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On (FC)-sequences and Mixed Multiplicities of Multi-graded Algebras

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Abstract. Let $S = \bigoplus_{n_1,\dots,n_s \geq 0} S_{(n_1,\dots,n_s)}$ be a finitely generated standard multi-graded algebra over a Noetherian local ring A. This paper investigates the positivity of mixed multiplicities of S and characterizes them in terms of Hilbert-Samuel multiplicities. As an application, we get some results on the mixed multiplicities of ideals that covers the main results in [Vi] and [TV].

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring of Krull dimension $d = \dim A > 0$ with maximal ideal $\mathfrak{m}$ and infinite residue $k = A/\mathfrak{m}$. Let

$$S = \bigoplus_{n_1,\dots,n_s \geq 0} S_{(n_1,\dots,n_s)}$$

$(s > 0)$ be a finitely generated standard s-graded algebra over A. Let J be an $\mathfrak{m}$-primary ideal of A. Set

$$D_J(S) = \bigoplus_{n \geq 0} \frac{J^n S_{(n,\dots,n)}}{J^{n+1} S_{(n,\dots,n)}}$$

and $\ell = \dim D_{\mathfrak{m}}(S)$. Then

$$\ell_A\left(\frac{J^{n_0} S_{(n_1,\dots,n_s)}}{J^{n_0+1} S_{(n_1,\dots,n_s)}}\right)$$

is a polynomial of total degree $\ell - 1$ in $n_0, n_1, \dots, n_s$ for all large $n_0, n_1, \dots, n_s$ (see Section 3). The terms of total degree $\ell - 1$ in this polynomial have the form

$$\sum_{k_0+k_1+\cdots+k_s=\ell-1} e(J, k_0, k_1, \dots, k_s, S) \frac{n_0^{k_0} n_1^{k_1} \cdots n_s^{k_s}}{k_0! k_1! \cdots k_s!}\,.$$

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Then $e(J, k_0, k_1, \ldots, k_s, S)$ are non-negative integers not all zero [HHRT] and called the *mixed multiplicity of S of type $(k_0, k_1, \ldots, k_s)$ with respect to J*.

In particular, when $S = A[I_1t_1, \ldots, I_st_s]$ is a multi-graded Rees algebra of ideals $I_1, \ldots, I_s$ in A, $e(J, k_0, k_1, \ldots, k_s, S)$ is the mixed multiplicity of ideals $J, I_1, \ldots, I_s$ (see [HHRT]).

Mixed multiplicities of $\mathfrak{m}$-primary ideals were introduced by Teissier and Risler in 1973 [Te] and by Rees in 1984 [Re]. In general, mixed multiplicities have been mentioned in the works of Verma [Ve], Katz and Verma [KV], Swanson [Sw], Trung [Tr], R. Callejas-Bedregal and V. H. Jorge Prez in 2007 [CJ]. Moreover, the positivity of mixed multiplicities of multi-graded modules over Artinian local rings was investigated by Kleiman and Thorup [KT1, KT2] in the geometric context. By using the concept of (FC)-sequences, Viet in 2000 expressed mixed multiplicities of arbitrary ideals in terms of Hilbert-Samuel multiplicities [Vi]. Trung and Verma in 2007 characterized mixed multiplicities of ideals via superficial sequences [TV]. Some another authors have extended mixed multiplicities of ideals to modules, e.g. Kirby and Rees in [KR1, KR2], Manh and Viet in [MV].

In this paper, we consider mixed multiplicities of multi-graded algebra S over Noetherian local ring. Our aim is to answer to question when mixed multiplicities of S are positive and to characterize these mixed multiplicities in terms of Hilbert-Samuel multiplicities (Theorem 3.3, Sect. 3). As an application, we get a version of Theorem 3.3 for mixed multiplicities of arbitrary ideals in local rings (Theorem 4.3, Sect. 4) that covers the main results in [Vi] and [TV].

The paper is divided in four sections. In Section 2, we investigate (FC)-sequences of multi-graded algebras. Section 3 gives some results on expressing mixed multiplicities of multi-graded algebras in terms of Hilbert-Samuel multiplicity. Section 4 devoted to the discussion of mixed multiplicities of arbitrary ideals in local rings.

2. Weak-(FC)-sequences of multi-graded algebras

The author in [Vi] built (FC)-sequences of ideals in local rings for calculating mixed multiplicities of ideals. In order to study mixed multiplicities of multi-graded algebras, this section introduces weak-(FC)-sequences in multi-graded algebras and gives some important properties of these sequences.

Set $\mathfrak{a} : \mathfrak{b}^\infty = \bigcup_{n=0}^\infty (\mathfrak{a} : \mathfrak{b}^n)$, and

$$(M : N)_A = \{a \in A \mid aN \subset M\};$$

$$S_+ = \bigoplus_{n_1+\cdots+n_s>0} S_{(n_1,\ldots,n_s)};$$

$$S_i = S_{(0,\ldots,\underset{(i)}{1},\ldots,0)};$$

$$S_i^+ = S_iS = \bigoplus_{n_i>0} S_{(n_1,\ldots,n_s)} (i = 1, 2, \ldots, s);$$

$$S_{++} = S_1^+ \cap \cdots \cap S_s^+ = \bigoplus_{n_1,\ldots,n_s>0} S_{(n_1,\ldots,n_s)} = S_{(1,\ldots,1)}S\,.$$

DEFINITION 2.1. Let $S = \bigoplus_{n_1,\ldots,n_s\geq 0} S_{(n_1,\ldots,n_s)}$ be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent, and let I be an ideal of A. A homogeneous element $x \in S$ is called a weak-(FC)-element of S with respect to I if there exists $i \in \{1, 2, \ldots, s\}$ such that $x \in S_i$ and

(FC$_1$): $xS_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0}S_{(n_1,\ldots,n_s)} = xI^{n_0}S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$ for all large $n_0, n_1, \ldots, n_s$.

(FC$_2$): x is a filter-regular element with respect to S_{++}, i.e., $0 : x \subseteq 0 : S_{++}^{\infty}$.

Let $x_1, \ldots, x_t$ be a sequence in S. We call that $x_1, \ldots, x_t$ is a weak-(FC)-sequence of S with respect to I if $\bar{x}_{i+1}$ is a weak-(FC)-element of $S/(x_1, \ldots, x_i)S$ with respect to I for all $i = 0, 1, \ldots, t-1$, where $\bar{x}_{i+1}$ is the image of x_{i+1} in $S/(x_1, \ldots, x_i)S$.

EXAMPLE 2.2. Let $R = A[X_1, X_2, \ldots, X_t]$ be the ring of polynomial in t indeterminates $X_1, X_2, \ldots, X_t$ with coefficients in A ($\dim A = d > 0$). Then

$$R = \bigoplus_{m\geq 0} R_m$$

is a finitely generated standard graded algebra over A, where R_m is the set of all homogeneous polynomials of degree m and the zero polynomial. It is well-known that $X_1, X_2, \ldots, X_t$ is a regular sequence of R. Let I be an ideal of A. It is easy to see that $X_1R_{m-1} \cap IR_m$ and IX_1R_{m-1} are both the set of all homogeneous polynomials of degree m with coefficients in I and divided by X_1. Hence

$$X_1R_{m-1} \cap IR_m = IX_1R_{m-1}$$

for any ideal I of A. Using the results just obtained and the fact that

$$R/(X_1, \ldots, X_i)R = A[X_{i+1}, \ldots, X_t]$$

for all $i < t$, we immediately show that $X_1, X_2, \ldots, X_t$ be a weak-(FC)-sequence of R with respect to I for any ideal I of A.

Now, we give some comments on weak-(FC)-sequences of a finitely generated standard multi-graded algebra over A by the following remark.

REMARK 2.3.

(i) By Artin-Rees lemma, there exist integers $u_1, u_2, \ldots, u_s$ such that

$$\begin{aligned}(0 : S_{++}^{\infty}) \cap S_{(n_1,\ldots,n_s)} &= S_{(n_1-u_1,\ldots,n_s-u_s)}((0 : S_{++}^{\infty}) \cap S_{(u_1,\ldots,u_s)})\\ &\subseteq S_{(n_1-u_1,\ldots,n_s-u_s)}(0 : S_{++}^{\infty})\end{aligned}$$

for all $n_1 \geq u_1, \ldots, n_s \geq u_s$. Since $S_{(n_1-u_1,\ldots,n_s-u_s)}(0 : S_{++}^{\infty}) = 0$ for all large enough $n_1, \ldots, n_s$, it follows that $(0 : S_{++}^{\infty})_{(n_1,\ldots,n_s)} = (0 : S_{++}^{\infty}) \bigcap S_{(n_1,\ldots,n_s)} = 0$ for all large enough $n_1, \ldots, n_s$.

(ii) Let $x \in S$ be a homogeneous element. If x is a filter-regular element with respect to S_{++} then $0 : x \subseteq 0 : S_{++}^{\infty}$. By (i),

$$(0 : x)_{(n_1,\ldots,n_s)} \subseteq (0 : S_{++}^{\infty})_{(n_1,\ldots,n_s)} = 0$$

for all large $n_1, \ldots, n_s$. Conversely, suppose that $(0 : x)_{(n_1,\ldots,n_s)} = 0$ for all large $n_1, \ldots, n_s$. Then we have $S_{(n,\ldots,n)}(0 : x)_{(v_1,\ldots,v_s)} \subseteq (0 : x)_{(n+v_1,\ldots,n+v_s)} = 0$ for all large n and all $v_1, \ldots, v_s$. It implies that

$$(0 : x)_{(v_1,\ldots,v_s)} \subseteq (0 : S_{++}^{n}) \subseteq (0 : S_{++}^{\infty})$$

for all large n and all $v_1, \ldots, v_s$. Hence $(0 : x) \subseteq (0 : S_{++}^{\infty})$. Therefore x is a filter-regular element with respect to S_{++} if and only if $(0 : x)_{(n_1,\ldots,n_s)} = 0$ for all large $n_1, \ldots, n_s$.

(iii) Suppose that $x \in S_i$ is a filter-regular element with respect to S_{++}. Consider

$$\lambda_x : S_{(n_1,\ldots,n_i,\ldots,n_s)} \longrightarrow xS_{(n_1,\ldots,n_i,\ldots,n_s)},\ y \mapsto xy.$$

It is clear that λ_x is surjective and $\ker \lambda_x = (0 : x) \cap S_{(n_1,\ldots,n_s)} = 0$ for all large $n_1, \ldots, n_s$. Therefore, $S_{(n_1,\ldots,n_i,\ldots,n_s)} \cong xS_{(n_1,\ldots,n_i,\ldots,n_s)}$. This follows that

$$IS_{(n_1,\ldots,n_i,\ldots,n_s)} \cong xIS_{(n_1,\ldots,n_i,\ldots,n_s)}$$

for all large $n_1, \ldots, n_s$ and for any ideal I of A.

(iv) If S_{++} is non-nilpotent then $S_{(n,\ldots,n)} \neq 0$ for all n. Hence, by Nakayama's lemma, $(D_{\mathfrak{m}}(S))_n = \frac{\mathfrak{m}^n S_{(n,\ldots,n)}}{\mathfrak{m}^{n+1} S_{(n,\ldots,n)}} \neq 0$ for all n. It implies that $\dim D_{\mathfrak{m}}(S) \geq 1$.

The following lemma will play a crucial role for showing the existence of weak-(FC)-sequences.

LEMMA 2.4 (Generalized Rees's lemma). *Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue $k = A/\mathfrak{m}$. Let $S = \bigoplus_{n_1,\ldots,n_s \geq 0} S_{(n_1,\ldots,n_s)}$ be a finitely generated standard s-graded algebra over A, and let I be an ideal of A. Let Σ be a finite collection of prime ideals of S not containing $S_{(1,\ldots,1)}$. Then for each $i = 1, \ldots, s$, there exists an element $x_i \in S_i \backslash \mathfrak{m}S_i$, x_i not contained in any prime ideal in Σ, and a positive integer k_i such that*

$$x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)} = x_i I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$$

for all $n_i > k_i$ and all non-negative integers $n_0, n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_s$.

PROOF. In the ring $S[t, t^{-1}]$ (t is an indeterminate), set

$$S^* = \bigoplus_{n_0 \in \mathbb{Z}} I^{n_0} S t^{n_0} = \bigoplus_{n_0 \in \mathbb{Z}; n_1,\ldots,n_s \geq 0} I^{n_0} S_{(n_1,\ldots,n_s)} t^{n_0}$$

where $I^n = A$ for $n \leq 0$. Then S^* is a Noetherian $(s+1)$-graded ring. Since $u = t^{-1}$ is non-zero-divisor in S^*, the set of prime associated with $u^n S^*$ is independent on $n > 0$ and so is finite by the corollary of [Lemma 2.7, Re]. We divide this set into two subsets: $\mathfrak{S}_1$ consisting of those containing S_i and $\mathfrak{S}_2$ those that do not (where $S_i = S_{(0,\ldots,\underset{(i)}{1},\ldots,0)} = S^*_{(0,0,\ldots,\underset{(i+1)}{1},\ldots,0)}$). Since $S_i/\mathfrak{m}S_i$ is a vector space over the infinite field k and the sets Σ, $\mathfrak{S}_2$ are both finite, we can choose $x_i \in S_i \backslash \mathfrak{m}S_i$ such that x_i is not contained in any prime ideal belonging to $\Sigma \cup \mathfrak{S}_2$. Set

$$M_n = \frac{(u^n S^* : x_i) \cap S^*}{u^n S^*}.$$

Then M_n is a S^*-module for any $n > 0$. We need must show that there exists a sufficiently large integer $N > 0$ such that $S_i^N M_n = 0$. Note that if $P \in \mathrm{Ass}_{S^*} M_n$ then $P \in \mathrm{Ass}_{S^*} S^*/u^n S^* = \mathfrak{S}_1 \cup \mathfrak{S}_2$, and there exists $z \in u^n S^* : x_i$ such that $P = u^n S^* : z$. Since $x_i z \in u^n S^*$, $x_i \in P$. So $P \in \mathfrak{S}_1$. Hence $S_i \subset P$. It follows that $S_i \subset \bigcap_{P \in \mathrm{Ass}_{S^*} M_n} P$. Therefore $S_i \subset \sqrt{\mathrm{Ann}_{S^*} M_n}$. Since S_i is finitely generated, there exists a sufficiently large integer $N > 0$ (how large depending on n) such that $S_i^N M_n = 0$. Hence $[M_n]_{(n_0, n_1, \ldots, n_s)} = 0$ for all $n_i > N$. This means that for each $n > 0$, we have

$$(u^n I^{n_0} S_{(n_1,\ldots,n_s)} t^{n_0} : x_i) \bigcap S^* = u^n I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)} t^{n_0} \tag{1}$$

for all large n_i and all non-negative integers $n_0, n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_s$.
Denote by $\mathfrak{b}$ an ideal of S^* consisting of all finite sums $\sum c_{n_0} t^{n_0}$ with

$$c_{n_0} \in x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)}.$$

Then $\mathfrak{b}$ has a finite generating set $U = \{x_i b_i t^{n_0}\}_{1 \leq i \leq v}$ with $b_i \in S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$. Note that if $0 \neq a \in I^m S$ and $m \geq n_0$ then $at^{n_0} \in S^*$, and if $n_0 < 0$ then $at^{n_0} \in S^*$ for all $a \in S$. Since U is finite, there exists an integer q such that $u^q b_i t^{n_0} = b_i t^{n_0 - q} \in S^*$ for all $1 \leq i \leq v$. Therefore $\mathfrak{b} \subseteq x_i S^* : u^q$.
Now, suppose that $z \in x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)}$. This means $zt^{n_0} \in \mathfrak{b}$. Since $\mathfrak{b} \subseteq x_i S^* : u^q$, $u^q z t^{n_0} = x_i w$ with $w \in S^*$. Note that $z \in I^{n_0} S_{(n_1,\ldots,n_s)}$, it follows that $x_i w = u^q z t^{n_0} \in u^q I^{n_0} S_{(n_1,\ldots,n_s)} t^{n_0}$. Hence by (1), we can find k_i such that

$$w \in (u^q I^{n_0} S_{(n_1,\ldots,n_s)} t^{n_0} : x_i) \cap S^* = u^q I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)} t^{n_0}$$

for all $n_i > k_i$. Thus $u^q z t^{n_0} = x_i w \in x_i u^q I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)} t^{n_0}$. Since u and t are non-zero-divisors in S^*, $z \in x_i I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$. Hence if $n_i > k_i$ then

$$x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)} \subset x_i I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}.$$

Consequently, $x_i S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)} = x_i I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}$. ■

The following proposition will show the existence of weak-(FC)-sequences.

PROPOSITION 2.5. *Suppose that S_{++} is non-nilpotent. Then for any $1 \leq i \leq s$, there exists a weak-(FC)-element $x \in S_i$ of S with respect to I.*

PROOF. Since S_{++} is non-nilpotent, $S/0 : S_{++}^{\infty} \neq 0$. Set

$$\Sigma = \mathrm{Ass}_S(S/0 : S_{++}^{\infty}) = \{P \in \mathrm{Ass} S \mid P \not\supseteq S_{(1,\ldots,1)}\}.$$

Then Σ is finite. By Lemma 2.4, for each $i = 1, \ldots, s$, there exists $x \in S_i \setminus \mathfrak{m} S_i$ such that $x \notin P$ for all $P \in \Sigma$ and

$$x S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap I^{n_0} S_{(n_1,\ldots,n_s)} = x I^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}.$$

Thus x satisfies the condition (FC_1). Since $x \notin P$ for all $P \in \Sigma$, $0 : x \subset 0 : S_{++}^{\infty}$. Hence x satisfies the condition (FC_2). ∎

3. Mixed multiplicities of multi-graded algebras

This section first determines mixed multiplicities of multi-graded algebras, next answers to the question when these mixed multiplicities are positive, and characterizes them in terms of Hilbert-Samuel multiplicities.

Let $S = \bigoplus_{n_1,\ldots,n_s \geq 0} S_{(n_1,\ldots,n_s)}$ be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent and an $\mathfrak{m}$-primary ideal J of A. Since

$$\bigoplus_{n_0,n_1,\ldots,n_s \geq 0} \frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}$$

is a finitely generated standard s-graded algebra over Artinian local ring A/J, by [HHRT, Theorem 4.1],

$$\ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right)$$

is a polynomial for all large $n_0, n_1, \ldots, n_s$. Denote by $P(n_0, n_1, \ldots, n_s)$ this polynomial. Set

$$D_J(S) = \bigoplus_{n \geq 0} \frac{J^n S_{(n,\ldots,n)}}{J^{n+1} S_{(n,\ldots,n)}}$$

and $\ell = \dim D_{\mathfrak{m}}(S)$. By Remark 2.3(iv), $\ell \geq 1$. Note that $\dim D_J(S) = \dim D_{\mathfrak{m}}(S)$ for any $\mathfrak{m}$-primary ideal J of A and $\deg P(n_0, n_1, \ldots, n_s) = \deg P(n, n, \ldots, n)$, and

$$P(n, n, \ldots, n) = \ell_A\left(\frac{J^n S_{(n,\ldots,n)}}{J^{n+1} S_{(n,\ldots,n)}}\right) = \ell_A(D_J(S)_n)$$

for all large n, it follows that $\deg P(n, n, \ldots, n) = \dim D_J(S) - 1 = \ell - 1$. Hence $\deg P(n_0, n_1, \ldots, n_s) = \ell - 1$.

If the terms of total degree $\ell - 1$ of $P(n_0, n_1, \ldots, n_s)$ have the form

$$\sum_{k_0+k_1+\cdots+k_s=\ell-1} e(J, k_0, k_1, \ldots, k_s, S) \frac{n_0^{k_0} n_1^{k_1} \cdots n_s^{k_s}}{k_0! k_1! \cdots k_s!},$$

then $e(J, k_0, k_1, \dots, k_s, S)$ are non-negative integers not all zero [HHRT] and called the *mixed multiplicity of S of type $(k_0, k_1, \dots, k_s)$ with respect to J*.

From now on, the notation $e_A(J, M)$ will mean the Hilbert-Samuel multiplicity of A-module M with respect to an $\mathfrak{m}$-primary ideal J of A. We shall begin this section with the following lemma.

LEMMA 3.1. *Let S be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent and an $\mathfrak{m}$-primary ideal J of A. Set $\ell = \dim D_{\mathfrak{m}}(S)$. Then $e(J, k_0, 0, \dots, 0, S) \neq 0$ if and only if*

$$\dim A/(0 : S^{\infty}_{(1,\dots,1)})_A = \ell.$$

In this case, $e(J, k_0, 0, \dots, 0, S) = e_A(J, S_{(n,\dots,n)})$ for all large n.

PROOF. Denote by $P(n_0, n_1, \dots, n_s)$ the polynomial of

$$\ell_A\left(\frac{J^{n_0} S_{(n_1,\dots,n_s)}}{J^{n_0+1} S_{(n_1,\dots,n_s)}}\right).$$

Then P is a polynomial of degree $\ell - 1$. By taking $n_1 = n_2 = \cdots = n_s = u$, where u is a sufficiently large integer, we get

$$e(J, k_0, 0, \dots, 0, S) = \lim_{n_0 \to \infty} \frac{(\ell - 1)! P(n_0, u, \dots, u)}{n_0^{\ell-1}}.$$

Since $P(n_0, u, \dots, u) = \ell_A\left(\frac{J^{n_0} S_{(u,\dots,u)}}{J^{n_0+1} S_{(u,\dots,u)}}\right)$, it follows that

$$\deg P(n_0, u, \dots, u) = \dim_A S_{(u,\dots,u)} - 1$$

and $e(J, k_0, 0, \dots, 0, S) \neq 0$ if and only if

$$\deg P(n_0, u, \dots, u) = \dim_A S_{(u,\dots,u)} - 1 = \ell - 1.$$

Since A is Noetherian, $(0 : S^{\infty}_{(1,\dots,1)})_A = (0 : S^{n}_{(1,\dots,1)})_A = (0 : S_{(n,\dots,n)})_A$ for all large n. Hence if u is chosen sufficiently large, we have

$$\dim_A S_{(u,\dots,u)} = \dim A/(0 : S_{(u,\dots,u)})_A = \dim A/(0 : S^{\infty}_{(1,\dots,1)})_A.$$

Therefore $e(J, k_0, 0, \dots, 0, S) \neq 0$ if and only if $\dim A/(0 : S^{\infty}_{(1,\dots,1)})_A = \ell$. Finally, if $\dim A/(0 : S^{\infty}_{(1,\dots,1)})_A = \ell$ then $\dim_A S_{(n,\dots,n)} - 1 = \ell - 1$ for all large n, and hence

$$e_A(J, S_{(n,\dots,n)}) = \lim_{n_0 \to \infty} \frac{(\ell - 1)! \ell_A\left(\frac{J^{n_0} S_{(n,\dots,n)}}{J^{n_0+1} S_{(n,\dots,n)}}\right)}{n_0^{\ell-1}}$$

$$= \lim_{n_0 \to \infty} \frac{(\ell - 1)! P(n_0, n, \dots, n)}{n_0^{\ell-1}} = e(J, k_0, 0, \dots, 0, S)$$

for all large integer n. ∎

PROPOSITION 3.2. *Let S be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent and an $\mathfrak{m}$-primary ideal J of A. Set $\ell = \dim D_{\mathfrak{m}}(S)$. Assume that $e(J, k_0, k_1, \ldots, k_s, S) \neq 0$, where $k_0, k_1, \ldots, k_s$ are non-negative integers such that $k_0 + k_1 + \cdots + k_s = \ell - 1$. Then*

(i) *If $k_i > 0$ and $x \in S_i$ is a weak-(FC)-element of S with respect to J then*

$$e(J, k_0, k_1, \ldots, k_s, S) = e(J, k_0, k_1, \ldots, k_i - 1, \ldots, k_s, S/xS),$$

and $\dim D_{\mathfrak{m}}(S/xS) = \ell - 1$.

(ii) *There exists a weak-(FC)-sequence of $t = k_1 + \cdots + k_s$ elements of S in $\bigcup_{i=1}^{s} S_i$ with respect to J consisting of k_1 elements of $S_1, \ldots, k_s$ elements of S_s.*

PROOF. The proof of (i): Denote by $P(n_0, n_1, \ldots, n_s)$ the polynomial of

$$\ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right).$$

Then $\deg P = \ell - 1$. Since x satisfies the condition (FC_1), for all large $n_0, n_1, \ldots, n_s$, we have

$$\ell_A\left(\frac{J^{n_0}(S/xS)_{(n_1,\ldots,n_s)}}{J^{n_0+1}(S/xS)_{(n_1,\ldots,n_s)}}\right) = \ell_A\left(\frac{J^{n_0}(S_{(n_1,\ldots,n_s)}\big/xS_{(n_1,\ldots,n_i-1,\ldots,n_s)})}{J^{n_0+1}(S_{(n_1,\ldots,n_s)}\big/xS_{(n_1,\ldots,n_i-1,\ldots,n_s)})}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)} + xS_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)} + xS_{(n_1,\ldots,n_i-1,\ldots,n_s)}}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{(J^{n_0+1} S_{(n_1,\ldots,n_s)} + xS_{(n_1,\ldots,n_i-1,\ldots,n_s)}) \cap J^{n_0} S_{(n_1,\ldots,n_s)}}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)} + xS_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap J^{n_0} S_{(n_1,\ldots,n_s)}}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)} + xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right) - \ell_A\left(\frac{J^{n_0+1} S_{(n_1,\ldots,n_s)} + xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right) - \ell_A\left(\frac{xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right) - \ell_A\left(\frac{xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{xS_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap J^{n_0} S_{(n_1,\ldots,n_s)} \cap J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right) - \ell_A\left(\frac{xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{xS_{(n_1,\ldots,n_i-1,\ldots,n_s)} \cap J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right)$$

$$= \ell_A\left(\frac{J^{n_0} S_{(n_1,\ldots,n_s)}}{J^{n_0+1} S_{(n_1,\ldots,n_s)}}\right) - \ell_A\left(\frac{xJ^{n_0} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}{xJ^{n_0+1} S_{(n_1,\ldots,n_i-1,\ldots,n_s)}}\right).$$

Since x is a filter-regular element of S,

$$J^{n_0}S_{(n_1,\dots,n_i,\dots,n_s)} \cong xJ^{n_0}S_{(n_1,\dots,n_i,\dots,n_s)}$$

for all n_0 and all large $n_1,\dots,n_s$ by Remark 2.3(iii). Then we have an isomorphism of A-modules

$$\frac{xJ^{n_0}S_{(n_1,\dots,n_i-1,\dots,n_s)}}{xJ^{n_0+1}S_{(n_1,\dots,n_i-1,\dots,n_s)}} \cong \frac{J^{n_0}S_{(n_1,\dots,n_i-1,\dots,n_s)}}{J^{n_0+1}S_{(n_1,\dots,n_i-1,\dots,n_s)}}$$

for all large $n_0, n_1,\dots,n_s$. From this it follows that

$$\ell_A\left(\frac{xJ^{n_0}S_{(n_1,\dots,n_i-1,\dots,n_s)}}{xJ^{n_0+1}S_{(n_1,\dots,n_i-1,\dots,n_s)}}\right) = \ell_A\left(\frac{J^{n_0}S_{(n_1,\dots,n_i-1,\dots,n_s)}}{J^{n_0+1}S_{(n_1,\dots,n_i-1,\dots,n_s)}}\right).$$

Hence

$$\ell_A\left(\frac{J^{n_0}(S/xS)_{(n_1,\dots,n_s)}}{J^{n_0+1}(S/xS)_{(n_1,\dots,n_s)}}\right) = \ell_A\left(\frac{J^{n_0}S_{(n_1,\dots,n_s)}}{J^{n_0+1}S_{(n_1,\dots,n_s)}}\right) - \ell_A\left(\frac{J^{n_0}S_{(n_1,\dots,n_i-1,\dots,n_s)}}{J^{n_0+1}S_{(n_1,\dots,n_i-1,\dots,n_s)}}\right)$$

for all large $n_0, n_1,\dots,n_s$. Denote by $Q(n_0, n_1,\dots,n_s)$ the polynomial of

$$\ell_A\left(\frac{J^{n_0}(S/xS)_{(n_1,\dots,n_s)}}{J^{n_0+1}(S/xS)_{(n_1,\dots,n_s)}}\right).$$

From the above fact, we get

$$Q(n_0, n_1,\dots,n_s) = P(n_0, n_1,\dots,n_i,\dots,n_s) - P(n_0, n_1,\dots,n_i-1,\dots,n_s).$$

Since $e(J, k_0, k_1,\dots,k_s, S) \neq 0$ and $k_i > 0$, it implies that $\deg Q = \deg P - 1$ and

$$e(J, k_0, k_1,\dots,k_i,\dots,k_s, S) = e(J, k_0, k_1,\dots,k_i-1,\dots,k_s, S/xS).$$

Note that $\deg Q = \dim D_{\mathfrak{m}}(S/xS) - 1$. Hence

$$\dim D_{\mathfrak{m}}(S/xS) = \deg Q + 1 = \deg P = \ell - 1.$$

The proof of (ii): The proof is by induction on $t = k_1 + \cdots + k_s$. For $t = 0$, the result is trivial. Assume that $t > 0$. Since $k_1 + \cdots + k_s = t > 0$, there exists $k_j > 0$. Since S_{++} is non-nilpotent, by Proposition 2.5, there exists a weak-(FC)-element $x_1 \in S_j$ of S with respect to J. By (i),

$$e(J, k_0, k_1,\dots,k_j-1,\dots,k_s, S/x_1S) = e(J, k_0, k_1,\dots,k_s, S) \neq 0.$$

This follows that

$$\frac{J^{n_0}(S/x_1S)_{(n_1,\dots,n_s)}}{J^{n_0+1}(S/x_1S)_{(n_1,\dots,n_s)}} \neq 0$$

and so $(S/x_1S)_{(n_1,\dots,n_s)} \neq 0$ for all large $n_1,\dots,n_s$. Hence $(S/x_1S)_{++}$ is non-nilpotent. Since $k_1 + \cdots + (k_j - 1) + \cdots + k_s = t - 1$, by the inductive assumption, there exist

$t-1$ elements $x_2, \dots, x_t$ consisting of k_1 elements of $S_1, \dots, k_j - 1$ elements of $S_j, \dots, k_s$ elements of S_s such that $\bar{x}_2, \dots, \bar{x}_t$ is a weak-(FC)-sequence of S/x_1S with respect to J ($\bar{x}_i$ is initial form of x_i in S/x_1S, $i = 2, \dots, t$). Remember that $x_1 \in S_j$ is a weak-(FC)-element of S with respect to J, $x_1, \dots, x_t$ is a weak-(FC)-sequence of S with respect to J consisting of k_1 elements of $S_1, \dots, k_s$ elements of S_s. ■

Now, we will give the criteria for the positivity of mixed multiplicities and characterize them in terms of Hilbert-Samuel multiplicity by the following theorem.

THEOREM 3.3. *Let S be a finitely generated standard s-graded algebra over a Noetherian local ring A such that S_{++} is non-nilpotent. Let J be an $\mathfrak{m}$-primary ideal of A. Set $\ell = \dim D_{\mathfrak{m}}(S)$. Then the following statements hold.*

(i) *$e(J, k_0, k_1, \dots, k_s, S) \neq 0$ if and only if there exists a weak-(FC)-sequence $x_1, \dots, x_t$ ($t = k_1 + \cdots + k_s$) of S with respect to J consisting of k_1 elements of $S_1, \dots, k_s$ elements of S_s and*

$$\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_t)S) = \dim A/((x_1, \dots, x_t)S : S^{\infty}_{(1,\dots,1)})_A = \ell - t.$$

(ii) *Suppose that $e(J, k_0, k_1, \dots, k_s, S) \neq 0$ and $x_1, \dots, x_t$ ($t = k_1 + \cdots + k_s$) is a weak-(FC)-sequence of S with respect to J consisting of k_1 elements of $S_1, \dots, k_s$ elements of S_s. Set $\bar{S} = S/(x_1, \dots, x_t)S$. Then*

$$e(J, k_0, k_1, \dots, k_s, S) = e_A(J, \bar{S}_{(n,\dots,n)})$$

for all large n.

PROOF. The proof of (i): First, we prove the necessary condition. By Proposition 3.2(ii), there exists a weak-(FC)-sequence $x_1, \dots, x_t$ of S with respect to J consisting of k_1 elements of $S_1, \dots, k_s$ elements of S_s. Set $\bar{S} = S/(x_1, \dots, x_t)S$. Applying Proposition 3.2(i) by induction on t, we get $\dim D_{\mathfrak{m}}(\bar{S}) = \ell - t$ and

$$0 \neq e(J, k_0, k_1, \dots, k_s, S) = e(J, k_0, 0, \dots, 0, \bar{S}).$$

Hence by Lemma 3.1, $\dim A/(0 : \bar{S}^{\infty}_{(1,\dots,1)})_A = \ell - t$. Since

$$\dim A/(0 : \bar{S}^{\infty}_{(1,\dots,1)})_A = \dim A/((x_1, \dots, x_t)S : S^{\infty}_{(1,\dots,1)})_A,$$

it follows that

$$\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_t)S) = \dim A/((x_1, \dots, x_t)S : S^{\infty}_{(1,\dots,1)})_A = \ell - t.$$

Now, we prove the sufficiently condition. We assume that $x_1 \in S_i$. Denote by $P(n_0, n_1, \dots, n_s)$ and $Q(n_0, n_1, \dots, n_s)$ the polynomials of

$$\ell_A\left(\frac{J^{n_0} S_{(n_1,\dots,n_s)}}{J^{n_0+1} S_{(n_1,\dots,n_s)}}\right) \quad \text{and} \quad \ell_A\left(\frac{J^{n_0}(S/x_1S)_{(n_1,\dots,n_s)}}{J^{n_0+1}(S/x_1S)_{(n_1,\dots,n_s)}}\right),$$

respectively. Then by the proof of Proposition 3.2(i) we have

$$Q(n_0, n_1, \dots, n_s) = P(n_0, n_1, \dots, n_i, \dots, n_s) - P(n_0, n_1, \dots, n_i - 1, \dots, n_s) .$$

This implies that $\deg Q \leq \deg P - 1$. Recall that $\deg Q = \dim D_{\mathfrak{m}}(S/x_1S) - 1$ and $\deg P = \dim D_{\mathfrak{m}}(S) - 1$. So $\dim D_{\mathfrak{m}}(S/x_1S) \leq \dim D_{\mathfrak{m}}(S) - 1$. Similarly, we have

$$\begin{aligned} \ell - t =& \dim D_{\mathfrak{m}}(S/(x_1, \dots, x_t)S) \leq \dim D_{\mathfrak{m}}(S/(x_1, \dots, x_{t-1})S) - 1 \\ &\leq \cdots \leq \dim D_{\mathfrak{m}}(S/x_1S) - (t-1) \leq \dim D_{\mathfrak{m}}(S) - t = \ell - t . \end{aligned}$$

This fact follows that $\dim D_{\mathfrak{m}}(S/x_1S) = \dim D_{\mathfrak{m}}(S) - 1$. Thus $\deg Q = \deg P - 1$. Hence

$$e(J, k_0, k_1, \dots, k_s, S) = e(J, k_0, k_1, \dots, k_i - 1, \dots, k_s, S/x_1S) .$$

By induction we have $e(J, k_0, k_1, \dots, k_s, S) = e(J, k_0, 0, \dots, 0, \bar{S})$. Since

$$\dim A/(0 : \bar{S}^{\infty}_{(1,\dots,1)})_A = \dim A/((x_1, \dots, x_t)S : S^{\infty}_{(1,\dots,1)})_A = \ell - t = \dim D_{\mathfrak{m}}(\bar{S}) ,$$

$e(J, k_0, 0, \dots, 0, \bar{S}) \neq 0$ by Lemma 3.1. Hence

$$e(J, k_0, k_1, \dots, k_s, S) \neq 0 .$$

The proof of (ii): Applying Proposition 3.2(i), by induction on t, we obtain

$$0 \neq e(J, k_0, k_1, \dots, k_s, S) = e(J, k_0, 0, \dots, 0, \bar{S}) .$$

On the other hand by Lemma 3.1, $e(J, k_0, 0, \dots, 0, \bar{S}) = e_A(J, \bar{S}_{(n,\dots,n)})$ for all large integer n. Hence

$$e(J, k_0, k_1, \dots, k_s, S) = e_A(J, \bar{S}_{(n,\dots,n)})$$

for all large n. ■

REMARK 3.4. From the proof of Theorem 3.3 we get the following comments.

(i) If $x_1, \dots, x_t$ is a weak-(FC)-sequence in $\bigcup_{i=1}^{s} S_i$ of S with respect to J satisfying the condition $\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_t)S) = \dim D_{\mathfrak{m}}(S) - t$, then
$\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_i)S) = \dim D_{\mathfrak{m}}(S) - i$ for all $1 \leq i \leq t$.

(ii) If $k_i > 0$ and $x \in S_i$ is a weak-(FC)-element of S with respect to J such that $\dim D_{\mathfrak{m}}(S/xS) = \dim D_{\mathfrak{m}}(S) - 1$ then

$$e(J, k_0, k_1, \dots, k_i, \dots, k_s, S) = e(J, k_0, k_1, \dots, k_i - 1, \dots, k_s, S/xS) .$$

(iii) If $e(J, k_0, k_1, \dots, k_s, S) \neq 0$ then for every weak-(FC)-sequence $x_1, \dots, x_t$ ($t = k_1 + \cdots + k_s$) of S with respect to J consisting of k_1 elements of $S_1, \dots, k_s$ elements of S_s we always have

$$\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_t)S) = \dim A/((x_1, \dots, x_t)S : S^{\infty}_{(1,\dots,1)})_A = \ell - t .$$

(iv) Suppose that $x_1, \dots, x_t$ is a weak-(FC)-sequence in $\bigcup_{i=1}^{s} S_i$ of S with respect to J. Then $\dim D_{\mathfrak{m}}(S/x_1S) \le \dim D_{\mathfrak{m}}(S) - 1$. And by induction we have

$$\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_t)S) \le \dim D_{\mathfrak{m}}(S) - t = \ell - t\,.$$

From this it follows that $\ell - t \ge 0$ or $t \le \ell$. Hence the length of any weak-(FC)-sequence in $\bigcup_{i=1}^{s} S_i$ of S with respect to J is not greater than ℓ.

EXAMPLE 3.5. Let $R = A[X, Y]$ be a polynomial rings of indeterminates X, Y and $\dim A = d > 2$. Then R is a finitely generated standard 2-graded algebra over A with $\deg X = (1, 0)$, $\deg Y = (0, 1)$ and

$$\dim D_{\mathfrak{m}}(R) = \dim\left[\oplus_{n\ge 0}\frac{\mathfrak{m}^n(XY)^n}{\mathfrak{m}^{n+1}(XY)^n}\right] = \dim\left(\oplus_{n\ge 0}\frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}}\right) = \dim A\,.$$

It can be verified that X, Y is a weak-(FC)-sequence of R with respect to $\mathfrak{m}$. Since $\dim D_{\mathfrak{m}}(R/(X)) = \dim(A/\mathfrak{m}) = 0$ and $d > 2$, $\dim D_{\mathfrak{m}}(R/(X)) < \dim D_{\mathfrak{m}}(R) - 1$.

REMARK 3.6. Example 3.5 showed that for any weak-(FC)-sequence $x_1, \dots, x_t$ of S with respect to J, one can get

$$\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_t)) < \dim D_{\mathfrak{m}}(S) - t\,.$$

In the case that $s = 1$, we get the following result for a graded algebra $S = \bigoplus_{n\ge 0} S_n$.

COROLLARY 3.7. *Let $S = \bigoplus_{n\ge 0} S_n$ be a finitely generated standard graded algebra over A such that $S_+ = \bigoplus_{n>0} S_n$ is non-nilpotent, and let J be an $\mathfrak{m}$-primary ideal of A. Set $D_J(S) = \bigoplus_{n\ge 0} J^n S_n / J^{n+1} S_n$ and $\dim D_{\mathfrak{m}}(S) = \ell$. Suppose that $x_1, \dots, x_q$ is a maximal weak-(FC)-sequence in S_1 of S with respect to J satisfying the condition $\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_q)S) = \ell - q$. Then*

(i) *$e(J, \ell - i - 1, i, S) \ne 0$ if and only if $i \le q$ and $\dim A/((x_1, \dots, x_i)S : S_1^{\infty})_A = \ell - i$.*

(ii) *If $e(J, \ell - i - 1, i, S) \ne 0$ then $e(J, \ell - i - 1, i, S) = e_A(J, S_n/(x_1, \dots, x_i)S_{n-1})$ for all large n.*

PROOF. By Theorem 3.3(ii) we immediately get (ii). Now let us to prove the part (i). The "if" part. Assume that $e(J, \ell - i - 1, i, S) \ne 0$. First, we show that $i \le q$. Assume the contrary that $i > q$. Since $x_1, \dots, x_q$ is a weak-(FC)-sequence in S_1 of S with respect to J, applying Proposition 3.2(i) by induction on q,

$$0 \ne e(J, \ell - i - 1, i, S) = e(J, \ell - i - 1, i - q, \bar{S})\,,$$

where $\bar{S} = S/(x_1, \dots, x_q)S$. Since $e(J, \ell - i - 1, i - q, \bar{S}) \ne 0$ and $i - q > 0$, there exists an element $x \in S_1$ such that $\bar{x}$ (the image of x in $\bar{S}$) is a weak-(FC)-element of $\bar{S}$ with respect to J by Proposition 3.2(ii). By Proposition 3.2(i), $\dim D_{\mathfrak{m}}(\bar{S}/x\bar{S}) = \ell - q - 1$. Hence

$x_1, \dots, x_q, x$ is a weak-(FC)-sequence in S_1 of S with respect to J satisfying the condition

$$\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_q, x)S) = \ell - q - 1 .$$

We thus arrive at a contradiction. Hence $i \leq q$. Since $e(J, \ell - i - 1, i, S) \neq 0$, $\dim A/((x_1, \dots, x_i)S : S_1^{\infty})_A = \ell - i$ by Remark 3.4(iii). We turn to the proof of sufficiency. Suppose that $i \leq q$ and

$$\dim A/((x_1, \dots, x_i)S : S_1^{\infty})_A = \ell - i .$$

Since $\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_q)S) = \ell - q$, $\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_i)S) = \ell - i$ for all $i \leq q$ by Remark 3.4(i). Since $x_1, \dots, x_i$ is a weak-(FC)-sequence of S with respect to J satisfying the condition

$$\dim D_{\mathfrak{m}}(S/(x_1, \dots, x_i)S) = \dim A/((x_1, \dots, x_i)S : S_1^{\infty})_A = \ell - i ,$$

$e(J, \ell - i - 1, i, S) \neq 0$ by Theorem 3.3(i). ■

EXAMPLE 3.8. Let $R = A[X_1, X_2, \dots, X_t]$ be the ring of polynomial in t indeterminates $X_1, X_2, \dots, X_t$ with coefficients in A ($\dim A = d > 0$). Then $R = \bigoplus_{m \geq 0} R_m$ is a finitely generated standard graded algebra over A (see Example 2.2). Let J is an $\mathfrak{m}$-primary ideal of A. By Example 2.2, $X_1, \dots, X_t \in R_1$ is a weak-(FC)-sequence of R with respect to J. Denote by $P(n, m)$ the polynomial of $\ell_A\big(\frac{J^n R_m}{J^{n+1} R_m}\big)$. We have

$$D_{\mathfrak{m}}(R) = \bigoplus_{T \geq 0} \frac{\mathfrak{m}^T R_T}{\mathfrak{m}^{T+1} R_T} = \frac{A[\mathfrak{m}X_1, \dots, \mathfrak{m}X_t]}{\mathfrak{m}A[\mathfrak{m}X_1, \dots, \mathfrak{m}X_t]} .$$

Since $\operatorname{ht}\mathfrak{m} > 0$, $\dim D_{\mathfrak{m}}(R) = \dim A + t - 1 = d + t - 1$. Hence $\deg P(n, m) = d + t - 2$. It is clear that $R/(X_1, \dots, X_i)R = A[X_{i+1}, \dots, X_t]$ for all $i \leq t$. Hence

$$\dim D_{\mathfrak{m}}(R/(X_1, \dots, X_i)R) = \dim D_{\mathfrak{m}}(R) - i$$

for all $i \leq t - 1$. Let us calculate $e(J, k_0, k_1, R)$ with $k_0 + k_1 = d + t - 1$. First, we consider the case $k_1 \geq t$. Since $X_1, \dots, X_{t-1}$ is a weak-(FC)-sequence of R with respect to J and $\dim D_{\mathfrak{m}}(R/(X_1, \dots, X_i)R) = \dim D_{\mathfrak{m}}(R) - i$ for all $i \leq t - 1$, by Remark 3.4(ii),

$$e(J, k_0, k_1, R) = e(J, k_0, k_1 - (t - 1), R/(X_1, \dots, X_{t-1})R) = e(J, k_0, k_1 - t + 1, A[X_t]) .$$

Denote by $Q(m, n)$ the polynomial of $\ell_A\big(\frac{J^n X_t^m A}{J^{n+1} X_t^m A}\big)$. Since X_t is regular element, $J^n X_t^m A \cong J^n A$. Thus, for all large n, m,

$$Q(n, m) = \ell_A\left(\frac{J^n X_t^m A}{J^{n+1} X_t^m A}\right) = \ell_A\left(\frac{J^n A}{J^{n+1} A}\right) .$$

Hence $Q(n, m)$ is independent on m. Note that $e(J, k_0, k_1 - t + 1, A[X_t])$ is the coefficient of $\frac{1}{k_0!(k_1 - t + 1)!} n^{k_0} m^{k_1 - t + 1}$ in $Q(n, m)$. Since $k_1 - t + 1 > 0$, it follows that

$$e(J, k_0, k_1, R) = e(J, k_0, k_1 - t + 1, A[X_t]) = 0 .$$

In the case $k_1 < t$, since $\dim D_{\mathfrak{m}}(R/(X_1, \ldots, X_{k_1})R) = \dim D_{\mathfrak{m}}(R) - k_1$, by Corollary 3.7(i), $e(J, k_0, k_1, R) \neq 0$ if and only if

$$\dim A/((X_1, \ldots, X_{k_1})R : R_1^{\infty})_A = d + t - 1 - k_1 .$$

Since $X_1, \ldots, X_t$ are independent indeterminates,

$$((X_1, \ldots, X_{k_1})R : R_1^{\infty})_A \subset ((X_1, \ldots, X_{k_1})R : ((X_{k_1+1}, \ldots, X_t)A)^{\infty})_A = 0 .$$

Hence $\dim A/((X_1, \ldots, X_{k_1})R : R_1^{\infty})_A = \dim A = d$. Therefore, $e(J, k_0, k_1, R) \neq 0$ if and only if $k_1 = t - 1$. For $k_1 = t - 1$ (then $k_0 = d - 1$), by Corollary 3.7(ii), we have

$$e(J, d-1, t-1, R) = e_A(J, R_u/(X_1, \ldots, X_{t-1})R_{u-1})$$

for all large u. Note that $R_u = (X_1, \ldots, X_t)^u A$ and $R_u/(X_1, \ldots, X_{t-1})R_{u-1} = X_t^u A$. Thus $e(J, R_u/(X_1, \ldots, X_{t-1})R_{u-1}) = e_A(J, X_t^u A)$. Since X_t^u is regular element in $A[X_t]$, $X_t^u A \cong A$. Hence $e(J, d-1, t-1, R) = e_A(J, X_t^u A) = e_A(J, A)$. From the above facts we get

$$e(J, k_0, k_1, R) = \begin{cases} 0 & \text{if } k_1 \neq t-1 \\ e_A(J, A) & \text{if } k_1 = t-1 \end{cases} .$$

Therefore

$$P(n, m) = \frac{e(J, A)}{(d-1)!(t-1)!} n^{d-1} m^{t-1} + \{\text{terms of lower degree}\} .$$

4. Applications

As an application of Theorem 3.3, this section devoted to the discussion of mixed multiplicities of arbitrary ideals in local rings.

Let J be an $\mathfrak{m}$-primary ideal and $I_1, \ldots, I_s$ ideals of A such that $I = I_1 \cdots I_s$ is non-nilpotent. Set $S = A[I_1 t_1, \ldots, I_s t_s]$. Then

$$D_J(S) = \bigoplus_{n \geq 0} \frac{(JI)^n}{J(JI)^n} \quad \text{and}$$

$$\ell_A\left(\frac{J^{n_0} S_{(n_1, \ldots, n_s)}}{J^{n_0+1} S_{(n_1, \ldots, n_s)}}\right) = \ell_A\left(\frac{J^{n_0} I_1^{n_1} \cdots I_s^{n_s}}{J^{n_0+1} I_1^{n_1} \cdots I_s^{n_s}}\right)$$

is a polynomial of total degree $\dim D_J(S) - 1$ for all large $n_0, n_1, \ldots, n_s$. By Proposition 3.1 in [Vi], the degree of this polynomial is $\dim A/0 : I^{\infty} - 1$. Hence $\dim D_J(S) = \dim A/0 : I^{\infty}$. Set $\dim A/0 : I^{\infty} = \ell$. In this case, $e(J, k_0, k_1, \ldots, k_s, S)$ for $k_0 + k_1 + \cdots + k_s = \ell - 1$ is called the mixed multiplicity of ideals $(J, I_1, \ldots, I_s)$ of type $(k_0, k_1, \ldots, k_s)$ and one put

$$e(J, k_0, k_1, \ldots, k_s, S) = e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}, A)$$

(see [Ve2] or [HHRT]). By using the concept of (FC)-sequences of ideals, one expressed mixed multiplicities of arbitrary ideals in terms of Hilbert-Samuel multiplicities [Vi].

DEFINITION 4.1 (see Definition Vi). Let $I_1, \dots, I_s$ be ideals such that $I = I_1 \cdots I_s$ is a non nilpotent ideal. A element $x \in A$ is called an (FC)-element of A with respect to $(I_1, \dots, I_s)$ if there exists $i \in \{1, 2, \dots, s\}$ such that $x \in I_i$ and

(FC_1): $(x) \cap I_1^{n_1} \cdots I_i^{n_i} \cdots I_s^{n_s} = x I_1^{n_1} \cdots I_i^{n_i - 1} \cdots I_s^{n_s}$ for all large $n_1, \dots, n_s$.
(FC_2): x is a filter-regular element with respect to I, i.e., $0 : x \subseteq 0 : I^\infty$.
(FC_3): $\dim A/[(x) : I^\infty] = \dim A/0 : I^\infty - 1$.

We call x a *weak-(FC)-element* with respect to $(I_1, \dots, I_s)$ if x satisfies conditions (FC_1) and (FC_2).

Let $x_1, \dots, x_t$ be a sequence in A. For each $i = 0, 1, \dots, t-1$, set $A_i = A/(x_1, \dots, x_i)S$, $\bar{I}_j = I_j[A/(x_1, \dots, x_i)]$, $\bar{x}_{i+1}$ the image of x_{i+1} in A_i. Then

$x_1, \dots, x_t$ is called a weak-(FC)-sequence of A with respect to $(I_1, \dots, I_s)$ if $\bar{x}_{i+1}$ is a weak-(FC)-element of A_i with respect to $(\bar{I}_1, \dots, \bar{I}_s)$ for all $i = 0, 1, \dots, t-1$.

$x_1, \dots, x_t$ is called an (FC)-sequence of A with respect to $(I_1, \dots, I_s)$ if $\bar{x}_{i+1}$ is an (FC)-element of A_i with respect to $(\bar{I}_1, \dots, \bar{I}_s)$ for all $i = 0, 1, \dots, t-1$.

A weak-(FC)-sequence $x_1, \dots, x_t$ is called a *maximal* weak-(FC)-sequence if IA_{t-1} is a non-nilpotent ideal of A_{t-1} and IA_t is a nilpotent ideal of A_t.

REMARK 4.2.

(i) The condition (FC_1) in Definition 4.1 is a weaker condition than the condition (FC_1) of definition of (FC)-element in [Vi].

(ii) If $x \in I_i$ is a weak-(FC)-element with respect to $(J, I_1, \dots, I_s)$, then it can be verified that xt_i is a weak-(FC)-element of S with respect to J as in Definition 2.1.

(iii) If $x_1, \dots, x_t$ is an (FC)-sequence with respect to $(J, I_1, \dots, I_s)$, then from the condition (FC_3) we follow that $\dim A/((x_1, \dots, x_t)S : S^\infty_{(1,\dots,1)})_A = \ell - t$. Hence

$$\dim D_J(S/(x_1, \dots, x_t)S) = \dim A/((x_1, \dots, x_t)S : S^\infty_{(1,\dots,1)})_A = \ell - t$$

that as in the state of Theorem 3.3(i).

(iv) By Lemma 3.1, $e(J, k_0, 0, \dots, 0, S) \neq 0$ if and only if $\dim A/(0 : S^\infty_{(1,\dots,1)})_A = \ell$. In this case, $e(J, k_0, 0, \dots, 0, S) = e_A(J, S_{(n,\dots,n)})$ for all large n. But since $\dim A/(0 : S^\infty_{(1,\dots,1)})_A = \dim A/0 : I^\infty$, $\dim A/(0 : S^\infty_{(1,\dots,1)})_A = \ell$. Hence $e(J, k_0, 0, \dots, 0, S) = e_A(J, S_{(n,\dots,n)})$ for all large n. It is a plain fact that $e_A(J, S_{(n,\dots,n)}) = e_A(J, I^n)$. On the other hand by the proof of Lemma 3.2 [Vi], $e_A(J, I^n) = e_A(J, A/0 : I^\infty)$ for all large n. Hence $e(J, k_0, 0, \dots, 0, S) = e_A(J, A/0 : I^\infty)$.

Then as an immediate consequence of Theorem 3.3, we obtained an improvement for the main result in [Theorem 3.4, Vi](see Remark 4.2 (i)) as follows.

THEOREM 4.3 (see Theorem 3.4, Vi). *Let $(A, \mathfrak{m})$ denote a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue $k = A/\mathfrak{m}$, and an $\mathfrak{m}$-primary ideal J, and $I_1, \dots, I_s$ ideals of A such that $I = I_1 \cdots I_s$ is non nilpotent. Then the following statements hold.*

(i) *$e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}, A) \neq 0$ if and only if there exists a weak-(FC)-sequence $x_1, \ldots, x_t$ with respect to $(J, I_1, \ldots, I_s)$ consisting of k_1 elements of $I_1, \ldots, k_s$ elements of I_s and* $\dim A/(x_1, \ldots, x_t) : I^\infty = \dim A/0 : I^\infty - t$.

(ii) *Suppose that $e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}, A) \neq 0$ and $x_1, \ldots, x_t$ is a weak-(FC)-sequence with respect to $(J, I_1, \ldots, I_s)$ consisting of k_1 elements of $I_1, \ldots, k_s$ elements of I_s. Set $\bar{A} = A/(x_1, \ldots, x_t) : I^\infty$. Then*

$$e(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}, A) = e_A(J, \bar{A}).$$

Recently, Trung and Verma in 2007 characterize also mixed multiplicities of ideals, in terms of superficial sequences [TV]. Now we prove that [Theorem 1.4, TV] is a consequence of Theorem 4.3.

Set $\mathfrak{T} = \bigoplus_{n_1,\ldots,n_s \geqslant 0} \frac{I_1^{n_1} \cdots I_s^{n_s}}{I_1^{n_1+1} \cdots I_s^{n_s+1}}$. Let ε be an index with $1 \leqslant \varepsilon \leqslant s$. An element $x \in A$ is an *ε-superficial element* for $I_1, \ldots, I_s$ if $x \in I_\varepsilon$ and the image x^* of x in $I_\varepsilon/I_1 \cdots I_{\varepsilon-1} I_\varepsilon^2 I_{\varepsilon+1} \cdots I_s$ is a filter-regular element in $\mathfrak{T}$, i.e., $(0 :_{\mathfrak{T}} x^*)_{(n_1,\ldots,n_s)} = 0$ for $n_1, \ldots, n_s \gg 0$. Let $\varepsilon_1, \ldots, \varepsilon_m$ be a non-decreasing sequence of indices with $1 \leqslant \varepsilon_i \leqslant s$. A sequence $x_1, \ldots, x_m$ is an *$(\varepsilon_1, \ldots, \varepsilon_m)$-superficial sequence* for $I_1, \ldots, I_s$ if for $i = 1, \ldots, m$, $\bar{x}_i$ is an ε_i-superficial element for $\bar{I}_1, \ldots, \bar{I}_s$, where $\bar{x}_i, \bar{I}_1, \ldots, \bar{I}_s$ are the images of $x_i, I_1, \ldots, I_s$ in $A/(x_1, \ldots, x_{i-1})$ [TV].

Then the relationship between $(\varepsilon_1, \ldots, \varepsilon_m)$-superficial sequences and weak-(FC)-sequences is given by the following proposition.

PROPOSITION 4.4 (Proposition 4.3, DV). *Let $I_1, \ldots, I_s$ be ideals in A. Let $x \in A$ be an ε-superficial element for $I_1, \ldots, I_s$. Then x is a weak-(FC)-element with respect to $(I_1, \ldots, I_s)$.*

PROOF. Assume that x is an ε-superficial element for $I_1, \ldots, I_s$. Without loss of generality, we may assume that $\varepsilon = 1$. Then

$$\left(I_1^{n_1+2} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\right) \cap I_1^{n_1} \cdots I_s^{n_s} = I_1^{n_1+1} I_2^{n_2+1} \cdots I_s^{n_s+1} \tag{2}$$

for $n_1, \ldots, n_s \gg 0$. (2) implies

$$\left(I_1^{n_1+2} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\right) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} = I_1^{n_1+1} I_2^{n_2+1} \cdots I_s^{n_s+1} \tag{3}$$

for $n_1, \ldots, n_s \gg 0$. We prove by induction on $k \geqslant 2$ that

$$\left(I_1^{n_1+k} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\right) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} = I_1^{n_1+k-1} I_2^{n_2+1} \cdots I_s^{n_s+1} \tag{4}$$

for $n_1, \ldots, n_s \gg 0$. The case $k = 2$ follows from (3). Assume now that

$$\left(I_1^{n_1+k} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\right) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} = I_1^{n_1+k-1} I_2^{n_2+1} \cdots I_s^{n_s+1}$$

for $n_1, \dots, n_s \gg 0$. Then

$$
\begin{aligned}
&\big(I_1^{n_1+k+1} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\big) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} \\
&= \big(I_1^{n_1+k+1} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\big) \cap \big(I_1^{n_1+k} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\big) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} \\
&= \big(I_1^{n_1+k+1} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\big) \cap I_1^{n_1+k-1} I_2^{n_2+1} \cdots I_s^{n_s+1} \\
&= I_1^{n_1+k} I_2^{n_2+1} \cdots I_s^{n_s+1}
\end{aligned}
$$

for $n_1, \dots, n_s \gg 0$. The last equality is derived from (3). Hence the induction is complete and we get (4). It follows that for $n_1, \dots, n_s \gg 0$,

$$
\begin{aligned}
&(0 : x) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} \\
&\quad = \Big(\bigcap_{k \geqslant 2} I_1^{n_1+k} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\Big) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} \\
&\quad = \Big(\bigcap_{k \geqslant 2} \big(I_1^{n_1+k} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\big)\Big) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1} \\
&\quad = \bigcap_{k \geqslant 2} \big(\big(I_1^{n_1+k} I_2^{n_2+1} \cdots I_s^{n_s+1} : x\big) \cap I_1^{n_1} I_2^{n_2+1} \cdots I_s^{n_s+1}\big) \\
&\quad = \bigcap_{k \geqslant 2} I_1^{n_1+k-1} I_2^{n_2+1} \cdots I_s^{n_s+1} = 0\,,
\end{aligned}
$$

that is, $(0 : x) \cap I^n = 0$ for $n \gg 0$, here $I = I_1 \cdots I_s$. Hence $0 : x \subseteq 0 : I^\infty$. So x is satisfies condition (FC2). Now we need to prove that $I_1^{n_1} \cdots I_s^{n_s} \cap (x) = x I_1^{n_1-1} I_2^{n_2} \cdots I_s^{n_s}$ for $n_1, \dots, n_s \gg 0$. But this has from the proof of [Lemma 1.3, TV]. Hence x is a weak-(FC)-element with respect to $(I_1, \dots, I_s)$. ■

REMARK 4.5. Assume that $Q = (x_1, \dots, x_m)$, where $x_1, \dots, x_m$ is an $(\varepsilon_1, \dots, \varepsilon_m)$-superficial sequence for $J, I_1, \dots, I_s$. Then $x_1, \dots, x_m$ is a weak-(FC)-sequence with respect to $(J, I_1, \dots, I_s)$ by Proposition 4.4. This fact proved that Theorem 4.3 covers a main result of Trung and Verma [Theorem 1.4, TV]. Hence our main result covers the main results in [Vi] and [TV].

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