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On Genelarized DS-diagram and Moves

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Abstract. DS-diagram and flow spine are good tools for studying 3-manifolds ([5], [8]). In this paper, we introduce the concept of generalized DS-diagram and study its properties. We define two types of moves that change generalized DS-diagrams but do not change their associated manifolds. We prove that any two generalized DS-diagrams such that their associated manifolds are homeomorphic to each other can be deformed into each other by a finite sequence of moves of the types.

1. Definitions and notations

For a graph G, we denote by $V(G)$ the set of vertices and by $E(G)$ the set of edges. For a fake surface P (see [4] for definition), we denote by $\mathfrak{S}'_i(P)$ the i–th singularity ($i = 2, 3$). Then $H = \mathfrak{S}'_2(P) \cup \mathfrak{S}'_3(P)$ is a 4-regular graph on P, $V(H)$ is $\mathfrak{S}'_3(P)$, and $E(H)$ is $\mathfrak{S}'_2(P)$.

We permit a 'graph' to have an 'edge' which is homeomorphic to 1-sphere, called *hoop*. We say $f : S \to P$ is a *local homeomorphism*, if for any point p in S there exists a neighborhood U of p in S such that $f|_U : U \to f(U)$ is a homeomorphism. We denote by $\overline{X}$ the closure of X and by $\sharp Z$ the number of all elements of a finite set Z.

DEFINITION 1.1. Let $S = S_1^2 \cup \cdots \cup S_k^2$ be a union of 2-spheres, G be a 3-regular graph on S and $f : (S, G) \to (P, H)$ be a map from S to a closed fake surface P. We call $\Sigma = (S, G, f)$ *generalized DS-diagram* if it satisfies the following conditions;

(1) The map $f : S \to P$ is an onto local homeomorphism.
(2) For any element $x \in V(G)$, $f^{-1} \circ f(x)$ consists of four elements.
(3) For any element $x \in E(G)$, $f^{-1} \circ f(x)$ consists of three elements.
(4) For any element $x \in S - G$, $f^{-1} \circ f(x)$ consists of two elements.

We call the number of spheres k *s-number* of Σ and denote by $s(\Sigma)$. Let $\mathcal{B} = B_1^3 \cup \cdots \cup B_k^3$ be a union of 3-balls and $\partial B_i^3 = S_i^2$ ($i = 1, \ldots, k$). We denote by $M(\Sigma)$ the identification space $\mathcal{B}/f$. The space $M(\Sigma)$ is a 3-manifold as in the case of as for DS-diagram. We call $M(\Sigma)$ *the manifold associated with the generalized DS-diagram* Σ. Generally, $M(\Sigma)$ may not be connected. Hereafter we assume $M(\Sigma)$ is connected.

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DEFINITION 1.2. Let S be a union of 2-spheres, G be a 3-regular graph on S and $g : G \to H$ be a map from a graph G to a graph H. We call $\Omega = \langle S, G, g\rangle$ *labeled graph* if it satisfies the following conditions;

(1) The map g is an onto local homeomorphism.

(2) For any element $x \in V(G)$, $g^{-1} \circ g(x)$ consists of four elements.

(3) For any element $x \in E(G)$, $g^{-1} \circ g(x)$ consists of three elements.

DEFINITION 1.3. For two generalized DS-diagrams $\Sigma = (S, G, f)$ and $\Sigma' = (S', G', f')$, we say *Σ is equivalent to Σ'* if there exist homeomorphisms $F : S \to S'$ and $\underline{F} : f(S) \to f'(S')$ such that $f' \circ F = \underline{F} \circ f$. Then we denote $\Sigma \equiv \Sigma'$.

For two labeled graphs $\Omega = \langle S, G, g\rangle$ and $\Omega' = \langle S', G', g'\rangle$, we say *$\Omega$ is equivalent to Ω'* if there exist homeomorphisms $F : S \to S'$ and $\underline{F} : g(G) \to g'(G')$ such that $g' \circ F|_G = \underline{F} \circ g$. Then we denote $\Omega \equiv \Omega'$.

For a generalized DS-diagram $\Sigma = (S, G, f)$, we define $g = f|_G$ and $\Omega = \langle S, G, g\rangle$. Then Ω is a labeled graph. We denote this labeled graph by $L(\Sigma)$ and we call $L(\Sigma)$ *the labeled graph associated with the generalized DS-diagram Σ*. If Σ is equivalent to Σ', $L(\Sigma)$ is equivalent to $L(\Sigma')$.

We can represent a labeled graph by a figure. Let A be a directed edge in $g(G)$. In the case $g^{-1}(A)$ consists of 3 components, we mark the 'label' A on each of them. For a directed edge A in $g(G)$, A^{-1} is the edge with the reverse direction.

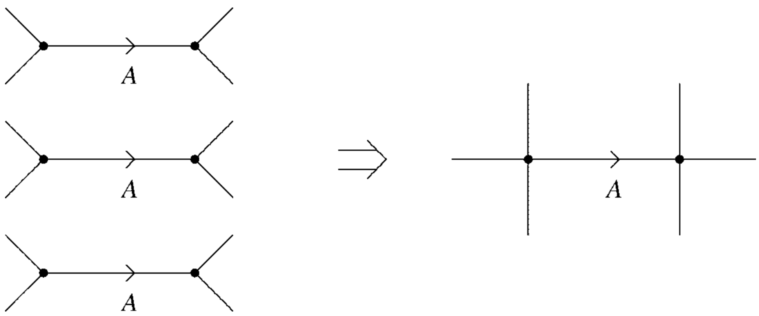

FIGURE 1

In the case $g^{-1}(A)$ consists of 2 components (say, e_1 and e_2), we assume that $g|_{e_1} : e_1 \to A$ is 2 to 1 and that $g|_{e_2} : e_2 \to A$ is 1 to 1. We mark the label $2A$ on e_1 and the label A on e_2. We call the 'edge' (hoop) e_1 *double type*. For a generalized DS-diagram Σ, $L(\Sigma)$ has no hoop of double type (see Lemma 2.8).

In the case $g^{-1}(A)$ is connected, we mark the label $3A$. We call the hoop $g^{-1}(A)$ *triple type*.

If a generalized DS-diagram $\Sigma = (S, G, f)$ satisfies the following conditions, Σ is a DS-diagram;

(1) The s-number of Σ is one.

(2) The graph G is connected and $V(G) \neq \emptyset$.

2. Relation between generalized DS-diagram and labeled graph

For any labeled graph Ω, there does not always exist a generalized DS-diagram Σ such that $L(\Sigma) = \Omega$. First, we consider the condition for existence.

For a generalized DS-diagram $\Sigma = (S, G, f)$, we can define an involution τ on $S - G$ as follows: Let p be any point in $S - G$. We put $f^{-1} \circ f(p) = \{ p, p' \}$. Then we define $\tau(p) = p'$. We call τ *the involution associated with the generalized DS-diagram* Σ.

The involution τ is fixed point free and satisfies the following property: Let p be any point in $E(G)$ and U be a small neighborhood of p in S. And $\{ p_n \}$ be any sequence converges to p such that every p_n is contained in the same component of $U - G$. Then $\tau(p_n)$ converges to some point $q \in E(G)$ such that $p \neq q$ and $f(p) = f(q)$.

Conversely, for a labeled graph $\Omega = \langle S, G, g \rangle$, we assume that there exists a fixed point free involution τ on $S - G$ which has the above property. We call τ *an involution compatible with* Ω. We can construct a closed fake surface P and a map f from S to P as follows: If $x' = x$, $x' = \tau(x)$ or $g(x') = g(x)$, we denote $x' \sim x$. This relation '$\sim$' is an equivalence relation on S. We define $P = S/\!\sim$ and $f : S \to S/\!\sim$ is the projection. We can easily check that P is a closed fake surface and that f satisfies the conditions for generalized DS-diagram. Thus the next two propositions hold.

PROPOSITION 2.1. *Let Σ be a generalized DS-diagram and τ be the involution associated with Σ. Then τ is compatible with $L(\Sigma)$.*

Conversely let Ω be a labeled graph and τ be an involution compatible with Ω. Then there exists a generalized DS-diagram Σ such that $L(\Sigma) = \Omega$ and the involution associated with Σ is τ.

PROPOSITION 2.2. *Let $\Sigma = (S, G, f)$ be a generalized DS-diagram, τ be the involution associated with Σ, $\Sigma' = (S', G', f')$ be a generalized DS-diagram and τ' be the involution associated with Σ'.*

Suppose that Σ is equivalent to Σ'. Let F and $\underline{F}$ be homeomorphisms which give the equivalence, namely $\underline{F} \circ f = f' \circ F$. Then $F \circ \tau = \tau' \circ F$.

Conversely, suppose that $L(\Sigma)$ is equivalent to $L(\Sigma')$. Let F and $\underline{F}$ be homeomorphisms which give the equivalence, namely $\underline{F} \circ f|_G = f'|_{G'} \circ F|_G$. If $F \circ \tau = \tau' \circ F$, then Σ is equivalent to Σ'.

For generalized DS-diagrams Σ and Σ', if $\Sigma \equiv \Sigma'$ then $L(\Sigma) \equiv L(\Sigma')$. The converse is not true generally. We consider the condition that the converse is true.

The first example of pair of non-equivalent generalized DS-diagrams whose associated labeled graphs are equivalent to each othe is shown in Figure 2. Edges with labels 'A' are hoops on some annulus $\mathcal{A} \subset S$. We call this type *Type I*.

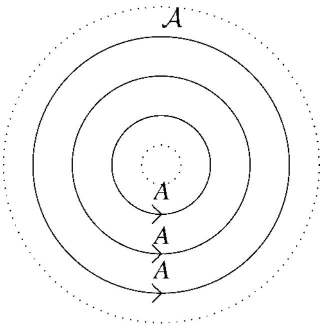

FIGURE 2

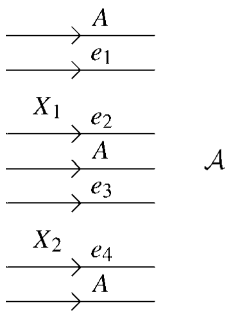

FIGURE 3

For a 3-manifold N, we denote by $\widehat{N}$ the manifold obtained from N by capping off each 2-sphere component of ∂N with 3-ball. Let $\Sigma = (S, G, f)$ be a generalized DS-diagram with an associated involution τ and $\Sigma' = (S, G, f')$ be a generalized DS-diagram with an associated involution τ'. We assume that the graph G contains hoops as in Figure 2, $f^{-1} \circ f(\mathcal{A}) = \mathcal{A}$ and $f'^{-1} \circ f'(\mathcal{A}) = \mathcal{A}$. Let e_1, e_2, e_3, e_4 be loops parallel to hoops with labels A and X_1 and X_2 be annuli as in Figure 3. We assume that $\tau(e_1) = e_2$, $\tau(e_3) = e_4$, $\tau'(e_1) = e_3$ and $\tau'(e_2) = e_4$. Let D_i be a proper 2-disk in $\mathcal{B}$ whose boundary is e_i $(i = 1, 2, 3, 4)$. Let V_1 be the closure of a component of $\mathcal{B} - (D_1 \cup D_2)$ that contains X_1. Then $f(D_1) \cup f(D_2)$ is a 2-sphere and $f(X_1)$ is a Möbius band. Thus the identification space $f(V_1)$ is homeomorphic to $P^3 - \operatorname{Int} D^3$, where P^3 is a projective space and D^3 is a 3-ball in P^3. Let V_2 be the closure of a component of $\mathcal{B} - (D_3 \cup D_4)$ that contains X_2. The identification space $f(V_2)$ is also homeomorphic to $P^3 - \operatorname{Int} D^3$. Let D_0 and D_5 be proper 2-disks whose boundaries are $\partial\mathcal{A}$. Let V be the closure of a component of $\mathcal{B} - (D_0 \cup D_5)$ that contains X_1 and $W = \mathcal{B} - \operatorname{Int} V$. Then $\widehat{f(V)}$ is homeomorphic to $P^3 \sharp P^3$, where $\sharp$ means connected sum. The identification space $f(W)$ is a 3-manifold whose boundary is a 2-sphere. We put $M_1 = \widehat{f(W)}$, then $M(\Sigma)$ is homeomorphic to $M_1 \sharp P^3 \sharp P^3$.

We assume that $f_{|S-\mathcal{A}} = f'_{|S-\mathcal{A}}$. Since $f'(W) = f(W)$, $\widehat{f'(W)}$ is M_1. The identification space $f'(V)$ is a non-orientable 3-manifold and contains a non-separating 2-sphere $f'(D_1) \cup f'(D_3)$. So $\widehat{f'(V)}$ is homeomorphic to $S^2{}_\tau\!\times S^1$, where $S^2{}_\tau\!\times S^1$ is a twisted S^2 bundle over S^1. Thus $M(\Sigma')$ is homeomorphic to $M_1 \sharp S^2{}_\tau\!\times S^1$.

Second example is shown in Figure 4. Edges with labels A and edges with labels B are hoops on some annuli $\mathcal{A}$ and $\mathcal{A}'$. We call this type *Type II*.

Let $\Sigma = (S, G, f)$ be a generalized DS-diagram with an associated involution τ and $\Sigma' = (S, G, f')$ be a generalized DS-diagram with an associated involution τ'. We assume that the graph G contains hoops as in Figure 4, $f(\mathcal{A} \cup \mathcal{A}') = \mathcal{A} \cup \mathcal{A}'$ and $f'(\mathcal{A} \cup \mathcal{A}') = \mathcal{A} \cup \mathcal{A}'$. Let e_1 be a loop parallel to a hoop with a label A and X_1 be an annulus as in Figure 5. We assume that $\tau(e_1)$ and $\tau'(e_1)$ are as in Figure 6.

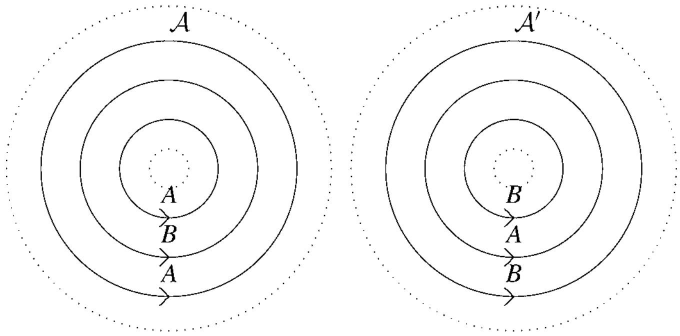

FIGURE 4

We assume that $f_{|S-(\mathcal{A}\cup\mathcal{A}')} = f'_{|S-(\mathcal{A}\cup\mathcal{A}')}$ and $f(S - (\mathcal{A} \cup \mathcal{A}'))$ is connected. Let D_1, D_2 and D_3 be proper 2-disks in $\mathcal{B}$ whose boundaries are e_1, $\tau(e_1)$ and $\tau'(e_1)$, respectively. Let D_0 and D_4 be proper 2-disks whose boundaries are $\partial\mathcal{A}$ and let D_5 and D_6 be proper 2-disks whose boundaries are $\partial\mathcal{A}'$.

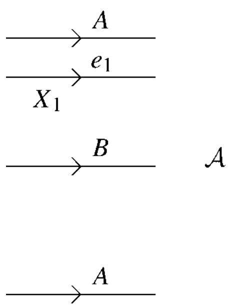

FIGURE 5

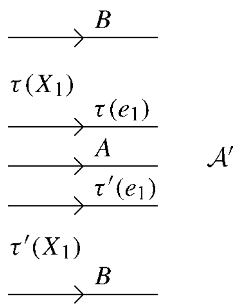

FIGURE 6

Let V_1 be the closure of a component of $\mathcal{B} - (D_0 \cup D_4)$ that contains X_1 and V_2 be the closure of a component of $\mathcal{B} - (D_5 \cup D_6)$ that contains $\tau(X_1)$. We put $W = \mathcal{B} - \mathrm{Int}(V_1 \cup V_2)$. Since $f(D_1) \cup f(D_2)$ and $f'(D_1) \cup f'(D_3)$ are non-separating 2-spheres, $f(V_1 \cup V_2)$ and

$f'(V_1 \cup V_2)$ are homeomorphic to $S^2 \times S^1 - \mathrm{Int}(D_1^3 \cup D_2^3)$ where D_1^3 and D_2^3 are 3-balls in $S^2 \times S^1$. The identification space $f(W)$ is a 3-manifold whose boundary consists of two 2-spheres and $f(W) = f'(W)$. We fix an orientation of V_1. Orientaions of $f(D_5) \cup f(D_6)$ and $f'(D_5) \cup f'(D_6)$ are induced by the orientation of V_1. Then the orientation of $f(D_5) \cup f(D_6)$ is reverse of the orientation of $f'(D_5) \cup f'(D_6)$. We put $M_1 = \widehat{f(W)}$. Then one of $M(\Sigma)$ and $M(\Sigma')$ is homeomorphic to $M_1 \sharp S^2 \times S^1 \sharp S^2 \times S^1$ and the other is homeomorphic to $M_1 \sharp S^2 \times S^1 \sharp S^2 {}_\tau\!\times S^1$. If M_1 is orientable, the two manifolds are not homeomorphic to each other.

Third example is shown in Figure 7. An edge with label '$3A$' is a hoop on some annulus $\mathcal{A}$. We call this type *Type III*.

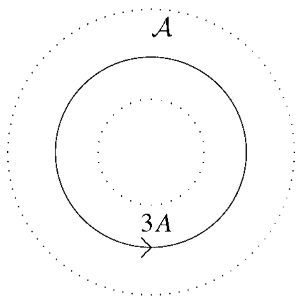

FIGURE 7

Let $\Sigma = (S, G, f)$ be a generalized DS-diagram with an associated involution τ and $\Sigma' = (S, G, f')$ be a generalized DS-diagram with an associated involution τ'. We assume that the graph G contains hoops as in Figure 7, $f(\mathcal{A}) = \mathcal{A}$, $f'(\mathcal{A}) = \mathcal{A}$ and $f_{|S-\mathcal{A}} = f'_{|S-\mathcal{A}}$. Further we assume that $\tau_{|\mathcal{A}}$ is the composition of the rotation of angle $\frac{2\pi}{3}$ along the hoop and the reflection about the hoop and $\tau'_{|\mathcal{A}}$ is the composition of the rotation of angle $\frac{4\pi}{3}$ along the hoop and the reflection about the hoop.

So one of $M(\Sigma)$ and $M(\Sigma')$ is homeomorphic to $M_1 \sharp L(3, 1)$ for some 3-manifold M_1 and the other is homeomorphic to $M_1 \sharp L(3, 2)$ where $L(p, q)$ is a lens space of type (p, q). If M_1 does not admit an orientation reversing self homeomorphism, the two manifolds are not homeomorphic to each other.

THEOREM 2.3. *Let Σ and Σ' be generalized DS-diagrams such that $L(\Sigma) \equiv L(\Sigma')$. If $L(\Sigma)$ does not admit Type I, II and III, then $\Sigma \equiv \Sigma'$.*

COROLLARY 2.4. *Let Σ and Σ' be generalized DS-diagrams such that $L(\Sigma) \equiv L(\Sigma')$. If $L(\Sigma)$ does not admit hoops, then $\Sigma \equiv \Sigma'$.*

COROLLARY 2.5. *Let Σ and Σ' be DS-diagrams such that $L(\Sigma) \equiv L(\Sigma')$, then $\Sigma \equiv \Sigma'$.*

For proving Theorem 2.3, we will prove the next two propositions.

PROPOSITION 2.6. *Let $\Sigma = (S, G, f)$ and $\Sigma' = (S', G', f')$ be generalized DS-diagrams such that $L(\Sigma) \equiv L(\Sigma')$ and assume that $L(\Sigma)$ does not admit Type I, II and III. Let $N(G; S)$ be a regular neighborhood of G in S such that $f^{-1} \circ f(N(G; S)) = N(G; S)$ and $N(G'; S')$ be a regular neighborhood of G' in S' such that $f'^{-1} \circ f'(N(G'; S')) = N(G'; S')$. Then there exist a homeomorphism $F : S \to S'$ and a homeomorphism $\underline{F} : f(N(G; S)) \to f'(N(G'; S'))$ such that $f' \circ F|_{N(G;S)} = \underline{F} \circ f|_{N(G;S)}$.*

PROPOSITION 2.7. *Let Σ and Σ' be generalized DS-diagrams such that $L(\Sigma) \equiv L(\Sigma')$ and assume that there exist homeomorphisms F and $\underline{F}$ such as in Proposition* 2.6. *Then Σ is equivalent to Σ'.*

For proving Propositions 2.6, we need some lemmata.

LEMMA 2.8. *Let Σ be a generalized DS-diagram. Then $L(\Sigma)$ does not have a hoop of double type.*

PROOF. We assume that Σ has a hoop e_1 with label $2C$. We put $e_2 = f^{-1}(C) - e_1$. Let τ be the involution associated with Σ and $N(G; S)$ be a regular neighborhood of G in S such that $f^{-1} \circ f(N(G; S)) = N(G; S)$. For the component X_1 of $N(G; S)$ which contains e_1, $\tau(X_1 - e_1) = X_1 - e_1$. So for the component X_2 of $N(G; S)$ which contains e_2, $\tau(X_2 - e_2) = X_2 - e_2$. This contradicts that τ is compatible. ∎

LEMMA 2.9. *Let τ and τ' be involutions associated with Σ and Σ', respectively. If there exist homeomorphisms F and $\underline{F}$ such as in Proposition* 2.6, *then $F \circ \tau|_{N(G;S)-G} = \tau' \circ F|_{N(G;S)-G}$. Conversely If there exists a homeomorphism $F : S \to S'$ such that $F \circ \tau|_{N(G;S)-G} = \tau' \circ F|_{N(G;S)-G}$, then there exists a homeomorphism $\underline{F} : f(N(G; S)) \to f'(N(G'; S'))$ such as in Proposition* 2.6.

The proof of Lemma 2.9 is easy. We omit the proof.

We begin to prove Proposition 2.6. Since $L(\Sigma) \equiv L(\Sigma')$, there exist homeomorphisms $F' : S \to S'$ and $\underline{F}' : f(G) \to f'(G)$ such that $f' \circ F'|_G = \underline{F}' \circ f$. By exchanging $\langle S', G', f'|_{G'}\rangle$ for $\langle F'^{-1}(S'), F'^{-1}(G'), \underline{F}'^{-1} \circ f' \circ F'|_G\rangle$, we assume that $L(\Sigma) = \langle S, G, g\rangle = L(\Sigma')$. Let τ be the involution associated with Σ and τ' be the involution associated with Σ'. We may assume that there exists a regular neighborhood $N(G; S)$ such that $\tau(N(G; S) - G) = N(G; S) - G$ and $\tau'(N(G; S) - G) = N(G; S) - G$.

Since a regular neighborhood $N(G; S)$ is a block bundle (see [9], [10] and [12] for definition) over G, there exists a projection map $\pi : N(G; S) \to G$. For each point $x \in G - V(G)$, $\pi^{-1}(x)$ is an arc. For each point $x \in V(G)$, $\pi^{-1}(x)$ is a graph with one degree 3 vertex and three degree 1 vertices as in Figure 8. Let X be a component of $N(G; S) - G$ and

$L = \partial\overline{X} - G$. If $\sharp\pi_{|L}^{-1}(\overline{X} \cap V(G))$ is n, we call $\overline{X}$ an *n-gon*. In Figure 8 $\overline{X}$ is a 4-gon. If $\overline{X}$ is a 0-gon, $\overline{X} \cap G$ is a hoop. Conversely, if $\overline{X} \cap G$ is a hoop, $\overline{X}$ is a 0-gon.

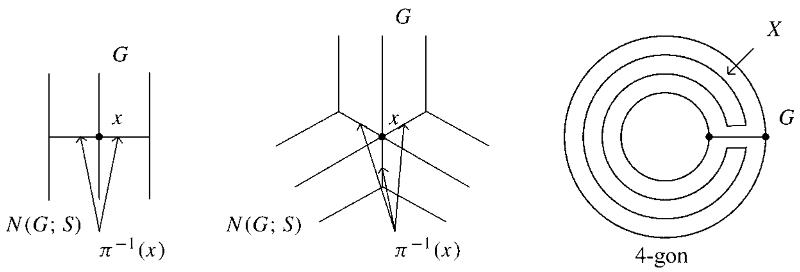

FIGURE 8

LEMMA 2.10. *If $\overline{X} \cap V(G) \neq \emptyset$ for a component X of $N(G; S) - G$, then $\tau(X) = \tau'(X)$.*

PROOF. We assume that there exists a component X such that $\tau(X) \neq \tau'(X)$. Since $\overline{X} \cap V(G)$ is not empty, $\overline{X}$ is an n-gon ($n \geq 1$).

First we consider the case $n = 1$. Let A be a label of $\overline{X} \cap G$. Because $\overline{X}$ is a 1-gon, $\tau(X) \neq X$ and $\tau'(X) \neq X$. There exist four 1-gons $\overline{X}$, $\overline{\tau(X)}$, $\overline{\tau'(X)}$ and $\overline{\tau' \circ \tau(X)}$ with labels A. This contradicts that there exist at most three edges with labels A.

Next we assume $n \geq 2$. Let e be any edge which is contained in $\overline{X} \cap G$. Let A be a label of e. We consider the subcase $\tau(X) = X$. Then $\overline{X} \cap G$ contains the other edge e_1 with the label A. So $\overline{\tau'(X)} \cap G$ contains two edges with labels A. Because there exist at most three edges with labels A, $\overline{X} \cap \overline{\tau'(X)}$ contains an edge e or e_1. Let B be a label of the next edge of e in $\overline{X} \cap G$. Then the situation is as in Figure 9. But this contradicts that f is a local homeomorphism.

We consider the subcase $\tau(X) \neq X$ and $\tau'(X) \neq X$. Then each of $\overline{X} \cap G$, $\overline{\tau(X)} \cap G$, $\overline{\tau'(X)} \cap G$ and $\overline{\tau' \circ \tau(X)} \cap G$ has an edge with label A. If a set of edges $\overline{X} \cap \overline{\tau(X)}$ or $\overline{X} \cap \overline{\tau'(X)}$ contains an edge e, an involution τ or τ' is not compatible. Because there exist at most three edges with label A, a set of edges $\overline{X} \cap \overline{\tau \circ \tau'(X)}$ contains an edge e with label A or a set of edges $\overline{\tau(X)} \cap \overline{\tau'(X)}$ contains an edge e_1 with label A. Let B be a label of the next edge of e in $\overline{X} \cap G$ or a label of the next edge of e_1 in $\overline{\tau(X)} \cap G$. Then the situation is as in Figure 10. But this contradicts that f is a local homeomorphism. This completes the proof of Lemma 2.10. ∎

Since there does not exist an edge of triple type, if $\tau(X) = \tau'(X)$ for any component X of $N(G; S) - G$, by isotopy, we may assume $\tau|_{N(G;S)-G} = \tau'|_{N(G;S)-G}$. We put $F = id$

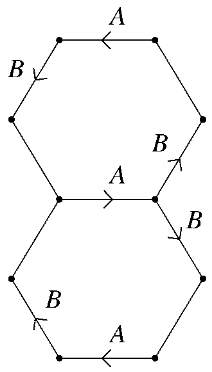

FIGURE 9

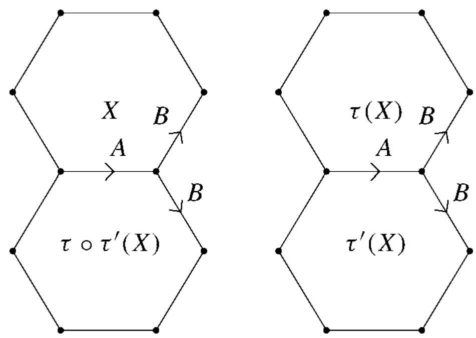

FIGURE 10

where id is the identity map on S, then $\tau \circ F = F \circ \tau'$. By Lemma 2.9, Proposition 2.6 has been proved in this case.

So we assume that there exists a component X of $N(G; S) - G$ such that $\tau(X) \neq \tau'(X)$. By Lemma 2.10, $\overline{X} \cap G$ is a hoop. Let A be a label of $\overline{X} \cap G$.

Let Y be a component of $S - G$ such that $X \subset Y$. If $\overline{Y} \cap V(G) \neq \emptyset$, then $\tau(Y) = \tau'(Y)$. The edges $\overline{\tau(X)} \cap G$ and $\overline{\tau'(X)} \cap G$ with labels A are contained in $\overline{\tau(Y)}$. So $\overline{Y}$ contains the edges $\overline{X} \cap G$ and $\overline{\tau' \circ \tau(X)} \cap G$ with labels A. If $\tau(Y) = Y$, $\partial\overline{Y}$ contains four hoops with labels A. If $\tau(Y) \neq Y$, $\overline{Y} \cap \overline{\tau(Y)}$ is non-empty and is a hoop with label A. This contradicts that involutions are compatible. So $\overline{Y} \cap V(G)$ is empty.

First we consider the case $\tau(Y) = Y$. Then $\overline{Y}$ has the edges $\overline{X} \cap G$ and $\overline{\tau(X)} \cap G$ with labels A. Sinece $\overline{\tau'(Y)}$ has edges $\overline{\tau'(X)} \cap G$ and $\overline{\tau' \circ \tau(X)} \cap G$ with labels A, we obtain $\tau'(Y) \neq Y$ and $\overline{Y} \cap \overline{\tau'(Y)} \neq \emptyset$.

If $\overline{Y} \cap G \neq (\overline{X} \cap G) \cup (\overline{\tau(X)} \cap G)$, the genus of S is positive. So $\overline{Y} \cap G = (\overline{X} \cap G) \cup (\overline{\tau(X)} \cap G)$ and $\overline{Y}$ is an annulus. If labels are as in Figure 11, τ has fixed points. In this case type I occurs, this is a contradiction.

Next we consider the case $\tau(Y) \neq Y$ and $\tau'(Y) \neq Y$. If $\tau(Y) = \tau'(Y)$, Y and $\tau(Y)$ have two edges with labels A. Then τ or τ' is not compatible, so $\tau(Y) \neq \tau'(Y)$.

We assume that $\overline{Y}$ is not homeomorphic to a 2-disk. There exists a component X_1 of $N(G; S) - G$ such that $X_1 \subset Y$ and $X \neq X_1$. Let B be the label of $\overline{X_1} \cap G$. Each of $\overline{Y}$, $\overline{\tau(Y)}$, $\overline{\tau'(Y)}$ and $\overline{\tau' \circ \tau(Y)}$ has an edge with label A and an edge with label B. So $\overline{Y} \cap \overline{\tau' \circ \tau(Y)} \neq \emptyset$ and $\overline{\tau(Y)} \cap \overline{\tau'(Y)} \neq \emptyset$. Then we obtain $\overline{Y} \cap G = (\overline{X} \cap G) \cup (\overline{X_1} \cap G)$, so type II occurs. This is a contradiction. Thus $\overline{Y}$ is homeomorphic to a 2-disk.

Then $\overline{Y}$, $\overline{\tau(Y)}$, $\overline{\tau'(Y)}$ and $\overline{\tau' \circ \tau(Y)}$ are 2-disks whose boundaries are hoops with labels A. Let Z_1 and Z_2 be the other components of $S - G$ that have an edge with label A. If $\overline{Z}_1$

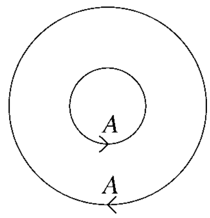

FIGURE 11

is 2-disk (then $\overline{Z}_2$ is also 2-disk), then $S = S_1^2 \cup S_2^2 \cup S_3^2$ and $H = \{ A \}$. If $\overline{Z}_1$ is not 2-disk, then either $Z_1 = Z_2$ has two edges with labels A or $\tau(Z_1) = \tau'(Z_1) = Z_2 \neq Z_1$. In all cases we can easily construct a homeomorphism F on S such that $F \circ \tau = \tau' \circ F$ on $N(G; S)$. This completes the proof of Proposition 2.6. ∎

Next we prove Proposition 2.7. Let F be the homeomorphism on S constructed above. We change F on $S - N(G; S)$. Let Y be any component of $S - G$ and $X_1, X_2, \dots, X_k$ be the components of $N(G; S) - G$ such that $X_i \subset Y$ $(i = 1, \dots, k)$. We divide the proof into two cases: (A); $\tau(Y) \neq Y$ and (B); $\tau(Y) = Y$.

In the case (A), for $x \in \tau(Y)$, we redefine $F(x) = \tau' \circ F \circ \tau(x)$. If x is in $\cup X_i$, $F(x)$ is not changed.

In the case (B), $\pi : Y \to Y/\tau$ and $\pi' : Y \to Y/\tau'$ are 2-fold coverings. If τ is orientation preserving, τ' is orientation preserving. Thus Y/τ is homeomorphic to Y/τ'. There exist a homeomorphsim $\underline{F} : \cup X_i/\tau \to \cup X_i/\tau'$. We can extend $\underline{F}$ to a homeomorphism from Y/τ to Y/τ', namely $\underline{F}$. We redefine F as a lift $F : Y \to Y$ of $\underline{F}$. We choose the lift which is not changed on $N(G; S) \cap Y$. This completes the proof of Proposition 2.7. ∎

3. G-move and S-move

Let $\Sigma = (S, G, f)$ be a generalized DS-diagram with an associated involution τ, ℓ be a loop on S and q be a point on G. We say that q is a *limit point* for ℓ if the following holds; There exist a point p in $\ell \cap G$ and a sequence $\{ p_n \}$ such that every p_n is contained the same component of $\ell - G$, $p = \lim_{n\to\infty} p_n$ and $q = \lim_{n\to\infty} \tau(p_n)$. We say that a loop ℓ is in *general position* for Σ if it satisfies the following conditions;

(1) The loop ℓ does not intersect $V(G)$.
(2) The loop ℓ is transversal with G and transversal with $\tau(\ell - G)$.
(3) Any limit point for ℓ is not contained in ℓ.

DEFINITION 3.1. Let $\Sigma = (S, G, f)$ be a generalized DS-diagram with an associated involution τ, where $S = S_1^2 \cup \dots \cup S_k^2$. Let ℓ be a loop on S_1^2 and suppose that ℓ is in general position for Σ. We put $\mathcal{B} = B_1^3 \cup \dots \cup B_k^3$ and $\partial\mathcal{B} = S$. We extend f to a map from $\mathcal{B}$ to

$M(\Sigma)$. There is a proper 2-disk D in $\mathcal{B}$ such that $\partial D = \ell$.

Let $\widetilde{S}_i$ be a 2-sphere ($i = 1, 2$). We put $S' = \widetilde{S}_1 \cup \widetilde{S}_2 \cup S_2^2 \cup \cdots \cup S_k^2$ and $\widetilde{D}_1 \cup \widetilde{D}_2 = S_1^2 - \ell$. So $\widetilde{D}_i \cup D$ is 2-sphere, we can define a homeomorphism $g_i : \widetilde{S}_i \to \widetilde{D}_i \cup D$ ($i = 1, 2$). We put $P' = P \cup f(D)$. We define a map $f' : S' \to P'$ as follows;

$$f'(x) = \begin{cases} f(x) & (x \in S_j^2, j \neq 1) \\ f(g_1(x)) & (x \in \widetilde{S}_1) \\ f(g_2(x)) & (x \in \widetilde{S}_2) \end{cases}$$

We define $G' = \{x \in S' \mid \sharp f'^{-1} \circ f'(x) \geq 3\}$. Then the set of vertices of G' is $\{x \in S' \mid \sharp f'^{-1} \circ f'(x) = 4\}$. $\Sigma' = (S', G', f')$ is a generalized DS-diagram and $M(\Sigma')$ is homeomorphic to $M(\Sigma)$. Then this operation $\Sigma \Rightarrow \Sigma'$ is called *S-move*, *S-move* along ℓ or *spoon cut*.

Suppose that $k \geq 2$. Let X_1 and X_2 be faces of Σ (components of $S - G$) such that $X_i \subset S_i^2$ ($i = 1, 2$). We assume that $\overline{X_1}$ and $\overline{X_2}$ are 2-disks and $\tau(X_1) = X_2$. Let τ_0 be the homeomorphism on $\overline{X_1} \cup \overline{X_2}$ which is the extension of $\tau|_{X_1 \cup X_2}$. We put that $S_1' = (S_1^2 - X_1) \cup_{\tau_0} (S_2^2 - X_2)$, $S' = S_1' \cup S_3^2 \cup \cdots \cup S_k^2$ and $P' = P - f(X_1)$. We define a map $f' : S' \to P'$ by $f'(x) = f(x)$. We define that $G' = \{x \in S' \mid \sharp f'^{-1} \circ f'(x) \geq 3\}$. $\Sigma' = (S', G', f')$ is a generalized DS-diagram and $M(\Sigma')$ is homeomorphic to $M(\Sigma)$. Then this operation $\Sigma \Rightarrow \Sigma'$ is called *G-move*, *G-move* along X_1 or *glue*.

If a generalized DS-diagram Σ' is obtained from a generalized DS-diagram Σ by G-move, Σ is obtained from Σ' by S-move. Conversely if a generalized DS-diagram Σ' is obtained from a generalized DS-diagram Σ by S-move, Σ is obtained from Σ' by G-move.

A successive application of a finite number of G-moves and S-moves is called *GS-deformation*. GS-deformation is an equivalence relation. If there exists a GS-deformation $\Sigma \Longrightarrow \Sigma'$, $M(\Sigma)$ is homeomorphic to $M(\Sigma')$.

Next is the main theorem for GS-deformation.

THEOREM 3.2. *Let Σ and Σ' be generalized DS-diagrams such that $M(\Sigma)$ is homeomorphic to $M(\Sigma')$. Then there exists a GS-deformation $\Sigma \Longrightarrow \Sigma'$.*

Proof of Theorem 3.2 depends on the following theorem ([11], [13], [1], [3]).

THEOREM 3.3 (Reidemeister–Singer–Chillingworth–Craggs). *Any two Heegaard splittings which give homeomorphic manifolds can be equivalent by stabilizing.*

A *Heegaard splitting* of 3-manifold M is a representation of M as $H_1 \cup H_2$, where H_1 and H_2 are homeomorphic to handlebodies of some fixed genus g and $H_1 \cap H_2 = \partial H_1 = \partial H_2 = F_g$ is the *Heegaard surface*. The splitting is denoted by (H_1, H_2) or (M, F_g).

Let $\vec{D}_i = D_{i,1} \cup D_{i,2} \cup \cdots \cup D_{i,g}$ be a complete system of meridian disks of H_i ($i = 1, 2$). A *Heegaard diagram* is a Heegarrd splitting (H_1, H_2) with complete systems of meridian disks $\vec{D}_1$ and $\vec{D}_2$. The diagram is denoted by $(H_1, H_2; \vec{D}_1, \vec{D}_2)$.

Let $V = D_{1,g+1} \times [-1, 1]$ and $W = D_{2,g+1} \times [-1, 1]$ be handles in $H_2 - \vec{D}_2$ where $D_{i,g+1}$ is a 2-disk ($i = 1, 2$) and $H_1 \cap V = \partial H_1 \cap V = D_{1,g+1} \times \{-1\} \cup D_{1,g+1} \times \{1\}$. Furthermore we assume that $\partial D_{2,g+1}$ is $L_1 \cup L_2$ where L_i is an arc ($i = 1, 2$), $L_1 \cap L_2 = \partial L_1 = \partial L_2$, $V \cap W = \partial V \cap \partial W = L_2 \times [-1, 1]$, $H_1 \cap W = \partial H_1 \cap W = L_1 \times [-1, 1]$ and $(V \cup W) \cap \vec{D}_1 = \emptyset$. We put $H_1' = H_1 \cup V$, $H_2' = H_2 - \operatorname{Int} V$, $\vec{D}_1' = \vec{D}_1 \cup D_{1,g+1} \times \{0\}$ and $\vec{D}_2' = \vec{D}_2 \cup D_{2,g+1} \times \{0\}$. Then $(H_1', H_2'; \vec{D}_1', \vec{D}_2')$ is a Heegaard diagram. This operation which change $(H_1, H_2; \vec{D}_1, \vec{D}_2)$ into $(H_1', H_2'; \vec{D}_1', \vec{D}_2')$ is called *attachings of trivial handles*. A finite application of attachings of trivial handles is called *stabilizing*.

For a Heegaard diagram $(H_1, H_2; \vec{D}_1, \vec{D}_2)$, we can construct a generalized DS-diagram Σ as follows; Let S_1^2 and S_2^2 be 2-spheres and $g_i : S_i^2 \to \partial H_i \cup \vec{D}_i$ be an onto local homeomorphism, where $\sharp g_i^{-1}(y) = 1$ fot $y \in \partial H_i - \vec{D}_i$ and $\sharp g_i^{-1}(y) = 2$ for $y \in \vec{D}_i$ ($i = 1, 2$). We put $S = S_1^2 \cup S_2^2$. Let $f = g_1 \cup g_2 : S \to \partial H_1 \cup \vec{D}_1 \cup \vec{D}_2$ be a local homeomorphism and $G = f^{-1}(\partial \vec{D}_1 \cup \partial \vec{D}_2)$. Then $\Sigma = (S, G, f)$ is a generalized DS-diagram. We call Σ the generalized DS-diagram *defined by* Heegaard diagram $(H_1, H_2; \vec{D}_1, \vec{D}_2)$.

DEFINITION 3.4. Let $\Sigma = (S, G, f)$ be a generalized DS-diagram. X is a face of Σ. X' is the face such that $f(X) = f(X')$ and $X \neq X'$. If $X \subset S_i^2$ and $X' \subset S_i^2$ for some i, we call X *self type*. Σ is called *type H* if it satisfies the following conditions;

(1) The s-number of Σ is equal to 2.

(2) If X is self type, $\overline{X}$ is homeomorphic to a 2-disk.

(3) If X and Y are self type and $X \neq Y$, then $\overline{X} \cap \overline{Y} = \emptyset$.

LEMMA 3.5. *If a generalized DS-diagram Σ is defined by Heegaard diagram, then Σ is of type H. Conversely if Σ is of type H, Σ is defined by some Heegaard diagram.*

PROOF. Suppose that $\Sigma = (S, G, f)$ be a generalized DS-diagram which is defined by Heegaard diagram, then the s-number of Σ is equal to 2. If a face X of Σ is self type, $f(\overline{X})$ is a meridian disk of Heegaard diagram. So $\overline{X}$ is a 2-disk. Suppose that faces X and Y are self type and $X \neq Y$. $f(\overline{X})$ and $f(\overline{Y})$ are meridian disks. So $f(\overline{X})$ and $f(\overline{Y})$ may intersect, but $\overline{X}$ and $\overline{Y}$ does not intersect. Thus Σ is of type H.

Suppose that $\Sigma = (S, G, f)$ be a generalized DS-diagram of type H. Let p be any vertex of G. There exist three faces X_1, X_2 and X_3 whose closure contain the vertex p. We show one of their faces is self type. We assume that none of the faces is self type. There exist three points p_1, p_2, p_3 on another sphere such that $f(p_1) = f(p_2) = f(p_3) = f(p)$. We assume that faces around vertices p, p_1, p_2 and p_3 are as in Figure 12. For a face X we denote X' by the face such that $f(X) = f(X')$ and $X \neq X'$. Faces Y_1, Y_2 and Y_3 are self type and $\overline{Y}_1 \cap \overline{Y}_2$ is not empty. So, by the condition (3) of type H, a face Y_1 is a face Y_2. Then $\overline{Y}_1$ is not a 2-disk. This contradicts the condition (2) of type H. Thus one of the faces is self type.

We put $S = S_1^2 \cup S_2^2$ and $\mathcal{B} = B_1^3 \cup B_2^3$. For i ($i = 1, 2$), we denote U_i by the union of closures of faces in S_i^2 which are selft type. We put $H_i = B_i^3 / f_{|U_i}$. Then H_1 and H_2 are

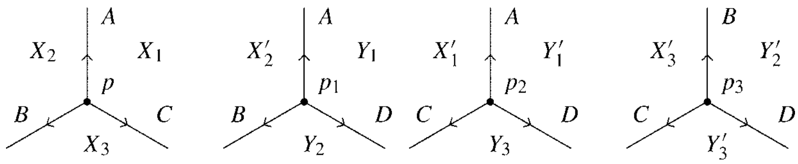

FIGURE 12

handlebodies and $(H_1, H_2; f(U_1), f(U_2))$ is a Heegaard diagram. ∎

In their papers ([6], [7]) elementary deformation of type I, elementary deformation of type II and piping are defined as deformations for DS-diagram. We can regard their deformations as for generalized DS-diagram. Piping or piping along L is the operation as in Figure 13. Elementary deformation of type I is the operation as in Figure 14. Elementary deformation of type II is the operation as in Figure 15.

PROPOSITION 3.6. *Elementary deformation of type I, elementary deformation of type II and piping are GS-deformations.*

PROOF. First we consider piping. Let ℓ be a loop whose labels are a_1, a_2, a_3 and a_4 as in Figure 16. We apply S-move along ℓ to given generalized DS-diagram. We obtain a generalized DS-diagram as in Figure 17.

Let ℓ' be a loop whose labels are b_1 and b_2 in Figure 17. We apply S-move along ℓ'. Next we apply G-move along the face whose labels are $a_1 A_2^{-1}$ and G-move along the face whose

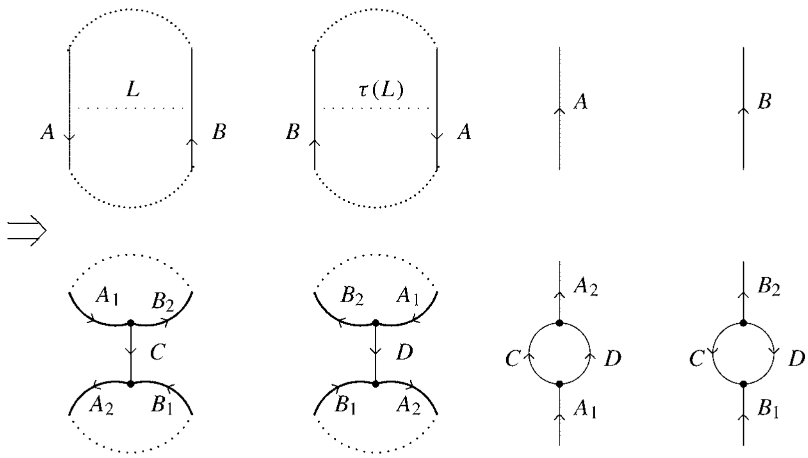

FIGURE 13

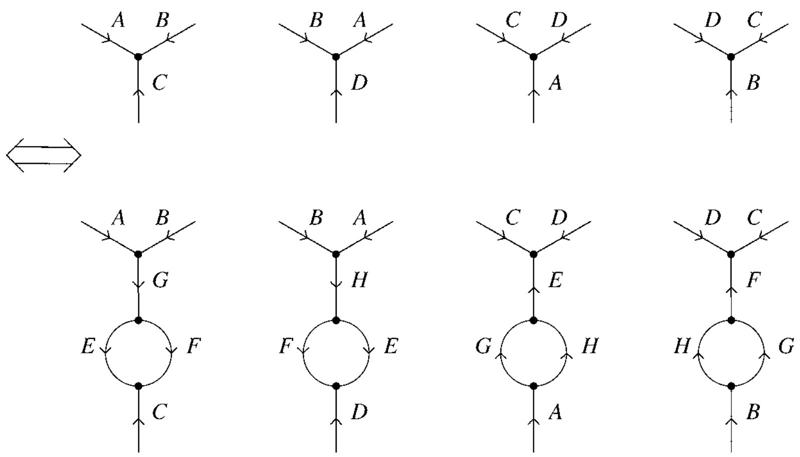

FIGURE 14

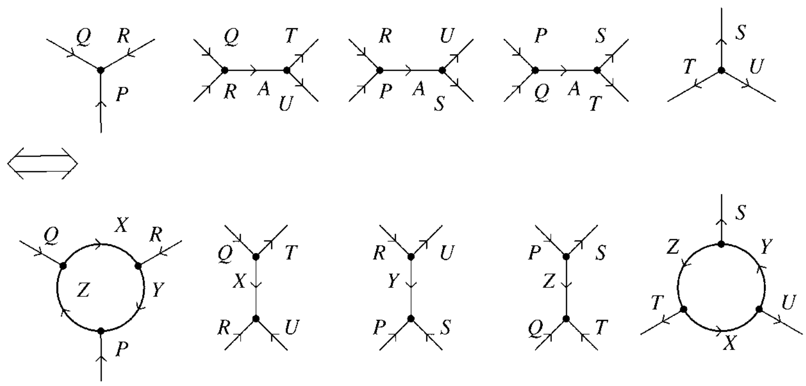

FIGURE 15

labels are $a_3 B_2^{-1}$. The generalized DS-diagram obtained by their operatons is the generalized DS-diagram obtained by piping along L.

Elementary deformation of type I is piping along L in Figure 18.

Let ℓ be a loop whose labels are a_1, a_2 and a_3, and W be a region in Figure 19. We apply S-moves along ℓ , and we apply G-move along the face W. The generalized DS-diagram obtained by their operations is the generalized DS-diagram obtained by elementary deformation of type II. ∎

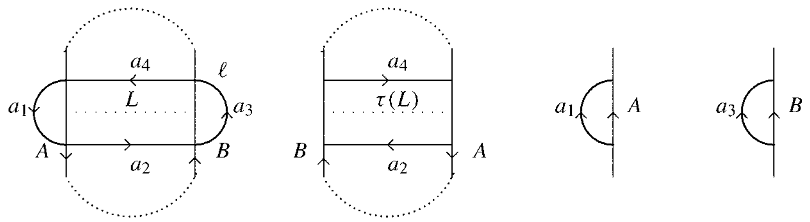

FIGURE 16

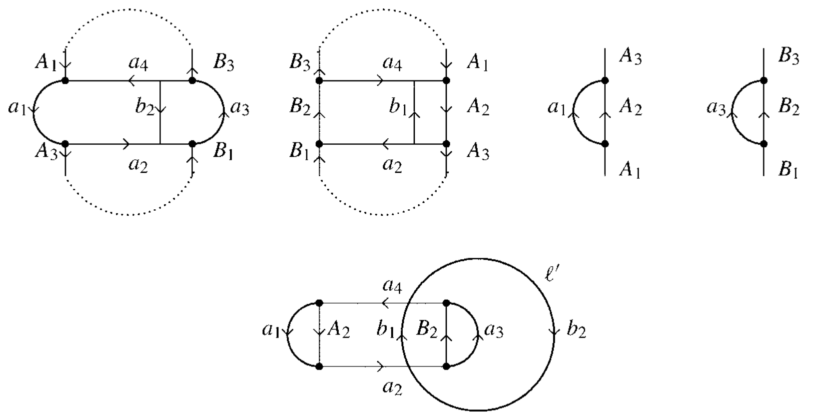

FIGURE 17

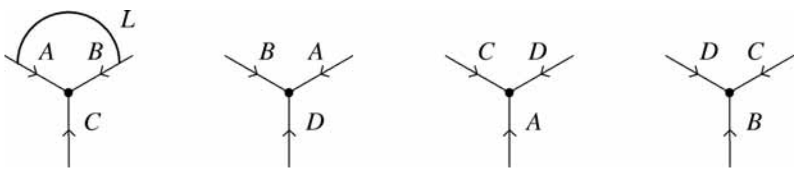

FIGURE 18

We note that their deformations do not change s-number.

LEMMA 3.7. *For a generalized DS-diagram Σ, there exists a GS-deformation $\Sigma \Longrightarrow \Sigma'$ such that the closure of any face of Σ' is homeomorphic to a 2-disk. We can choose a GS-deformation as finite application of pipings.*

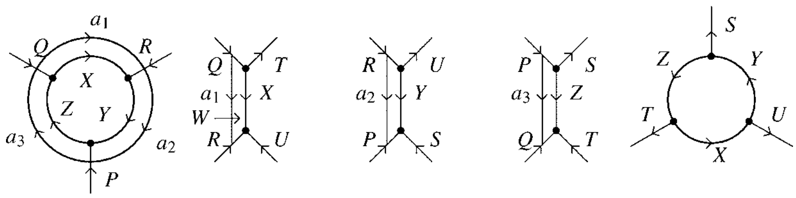

FIGURE 19

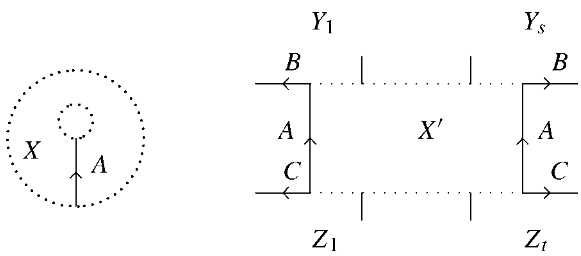

FIGURE 20

PROOF. Let X be a face of Σ which is not homeomorphic to an open 2-disk. By piping we can change X into an open 2-disk. So we assume that all faces of Σ are open 2-disks.

Suppose that $\overline{X}$ is not homeomorphic to a 2-disk. X is as in Figure 20 where X' is a face such that $f(X) = f(X')$ and $X \neq X'$. Let $Y_1, \ldots, Y_s, Z_1, \ldots, Z_t$ be adjacent faces to X'. If there exist Y_i and Z_j such that $Y_i \neq Z_j$, we choose an arc L on X' connecting Y_i and Z_j. By piping along L, we change X into two faces $\widetilde{X_1}$ and $\widetilde{X_2}$ where the number of boundary components of $\overline{\widetilde{X_i}}$ is less than that of $\overline{X}$ $(i = 1, 2)$. This operation does not generate a face whose closure is not homeomorphic to a 2-disk.

If there does not exist such Y_i and Z_j, then $Y_1 = Y_2 = \cdots = Y_s = Z_1 = \cdots = Z_t$. There exists an edge e of $\overline{X'}$ such that a label of e is not a label of the other edge of $\overline{X'}$. Let L_1 be an arc on $\overline{Y_1}$ connecting an edge with a label B and the edge e. By piping along L_1, Y_1 changes into two faces U_1 and U_2. This operation does not generate a face whose closure is not homeomorphic to a 2-disk. There exists an edges e_1 such that is contained in $\overline{U_1}$ and is contained in $\overline{Y_i} \cap \overline{X'}$ for some i. And there exists an edges e_2 such that is contained in $\overline{U_2}$ and is contained in $\overline{Z_j} \cap \overline{X'}$ for some j. Let L an arc on X' connecting e_1 and e_2. By piping along L, we change X into $\widetilde{X}$ where the number of boundary components of $\overline{\widetilde{X}}$ is less than of $\overline{X}$. This operation does not generate a face whose closure is not homeomorphic to a 2-disk. This completes the proof. ∎

LEMMA 3.8. *For a generalized DS-diagram Σ, there exist a generalized DS-diagram Σ' which is of type H and a GS–deformation $\Sigma \Longrightarrow \Sigma'$.*

PROOF. By Lemma 3.7, we may assume that the closure of any face of Σ is homeomorphic to a 2-disk. If $s(\Sigma)$ is greater than two, by G-move, we may assume $s(\Sigma)$ is equal to two. If $s(\Sigma)$ is equal to one, by S-move, we may assume $s(\Sigma)$ is equal to two. If this operation generates a face X such that X is not an open 2-disk or $\overline{X}$ is not homeomorphic to a 2-disk, we apply Lemma 3.7 once more.

We put $S = S_1^2 \cup S_2^2$. Suppose that there exist two different faces X and Y on S_1^2 such that X and Y are self type and $\overline{X} \cap \overline{Y} \neq \emptyset$. We assume that there does not exist a face W such that W is not self type, $\overline{W} \cap \overline{X} \neq \emptyset$ and $\overline{W} \cap \overline{Y} \neq \emptyset$ as in Figure 21 for any X and Y such that X and Y are self type and $\overline{X} \cap \overline{Y} \neq \emptyset$. Then all faces on S_1^2 are self type. So $M(\Sigma)$ is not connected, this is a contradiction. There exists a face W such that W is not self type, $\overline{W} \cap \overline{X} \neq \emptyset$ and $\overline{W} \cap \overline{Y} \neq \emptyset$ as in Figure 21 for some X and Y.

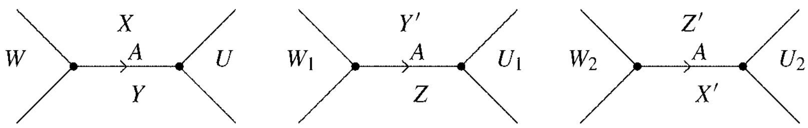

FIGURE 21

Let W_1, W_2, U, U_1 and U_2 be faces as in Figure 21. Since W is not self type, W_1 and W_2 are not self type. If U is self type, then U_1 and U_2 are self type and $W \neq U$, $W_1 \neq U_1$ and $W_2 \neq U_2$. By elementary deformation of type II, we obtain Figure 22.

This operation may change a face whose closure is a 2-disk into a face whose closure is not homeomorphic to a 2-disk. If $W = U$, this operation change a face W into a face whose closure is not homeomorphic to a 2-disk. But then U is not self type. A face which is not self type can be a face whose closure is not homeomorphic to a 2-disk. It is the same as in the cases $W_1 = U_1$ and $W_2 = U_2$. Thus we obtain a generalized DS-diagram of type H. ∎

LEMMA 3.9. *Let Σ be a generalized DS-diagram defined by Heegaard diagram $(H_1, H_2; \vec{D}_1, \vec{D}_2)$ and Σ' be a generalized DS-diagram defined by Heegaard diagram $(H_1', H_2'; \vec{D}_1', \vec{D}_2')$. $(H_1', H_2'; \vec{D}_1', \vec{D}_2')$ is obtained from $(H_1, H_2; \vec{D}_1, \vec{D}_2)$ by attachings of trivial handles. Then there exists a GS–deformation $\Sigma \Longrightarrow \Sigma'$.*

PROOF. Let $V = D_{1,g+1} \times [-1, 1]$ and $W = D_{2,g+1} \times [-1, 1]$ be handles as in the definition of attachings of trivial handles. So $(V \cup W) \cap H_1$ is a 2-disk, $\ell = \partial\big((V \cup W) \cap H_1\big)$ is a loop on S. So $L_2 \times [-1, 1]$ is a 2-disk, $\ell_1 = \partial\big(L_2 \times [-1, 1]\big)$ is a loop on $\partial\big(V \cup W\big)$.

We apply S-moves along a loop corresponding to ℓ and S-moves along a loop corresponding to ℓ_1. Next we apply G-moves along the face $D_{1,g+1} \times \{ 1 \}$ and G-moves along the

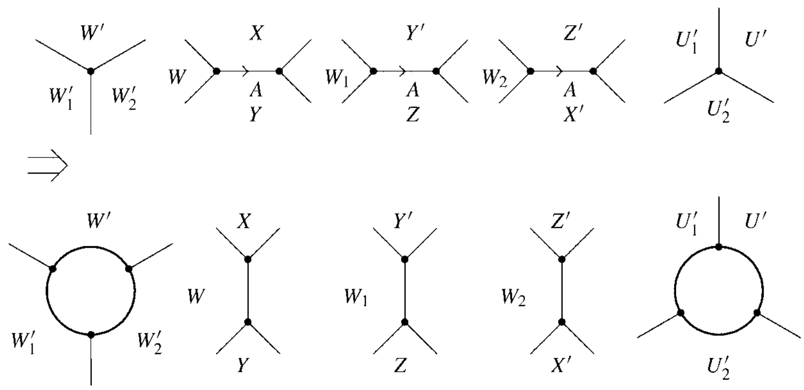

FIGURE 22

face $D_{2,g+1} \times \{1\}$. The generalized DS-diagram obtained by their operations is a generalized DS-diagram defined by Heegaard diagram $(H_1', H_2'; \vec{D}_1', \vec{D}_2')$. ∎

LEMMA 3.10. *Let Σ be a generalized DS-diagram defined by Heegaard diagram $(H_1, H_2; \vec{D}_1, \vec{D}_2)$ and Σ' be a generalized DS-diagram defined by Heegaard diagram $(H_1, H_2; \vec{D}_1', \vec{D}_2')$. Then there exists GS-deformation $\Sigma \Rightarrow \Sigma'$.*

PROOF. First we consider the case $\vec{D}_1 \cap \vec{D}_1' = \emptyset$. If each component of $\vec{D}_1'$ is parallel to some component of $\vec{D}_1$, then $\Sigma \equiv \Sigma'$. Suppose that there exists $D_{1,j}'$ which does not parallel to any component of $\vec{D}_1$. Let $D_{1,i}^+$ and $D_{1,i}^-$ be 2-disks on S_1^2 corresponding to $D_{1,i}$ $(i = 1, \dots, g)$. We put $\tilde{D}_1 \cup \tilde{D}_2 = S_1^2 - g_1^{-1}(\partial D_{1,j}')$. For all i, if both $D_{1,i}^+$ and $D_{1,i}^-$ are contained in $\tilde{D}_1$ or are contained in $\tilde{D}_2$, $D_{1,j'}$ splits H_1. This is a contradiction. So there exist $D_{1,i}^+$ and $D_{1,i}^-$ such that $D_{1,i}^+ \subset \tilde{D}_1$ and $D_{1,i}^- \subset \tilde{D}_2$. We apply S-move along $g_1^{-1}(\partial D_{1,j}')$ and G-move along $D_{1,i}^+$. We put $\vec{D}_1'' = D_{1,1} \cup \cdots \cup D_{1,i-1} \cup D_{1,j}' \cup \cdots \cup D_{1,g}$. The Heegaard diagram which is corresponding to the generalized DS-diagram is $(H_1, H_2; \vec{D}_1'', \vec{D}_2)$. So in this case, we obtain a GS-deformation from a generalized DS-diagram defined by $(H_1, H_2; \vec{D}_1, \vec{D}_2)$ to a generalized DS-diagram defined by $(H_1, H_2; \vec{D}_1', \vec{D}_2')$.

Next we consider the case $\vec{D}_1 \cap \vec{D}_1' \neq \emptyset$. By isotopy we may assume that $\vec{D}_1 \cap \vec{D}_1'$ consists of arcs. We show that the number of arcs can be decrease by GS-deformation. There exists a outermost 2-disk d in $\vec{D}_1'$. We suppose that $d \subset D_{1,1}'$, $\partial d = \alpha \cup \beta$ where β is a arc in $\partial D_{1,1}'$ and $\alpha = d \cap \vec{D}_1 = d \cap D_{1,1}$. We put $D_a \cup D_b = D_{1,1} - \alpha$. Then $(D_a \cup d, D_{1,2}, \dots, D_{1,g})$ or $(D_a \cup d, D_{1,2}, \dots, D_{1,g})$ is a complete system of meridian disks. Suppose that $(D_a \cup d, D_{1,2}, \dots, D_{1,g})$ is a complete system of meridian disks. We apply G-move along the loop corresponding to ∂d and S-move along the face corresponding to $\overline{D_b}$. Thus we can decrease the number of arcs. We can prove similarly for $\vec{D}_2 \cap \vec{D}'_2$. This completes the proof. ∎

By combining Theorem 3.3 and Lemmas 3.8, 3.9 and 3.10, we have Theorem 3.2.

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