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The Best Constant of L^p Sobolev Inequality Corresponding to Dirichlet Boundary Value Problem II

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Abstract. Let $M = 2m$ $(m = 1, 2, \dots)$. In [1] the best constant of L^p Sobolev inequality

$$\sup_{-1 \le x \le 1} |u(x)| \le C\left(\int_{-1}^{1} |u^{(M)}(x)|^p dx\right)^{1/p}$$

was obtained for u satisfying $u, u^{(M)} \in L^p(-1, 1)$ and $u^{(2i)}(\pm 1) = 0$ $(0 \le i \le [(M-1)/2])$. On the other hand, for the case M is odd, up to now, only the case $M = 1$ was treated for technical difficulty; see [2]. This paper treats the case $M = 3$ with different two approach, one is based on the property of the function associated with certain Green function and another is on the property of function space. For the latter approach, symmetrizations of functions play an important role.

1. Introduction

Let M be a natural number and $p > 1$ be a real number and let q be its conjugate satisfying $1/p + 1/q = 1$. Moreover, let us introduce Sobolev space $W(-1, 1)$ as

$$W(-1, 1) := \left\{u \mid u, u^{(M)} \in L^p(-1, 1),\ u^{(2i)}(\pm 1) = 0\, (0 \le i \le [(M-1)/2]\right\} \tag{1}$$

In [1] the best constant $C(M, p)$ of L^p Sobolev inequality

$$\sup_{-1 \le x \le 1} |u(x)| \le C\left(\int_{-1}^{1} |u^{(M)}(x)|^p dx\right)^{1/p} \tag{2}$$

was obtained for $u \in W(-1, 1)$ and $M = 2m$ $(m = 1, 2, \dots)$. On the other hand, for the case M is odd, up to now, only the case $M = 1$ (see Oshime [2, Theorem 9]) was treated for technical reason (the difference for the case M is even and M is odd will be explained in the next section). This paper studies the case $M = 3$ with different two approach, one is based on the property of certain Green function and another is on the property of function space. The main result is as follows:

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THEOREM 1. *Let $u \in W(-1,1)$, where*

$$W(-1,1) := \left\{ u \mid u, u''' \in L^p(-1,1),\ u(\pm 1) = u''(\pm 1) = 0 \right\} \tag{3}$$

Then the best constant of

$$\sup_{-1 \le x \le 1} |u(x)| \le C \|u'''\|_{L^p(-1,1)} \tag{4}$$

is given by

$$C(3,q) = \frac{1}{4}\left(B\left(q+1, \frac{1}{2} \right) \right)^{\frac{1}{q}} = \left(\frac{2\Gamma(q+1)^2}{\Gamma(2q+2)} \right)^{\frac{1}{q}} \tag{5}$$

where $B(\cdot,\cdot)$ is a beta function. The equality in (4) *holds for*

$$u(x) = \int_{-1}^{1} F(x,y)\phi(y)dy \tag{6}$$

where

$$F(x,y) = \begin{cases} -\dfrac{1}{12}(x-1)(3y^2+6y+x^2-2x)\,, & (-1 \le y \le x \le 1) \\ -\dfrac{1}{12}(x+1)(3y^2-6y+x^2+2x)\,, & (-1 \le x \le y \le 1) \end{cases} \tag{7}$$

and

$$\phi(y) = \begin{cases} -\dfrac{1}{4^{q-1}}(-y^2-2y)^{q-1}\,, & -1 \le y \le 0 \\ \dfrac{1}{4^{q-1}}(-y^2+2y)^{q-1}\,, & 0 \le y \le 1 \end{cases} \tag{8}$$

In Section 3, we prove this theorem by studying the property of the function associated with Green Function of Dirichlet boundary value problem [3]. On the other hand, in Section 4, an alternative proof is presented which is based on the following lemma with respect to the symmetrizations of the functions.

LEMMA 1. *Let S be the functional*:

$$S(u) := \frac{\sup_{-1 \le x \le 1} |u(x)|}{\left(\int_{-1}^{1} |u'''(x)|^p dx \right)^{\frac{1}{p}}} \quad (u \in W(-1,1), u \ne 0) \tag{9}$$

Then, for an arbitrary $u \in W(-1,1)$, there exists an element u_* which belongs to the following sub-set $W_*(-1,1)$ of $W(-1,1)$:

$$W_*(-1,1) := \left\{ u \in W(-1,1) \mid \max_{-1 \le x \le 1} |u(x)| = u(0),\ u(x) = u(-x)\ (-1 \le x \le 1) \right\}$$

such that

$$S(u) \le S(u_*) \tag{10}$$

Thus, to obtain the best constant of (4), it is enough to maximize (9) in $W_*(-1,1)$. This type of lemma also holds for the functions which have clamped boundary condition i.e. $u^{(i)}(\pm 1) = 0$ $(0 \le i \le M-1)$; see [4].

Finally, we would like to mention some remarks for the case $p=2$ (Hilbertian Sobolev space case). For this case, regardless of the parity of M, the best constant of (4) was obtained in [3] as $C(M,2) = (2^{2M}-1)\pi^{-2M}\zeta(2M)$, where $\zeta(\cdot)$ is the Riemann zeta function. This seems mainly due to the fact that for the case $p=2$, reproducing kernel (see for example [5]) of the Sobolev space uniquely exists, and constructing it concretely works very effective for obtaining the best constant. The following articles, Richardson [6], Kalyabin [7], [8, 9, 10, 11] also studies the best constant of the embedding inequality of the Hilbertian Sobolev space into L^∞, and most of them are obtained via the construction of the corresponding reproducing kernel.

REMARK 1. Proof of Lemma 1 does not apply to the case $M \ge 5$, since the relation

$$\tilde{u}^{(i)}(y-0) = \tilde{u}^{(i)}(y+0)$$

may fail to hold for $i \ge 3$ (see; (39)–(41)).

2. Brief review of the case M is even

As a preparation, first we briefly explain about Bernoulli polynomials and their properties which are required. Bernoulli polynomials $\{b_j\}$ are defined by the following relations:

$$\begin{cases} b_0(x) = 1 \\ b_j'(x) = b_{j-1}(x)\,, \quad \displaystyle\int_0^1 b_j(x)\,dx = 0 \quad (j=1,2,3,\ldots)\,. \end{cases}$$

Here we list explicit forms of $b_j(x)$ $(j=0,1,\ldots,6)$.

$$\begin{aligned}
&b_0(x) = 1\,, \qquad b_1(x) = x - \frac{1}{2}\,, \qquad b_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}\,, \\
&b_3(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{12}x\,, \quad b_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{24}x^2 - \frac{1}{720}\,, \\
&b_5(x) = \frac{1}{120}x^5 - \frac{1}{48}x^4 + \frac{1}{72}x^3 - \frac{1}{720}x\,, \\
&b_6(x) = \frac{1}{720}x^6 - \frac{1}{240}x^5 + \frac{1}{288}x^4 - \frac{1}{1440}x^2 + \frac{1}{30240}\,, \\
&\ldots
\end{aligned}$$

Next, we prepare the lemma concerning the existence and uniqueness of the solution of the following boundary value problems.

LEMMA 2 ([12, Theorem 3.1]). *Let f be a bounded continuous function. We consider the following boundary value problems*:

$BVP(D, m)$ (Dirichlet):

$$\begin{cases} (-1)^m u^{(2m)}(x) = f(x), & (-1 < x < 1) \\ u^{(2i)}(\pm 1) = 0, & (0 \le i \le m-1) \end{cases}$$

$BVP(N, m)$ (Neumann):

$$\begin{cases} \int_{-1}^{1} f(x)dx = 0 \\ (-1)^m u^{(2m)}(x) = f(x), & (-1 < x < 1) \\ \int_{-1}^{1} u(x)dx = 0 \\ u^{(2i+1)}(\pm 1) = 0, & (0 \le i \le m-1) \end{cases}$$

Then $BVP(D, m)$ and $BVP(N, m)$ have a unique classical solution u expressed by

$$u(x) = \int_{-1}^{1} G(\mathrm{D}, m; x, y) f(y) dy \tag{11}$$

and

$$u(x) = \int_{-1}^{1} G(\mathrm{N}, m; x, y) f(y) dy \tag{12}$$

respectively, where $G(\mathrm{D}, m, x, y)$ and $G(\mathrm{N}, m, x, y)$ are Green functions expressed by Bernoulli polynomials:

$$G(\mathrm{D}, m; x, y) = (-1)^{m+1} 4^{2m-1} \left[b_{2m}\left(\frac{|x-y|}{4}\right) - b_{2m}\left(\frac{2-x-y}{4}\right) \right] \tag{13}$$

$$G(\mathrm{N}, m; x, y) = (-1)^{m+1} 4^{2m-1} \left[b_{2m}\left(\frac{|x-y|}{4}\right) + b_{2m}\left(\frac{2-x-y}{4}\right) \right] \tag{14}$$

Using this lemma, we obtain the following lemma.

LEMMA 3. *For an arbitrary $u \in W(-1, 1)$ and $x \in [-1, 1]$, there exists an element $H(x, \cdot) \in L^q(-1, 1)$ satisfying*

$$u(x) = \int_{-1}^{1} H(x, y)\, u^{(M)}(y) dy \tag{15}$$

PROOF. We can take

$$\begin{aligned} &\partial_y^M G(\mathrm{D}, M, x, y) \qquad (16) \\ &= (-1)^{M+1} 4^{M-1} \left[\left(\operatorname{sgn}(y-x)\right)^M b_M\left(\frac{|y-x|}{4}\right) + (-1)^{M+1} b_M\left(\frac{2-y-x}{4}\right) \right] \end{aligned}$$

as $H(x,y)$, see Theorem 3.1 [1]. □

Let us see Lemma 3 by an example. Assume $M = 3$, then

$$\partial_y^3 G(\mathrm{D}, 3; x, y) = 4^2\left[(\mathrm{sgn}(y-x))b_3\left(\frac{|y-x|}{4}\right) + b_3\left(\frac{2-y-x}{4}\right)\right] \tag{17}$$
$$= \begin{cases} -\dfrac{1}{12}(x-1)(3y^2+6y+x^2-2x)\,, & (-1 \le y \le x \le 1) \\ -\dfrac{1}{12}(x+1)(3y^2-6y+x^2+2x)\,, & (-1 \le x \le y \le 1) \end{cases}$$

So,

$$\int_{-1}^{1} \partial_y^3 G(\mathrm{D}, 3; x, y) u''' dy$$
$$= -\frac{x-1}{12}\int_{-1}^{x}(3y^2+6y+x^2-2x)u'''(y)dy - \frac{x+1}{12}\int_{x}^{1}(3y^2-6y+x^2+2x)u'''(y)dy$$
$$= -\frac{x-1}{12}\big[(3y^2+6y+x^2-2x)u''(y)\big]_{-1}^{x} + \frac{x-1}{12}\int_{-1}^{x}(6y+6)u''(y)dy$$
$$-\frac{x+1}{12}\big[(3y^2-6y+x^2+2x)u''(y)\big]_{x}^{1} + \frac{x+1}{12}\int_{x}^{1}(6y-6)u''(y)dy$$
$$= \frac{x-1}{2}\int_{-1}^{x}(y+1)u''(y)dy + \frac{x+1}{2}\int_{x}^{1}(y-1)u''(y)dy$$
$$= \frac{x-1}{2}\big[(y+1)u'(y)\big]_{-1}^{x} - \frac{x-1}{2}\int_{-1}^{x}u'(y)dy + \frac{x+1}{2}\big[(y-1)u'(y)\big]_{x}^{1}$$
$$-\frac{x+1}{2}\int_{x}^{1}u'(y)dy$$
$$= -\frac{x-1}{2}\int_{-1}^{x}u'(y)dy - \frac{x+1}{2}\int_{x}^{1}u'(y)dy = u(x)$$

where $u(\pm 1) = 0$, $u''(\pm 1) = 0$ were used.

REMARK 2. $H(x,y)$ in Lemma 3 is not unique. Indeed, when $M = 3$,

$$H(x,y) = \begin{cases} c + \dfrac{1-x}{2}y + \dfrac{1-x}{4}y^2\,, & (-1 \le y \le x \le 1) \\ (c - \dfrac{x^2}{2}) + \frac{1+x}{2}y - \dfrac{1+x}{4}y^2\,, & (-1 \le x \le y \le 1) \end{cases} \tag{18}$$

also satisfies (15), where $c = c(x)$ is an arbitrary function in x, i.e., a constant with respect to y. If we take $c = -(x-1)(x^2-2x)/12$, this coincide with $\partial_y^3 G(\mathrm{D}, 3; x, y)$.

Now, from, (15), we have by Hölder's inequality

$$|u(x)| = \left| \int_{-1}^{1} H(x,y)u^{(M)}(y)dy \right| \le \|H(x,\cdot)\|_{L^q(-1,1)} \|u^{(M)}\|_{L^p(-1,1)} \tag{19}$$

For $M = 2m$, we have the following proposition which is the key result of [1].

PROPOSITION 1 ([1, Theorem 4.1]). *If $M = 2m$, then it holds that*

$$\sup_{-1\le x\le 1} \|H(x,\cdot)\|_{L^q(-1,1)} = \|H(0,\cdot)\|_{L^q(-1,1)} \tag{20}$$

Thus, if there exists $u \in W(-1,1)$ satisfying the equality

$$|u(0)| = \|H(0,\cdot)\|_{L^q(-1,1)} \|u^{(M)}\|_{L^p(-1,1)} \tag{21}$$

such u attains the best constant of (4), and $C(M,p) = \|H(0,\cdot)\|_{L^q(-1,1)}$. Clearly, necessary and sufficient conditions for the existence of such u is that $u \in W(-1,1)$ satisfies the equality condition of Hölder's inequality in (19), so it must be the solution of the following boundary value problem:

$$\begin{cases} u^{(2m)}(x) = \big(\mathrm{sgn} H(0,x)\big)|H(0,x)|^{q-1}, & (-1 < x < 1) \\ u^{(2i)}(\pm 1) = 0, & (0 \le i \le [(M-1)/2] = m-1) \end{cases} \tag{22}$$

This is the boundary value problem BVP(D,m) of Lemma 2, so the existence of $u \in W(-1,1)$ satisfying the equality (21) is assured.
Now, let us consider the case $M = 2m-1$. Assume the assertion of Proposition 1 holds for $M = 2m-1$, and

$$\int_{-1}^{1} (\mathrm{sgn} H(0,x))|H(0,x)|^{q-1} dx = 0 \tag{23}$$

Then the best constant of (4) will be attained by the function u satisfying

$$\begin{cases} \int_{-1}^{1} (\mathrm{sgn} H(0,x))|H(0,x)|^{q-1} dx = 0 \\ u^{(2m-1)}(x) = \big(\mathrm{sgn} H(0,x)\big)|H(0,x)|^{q-1}, & (-1 < x < 1) \\ u^{(2i)}(\pm 1) = 0, & (0 \le i \le [(M-1)/2] = m-1) \end{cases} \tag{24}$$

if it exists. The existence of such u is assured by the boundary value problem BVP(N,m). To see this, let us put $u(x) = v'(x)$ in (24). Then v satisfies

$$\begin{cases} \int_{-1}^{1} (\mathrm{sgn} H(0,x))|H(0,x)|^{q-1} dx = 0 \\ v^{(2m)}(x) = \big(\mathrm{sgn} H(0,x)\big)|H(0,x)|^{q-1}, & (-1 < x < 1) \\ v^{(2i+1)}(\pm 1) = 0, & (0 \le i \le [(M-1)/2] = m-1) \end{cases} \tag{25}$$

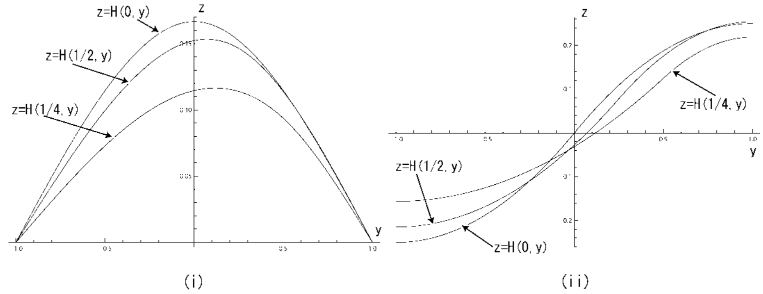

FIGURE 1. (The graphs of (i) $z = H(x, y) = \partial_y^4 G(\mathrm{D}, 4; x, y)$ and (ii) $z = H(x, y) = \partial_y^3 G(\mathrm{D}, 3; x, y)$)

Thus, from BVP(N,m), v exists and hence u is expressed as

$$u(x) = \int_{-1}^{1} \partial_x G(\mathrm{N}, m, x, y)\big(\mathrm{sgn} H(0, y)\big)|H(0, y)|^{q-1} dy \tag{26}$$

Hence, all we have to do is to show the Proposition 1 for $M = 2m - 1$ and the equality (23). However, different from the case $M = 2m$, this seems to be difficult if we take $\partial_y^{2m-1} G(\mathrm{D}, 2m-1; x, y)$ as $H(x, y)$ (although $\partial_y^{2m-1} G(\mathrm{D}, 2m-1; 0, y)$ satisfies the equality (23); see the graph of $z = H(0, y)$ in Fig. 1-(ii)). One of the reasons for its difficulty is failure of positivity. For the case $M = 2m$, $H(x, y) = \partial_y^{2m} G(\mathrm{D}, 2m; x, y)$ behaves like Fig. 1-(i), on the other hand, for the case $M = 2m - 1$, $H(x, y) = \partial_y^{2m-1} G(\mathrm{D}, 2m-1; x, y)$ is like Fig. 1-(ii), thus fails to hold the positivity. For this reason, the method used in [1] to prove Proposition 1 can not be applicable.

3. Proof of Theorem 1 (Based on Green function method)

From now on, we fix $M = 3$. Let us put $c = -x(1 - x)/4$ in (18). Then $H(x, y)$ becomes

$$H(x, y) = \begin{cases} \dfrac{1}{4}(1 - x)(y^2 + 2y - x)\,, & (-1 \le y \le x \le 1) \\ -\dfrac{1}{4}(1 + x)(y^2 - 2y + x)\,, & (-1 \le x \le y \le 1) \end{cases} \tag{27}$$

Further, we have the following proposition which correspond to Proposition 1.

PROPOSITION 2. *Let $H(x, y)$ be* (27), *then* (20) *holds.*

From this proposition, Theorem 1 is proved as follows. First, we see that (23) is satisfied. This follows from the fact that

$$H(0,y)=\begin{cases}\dfrac{1}{4}(y^2+2y)\,, & (-1\le y\le 0)\\ \dfrac{1}{4}(-y^2+2y)\,, & (0\le y\le 1)\end{cases} \tag{28}$$

is an odd function. Moreover, from (17), we have

$$\begin{aligned}\partial_x G(\mathrm{N},2;x,y)&=4^2\left[(\mathrm{sgn}(y-x))b_3\left(\frac{|y-x|}{4}\right)+b_3\left(\frac{2-y-x}{4}\right)\right]\\ &=\partial_y^3 G(\mathrm{D},3;x,y)\end{aligned} \tag{29}$$

Substituting (28) and (29) into (26), we obtain the expression (6), i.e. the function which attains the best constant. Next, we compute the best constant $C(3,p)$:

$$\begin{aligned}C(3,p)=\|H(0,\cdot)\|_{L^q(-1,1)}&=\frac{1}{4}\left(2\int_0^1(y(2-y))^q dy\right)^{\frac{1}{q}}=\frac{1}{4}\left(\int_0^1(1-t)^q t^{-\frac{1}{2}}dt\right)^{\frac{1}{q}}\\ &=\frac{1}{4}\left(B\left(q+1,\frac{1}{2}\right)\right)^{\frac{1}{q}}=\frac{1}{4}\left(\frac{\Gamma(q+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(q+\frac{3}{2}\right)}\right)^{\frac{1}{q}}=\left(\frac{2\Gamma(q+1)^2}{\Gamma(2q+2)}\right)^{\frac{1}{q}}\end{aligned}$$

Now, let us prove Proposition 2. Since $H(-x,-y)=-H(x,y)$, we have $\|H(-x,\cdot)\|_{L^q(-1,1)}=\|H(x,\cdot)\|_{L^q(-1,1)}$, so, we may only consider the case $x\ge 0$.

LEMMA 4. *Let $\eta=-1+\sqrt{x+1}$ be a zero point of $H(x,\cdot)$ and let*

$$K(x,y)=\begin{cases}H(x,y)=\dfrac{1}{4}(1-x)(y^2+2y-x)\,, & (-1\le y\le\eta)\\ -\dfrac{(1-x)(1+x)}{4(1-\eta)^2}(y^2-2y-\eta^2+2\eta)\,, & (\eta\le y\le 1)\end{cases} \tag{30}$$

Then it holds that $K(0,y)=H(0,y)$ $(-1\le y\le 1)$ and for arbitrary $x\in[-1,1]$

$$|H(x,y)|\le|K(x,y)|\quad(-1\le y\le 1) \tag{31}$$

PROOF. $K(0,y)=H(0,y)$ $(-1\le y\le 1)$ is clear. As noted, we can assume $x\ge 0$, so, it holds that $0\le\eta\le x/2$ since $\sqrt{1+x}\le 1+x/2$. So, we have

$$\frac{1-x}{(1-\eta)^2}\le\frac{1-x}{(1-x/2)^2}\le\frac{1-x}{1-x}=1$$

Therefore, for $y\in[x,1]$ we have

$$-\frac{(1-x)(1+x)}{4(1-\eta)^2}(y-1)^2+\frac{(1-x)(1+x)}{4}\ge-\frac{1+x}{4}(y-1)^2+\frac{(1-x)(1+x)}{4}$$

So,

$$-\frac{(1-x)(1+x)}{4(1-\eta)^2}(y^2-2y+2\eta-\eta^2) \geq -\frac{1+x}{4}(y^2-2y+x) \tag{32}$$

That is

$$K(x,y) \geq H(x,y) \geq 0\,, \quad (x \leq y \leq 1) \tag{33}$$

For the case $\eta \leq y \leq x$, if $x = 0$ then $\eta = x$, so we have nothing to do, thus we can assume $x > 0$. In this case, note that $K(x,x) \geq H(x,x)$ holds from (33) and $K(x,\eta) = H(x,\eta) = 0$. Moreover, on the interval $[\eta, x]$, $H(x,\cdot)$ is convex and $K(x,\cdot)$ is a concave function, and hence

$$K(x,y) \geq H(x,y) \geq 0\,, \quad (\eta \leq y \leq x) \tag{34}$$

holds. Recalling $K(x,y) = H(x,y)$ on $(-1 \leq y \leq \eta)$, we obtain the assertion. □

LEMMA 5. *For $x \in (0,1]$, we have*

$$\|K(x,\cdot)\|_{L^q(-1,1)} < \|K(0,\cdot)\|_{L^q(-1,1)} = \|H(0,\cdot)\|_{L^q(-1,1)} \tag{35}$$

PROOF.

$$\begin{aligned}
\int_{-1}^{1} |K(x,y)|^q dy &= 4^{-q}(1-x)^q \int_{-1}^{\eta} |y^2+2y-x|^q dy \\
&\quad + 4^{-q}(1-x)^q(1+x)^q(1-\eta)^{-2q} \int_{\eta}^{1} |y^2-2y-\eta^2+2\eta|^q dy \\
&= 2^{-1}4^{-q}(1-x)^q \int_{-2-\eta}^{\eta} |y^2+2y-x|^q dy \\
&\quad + 2^{-1}4^{-q}(1-x)^q(1+x)^q(1-\eta)^{-2q} \int_{\eta}^{2-\eta} |y^2-2y-\eta^2+2\eta|^q dy \\
&= 2^{-1}4^{-q}(1-x)^q \int_{-2-\eta}^{\eta} |(y+2+\eta)(y-\eta)|^q dy \\
&\quad + 2^{-1}4^{-q}(1-x)^q(1+x)^q(1-\eta)^{-2q} \int_{\eta}^{2-\eta} |(y-2+\eta)(y-\eta)|^q dy
\end{aligned}$$

where we used the symmetry of the integrands around $y = -1$ and $y = 1$. Changing the variable in the integrands, above integrands reduces to

$$\begin{aligned}
&\left[(1-x)^q(1+\eta)^{2q+1} + (1-x)^q(1+x)^q(1-\eta)\right] \int_0^1 y^q(1-y)^q dy \\
&= 2(1-x)^q(1+x)^q \int_0^1 y^q(1-y)^q dy = 2(1-x^2)^q \int_0^1 y^q(1-y)^q dy
\end{aligned}$$

where the relation $\eta = -1 + \sqrt{1+x}$ was used. Thus, we have proven the lemma. □

From Lemmas 4 and 5, we obtain for $-1 \le x \le 1$ and $x \neq 0$,

$$\|H(x,\cdot)\|_{L^q(-1,1)} \le \|K(x,\cdot)\|_{L^q(-1,1)} < \|K(0,\cdot)\|_{L^q(-1,1)} = \|H(0,\cdot)\|_{L^q(-1,1)} \quad (36)$$

Thus Proposition 2 was proved, and hence, the proof of Theorem 1 is finished.

4. Proof of Theorem 1 (Based on the symmetrizations of functions)

In this section, an alternative proof of Theorem 1, which is based on Lemma 1 (that seems to reflect the geometric nature of the function space) is considered.
From, Lemma 1, the best constant of (4) is attained by an element $u \in W_*(-1,1)$. Thus, if there exists an element $u \in W_*(-1,1)$ such that

$$|u(0)| = \|H(0,\cdot)\|_{L^q(-1,1)}\|u'''\|_{L^p(-1,1)}$$

in other words, if there exists an element satisfying

$$\begin{cases} u'''(x) = (\mathrm{sgn} H(0,x))|H(0,x)|^{q-1}\,, & (-1 < x < 1) \\ u(\pm 1) = u''(\pm 1) = 0\,, & \\ u(-x) = u(x)\,, & (-1 < x < 1) \\ \max_{-1 \le x \le 1} |u(x)| = u(0) & \end{cases} \quad (37)$$

then, such u attains the best constant. Let us see u given by (6) satisfies above properties (Note that $H(x,y) \equiv F(x,y)$ in Theorem 1). As shown in Section 3, u given by (6) satisfies above differential equation and boundary conditions. Next, we see that u is an even function. Since $F(-x,-y) = -F(x,y)$, we have

$$\begin{aligned} u(-x) &= \int_{-1}^{1} F(-x,y)\phi(y)dy = \int_{-1}^{1} F(-x,-y)\phi(-y)dy \\ &= \int_{-1}^{1} -F(x,y)\phi(-y)dy = \int_{-1}^{1} F(x,y)\phi(y)dy = u(x) \end{aligned}$$

Finally, we show that the fourth property of (37) holds. From (6), and relation

$$\int_{-1}^{1} \phi(x)dx = 0$$

we have

$$\begin{aligned} u''(x) &= \int_{-1}^{x} -\frac{x-1}{2}\phi(y)dy + \int_{x}^{1} -\frac{x+1}{2}\phi(y)dy \\ &= \frac{1}{2}\int_{-1}^{x} \phi(x)dx - \frac{1}{2}\int_{x}^{1} \phi(y)dy = \int_{-1}^{x} \phi(y)dy \end{aligned}$$

Hence, for some constant c,

$$u'(x) = \int_0^x \int_{-1}^s \phi(t)dtds + c$$

Computing $u'(0)$ from (6), we have

$$\begin{aligned} u'(0) &= \int_{-1}^0 -\frac{1}{12}(3y^2+6y+2)\phi(y)dy + \int_0^1 -\frac{1}{12}(3y^2-6y+2)\phi(y)dy \\ &= -\frac{1}{2}\int_{-1}^0 y\phi(y)dy + \frac{1}{2}\int_0^1 y\phi(y)dy = 0 \end{aligned}$$

Thus, u' is expressed as

$$u'(x) = \int_0^x \int_{-1}^s \phi(t)dtds$$

Noting that

$$\int_{-1}^s \phi(t)dt\,, \quad (-1 \le s \le 1)$$

is an even function and negative on $(-1, 1)$, we obtain that

$$\begin{cases} u'(x) < 0\,, & (0 < x \le 1) \\ u'(x) = -u'(-x) > 0\,, & (-1 \le x < 0) \end{cases}$$

So, fourth property of (37) was shown

Now, all we have to do is to prove Lemma 1.

PROOF OF LEMMA 1. Let u be an arbitrary element of $W(-1, 1)$, and let

$$\max_{-1\le x\le 1} |u(x)| = u(y)$$

Without loss of generality, we can assume $y \ge 0$, since if it does not, $u(x)$ can be replaced by $u(-x)$.

Case (i)

$$\int_{-1}^y |u'''(x)|^p dx \ge \int_y^1 |u'''(x)|^p dx \tag{38}$$

Let us define $\tilde{u}$ as

$$\tilde{u}(x) := \begin{cases} 0\,, & (-1 \le x < -1+2y) \\ u(2y-x)\,, & (-1+2y \le x < y) \\ u(x)\,, & (y \le x \le 1) \end{cases}$$

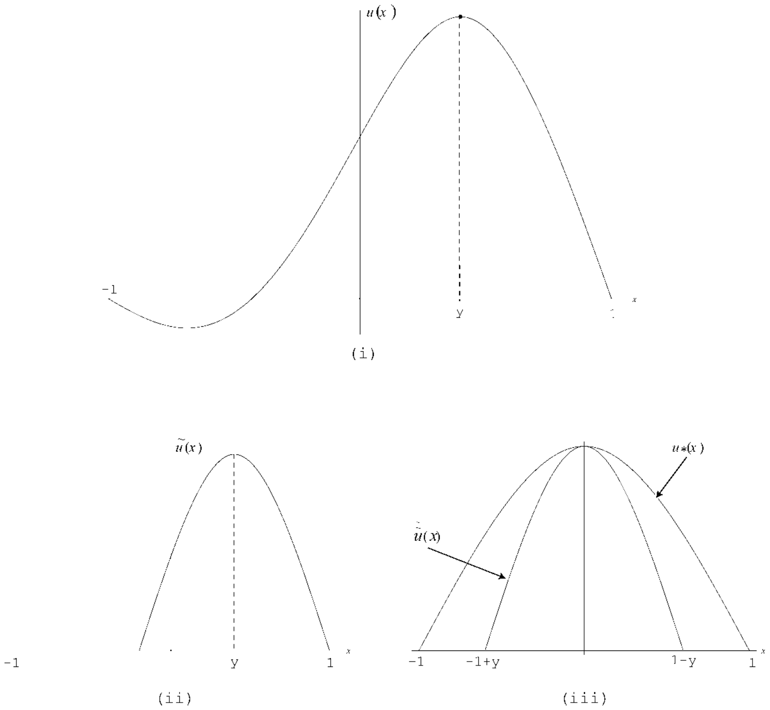

FIGURE 2. (The graph of (i) u, (ii) $\tilde{u}(x)$, (iii) $\tilde{\tilde{u}}$ and u_*)

We have

$$\tilde{u}(y-0) = \tilde{u}(y+0) = u(y) \tag{39}$$

$$\tilde{u}'(y-0) = \tilde{u}'(y+0) = 0 \tag{40}$$

$$\tilde{u}''(y-0) = \tilde{u}''(y+0) = u''(y)\,. \tag{41}$$

Further, let us define

$$\tilde{\tilde{u}}(x) := \begin{cases} \tilde{u}(x+y)\,, & (-1+y \le x \le 1-y) \\ 0\,, & (1-y < |x| \le 1)\,, \end{cases}$$

and $u_*(x) := \tilde{\tilde{u}}((1-y)x)$. Then $u_* \in W_*(-1,1)$, since by (39), (40), (41), $\max_{-1\le x\le 1} |u_*(x)| = u_*(0)$, and $u_*^{(2i)}(\pm 1) = 0$ $(i = 0, 1)$. Moreover, from (38), we have $\|u'''\|^p_{L^p(-1,1)} \ge \|\tilde{u}'''\|^p_{L^p(-1+2y,1)} = \|\tilde{\tilde{u}}'''\|^p_{L^p(-1+y,1-y)} \ge (1-y)^{3p-1}\|\tilde{\tilde{u}}'''\|^p_{L^p(-1+y,1-y)} = \|u_*\|^p_{L^p(-1,1)}$ In addition, clearly $u_*(0) = \max_{-1\le x\le 1} |u_*(x)| = u(y) = \max_{-1\le x\le 1} |u(x)|$. So, in the case (i), we have proven the lemma; see Fig. 2.

Case (ii)

$$\int_{-1}^{y} |u'''(x)|^p dx < \int_{y}^{1} |u'''(x)|^p dx \tag{42}$$

Let t be an element satisfying $0 \le t \le y$, and let

$$x' = -1 + \frac{t+1}{y+1}(x+1) = -1 + a(x+1) \tag{43}$$

Further, let $U_t(x')$ be

$$U_t(x') := u\left(\frac{x'+1}{a} - 1\right), \quad (-1 \le x' \le t) \tag{44}$$

So,

$$\partial_{x'}^3 U_t(x') = a^{-3}\partial_x^3 u(x)|_{x=\frac{x'+1}{a}-1} = a^{-3}u'''\left(\frac{x'+1}{a} - 1\right)$$

and hence we obtain

$$\int_{-1}^{t} |\partial_{x'}^3 U_t(x')|^p dx' = a^{-3p}\int_{-1}^{t} |u'''\left(\frac{x'+1}{a} - 1\right)|^p dx' \tag{45}$$

By putting $x' = -1 + a(x+1)$ the right hand side of (45) becomes

$$= a^{-3p+1}\int_{-1}^{y} |u'''(x)|^p dx \tag{46}$$

Similarly, let us put

$$x' = 1 + \frac{1-t}{1-y}(x-1) = 1 + b(x-1) \tag{47}$$

and define $U_t(x')$ as

$$U_t(x') := u\left(\frac{x'-1}{b} + 1\right), \quad (t \le x' \le 1) \tag{48}$$

So,

$$\partial_{x'}^3 U_t(x') = b^{-3}\partial_x^3 u(x)|_{x=\frac{x'-1}{b}+1} = b^{-3}u'''\left(\frac{x'-1}{b} + 1\right)$$

and hence we obtain

$$\int_{t}^{1} |\partial_{x'}^3 U_t(x')|^p dx' = b^{-3p}\int_{t}^{1} |u'''\left(\frac{x'-1}{b} + 1\right)|^p dx' \tag{49}$$

By putting $x' = 1 + b(x-1)$ the right hand side of (49) becomes to

$$= b^{-3p+1} \int_y^1 |u'''(x)|^p dx \tag{50}$$

Let us put

$$A := \int_{-1}^y |u'''(x)|^p dx\,, \quad B := \int_y^1 |u'''(x)|^p dx$$

and define

$$\begin{aligned} f(t) :&= \int_{-1}^t |U_t'''(x)|^p dx + \int_t^1 |U_t'''(x)|^p dx \\ &= \left(\frac{t+1}{y+1}\right)^{-3p+1} A + \left(\frac{1-t}{1-y}\right)^{-3p+1} B \end{aligned} \tag{51}$$

Note that

$$f(y) = A + B = \|u'''\|_{L^p(-1,1)}^p \tag{52}$$

The derivative of f is

$$f'(t) = (3p-1)\left\{-\frac{1}{y+1}\left(\frac{t+1}{y+1}\right)^{-3p} A + \frac{1}{1-y}\left(\frac{1-t}{1-y}\right)^{-3p} B\right\} \tag{53}$$

<u>Case (ii)-(a)</u>

$$1 \le \left(\frac{1-y}{1+y}\right)^{3p-1} \frac{B}{A} \tag{54}$$

In this case, we have

$$\begin{aligned} & f'(t) \ge 0 \\ & \quad \leftrightarrow (y+1)^{3p-1}(t+1)^{-3p} A \le (1-y)^{3p-1}(1-t)^{-3p} B \\ & \quad \leftrightarrow \left(\frac{1-t}{1+t}\right)^{3p} \le \left(\frac{1-y}{1+y}\right)^{3p-1} \frac{B}{A} \end{aligned}$$

Since

$$\max_{0 \le t \le y} \left(\frac{1-t}{1+t}\right)^{3p} = 1$$

from the assumption (54), f is monotone increasing. So, we have

$$\min_{0 \le t \le y} f(t) = f(0) = \int_{-1}^0 |U_0'''(x)|^p dx + \int_0^1 |U_0'''(x)|^p dx$$

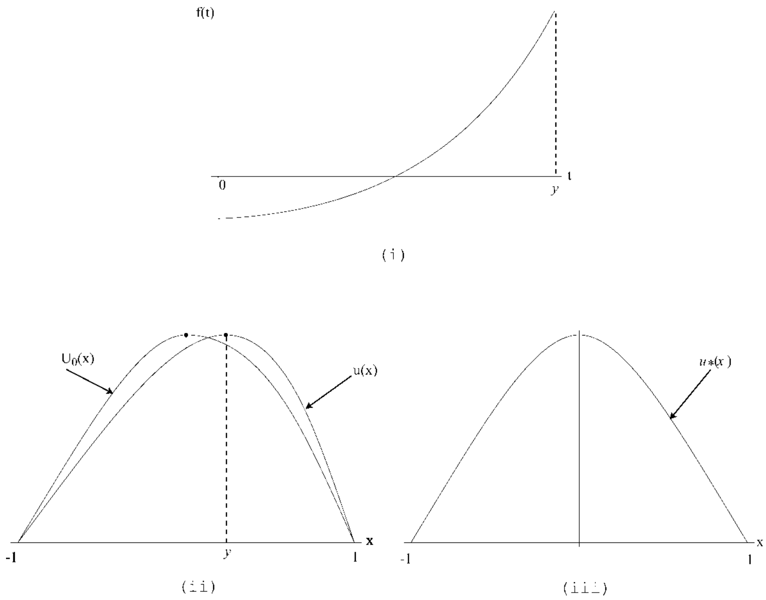

FIGURE 3. (The graph of (i) f, (ii) u and U_0, (iii) u_*)

But, from (54), it holds that

$$\int_{-1}^{0} |U_0'''(x)|^p dx = \left(\frac{1}{y+1}\right)^{-3p+1} A \leq \left(\frac{1}{1-y}\right)^{-3p+1} B = \int_0^1 |U_0'''(x)|^p dx$$

So, if we put

$$u_*(x) := \begin{cases} U_0(x)\,, & (-1 \leq x \leq 0) \\ U_0(-x)\,, & (0 \leq x \leq 1) \end{cases}$$

as Case (i), we have $u_* \in W_*(-1,1)$ and $\|u_*'''\|^p_{L^p(-1,1)} \leq \|U_0'''\|^p_{L^p(-1,0)} + \|U_0'''\|^p_{L^p(0,1)} = f(0) \leq f(y) = \|u'''\|^p_{L^p(-1,1)}$. In addition, $u_*(0) = \max_{-1 \leq x \leq 1} |u_*(x)| = u(y) = \max_{-1 \leq x \leq 1} |u(x)|$. So, we have proven the case (ii)-(a). The shapes of the graph f, U_0 and u_* are like Fig. 3.

Case (ii)-(b)

$$\left(\frac{1-y}{1+y}\right)^{3p-1} \frac{B}{A} < 1 \tag{55}$$

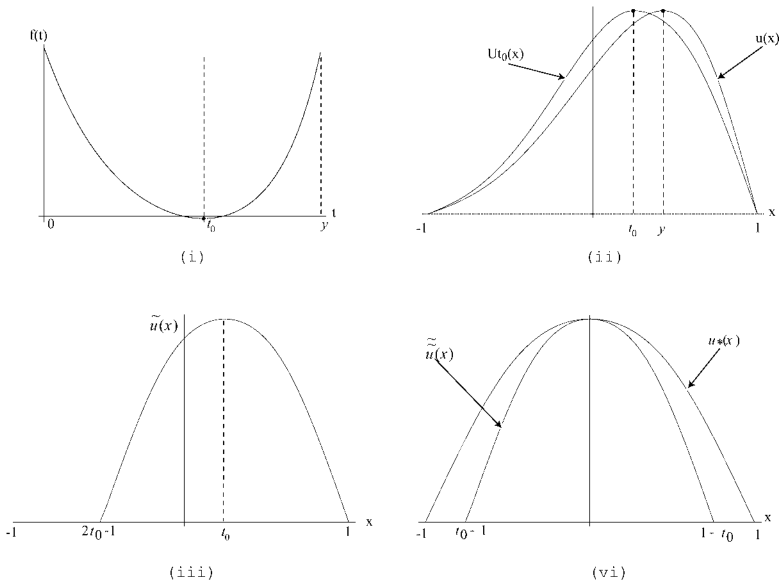

FIGURE 4. (The graph of (i) f, (ii) u and U_{t_0}, (iii) $\tilde{u}$, (iv) $\tilde{\tilde{u}}$ and u_*)

In this case, we have

$$f''(t) = 3p(3p-1)\left\{\frac{1}{y+1}\left(\frac{t+1}{y+1}\right)^{-3p-1} A + \frac{1}{1-y}\left(\frac{1-t}{1-y}\right)^{-3p-1} B\right\} > 0$$

Moreover

$$f'(0) = (3p-1)\left\{-(1+y)^{3p-1}A + (1-y)^{3p-1}B\right\} < 0$$

$$f'(y) = (3p-1)\left(-\frac{1}{y+1}A + \frac{1}{1-y}B\right) > 0$$

since (55) and the assumption (42) ($A < B$) respectively; see Fig. 4-(i). Therefore, there exists t_0 ($0 < t_0 < y$) such that $f'(t_0) = 0$. Let us define the constant M as

$$M := \left(\frac{1-y}{1+y}\right)^{\frac{3p-1}{3p}} \left(\frac{B}{A}\right)^{\frac{1}{3p}},$$

then t_0 can be expressed as $t_0 = (1-M)/(1+M)$. Now we have

$$\int_{t_0}^{1} |U_{t_0}'''(x)|^p dx < \int_{-1}^{t_0} |U_{t_0}'''(x)|^p dx \tag{56}$$

since

$$\int_{t_0}^{1} |U_{t_0}'''(x)|^p dx < \int_{-1}^{t_0} |U_{t_0}'''(x)|^p dx$$
$$\leftrightarrow \left(\frac{1-t_0}{1-y}\right)^{-3p+1} B < \left(\frac{t_0+1}{y+1}\right)^{-3p+1} A \leftrightarrow \left(\frac{1-y}{1+y}\right)^{3p-1} \frac{B}{A} < \left(\frac{1-t_0}{1+t_0}\right)^{3p-1}$$
$$\leftrightarrow M^{3p} < M^{3p-1} \leftrightarrow M < 1 \leftrightarrow (55)$$

So, let $t = t_0$ and define $\tilde{u}$ as

$$\tilde{u}(x) := \begin{cases} 0\,, & (-1 \le x \le 2t_0 - 1) \\ U_{t_0}(2t_0 - x)\,, & (2t_0 - 1 \le x \le t_0) \\ U_{t_0}(x)\,, & (t_0 \le x \le 1) \end{cases}$$

In this case, the shapes of U_{t_0} and $\tilde{u}$ are like Fig. 4-(ii) and (iii). Further, let us define $\tilde{\tilde{u}}(x) := \tilde{u}(x + t_0)$ and $u_*(x) = \tilde{\tilde{u}}((1 - y)x)$ as Case (i). Then, again as Case (i), we have $u_* \in W_*(-1, 1)$ and by (56), $\|u_*'''\|^p_{L^p(-1,1)} = (1 - t_0)^{3p-1}\|\tilde{\tilde{u}}'''\|^p_{L^p(-1+t_0,1-t_0)} \le \|\tilde{\tilde{u}}'''\|^p_{L^p(-1+t_0,1-t_0)} = \|\tilde{u}'''\|^p_{L^p(-1+2t_0,1)} \le \|U_{t_0}'''\|^p_{L^p(-1,t_0)} + \|U_{t_0}'''\|^p_{L^p(t_0,1)} = f(t_0) < f(y) = \|u'''\|^p_{L^p(-1,1)}$. In addition, $u_*(0) = \max_{-1\le x\le 1} |u_*(x)| = u(y) = \max_{-1\le x\le 1} |u(x)|$. So, we have proven the case (ii)-(b). This completes the proof.

5. Numerical Experiments

In this section, numerical experiments for Lemma 1 are demonstrated.

Case (i)

The function u which satisfies this case is

$$u(x) = \begin{cases} -\dfrac{(x+1)\left(8750x^3 + 9250x^2 - 7750x - 4337\right)}{17500}\,, & x \le 0.4 \\ \dfrac{(x-1)\left(2500x^2 - 5000x - 199\right)}{2500}\,, & x > 0.4 \end{cases}$$

The shape of u is like Fig. 2-(i). Let $p = 3$, then we have:

$y = 0.4001\,,$

$\|u'''\|^p_{L^p(-1,1)} = 455.508, \quad \|u'''\|^p_{L^p(-1,y)} = 325.932 > 129.576 = \|u'''\|^p_{L^p(y,1)}\,,$

(Eq. (38) is satisfied)

$\|\tilde{\tilde{u}}'''\|^p_{L^p(-1,1)} = 259.152\,, \quad (1 - y)^{3p-1}\|\tilde{\tilde{u}}'''\|^p_{L^p(-1,1)} = 4.346 = \|u_*'''\|^p_{L^p(-1,1)}$

Especially, we have $\|u_*'''\|^p_{L^p(-1,1)} = 4.346 < 455.508 = \|u'''\|^p_{L^p(-1,1)}$.

Case (ii)-(a)

The function u which satisfies this case is

$$u(x) = \begin{cases} -\dfrac{(x+1)\left(340000x^3 + 1064000x^2 + 1108000x - 2387017\right)}{3400000}, & x \le 0.7 \\ \dfrac{(x-1)\left(400000x^2 - 800000x + 39521\right)}{200000}, & x > 0.7 \end{cases}$$

The shape of u is like Fig. 3-(ii). Let $p = 3$, then we have:

$y = 0.2362$,

$\|u'''\|^p_{L^p(-1,1)} = 549.526$, $A = \|u'''\|^p_{L^p(-1,y)} = 8.950$, $B = \|u'''\|^p_{L^p(y,1)} = 540.576$,

(Eq. (42) is satisfied)

$\left(\dfrac{1-y}{1+y}\right)^{3p-1} \dfrac{B}{A} = 1.283$, (Eq. (54) is satisfied)

$\|U'''\|^p_{L^p(-1,0)} = 48.815 < 62.612 = \|U'''\|^p_{L^p(0,1)}$,

$\|u'''_*\|^p_{L^p(-1,1)} = 2\|U'''\|^p_{L^p(-1,0)} = 97.630$

Especially, we have $\|u'''_*\|^p_{L^p(-1,1)} = 97.630 < 549.526 = \|u'''\|^p_{L^p(-1,1)}$.

Case (ii)-(b)

The function u which satisfies this case is

$$u(x) = \begin{cases} -\dfrac{(x+1)\left(340000x^3 + 404000x^2 - 212000x - 466417\right)}{680000}, & x \le 0.7 \\ \dfrac{(x-1)\left(180000x^2 - 360000x + 63721\right)}{40000}, & x > 0.7 \end{cases}$$

The shape of u is like Fig. 4-(ii). Let $p = 3$, then we have:

$y = 0.4275$, $t_0 = 0.2466$

$\|u'''\|^p_{L^p(-1,1)} = 6967.88$, $A = \|u'''\|^p_{L^p(-1,y)} = 407.916$, $B = \|u'''\|^p_{L^p(y,1)} = 6559.96$,

(Eq. (42) is satisfied)

$\left(\dfrac{1-y}{1+y}\right)^{3p-1} \dfrac{B}{A} = 0.0108$, (Eq. (55) is satisfied)

$\|U'''\|^p_{L^p(-1,t_0)} = 1206.23 > 729.002 = \|U'''\|^p_{L^p(t_0,1)}$,

$\|\tilde{\tilde{u}}'''\|^p_{L^p(-1,1)} = 2 * \|U'''\|^p_{L^p(t_0,1)} = 1458.0$,

$(1-t_0)^{3p-1}\|\tilde{\tilde{u}}'''\|^p_{L^p(-1,1)} = 151.342 = \|u'''_*\|^p_{L^p(-1,1)}$

Especially, we have $\|u'''_*\|^p_{L^p(-1,1)} = 151.342 < 6967.88 = \|u'''\|^p_{L^p(-1,1)}$.

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