_{l'}$ is equivariantly diffeomorphic to $(\beta, M(B_1))_{p'}$ where $1 \leq l < p \leq k+1$ $(l = 1, \dots, k;\ p = 2, \dots, k+1)$. Then by (4.20), we have

$$
\begin{aligned}
&(\ell_2\bar{z}_2, \ell_3 z_3, \ell_4\overset{\alpha}{\widehat{z_4}}, \dots, \ell_{n-k}\overset{\alpha}{\widehat{z_{n-k}}}, c_{n-k+1}w'_{n-k+1}, \dots, c_n w'_n)\\
&= g(\overline{\ell_2 z_2}, \ell_3 z_3, \overset{\beta}{\widehat{\ell_4 z_4}}, \dots, \overset{\beta}{\widehat{\ell_{n-k} z_{n-k}}},\\
&\overline{c_{n-k+1}w_{n-k+1}}, \dots, \overline{c_{n-k+p-1}w_{n-k+p-1}}, c_{n-k+p}w_{n-k+p}, \dots, c_n w_n)\,,
\end{aligned}
\tag{4.22}
$$

for some $g \in (\mathbb{Z}_2)^{n-1} = \langle g_2, \dots, g_n\rangle$, where

$$(w'_{n-k+1}, \dots, w'_n) = p((-x_{n-k+1}, \dots, -x_{n-k+l-1}, x_{n-k+l}, \dots, x_n)\,{}^tD)\,.$$

Now we check that there is no $g \in \langle g_2, \dots, g_n\rangle = (\mathbb{Z}_2)^{n-1}$ satisfying (4.22) so that we have a contradiction. To show this, we consider the following cases for such $g$.
*Case 1*. It is easy to see that $g \in \langle g_2, g_3\rangle$ does not satisfy (4.22).
*Case 2*. Let $g = g_t g'$ where $g_t \in \{g_4, \dots, g_{n-k}\}$ $(t = 4, \dots, n-k)$, $g' \in \langle g_{t+1}, \dots, g_n\rangle$.

Since $g_t(z_t) = -z_t$, $g(z_t) = -z_t$. If $\overset{\alpha}{\widehat{z_t}} \neq \overset{\beta}{\widehat{z_t}}$ (resp. $\overset{\alpha}{\widehat{z_t}} = \overset{\beta}{\widehat{z_t}} = z_t$) then $g = g_t g'$ does not satisfy (4.22), because it implies that $\ell_t z_t = g(\overline{\ell_t z_t}) = -\overline{\ell_t z_t}$ (resp. $\ell_t z_t = g(\ell_t z_t) = -\ell_t z_t$). If $\overset{\alpha}{\widehat{z_t}} = \overset{\beta}{\widehat{z_t}} = \bar{z}_t$ then $\ell_t\bar{z}_t = g(\overline{\ell_t z_t}) = -\overline{\ell_t z_t}$. This implies that $\ell_t = \pm\mathbf{i}$. Therefore

$$\tilde{\varphi}(z_2, \dots, z_n) = (\ell_2 z_2, \dots, \ell_{t-1}z_{t-1}, \pm\mathbf{i}z_t, \ell_{t+1}z_{t+1}, \dots, \ell_{n-k}z_{n-k}, p\,{}^t(\mathbf{b} + DX)).$$

Now, from (4.21), we consider

$$
\begin{aligned}
\tilde{\varphi}g_2\tilde{\varphi}^{-1}(z_2, \dots, z_n) &= g_2(z_2, \overline{\ell_3^2}z_3, \dots, \overline{\ell_{t-1}^2}z_{t-1}, -z_t, \overline{\ell_{t+1}^2}z_{t+1}, \dots, \overline{\ell_{n-k}^2}z_{n-k},\\
&\qquad exp(4\pi(-\mathbf{i})b_{n-k+1})z_{n-k+1}, \dots, exp(4\pi(-\mathbf{i})b_n)z_n)\\
&= g_2 h(z_2, \dots, z_n)
\end{aligned}
$$

where

$$
\begin{aligned}
h(z_2, \dots, z_n) = (&z_2, \overline{\ell_3^2}z_3, \overline{\ell_4^2}z_4, \dots, \overline{\ell_{t-1}^2}z_{t-1}, -z_t, \overline{\ell_{t+1}^2}z_{t+1}, \dots, \overline{\ell_{n-k}^2}z_{n-k},\\
&exp(4\pi(-\mathbf{i})b_{n-k+1})z_{n-k+1}, \dots, exp(4\pi(-\mathbf{i})b_n)z_n)\,.
\end{aligned}
\tag{4.23}
$$

We check that $h \notin \langle g_3, \dots, g_n\rangle$ (i.e., $g_2 h \notin (\mathbb{Z}_2)^{n-1}$).

Suppose that $h \in \langle g_3, \dots, g_n\rangle$. Since

$$h(z_{n-k+1}, \dots, z_n) = (exp(4\pi(-\mathbf{i})b_{n-k+1})z_{n-k+1}, \dots, exp(4\pi(-\mathbf{i})b_n)z_n)$$

in (4.23), we may write $h = h'_{(even)}h''$ where $h' \in \langle g_3, \dots, g_{n-k}\rangle$ and $h'' \in \langle g_{n-k+1}, \dots, g_n\rangle$. Here $h'_{(even)}$ means a composition of an even number of generators $\{g_3, \dots, g_{n-k}\}$. On the other hand, since $h(z_t) = -z_t$ in (4.23), we may write $h'_{(even)} = \hat{h}g_t\check{h}$, where $\hat{h}$ (resp. $\check{h}$) is a composition of an even (resp. odd) number of generators $\{g_3, \dots, g_{t-1}\}$ (resp. $\{g_{t+1}, \dots, g_{n-k}\}$). For $t = 4, 5, \dots, n-k-1$, such a $h'_{(even)}$ implies that

$$h(z_{t+1}) = h'_{(even)}(z_{t+1}) = \begin{cases} \bar{z}_{t+1} & \text{if } h'_{(even)} = \hat{h} g_t \ddot{g},\ \ddot{g} \in \langle g_{t+2}, \dots, g_{n-k}\rangle \\ -\bar{z}_{t+1} & \text{if } h'_{(even)} = \hat{h} g_t g_{t+1} \ddot{g}. \end{cases}$$

Hence this contradicts (4.23). Similarly for $h'_{(even)} = g_t \check{h}$ $(t = 4, 5, \dots, n-k-1)$.

Now let us consider $t = n-k$. Since $h(z_t) = -z_t$ in (4.23), $h = \dot{h} g_{n-k} h''$ where $\dot{h}$ is a composition of an even number of generators $\{g_3, \dots, g_{n-k-1}\}$. This implies that $h(z_{n-k+1}, \dots, z_n) = (\pm\bar{z}_{n-k+1}, \dots, \pm\bar{z}_n)$. This also contradicts (4.23). Similarly for $h = g_{n-k}h''$. Thus $g_2 h \notin (\mathbb{Z}_2)^{n-1}$. Hence *Case 2* cannot occur.

*Case 3*. Let $g = g''$ where $g'' \in \langle g_{n-k+1}, \dots, g_n\rangle$.

If $g = g''$ satisfies (4.22), this implies that $(\overset{\alpha}{\widehat{z_4}}, \dots, \overset{\alpha}{\widehat{z_{n-k}}}) = (\overset{\beta}{\widehat{z_4}}, \dots, \overset{\beta}{\widehat{z_{n-k}}})$. Then we obtain that the fixed point set of $(\alpha, T^{n-1})$ is

$$Fix\,\alpha = (V, \{\pm1\}_{n-k+1}, \dots, \{\pm1\}_{n-k+l-1}, z_{n-k+l}, \dots, z_n)$$

with $V = \{(z_2, \dots, z_{n-k}) | \alpha(z_2, \dots, z_{n-k}) = (z_2, \dots, z_{n-k})\}$ and that of $(g''\beta, T^{n-1})$ is

$$Fix\, g''\beta = (W, \{\star\}_{n-k+1}, \dots, \{\star\}_{n-k+p-1}, z_{n-k+p}, \dots, z_n)$$

with

$$W = \{(z_2, \dots, z_{n-k}) | g''\beta(z_2, \dots, z_{n-k}) = \beta(z_2, \dots, z_{n-k}) = (z_2, \dots, z_{n-k})\}$$

and $\star \in \{\pm1, \pm\mathbf{i}\}$. Since $(\overset{\alpha}{\widehat{z_4}}, \dots, \overset{\alpha}{\widehat{z_{n-k}}}) = (\overset{\beta}{\widehat{z_4}}, \dots, \overset{\beta}{\widehat{z_{n-k}}})$, $dim V = dim W$. Then by (4.20), we have $dim V + (k - l + 1) = dim W + (k - p + 1)$. Hence we get a contradiction.

CLAIM 2. Bott matrices derived from each $A_{j'1}$ $(j = 1, \dots, k)$ are not equivalent to each other, (i.e., there are $(2^{n-k-3} - 1)$ nonequivalent Bott matrices derived from each $A_{j'1}$ $(j = 1, \dots, k)$).

Associated with the entries $(\{\hat{1}\}_4, \dots, \{\hat{1}\}_{n-k})$ in each $A_{j'1}$, there are $2^{(n-k-3)} - 1$ different actions $(\mathbb{Z}_2, M(B_1))_{j'}$. (Note that $(\{\hat{1}\}_4, \dots, \{\hat{1}\}_{n-k}) \neq (0, \dots, 0)$.)

We prove that every two different actions $(\mathbb{Z}_2, M(B_1))_{j'}$ derived from $A_{j'1}$ (denoted by $(\alpha, M(B_1))_{j'_\alpha}$ and $(\beta, M(B_1))_{j'_\beta}$, respectively), the corresponding Bott matrices (denoted by $A_{j'_\alpha 1}$ and $A_{j'_\beta 1}$, respectively) are not equivalent.

Since the $\alpha$-action and $\beta$-action are different, we may assume that $\overset{\alpha}{\widehat{z_i}} = z_i$, $\overset{\beta}{\widehat{z_i}} = \bar{z}_i$, for some $i \in \{4, \dots, n-k\}$. As before, if $A_{j'_\alpha 1}$ and $A_{j'_\beta 1}$ are equivalent, by (4.20), we have

$$\begin{aligned} &(\ell_2\bar{z}_2, \ell_3 z_3, \ell_4\overset{\alpha}{\widehat{z_4}}, \dots, , \ell_{i-1}\overset{\alpha}{\widehat{z_{i-1}}}, \ell_i z_i, \ell_{i+1}\overset{\alpha}{\widehat{z_{i+1}}}, \dots, \ell_{n-k}\overset{\alpha}{\widehat{z_{n-k}}}, c_{n-k+1}v'_{n-k+1}, \dots, c_n v'_n) \\ &= g(\overline{\ell_2 z_2}, \ell_3 z_3, \overset{\beta}{\widehat{\ell_4 z_4}}, \dots, \overset{\beta}{\widehat{\ell_{i-1} z_{i-1}}}, \overline{\ell_i z_i}, \overset{\beta}{\widehat{\ell_{i+1} z_{i+1}}}, \dots, \overset{\beta}{\widehat{\ell_{n-k} z_{n-k}}}, \\ &\quad \overline{c_{n-k+1}w_{n-k+1}}, \dots, \overline{c_{n-k+j-1}w_{n-k+j-1}}, c_{n-k+j}w_{n-k+j}, \dots, c_n w_n), \end{aligned} \tag{4.24}$$

for some $g \in (\mathbb{Z}_2)^{n-1} = \langle g_2, \dots, g_n \rangle$, where

$$(v'_{n-k+1}, \dots, v'_n) = p((-x_{n-k+1}, \dots, -x_{n-k+j-1}, x_{n-k+j}, \dots, x_n)\, {}^t D)\,. \tag{4.25}$$

It is easy to check that $g \in \langle g_2, g_3 \rangle$ does not satisfy (4.24), so does $g \in \langle g_i, \dots, g_n \rangle$. So, if $i = 4$ then there is no $g \in (\mathbb{Z}_2)^{n-1}$ satisfying (4.24).

Now let us consider $i \in \{5, \dots, n-k\}$. Since $\overset{\alpha}{\widehat{z_i}} = z_i$ and $\overset{\beta}{\widehat{z_i}} = \bar{z}_i$, for some $i \in \{5, \dots, n-k\}$, we may write $g = g_t \dot{g}^{(even)} \ddot{g}$ with $g_t \in \{g_4, \dots, g_{i-1}\}$ $(t = 4, \dots, i-1)$, $\dot{g} \in \langle g_{t+1}, \dots, g_{i-1} \rangle$, $\ddot{g} \in \langle g_{i+1}, \dots, g_n \rangle$ (for $t = i-1$, $g = g_t \ddot{g}$). Here $\dot{g}^{(even)}$ means a composition of an even number of generators of $\dot{g}$. Since $g_t(z_t) = -z_t$, $g(z_t) = -z_t$. If $\overset{\alpha}{\widehat{z_t}} \neq \overset{\beta}{\widehat{z_t}}$ (resp. $\overset{\alpha}{\widehat{z_t}} = \overset{\beta}{\widehat{z_t}} = z_t$) then $g = g_t \dot{g}^{(even)} \ddot{g}$ does not satisfy (4.24), because it implies that $\ell_t z_t = g(\overline{\ell_t z_t}) = -\overline{\ell_t z_t}$ (resp. $\ell_t z_t = g(\ell_t z_t) = -\ell_t z_t$). If $\overset{\alpha}{\widehat{z_t}} = \overset{\beta}{\widehat{z_t}} = \bar{z}_t$ then $\ell_t \bar{z}_t = g(\overline{\ell_t z_t}) = -\overline{\ell_t z_t}$. This implies that $\ell_t = \pm\mathbf{i}$. Therefore

$$\tilde{\varphi}(z_2, \dots, z_n) = (\ell_2 z_2, \dots, \ell_{t-1} z_{t-1}, \pm\mathbf{i} z_t, \ell_{t+1} z_{t+1}, \dots, \ell_{n-k} z_{n-k}, p\, {}^t(\mathbf{b} + DX))\,.$$

Similarly to the proof of Claim 1, one can check that $\tilde{\varphi} g_2 \tilde{\varphi}^{-1}(z_2, \dots, z_n) \notin (\mathbb{Z}_2)^{n-1}$. Hence we have a contradiction.

Since all combinations of $(\{\hat{1}\}_4, \dots, \{\hat{1}\}_{n-k})$ with $(\{\hat{1}\}_4, \dots, \{\hat{1}\}_{n-k}) \neq (0, \dots, 0)$ are different for each $A_{j'1}$, there are $(2^{n-k-3} - 1)$ nonequivalent Bott matrices derived from each $A_{j'1}$ $(j = 1, \dots, k)$.

CLAIM 3. Bott matrices derived from $A_{(k+1)'1}$ are not equivalent to each other (i.e., there are $2^{n-k-3} - (n-k-2)$ nonequivalent Bott matrices derived from $A_{(k+1)'1}$).

Since there are $2^{n-k-3} - (n-k-2)$ different combination of $(\{\hat{1}\}_4, \dots, \{\hat{1}\}_{n-k})$ with $(\{\hat{1}\}_4, \dots, \{\hat{1}\}_{n-k}) \neq (\overbrace{0, \dots, 0}^{l}, 1, \dots, 1)$ (that is, there are $2^{n-k-3} - (n-k-2)$ different actions $(\mathbb{Z}_2, M(B_1))_{(k+1)'}$), by using the argument in the proof of Claim 2 above, there are $2^{n-k-3} - (n-k-2)$ nonequivalent Bott matrices derived from $A_{(k+1)'1}$.

According to Claims 1, 2 and 3, we obtain that there are $k(2^{n-k-3} - 1) + (2^{n-k-3} - (n-k-2))$ nonequivalent Bott matrices derived from (4.16). □

In view of the argument in the proof of Lemma 4, one can prove the following lemmas.

LEMMA 5. *Let $M(A_{i1}) = S^1 \times_{\mathbb{Z}_2} M(B_1)$ $(i = 1, \dots, n-2)$ be $n$-dimensional real Bott manifolds created from an $(n-1)$-dimensional real Bott manifold $M(B_1)$ where $B_1$ is as in* (4.15), *$k \geq 2$, and $n-k \geq 3$. Such real Bott manifolds $M(A_{i1})$ corresponding to $A_{i1}$ in* (4.26) *are not diffeomorphic to each other.*

$$A_{i1} = \left( \begin{array}{c|ccccccc} 1 & \{1\}_2 & \{1\}_3 & \dots & \{1\}_{1+i} & \{0\}_{2+i} & \dots & \{0\}_n \\ \hline 0 & & & & B_1 & & & \end{array} \right), \tag{4.26}$$

*for $i = 1, \dots, n-2$.*

LEMMA 6. *Each Bott matrix in* (4.26) *is not equivalent to each Bott matrix derived from* (4.16).

REMARK 4. Consider Bott matrices $A_{i1}$ $(i = 1, \ldots, n-2)$ in (4.26) and $A_{j'1}$ $(j = 1, \ldots, k, k+1)$ in (4.16).

(i) Associated with the entries $(\{\hat{1}\}_4, \ldots, \{\hat{1}\}_{n-k})$ in $A_{j'1}$ $(j = 1, \ldots, k)$, if $(\{\hat{1}\}_4, \ldots, \{\hat{1}\}_{n-k}) = (0, \ldots, 0)$ then $A_{1'1} = A_{11}$ and $A_{j'1} \sim A_{(n-j)1}$ $(j = 2, \ldots, k)$ by the equivariant diffeomorphism $\varphi \colon (\mathbb{Z}_2, M(B_1))_{j'} \to (\mathbb{Z}_2, M(B_1))_{n-j}$ defined by

$$[z_2, \ldots, z_{n-k}, z_{n-k+1}, \ldots, z_{n-k+(j-1)}, z_{n-k+j}, \ldots, z_n] \overset{\varphi}{\longmapsto}$$
$$[\mathbf{i}z_2, \ldots, z_{n-k}, z_{n-k+j}, \ldots, z_n, z_{n-k+1}, \ldots, z_{n-k+(j-1)}]\,.$$

(ii) Associated with the entries $(\{\hat{1}\}_4, \ldots, \{\hat{1}\}_{n-k})$ in $A_{(k+1)'1}$, if $(\{\hat{1}\}_4, \ldots, \{\hat{1}\}_{n-k}) = (\overbrace{0, \ldots, 0}^{l}, 1, \ldots, 1)$ $(l = 0, 1, \ldots, n-k-3)$, then $A_{(k+1)'1} \sim A_{(l+2)1}$ $(l = 0, 1, \ldots, n-k-3)$ by the equivariant diffeomorphism $\varphi \colon (\mathbb{Z}_2, M(B_1))_{(k+1)'} \to (\mathbb{Z}_2, M(B_1))_{(l+2)}$ defined by $\varphi([z_2, \ldots, z_n]) = [\mathbf{i}z_2, \ldots, z_n]$.

THEOREM 9. *Let $M(A) = S^1 \times_{\mathbb{Z}_2} M(B)$ be an $n$-dimensional real Bott manifold. Suppose that $B$ is either one of the list in* (4.27)*. Then $M(B)$ are diffeomorphic to each other and the number of diffeomorphism classes of such real Bott manifolds $M(A)$ above is* $(k+1)2^{n-k-3}$ $(k \geq 2, n-k \geq 3)$.

$$B_1 = \begin{pmatrix} 1 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ & & \ddots & & & & \vdots \\ & & & 1 & 1 & \cdots & 1 \\ & 0 & & & & I_k & \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ & & \ddots & & & & \vdots \\ & & & 1 & 1 & \cdots & 1 \\ & 0 & & & & I_k & \end{pmatrix}, \ldots,$$

$$B_{n-k-1} = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ & 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & & \ddots & & & & & \vdots \\ & & & 1 & 1 & 0 & \cdots & 0 \\ & & & & 1 & 1 & \cdots & 1 \\ & 0 & & & & & I_k & \end{pmatrix},$$

$$B_{n-k} = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & & \ddots & & & & & & & \vdots \\ & & & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ & & & & 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & & & 1 & 1 & 0 & \cdots & 0 \\ & & & & & & 1 & 1 & \cdots & 1 \\ & 0 & & & & & & & I_k & \end{pmatrix}, \ldots, \tag{4.27}$$

$$B_{n-k+(n-k-4)} = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ & & \ddots & & \vdots & \vdots & & \vdots \\ & & & 1 & 1 & 0 & \cdots & 0 \\ & & & & 1 & 1 & \cdots & 1 \\ & 0 & & & & & I_k & \end{pmatrix}.$$

PROOF. Note that each $B_j$ is of size $n-1$ and $M(B_j) = S^1 \times_{\mathbb{Z}_2} M(B'_j)$ where $B'_j$ is the submatrix of $B_j$ obtained by deleting the first row and the first column of $B_j$. $M(B_{j-1})$ is diffeomorphic to $M(B_j)$ ($j = 2, 3, \ldots, n-k-1$ ) by the equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B'_{j-1})) \to (\mathbb{Z}_2, M(B'_j))$ which is defined by $\varphi([z_2, \ldots, z_{n-1}]) = [z_2, \ldots, \mathbf{i}z_j, \ldots, z_{n-1}]$. Likewise $M(B_a)$ is diffeomorphic to $M(B_b)$ ($a = n-k+(n-k-3)-(j-1)$; $b = n-k+(n-k-3)-(j-2)$; $j = n-k-1, n-k-2, \ldots, 4, 3$) by the equivariant diffeomorphism $\varphi : (\mathbb{Z}_2, M(B'_a)) \to (\mathbb{Z}_2, M(B'_b))$ which is defined by $\varphi([z_2, \ldots, z_{n-1}]) = [z_2, \ldots, \mathbf{i}z_{j-1}, \ldots, z_{n-1}]$. Therefore $M(B_j)$ are diffeomorphic to each other.

By the hypothesis, there are $2^{n-2}$ possible $\mathbb{Z}_2$-actions on each $M(B_j)$, $j = 1, \ldots, 2(n-k)-4$. We shall prove that among $2^{n-2}(2(n-k)-4)$ real Bott manifolds created from $M(B_j)$ ($j = 1, \ldots, 2(n-k)-4$), there are only $(k+1)2^{n-k-3}$ diffeomorphism classes.

First of all we show that there are $(k+1)2^{n-k-3}$ diffeomorphism classes of real Bott manifolds $M(A_{i1})$ created from $M(B_1)$ by $\mathbb{Z}_2$-actions.

Let $A'$ be a Bott matrix, other than the Bott matrices in (4.26) and (4.16), created from $B_1$. It is easy to check that such a $A'$ is equivalent to one of the Bott matrices in (4.26) or (4.16) by using move I or the equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_1)) \to (\mathbb{Z}_2, M(B_1))$ defined by $\varphi([z_2, \ldots, z_n]) = [\mathbf{i}z_2, \ldots, z_n]$. Then, because of Lemmas 4, 5, 6 and Remark 4, there are $(n-2) + (2^{n-k-3}-1)k + 2^{n-k-3} - (n-k-2) = (k+1)2^{n-k-3}$ distinct diffeomorphism classes of $M(A_{i1})$ ($i = 1, \ldots, 2^{n-2}$) created from $M(B_1)$.

Next, we show that $M(A_{ij})$ ($i = 1, \ldots, 2^{n-2}$) created from $M(B_j)$ ($j = 2, \ldots, n-k+(n-k-4)$) is diffeomorphic to one of the real Bott manifolds corresponding to Bott matrices in (4.26) or (4.16).

For brevity we can consider

$$A_{\ell 1} = \left(\begin{array}{c|ccccc} 1 & 1 & 0 & \{\hat{1}\}_{4_{(1)}} & \cdots & \{\hat{1}\}_{n_{(1)}} \\ \hline 0 & & & B_1 & & \end{array}\right) \tag{4.28}$$

($\ell = 1, \ldots, n-2, 1', \ldots, k', (k+1)'$) representing $A_{j'1}$ ($j = 1, \ldots, k, (k+1)$) in (4.16), and $A_{i1}$ ($i = 1, \ldots, n-2$) in (4.26), where $A_{i1}$ ($i = 2, \ldots, n-2$) is equivalent to (4.28) by the equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_1)) \to (\mathbb{Z}_2, M(B_1))$ which is defined by $\varphi([z_2, \ldots, z_n]) = [\mathbf{i}z_2, \ldots, z_n]$. Note that $\{\hat{1}\}_{l_{(j)}}$ means $\hat{1} (\in \{0, 1\})$ in the $l$-th spot where the corresponding Bott matrix is created from $B_j$.

Now we define an equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_1)) \to (\mathbb{Z}_2, M(B_2))$ by

$\varphi([z_2, \dots, z_n]) = [z_2, \mathbf{i}z_3, z_4, \dots, z_n]$, then (4.28) is equivalent to

$$A_{\ell 2} = \left( \begin{array}{c|ccccc} 1 & 1 & 0 & \{\hat{1}\}_{4_{(2)}} & \dots & \{\hat{1}\}_{n_{(2)}} \\ \hline 0 & & & B_2 & & \end{array} \right). \tag{4.29}$$

Next we define an equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_2)) \to (\mathbb{Z}_2, M(B_3))$ by $\varphi([z_2, \dots, z_n]) = [z_2, z_3, \mathbf{i}z_4, z_5, \dots, z_n]$ so that Bott matrix (4.29) is equivalent to

$$A_{\ell 3} = \left( \begin{array}{c|ccccc} 1 & 1 & 0 & \{\hat{1}\}_{4_{(3)}} & \dots & \{\hat{1}\}_{n_{(3)}} \\ \hline 0 & & & B_3 & & \end{array} \right).$$

In general, let us consider

$$A_{\ell(j-1)} = \left( \begin{array}{c|ccccccccccc} 1 & 1 & 0 & \{\hat{1}\}_{4_{(j-1)}} & \dots & \dots & \dots & \dots & \dots & \dots & \{\hat{1}\}_{n_{(j-1)}} \\ \hline & 1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ & & 1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ & & & \ddots & & & & & & & \vdots \\ & & & & 1 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & & & & & 1 & 1 & \dots & \dots & \dots & 1 \\ & & & & & & 1 & 1 & \dots & \dots & 1 \\ & & 0 & & & & \ddots & & & & \vdots \\ & & & & & & & 1 & 1 & \dots & 1 \\ & & & & & & & & & I_k & \end{array} \right) \begin{array}{c} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{j-1} \\ g_j \\ g_{j+1} \\ \vdots \\ g_{n-k} \\ \\ \end{array}$$

$(j = 2, \dots, n-k-1)$. Defining an equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_{j-1})) \to (\mathbb{Z}_2, M(B_j))$ by $\varphi([z_2, \dots, z_n]) = [z_2, \dots, z_j, \mathbf{i}z_{j+1}, z_{j+2}, \dots, z_n]$,

we obtain that $A_{\ell(j-1)}$ is equivalent to

$$A_{\ell(j)} = \left( \begin{array}{c|ccccccccccc} 1 & 1 & 0 & \{\hat{1}\}_{4_{(j)}} & \dots & \dots & \dots & \dots & \dots & \dots & \{\hat{1}\}_{n_{(j)}} \\ \hline & 1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ & & 1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ & & & \ddots & & & & & & & \vdots \\ & & & & 1 & 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & & & & & 1 & 1 & 0 & \dots & \dots & 0 \\ & & & & & & 1 & 1 & \dots & \dots & 1 \\ & & 0 & & & & \ddots & & & & \vdots \\ & & & & & & & 1 & 1 & \dots & 1 \\ & & & & & & & & & I_k & \end{array} \right) \begin{array}{c} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{j-1} \\ g_j \\ g_{j+1} \\ \vdots \\ g_{n-k} \\ \\ \end{array}.$$

Therefore the previous Bott matrix is equivalent to

$$A_{\ell(n-k-1)} = \left( \begin{array}{c|ccccc} 1 & 1 & 0 & \{\hat{1}\}_{4_{(n-k-1)}} & \dots & \{\hat{1}\}_{n_{(n-k-1)}} \\ \hline 0 & & & B_{n-k-1} & & \end{array} \right). \tag{4.30}$$

Next, (4.30) is equivalent to the following one by the equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_{n-k-1})) \to (\mathbb{Z}_2, M(B_{n-k}))$ defined by $\varphi([z_2, \dots, z_n]) = [z_2, \dots, z_{n-k-2}, \mathbf{i}z_{n-k-1}, z_{n-k}, \dots, z_n]$

$$A_{\ell(n-k)} = \left(\begin{array}{c|ccccc} 1 & 1 & 0 & \{\hat{1}\}_{4_{(n-k)}} & \dots & \{\hat{1}\}_{n_{(n-k)}} \\ \hline 0 & \multicolumn{5}{c}{B_{n-k}} \end{array}\right).$$

In general, let us consider the following Bott matrix

$$A_{\ell a} = \left(\begin{array}{c|ccccccccccc}
1 & 1 & 0 & \{\hat{1}\}_{4_{(a)}} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \{\hat{1}\}_{n_{(a)}} \\ \hline
 & 1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & & 1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & & & \ddots & & & & & & & & \vdots \\
 & & & & 1 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\
 & & & & & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\
0 & & & & & & 1 & \dots & 1 & 0 & \dots & 0 \\
 & & & & & & \ddots & & & & & \vdots \\
 & & & 0 & & & & 1 & 1 & 0 & \dots & 0 \\
 & & & & & & & & 1 & 1 & \dots & 1 \\
 & & & & & & & & & & I_k &
\end{array}\right)
\begin{array}{c} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{j-1} \\ g_j \\ g_{j+1} \\ \vdots \\ g_{n-k-1} \\ g_{n-k} \\ \\ \end{array}$$

where $a = n-k+(n-k-3)-(j-1)$ $(j = n-k-1, n-k-2, \dots, 4, 3)$. Defining an equivariant diffeomorphism $\varphi\colon (\mathbb{Z}_2, M(B_a)) \to (\mathbb{Z}_2, M(B_b))$ by $\varphi([z_2, \dots, z_n]) = [z_2, \dots, z_{j-1}, \mathbf{i}z_j, z_{j+1}, \dots, z_n]$, we obtain that $A_{\ell a}$ is equivalent to

$$A_{\ell b} = \left(\begin{array}{c|ccccccccccc}
1 & 1 & 0 & \{\hat{1}\}_{4_{(b)}} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \{\hat{1}\}_{n_{(b)}} \\ \hline
 & 1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & & 1 & 1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & & & \ddots & & & & & & & & \vdots \\
 & & & & 1 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\
 & & & & & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\
0 & & & & & & 1 & \dots & 1 & 0 & \dots & 0 \\
 & & & & & & \ddots & & & & & \vdots \\
 & & & 0 & & & & 1 & 1 & 0 & \dots & 0 \\
 & & & & & & & & 1 & 1 & \dots & 1 \\
 & & & & & & & & & & I_k &
\end{array}\right)
\begin{array}{c} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{j-2} \\ g_{j-1} \\ g_j \\ \vdots \\ g_{n-k-1} \\ g_{n-k} \\ \\ \end{array}$$

where $b = n-k+(n-k-3)-(j-2)$. Therefore the previous Bott matrix is equivalent to

$$A_{\ell c} = \left(\begin{array}{c|ccccc} 1 & 1 & 0 & \{\hat{1}\}_{4_{(c)}} & \dots & \{\hat{1}\}_{n_{(c)}} \\ \hline 0 & \multicolumn{5}{c}{B_{2(n-k)-4}} \end{array}\right)$$

where $c = n-k+(n-k-3)-1$.

Finally each Bott matrix

$$\left(\begin{array}{c|ccccc} 1 & 1 & 1 & \{\hat{1}\}_{4_{(j)}} & \dots & \{\hat{1}\}_{n_{(j)}} \\ \hline 0 & \multicolumn{5}{c}{B_j} \end{array}\right)$$

for $j = 2, \dots, 2(n-k)-4$ is equivalent to one of Bott matrices in (4.26) or (4.16) by the equivariant diffeomorphism $\varphi([z_2, \dots, z_n]) = [\mathbf{i}z_2, z_3, \dots, z_n]$. This completes the proof of the theorem. $\square$

ACKNOWLEDGMENT. I would like to express my deepest gratitude to Professor Yoshinobu Kamishima who guided me during the preparation of this paper. I also would like to thank Professor Mikiya Masuda for his useful suggestions, the referee for giving me some valuable comments, and Omolola Odebiyi for useful comments on the paper.

## References

[ 1 ] M. GROSSBERG and Y. KARSHON, Bott towers, complete integrability, and the extended character of representations, Duke Math. J. **76** (1994), 23–58.

[ 2 ] Y. KAMISHIMA and M. MASUDA, Cohomological rigidity of real Bott manifolds, Algebr. & Geom. Topol. **9** (2009), 2479–2502.

[ 3 ] Y. KAMISHIMA and A. NAZRA, Seifert fibred structure and rigidity on real Bott towers, Contemp. Math. **501** (2009), 103–122.

[ 4 ] M. MASUDA, Classification of real Bott manifolds, arXiv: 0809.2178v2.

[ 5 ] A. NAZRA, Real Bott Tower, Tokyo Metropolitan University, Master Thesis, 2008.

[ 6 ] A. NAZRA, Geometry of Transformation Groups and Related Topics, Diffeomorphism Type of Real Bott Towers, RIMS Kokyuroku 1612, Kyoto University, September 2008, 165–176.

[ 7 ] J. A. WOLF, J. A. *Spaces of Constant Curvature*, McGraw-Hill Book Company, 1967.

*Present Address*:
DEPARTMENT OF MATHEMATICS,
ANDALAS UNIVERSITY,
KAMPUS UNAND LIMAU MANIS PADANG 25163, INDONESIA.
*e-mail*: admi@fmipa.unand.ac.id