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Heat Kernel Estimates for Random Walks on Some Kinds of One-dimensional Continuum Percolation Clusters

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Abstract. We consider random walks on random graphs determined by a some kind of continuum percolation on **R**. The vertex set of the random graph is given by the Poisson points conditioned that all points of **Z** are contained. The edge set of the random graph is determined by the random radii of the spheres centered at each points. We give heat kernel estimates for the random walks under the condition on the moment of the random radii. We will also discuss random walks on continuum percolation clusters in $\mathbf{R}^d$, $d \geq 2$.

1. Introduction and Main results

There have been many studies on random walks on random graphs. The heat kernel estimate for random walks on percolation clusters is one of the topics which have been well studied. Among them, in [1], the detailed Gaussian heat kernel estimates are shown for the random walks on the supercritical bond percolation clusters in $\mathbf{Z}^d$. For the studies on the transition densities of the random walks on the critical percolation clusters in trees and other graphs, see for example [3], [2]. Also, in [7], strongly recurrent random walks on random media are discussed in some general settings, and as an application, on-diagonal heat kernel estimates are given for the random walks on the long-range percolation clusters in **Z**. For the heat kernel estimates on the long-range percolation clusters, see also [5]. In this paper, we apply the argument in [7] to random walks on random graphs determined by a some kind of continuum percolation on **R**, and give heat kernel estimates.

The continuum percolation (or the Poisson Boolean model) is a stochastic model given by random points in $\mathbf{R}^d$ and spheres with random radii centered at each points. Two points x and y are adjacent if the sphere centered at x and the sphere centered at y intersect; see Subsection 1.1 for more precise definitions. For the basic results on the continuum percolation, see [8] and references therein. Among the previous works, in [9], the homogenization of the reflecting Brownian motions in the continuum percolation clusters is shown. Also, in [10], the type problem for the random walks on the continuum percolation clusters is studied.

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For the long-range percolation discussed in [7], one can observe a some kind of discontinuity for a given parameter, and such a phenomenon appears if and only if $d = 1$. On the other hand, for the continuum percolation, the picture is relatively simple for $d = 1$. But, it seems that known results on random walks or diffusions on continuum percolation clusters are not so many, especially in the case that the radii of the spheres are unbounded. In this sense, the result of this paper for $d = 1$ may be a one attempt for further studies on such important problems. In Section 3, we will also give a brief discussion for random walks on continuum percolation clusters in $\mathbf{R}^d$, $d \geq 2$.

1.1. Random graphs determined by Continuum percolation. First, we define the random graph $\Gamma = (G, E)$ in the following way. Let $\{M = \{m_n\}_{n=1}^{\infty}, \{r_n\}_{n=1}^{\infty}\}$ be the Poisson Boolean model in $\mathbf{R}$, arising from an underlying point process $M = \{m_n\}_{n=1}^{\infty}$ and random radii $\{r_n\}_{n=1}^{\infty}$. Here, M is the Poisson points in $\mathbf{R}$ with intensity $\eta \in [0, \infty)$, conditioned that there is a point at x for all $x \in \mathbf{Z}$. Note that $M = \mathbf{Z}$ for $\eta = 0$, and $M \supset \mathbf{Z}$ for $\eta > 0$. For any disjoint subsets $A_1, A_2, \ldots, A_m \in \mathcal{B}(\mathbf{R})$ satisfying $A_i \cap \mathbf{Z} = \emptyset$ and $|A_i| < \infty$ for $i = 1, 2, \ldots, m$, the number of points in $A_1, \ldots, A_m$ are independent, and

$$\mathbf{P}[\sharp(M \cap A_i) = n] = \frac{(\eta|A_i|)^n}{n!} \exp(-\eta|A_i|)$$

for $1 \leq i \leq m$ and $n = 0, 1, 2, \ldots$. In the above, $|A_i|$ is a one-dimensional Lebesgue measure of A_i. Further, $\{r_n\}_{n=1}^{\infty}$ are i.i.d. random variables which take values in $[0, \infty)$, and also independent of the underlying point process. The value of r_n stands for the radius of the sphere centered at m_n. We define the vertex set G by $G = M$, and the edge set E by

$$E = \left\{ \langle m_i, m_j \rangle : i \neq j, r_i + r_j \geq |m_i - m_j| \right\} .$$

Here, $| \cdot |$ stands for the Euclidean metric on $\mathbf{R}$. In other words, two points m_i and m_j are connected by a bond if and only if the intersection of two balls

$$\{y \in \mathbf{R} : |m_i - y| \leq r_i\} ,$$
$$\{y \in \mathbf{R} : |m_j - y| \leq r_j\}$$

is not empty. Here, we are considering bonds which are not oriented, and we identify $\langle x, y \rangle$ and $\langle y, x \rangle$. We denote Ω as the probability space on which Γ is defined, and denote $\mathbf{P}$ as the corresponding probability measure. We denote $\mathbf{E}$ as the expectation under $\mathbf{P}$. We write ω to denote elements of Ω. We denote $x \sim y$ if $\langle x, y \rangle \in E$. Let μ_{xy} be a $\{0, 1\}$-valued random variable, which takes 1 if $x \sim y$ and takes 0 otherwise. We note $\mu_{xy} = \mu_{yx}$ and $\mu_{xx} = 0$. Let $\mu_x = \sum_{y \in G} \mu_{xy}$ be the number of the bonds which contain x. For $A \subset G$, set $\mu(A) = \sum_{x \in A} \mu_x$. In Sections 1 and 2, we always assume the followings.

ASSUMPTION 1.1. There exists $\varepsilon > 0$ such that

$$\mathbf{E}[{r_n}^{1+\varepsilon}] < \infty , \tag{1.1}$$

and

$$\mathbf{P}[r_n \geq 1/2] = 1\,. \tag{1.2}$$

REMARK 1.2. By (1.2), all points of G are in the same connected component since $G \supset \mathbf{Z}$. Moreover, by a direct calculation or by volume estimates which will be given in Proposition 1.3, we have that $\mu_x < \infty$ for all $x \in G$. Thus, under Assumption 1.1, Γ is an infinite, connected, locally finite graph which contains the origin 0, almost surely.

1.2. Random walks on Continuum percolation clusters. Next, we define the random walks on the continuum percolation clusters. Fix $\omega \in \Omega$. Let $\Gamma = (G, E)$ be the random graph constructed as in Subsection 1.1. Let $X = (X_n, n \in \mathbf{Z}_+, P^x, x \in G)$ be the discrete-time simple random walk on Γ. This is the Markov process in which a particle at the point of G jumps to one of the points connected by a bond with an equal probability. To define X, we introduce the second probability space $\bar{\Omega}$, and define X on the product $\Omega \times \bar{\Omega}$. We write $\bar{\omega}$ to denote the element of $\bar{\Omega}$. In the above, P^x is the probability measure for the random walk starting from $x \in G$. We write E^x as the expectation under P^x.

The random walk X has transition probabilities

$$P^x(X_1 = y) = {\mu_x}^{-1}\,, \quad y \sim x\,.$$

For $n \geq 0$ and $x, y \in G$, we define the transition density (or discrete-time heat kernel) of X with respect to the reversible measure μ by

$$p_n(x, y) = {\mu_y}^{-1} P^x(X_n = y)\,.$$

We have $p_n(x, y) = p_n(y, x)$. Now, let

$$B(x, R) = \{y \in G : |x - y| < R\}$$

be the Euclidean ball with center $x \in G$ and radius $R > 0$. We denote $B_R = B(0, R)$. We define

$$\tau_R = \tau_{B(0,R)} = \min\{n \geq 0 : X_n \notin B_R\}\,.$$

Further, let $Y_n = \max_{0 \leq k \leq n} |X_k|$, and for $W_n = \{X_0, X_1, \ldots, X_n\}$, let $S_n = \mu(W_n) = \sum_{x \in W_n} \mu_x$. We will give several estimates for the random walk X in Theorem 1.4.

1.3. Main results. In this subsection, we state the main results of this paper. We will use the notation c_i as positive constants which depend on η and the distribution of r_n. We note that the values of c_i's may change from line to line.

We call

$$V(x, R) = \mu(B(x, R))$$

the volume of $B(x, R)$. We denote $V_R = V(0, R)$. For $f, g : G \to \mathbf{R}$, we define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x,y \in G} (f(x) - f(y))(g(x) - g(y))\mu_{xy} .$$

Set $H^2 = \{f \in \mathbf{R}^G : \mathcal{E}(f, f) < \infty\}$. Let A, B be disjoint subsets of G. We define the effective resistance between A and B by

$$R_{\mathrm{eff}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f \in H^2, f|_A = 1, f|_B = 0\} .$$

We simply denote $R_{\mathrm{eff}}(x, y) = R_{\mathrm{eff}}(\{x\}, \{y\})$. To obtain the heat kernel estimates, estimates on the volumes and the effective resistances in the next proposition are essential.

PROPOSITION 1.3. (1) *There exist* $\lambda_0 > 1$ *and* $q_0, c_1 > 0$ *such that*

$$\mathbf{P}(R \in J(\lambda)) \geq 1 - \frac{c_1}{\lambda^{q_0}} \qquad \text{for } R \geq 1, \lambda \geq \lambda_0 ,$$

where

$$\begin{aligned} J(\lambda) = \{R \in [1, \infty] : \lambda^{-1} R \leq V_R \leq \lambda R,\ & R_{\mathrm{eff}}(0, B_R^c) \geq \lambda^{-1} R, \\ & R_{\mathrm{eff}}(0, y) \leq \lambda |y|,\ \forall y \in B_R\} . \end{aligned}$$

(2) $\mathbf{E}[R_{\mathrm{eff}}(0, B_R^c) V_R] \leq c_2 R^2$.

Then, using Proposition 1.4 and Theorem 1.5 in [7], we have the followings.

THEOREM 1.4. (1) *The following estimates hold*:

$$\begin{aligned} c_1 R^2 \leq \mathbf{E}(E_\omega^0 \tau_R) \leq c_2 R^2 \text{ for } R \geq 1 , \\ c_3 n^{-1/2} \leq \mathbf{E}(p_{2n}^\omega(0, 0)) \text{ for } n \geq 1 , \\ c_4 n^{1/2} \leq \mathbf{E}(E_\omega^0 |X_n|) \text{ for } n \geq 1 . \end{aligned} \tag{1.3}$$

(2) *There exist* $\beta_1, \beta_2, \beta_3, \beta_4 < \infty$, *and a subset* Ω_0 *with* $\mathbf{P}(\Omega_0) = 1$ *such that the following statements hold.*
(a) *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$, *there exists* $U_x(\omega) < \infty$ *such that*

$$(\log n)^{-\beta_1} n^{-1/2} \leq p_{2n}^\omega(x, x) \leq (\log n)^{\beta_1} n^{-1/2} , \quad n \geq U_x(\omega) .$$

Especially, the random walk is recurrent.
(b) *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$, *there exists* $R_x(\omega) < \infty$ *such that*

$$(\log R)^{-\beta_2} R^2 \leq E_\omega^x \tau_R \leq (\log R)^{\beta_2} R^2, \quad R \geq R_x(\omega) .$$

(c) *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$, *there exist* $U_x(\omega, \overline{\omega})$, $R_x(\omega, \overline{\omega})$ *such that* $P_\omega^x(U_x < \infty) = P_\omega^x(R_x < \infty) = 1$, *and*

$$(\log n)^{-\beta_3} n^{1/2} \leq Y_n(\omega, \overline{\omega}) \leq (\log n)^{\beta_3} n^{1/2} , \quad n \geq U_x(\omega, \overline{\omega}) ,$$

$$(\log R)^{-\beta_4} R^2 \le \tau_R(\omega, \overline{\varpi}) \le (\log R)^{\beta_4} R^2, \quad R \ge R_x(\omega, \overline{\varpi}) .$$

(d) *For each $\omega \in \Omega_0$ and $x \in G(\omega)$,*

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = \frac{1}{2}, \quad P_\omega^x\text{–a.s.}.$$

REMARK 1.5. We define

$$\hat{J}(\lambda) = \{R \in [1, \infty] : \lambda^{-1} R \le V_R \le \lambda R,\, R_{\rm eff}(0, B_R^c) \ge \lambda^{-1} R,\, R_{\rm eff}(0, y) \le |y|,\ \forall y \in B_R\}.$$

For $\eta = 0$, we can replace $J(\lambda)$ by $\hat{J}(\lambda)$ in Proposition 1.3, and consequently, by Remark 1.6 (1) in [7], we have also

$$\mathbf{E}(p_{2n}^\omega(0,0)) \le c_5 n^{-1/2} \text{ for } n \ge 1, \tag{1.4}$$

which is the opposite side bound of (1.3). For $\eta > 0$, we do not have a proof of (1.4).

1.4. Additional remarks. Here, we give some additional comments.

REMARK 1.6. (1) If we consider the Poisson point process in $\mathbf{R}$ without the condition that there is a point at x for all $x \in \mathbf{Z}$, it is known that the random graph determined by the one-dimensional continuum percolation is not locally finite if $\mathbf{E}[r_n] = \infty$, and that the random graph has no connected component with infinite size if $\mathbf{E}[r_n] < \infty$. So, the condition that G contains all points of $\mathbf{Z}$ is essential in the above discussions.

(2) In Assumption 1.1, the condition (1.1) is a little stronger than $\mathbf{E}[r_n] < \infty$. Whether this condition can be weaken or not is a remaining problem.

EXAMPLE 1.7. Assume that the random variable r_n is subject to the Pareto distribution, that is, its probability density function $p(t)$ is as follows for given $s > 0$, $t_0 > 0$;

$$p(t) = \begin{cases} 0 & t < t_0, \\ \frac{s}{t_0}(\frac{t_0}{t})^{s+1} & t \ge t_0. \end{cases}$$

Then, for $s > 1$, $t_0 \ge 1/2$, Assumption 1.1 is satisfied and we have the heat kernel estimates.

REMARK 1.8. Let us consider the case $\eta = 0$ (i.e. $G = \mathbf{Z}$). In Example 1.7, by a simple calculation, we have

$$c_1 |x-y|^{-s} \le \mathbf{P}[\mu_{xy} = 1] \le c_2 |x-y|^{-s} \tag{1.5}$$

for $x, y \in \mathbf{Z}$. Recall that the random variable μ_{xy} is determined by the random radii of the spheres centered at each points. On the other hand, the long-range percolation discussed in [7] is a model in which the order of the connecting probabilities are as in (1.5), and $\{\mu_{xy}\}$ is determined for each pair of two points $x, y \in \mathbf{Z}$ independently. In the long-range percolation, the corresponding heat kernel estimates hold for $s > 2$. So, we can observe that the critical value of s becomes smaller in the continuum percolation because of the dependence of each bonds.

We will give the proof of main results in Section 2.

2. Proof of Main results

2.1. Estimates on volumes and effective resistances. In this subsection, we state the proof of Proposition 1.3. Let $\Gamma = (G, E)$ be the random graph given in Section 1. For each $\omega \in \Omega$, we define the random graph $\Gamma_1 = (G_1, E_1)$ in the following way. For $x \in \mathbf{Z}$, let

$$L_x = \begin{cases} (x-1, x] & x \geq 1\,, \\ \{0\} & x = 0\,, \\ [x, x+1) & x \leq -1\,. \end{cases}$$

Note that $\mathbf{R} = \cup_{x \in \mathbf{Z}} L_x$, which is a disjoint union. We denote

$$N_x = \sharp(G \cap L_x)\,.$$

Note that $\{N_x\}_{x \in \mathbf{Z}}$ are independent random variables, and $N_x - 1$ is subject to the Poisson distribution with parameter η for $x \in \mathbf{Z} \setminus \{0\}$, and $N_0 \equiv 1$. For $x \in \mathbf{Z}$, we define

$$\xi_x = 1 + \sup\{r_i : m_i \in L_x\}\,,$$

where $\{m_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ are the random points and the random radii given in Subsection 1.1.

LEMMA 2.1. *Under Assumption* 1.1, *we have*

$$\sup_{x \in \mathbf{Z}} \mathbf{E}[{\xi_x}^{1+\varepsilon}] < \infty\,,$$

where $\varepsilon > 0$ *is the constant given in Assumption* 1.1.

PROOF. By the definition, $\mathbf{E}[{\xi_0}^{1+\varepsilon}] < \infty$ is obvious. It is enough to show that

$$\mathbf{E}[\sup\{r_i : m_i \in L_x\}^{1+\varepsilon}] < \infty$$

for $x \neq 0$. We have

$$\begin{aligned} \mathbf{E}[\sup\{r_i : m_i \in L_x\}^{1+\varepsilon}] &= \sum_{k=1}^{\infty} \mathbf{P}[N_x = k]\mathbf{E}\left[\sup\{r_i : m_i \in L_x\}^{1+\varepsilon} | N_x = k\right] \\ &= \sum_{k=1}^{\infty} \mathbf{P}[N_x = k]\mathbf{E}[\sup\{r_i^{1+\varepsilon} : 1 \leq i \leq k\}] \\ &\leq \sum_{k=1}^{\infty} \mathbf{P}[N_x = k]\mathbf{E}\left[\sum_{i=1}^{k} r_i^{1+\varepsilon}\right] \end{aligned}$$

$$= \sum_{k=1}^{\infty} \mathbf{P}[N_x = k] k \mathbf{E}[{r_1}^{1+\varepsilon}] = \mathbf{E}[N_x]\mathbf{E}[{r_1}^{1+\varepsilon}] < \infty \,.$$

So, the assertion holds. □

For each $m_n \in G$, we define $\tilde{m}_n$ as follows. If $m_n > 0$, let $\tilde{m}_n$ be the first integer which is not smaller than m_n. If $m_n \leq 0$, let $\tilde{m}_n$ be the last integer which is not larger than m_n. We set

$$\tilde{r}_n = \xi_{\tilde{m}_n} \,.$$

Then, we define the graph $\Gamma_1 = (G_1, E_1)$ by $G_1 = G$ and

$$E_1 = \left\{ \langle m_i, m_j \rangle : i \neq j, \tilde{r}_i + \tilde{r}_j \geq |\tilde{m}_i - \tilde{m}_j| \right\}.$$

By the definition, Γ is a subgraph of Γ_1.

Now, we consider the estimates on the volumes. The lower bound for V_R is obvious, because $V_R \geq 2R - 1$ for each ω. We show the upper bound for V_R in the next lemma.

LEMMA 2.2. *We have*

$$\mathbf{P}[V_R \leq \lambda R] \geq 1 - \frac{c_1}{\lambda} \,.$$

PROOF. Since $\mathbf{P}[V_R \geq \lambda R] \leq \frac{1}{\lambda R}\mathbf{E}[V_R]$, it is enough to show that

$$\mathbf{E}[V_R] \leq c_2 R \,. \tag{2.1}$$

We denote $\tilde{V}_R$ for the volume corresponding to Γ_1. Then, $V_R(\omega) \leq \tilde{V}_R(\omega)$ for each ω, and

$$\tilde{V}_R \leq \sum_{x=-R}^{R} N_x^2 + \sum_{x=-R}^{R} N_x \sum_{y \in \mathbf{Z} \setminus \{x\}} N_y 1_{\{\langle x,y \rangle \in E_1\}} \,.$$

Furthermore,

$$\mathbf{E}\left[\sum_{x=-R}^{R} N_x^2 \right] = \sum_{x=-R}^{R} \mathbf{E}[N_x^2] \leq c_3 R \,,$$

and

$$\begin{aligned}
\mathbf{E}\left[\sum_{x=-R}^{R} N_x \sum_{y \in \mathbf{Z} \setminus \{x\}} N_y 1_{\{\langle x,y \rangle \in E_1\}} \right] &= \sum_{x=-R}^{R} \sum_{y \in \mathbf{Z} \setminus \{x\}} \mathbf{E}[N_x N_y 1_{\{\langle x,y \rangle \in E_1\}}] \\
&\leq \sum_{x=-R}^{R} \sum_{y \in \mathbf{Z} \setminus \{x\}} \mathbf{E}[N_x^p N_y^p]^{1/p} \mathbf{P}[\langle x, y \rangle \in E_1]^{1/q}
\end{aligned}$$

$$= \sum_{x=-R}^{R} \sum_{y \in \mathbf{Z} \setminus \{x\}} c_4 \mathbf{P}[\langle x, y\rangle \in E_1]^{1/q} .$$

We have used the Hölder inequality for p, q, such that $1 < q < 1 + \varepsilon$, $\frac{1}{p} + \frac{1}{q} = 1$, where $\varepsilon > 0$ is the constant given in Assumption 1.1. By Lemma 2.1, note that

$$\begin{aligned}
\mathbf{P}[\langle x, y\rangle \in E_1] &= \mathbf{P}[\xi_x + \xi_y \geq |x-y|] \\
&\leq \mathbf{P}\left[\xi_x \geq \frac{|x-y|}{2}\right] + \mathbf{P}\left[\xi_y \geq \frac{|x-y|}{2}\right] \\
&\leq \frac{2^{1+\varepsilon}}{|x-y|^{1+\varepsilon}} (\mathbf{E}[{\xi_x}^{1+\varepsilon}] + \mathbf{E}[{\xi_y}^{1+\varepsilon}]) \\
&\leq c_5 |x-y|^{-(1+\varepsilon)} .
\end{aligned}$$

So,

$$\sum_{x=-R}^{R} \sum_{y \in \mathbf{Z} \setminus \{x\}} c_4 \mathbf{P}[\langle x, y\rangle \in E_1]^{1/q} \leq \sum_{x=-R}^{R} \sum_{y \in \mathbf{Z} \setminus \{x\}} c_6 |x-y|^{-(1+\varepsilon)/q} \leq c_7 R .$$

From these, (2.1) holds. □

Next, we would like to prove the lower bound for the effective resistance. Before this, we prepare some technical tools.

LEMMA 2.3. *There exists sufficiently small* $\alpha > 0$, *such that*

$$\sup_{x \in \mathbf{Z}} \mathbf{E}\left[\left(\sum_{k=1}^{[\gamma \xi_x]} N_x N_{x-k} k\right)^{\alpha}\right] < \infty,$$

for all $\gamma \geq 1$. *Here*, $[\cdot]$ *stands for the integer part.*

PROOF. We prove the estimate for $\gamma = 1$. The general case can be proved by the same way. For suitable $\alpha \in (0, 1]$, we have

$$\begin{aligned}
\mathbf{E}\left[\left(\sum_{k=1}^{[\xi_x]} N_x N_{x-k} k\right)^{\alpha}\right] &\leq \mathbf{E}\left[\left(\sup\{N_x N_{x-k} : 1 \leq k \leq [\xi_x]\} \sum_{k=1}^{[\xi_x]} k\right)^{\alpha}\right] \\
&\leq \mathbf{E}\left[\left(\sum_{k=1}^{[\xi_x]} N_x N_{x-k} c_1 \xi_x^2\right)^{\alpha}\right] \\
&\leq c_2 \mathbf{E}\left[\left(\sum_{k=1}^{\infty} 1_{\{\xi_x \geq k\}} N_x N_{x-k}\right)^{\alpha} \xi_x^{2\alpha}\right]
\end{aligned}$$

$$
\begin{aligned}
&\le c_2 \mathbf{E}\left[\sum_{k=1}^{\infty} 1_{\{\xi_x \ge k\}} (N_x N_{x-k})^{\alpha} \xi_x^{2\alpha}\right] \\
&= c_2 \sum_{k=1}^{\infty} \mathbf{E}\left[1_{\{\xi_x \ge k\}} (N_x N_{x-k})^{\alpha} \xi_x^{2\alpha}\right] \\
&\le c_2 \sum_{k=1}^{\infty} \mathbf{P}[\xi_x \ge k]^{1/u} \mathbf{E}[(N_x N_{x-k})^{\alpha v} \xi_x^{2\alpha v}]^{1/v} \\
&\le c_2 \sum_{k=1}^{\infty} (k^{-(1+\varepsilon)} \mathbf{E}[\xi_x^{1+\varepsilon}])^{1/u} \mathbf{E}[(N_x N_{x-k})^{\alpha v} \xi_x^{2\alpha v}]^{1/v} \equiv I \,.
\end{aligned}
$$

We have used the Hölder inequality for u, v, such that $1 < u < 1+\varepsilon$, $1 < v \le \frac{1+\varepsilon}{4\alpha}$, $\frac{1}{u} + \frac{1}{v} = 1$. We can find such u, v, if we choose α satisfying $0 < \alpha < \varepsilon/4$. Furthermore,

$$
(k^{-(1+\varepsilon)} \mathbf{E}[\xi_x^{1+\varepsilon}])^{1/u} \le c_3 k^{-(1+\varepsilon)/u} \,,
$$

and

$$
\begin{aligned}
\mathbf{E}[(N_x N_{x-k})^{\alpha v} \xi_x^{2\alpha v}] &\le \mathbf{E}[(N_x N_{x-k})^{2\alpha v}]^{1/2} \mathbf{E}[\xi_x^{4\alpha v}]^{1/2} \\
&\le c_4 \mathbf{E}[{N_x}^{2\alpha v}]^{1/2} \mathbf{E}[{N_{x-k}}^{2\alpha v}]^{1/2} \le c_5 \,.
\end{aligned}
$$

So, we have

$$
I \le c_6 \sum_{k=1}^{\infty} k^{-(1+\varepsilon)/u} < \infty \,.
$$

□

Let $\Gamma_1 = (G_1, E_1)$ be the graph constructed as the above. For each $\omega \in \Omega$, we construct the new weighted graph $\Gamma_2 = (G_2, E_2)$, by shorting all bonds of Γ_1 which connect the points in $G_1 \cap L_x$ for each $x \in \mathbf{Z}$. More precisely, we define $G_2 = \mathbf{Z}$,

$$
E_2 = \{\langle x, y\rangle : x, y \in \mathbf{Z}, x \ne y \,, \xi_x + \xi_y \ge |x-y|\} \,,
$$

and we set that each $\langle x, y\rangle \in E_2$ has a weight $N_x N_y$.

We have the following lemma.

LEMMA 2.4. *For $i \in \mathbf{Z}$, we define*

$$
A_i = \{\langle u, v\rangle \in E_2 : u, v \in \mathbf{Z}, u < v, [u, v] \supset [i-1, i]\} \,.
$$

Then, there exists sufficiently small $\beta > 0$ such that

$$
\sup_{i \in \mathbf{Z}} \mathbf{E}\left[\left(\sum_{\langle u,v\rangle \in A_i} N_u N_v |u-v|\right)^{\beta}\right] < \infty \,.
$$

PROOF. For suitable $\beta \in (0, 1]$, we have

$$
\begin{aligned}
&\mathbf{E}\left[\left(\sum_{\langle u,v\rangle\in A_i} N_u N_v |u-v|\right)^{\beta}\right] \\
&= \mathbf{E}\left[\left(\sum_{u=-\infty}^{i-1}\sum_{v=i}^{\infty} 1_{\{\xi_u+\xi_v\ge|u-v|\}} N_u N_v |u-v|\right)^{\beta}\right] \\
&\le \mathbf{E}\left[\left\{\sum_{u=-\infty}^{i-1}\sum_{v=i}^{\infty} (1_{\{2\xi_u\ge|u-v|\}} + 1_{\{2\xi_v\ge|u-v|\}}) N_u N_v |u-v|\right\}^{\beta}\right] \\
&\le \mathbf{E}\Bigg[\Bigg(\sum_{x=i}^{\infty} 1_{\{2\xi_x\ge x-i+1\}} \sum_{j=[x-2\xi_x]}^{i-1} N_x N_j |x-j| \\
&\qquad + \sum_{x=-\infty}^{i-1} 1_{\{2\xi_x\ge i-x\}} \sum_{j=i}^{[x+2\xi_x]} N_x N_j |x-j|\Bigg)^{\beta}\Bigg] \\
&\le \mathbf{E}\left[\left(\sum_{x=i}^{\infty} 1_{\{2\xi_x\ge x-i+1\}} \sum_{j=[x-2\xi_x]}^{i-1} N_x N_j |x-j|\right)^{\beta}\right] \\
&\qquad + \mathbf{E}\left[\left(\sum_{x=-\infty}^{i-1} 1_{\{2\xi_x\ge i-x\}} \sum_{j=i}^{[x+2\xi_x]} N_x N_j |x-j|\right)^{\beta}\right] \\
&\equiv J_1 + J_2
\end{aligned}
$$

for each $i \in \mathbf{Z}$. Then,

$$
\begin{aligned}
J_1 &\le \mathbf{E}\left[\sum_{x=i}^{\infty} 1_{\{2\xi_x\ge x-i+1\}} \left(\sum_{j=[x-2\xi_x]}^{i-1} N_x N_j |x-j|\right)^{\beta}\right] \\
&= \sum_{x=i}^{\infty} \mathbf{E}\left[1_{\{2\xi_x\ge x-i+1\}} \left(\sum_{j=[x-2\xi_x]}^{i-1} N_x N_j |x-j|\right)^{\beta}\right] \\
&\le \sum_{x=i}^{\infty} \mathbf{E}\left[1_{\{2\xi_x\ge x-i+1\}} \left(\sum_{k=x-i+1}^{[2\xi_x+1]} N_x N_{x-k} k\right)^{\beta}\right] \\
&\le \sum_{x=i}^{\infty} \mathbf{P}[2\xi_x \ge x-i+1]^{1/u'} \mathbf{E}\left[\left(\sum_{k=x-i+1}^{[2\xi_x+1]} N_x N_{x-k} k\right)^{\beta v'}\right]^{1/v'} \equiv J_3 .
\end{aligned}
$$

We have used the Hölder inequality for u', v', such that $1 < u' < 1+\varepsilon$, $v' = \frac{\alpha}{\beta}$, $\frac{1}{u'} + \frac{1}{v'} = 1$, where $\alpha > 0$ is the constant given in Lemma 2.3. We can find such u', v', if we choose β satisfying $0 < \beta < \frac{\alpha\varepsilon}{1+\varepsilon}$. Then,

$$\mathbf{P}[2\xi_x \geq x - i + 1] \leq \left(\frac{2}{x-i+1}\right)^{1+\varepsilon} \mathbf{E}[\xi_x^{1+\varepsilon}] \leq c_1(x-i+1)^{-(1+\varepsilon)},$$

and by Lemma 2.3,

$$\mathbf{E}\left[\left(\sum_{k=x-i+1}^{[2\xi_x+1]} N_x N_{x-k} k\right)^{\beta v'}\right] \leq \mathbf{E}\left[\left(\sum_{k=1}^{[2\xi_x+1]} N_x N_{x-k} k\right)^{\beta v'}\right] \leq c_2 < \infty .$$

So, $J_3 \leq \sum_{x=i}^{\infty} c_3(x-i+1)^{-(1+\varepsilon)/u'} \leq c_4 < \infty$. We can check $J_2 \leq c_5 < \infty$ similarly. □

Now, we show the lower bound for the effective resistance.

PROPOSITION 2.5. *There exist $q > 0$ and $c_1 > 0$, such that*

$$\mathbf{P}[R_{\text{eff}}(0, B_R^c) \geq \lambda^{-1} R] \geq 1 - c_1 \lambda^{-q},$$

for all $R \geq 1$.

PROOF. Let $\Gamma_2 = (G_2, E_2)$ be the weighted graph constructed as the above. Further, for each $\omega \in \Omega$, we construct another weighted graph $\Gamma_3 = (G_3, E_3)$ from $\Gamma_2 = (G_2, E_2)$ in the following way.

(1) If a bond $\langle x, y\rangle \in E_2$ such that $x, y \in \mathbf{Z}$, $x + 2 \leq y$ exists, then, divide $\langle x, y\rangle$ into $y - x$ short bonds with weight $N_x N_y (y - x)$.

(2) For each $i = 1, \dots, y - x$, replace the ith short bond by a bond which has $x + i - 1$ and $x + i$ as its endpoints and has a weight $N_x N_y (y - x)$.

(3) Repeat (1), (2) for all bonds of E_2 except nearest-neighbor bonds. We denote $\tilde{R}_{\text{eff}}$ for the effective resistance corresponding to Γ_3. By the way of construction, we can see that

$$\tilde{R}_{\text{eff}}(0, R) = \sum_{i=1}^{R} \left(\sum_{\langle u,v\rangle \in A_i} N_u N_v |u - v|\right)^{-1}.$$

We may consider under the case that B_R is defined as a closed ball. Then, by the cutting law, the effective resistance does not increase when we construct Γ_1 from Γ. Moreover, by the shorting law, the effective resistance does not increase when we construct Γ_2 from Γ_1 and construct Γ_3 from Γ_2. Thus, for suitable $q \in (0, 1]$,

$$\begin{aligned}
\mathbf{E}[R_{\text{eff}}(0, B_R^c)^{-q}] &\leq \mathbf{E}[\tilde{R}_{\text{eff}}(0, B_R^c)^{-q}] \\
&= \mathbf{E}[\{\tilde{R}_{\text{eff}}(0, R)^{-1} + \tilde{R}_{\text{eff}}(0, -R)^{-1}\}^q] \\
&\leq \mathbf{E}[\tilde{R}_{\text{eff}}(0, R)^{-q} + \tilde{R}_{\text{eff}}(0, -R)^{-q}] = 2\mathbf{E}[\tilde{R}_{\text{eff}}(0, R)^{-q}].
\end{aligned}$$

Furthermore,

$$\mathbf{E}[\tilde{R}_{\mathrm{eff}}(0,R)^{-q}] = \mathbf{E}\left[\left\{\sum_{i=1}^{R}\left(\sum_{\langle u,v\rangle\in A_i} N_u N_v|u-v|\right)^{-1}\right\}^{-q}\right]$$
$$\le R^{-q-1}\sum_{i=1}^{R}\mathbf{E}\left[\left(\sum_{\langle u,v\rangle\in A_i} N_u N_v|u-v|\right)^{q}\right] \le c_2 R^{-q}\,.$$

We have used the Hölder inequality in the first inequality and Lemma 2.4 in the second inequality. We have taken $q=\beta$, where $\beta>0$ is the constant given in Lemma 2.4. Hence,

$$\mathbf{P}[R_{\mathrm{eff}}(0,B_R^c)\le\lambda^{-1}R]\le\lambda^{-q}R^q\mathbf{E}[R_{\mathrm{eff}}(0,B_R^c)^{-q}]\le c_3\lambda^{-q}\,,$$

which completes the proof. □

Next, we see the upper bound for the effective resistance. For $\eta=0$, it is obvious that $R_{\mathrm{eff}}(0,y)\le|y|, \forall y\in B_R$ for each $\omega\in\Omega$. We consider the case $\eta>0$.

LEMMA 2.6. *For $\eta>0$, we have*

$$\mathbf{P}[R_{\mathrm{eff}}(0,y)\le\lambda|y|,\ \forall y\in B_R]\ge 1-\frac{c_1}{\lambda}\,.$$

PROOF. It is obvious that $R_{\mathrm{eff}}(0,y)\le|y|+1$ for $y\in B_R\cap(-1,1)^c$. Also,

$$\mathbf{P}[R_{\mathrm{eff}}(0,y)>\lambda|y|,\exists y\in B_R\cap(-1,1)]\le\mathbf{P}[1>\lambda|y|,\exists y\in G\cap(-1,1)\setminus\{0\}]$$
$$\le\mathbf{P}\left[G\cap\left(-\frac{1}{\lambda},\frac{1}{\lambda}\right)\setminus\{0\}\neq\emptyset\right]$$
$$=1-e^{-2\eta/\lambda}\le 2\eta/\lambda\,.$$

So, the assertion holds. □

From Lemma 2.2, Proposition 2.5 and Lemma 2.6, we obtain Proposition 1.3 (1). Moreover, Proposition 1.3 (2) follows by (2.1) and the trivial bound $R_{\mathrm{eff}}(0,B_R^c)\le R/2$ which holds for each $\omega\in\Omega$. We have proved Proposition 1.3.

2.2. Heat kernel estimates.

PROOF OF THEOREM 1.4. For the continuum percolation clusters in $\mathbf{R}$, we have shown suitable estimates on the volumes and the effective resistances in Proposition 1.3. So, by using Proposition 1.4 and Theorem 1.5 in [7] (see also Proposition 3.5 and Theorem 3.6 in Appendix, which explain the corresponding assertions) with $D=\alpha=1$, we obtain the desired results. Here, R^D stands for the order of the volume V_R, and R^α stands for the order of the effective resistance $R_{\mathrm{eff}}(0,B_R^c)$. □

3. Discussion in High dimensional case

In Sections 1 and 2, we have considered random walks on continuum percolation clusters in $\mathbf{R}$. In this section, we discuss corresponding problems in $\mathbf{R}^d$, $d \geq 2$. We consider the Poisson points $M = \{m_n\}_{n=1}^{\infty}$ with intensity $\eta \in (0, \infty)$, and i.i.d. random variables $\{r_n\}_{n=1}^{\infty}$ which take values in $[0, \infty)$ and also independent of the underlying point process. The value of r_n stands for the radius of the sphere centered at m_n. Note that, unlike the case $d = 1$, we do not need the condition that M contains all points of $\mathbf{Z}^d$. Then, we define the random graph $\Gamma = (G, E)$ as follows. Let the vertex set $G = M$, and the edge set

$$E = \{\langle m_i, m_j \rangle : i \neq j, r_i + r_j \geq |m_i - m_j|_{\mathbf{R}^d}\}.$$

The graph is locally finite when $\mathbf{E}[{r_n}^d] < \infty$. When Γ is locally finite and there exists an ∞-cluster, say, a connected subgraph of Γ which has infinitely many points, we will consider the simple random walk on it. It is known that, almost surely, there is at most one ∞-cluster; see Theorem 3.6 in [8]. Among the important problems, let us discuss the type problem in the followings. For $d = 1$, we have shown in Theorem 1.4 that, under some technical assumptions, the random walk is recurrent. For $d \geq 3$, by analogy with the result of the bond percolation, it seems natural that the random walk may be transient, though we need to check carefully. The case $d = 2$ is most interesting. By using a result in [4], we obtain a partial result as follows.

THEOREM 3.1. *Let $d = 2$. Assume that there exists $\varepsilon > 0$ such that $\mathbf{E}[{r_n}^{4+\varepsilon}] < \infty$. We also assume that, almost surely, there exists an ∞-cluster under the Poisson Boolean model. Then, almost surely, the random walk on the ∞-cluster is recurrent.*

REMARK 3.2. In the bond percolation, it is shown that the random walk is transient if and only if $d \geq 3$ ([6]). In the long range percolation, it is shown that, for $d = 1$, the random walk is transient if $1 < s < 2$ and recurrent if $s = 2$, and for $d = 2$, the random walk is transient if $2 < s < 4$ and recurrent if $s \geq 4$ ([4]). Here, the long range percolation is the model in which each pair of distinct points $x, y \in \mathbf{Z}^d$ is connected by a bond with probability $p(x, y) \sim {|x - y|_{\mathbf{Z}^d}}^{-s}$, independently of other pairs. Let us return to the continuum percolation. For $d = 2$, the case $\mathbf{E}[{r_n}^2] < \infty$ and $\mathbf{E}[{r_n}^{4+\varepsilon}] = \infty$ is still open. Is there any possibility that the random walk becomes transient? Is the random walk always recurrent? It seems to be an interesting problem.

PROOF OF THEOREM 3.1. For each $x = (x_1, x_2) \in \mathbf{Z}^2$, we define $L_x = [x_1, x_1 + 1) \times [x_2, x_2 + 1)$, and $N_x = \sharp\{G \cap L_x\}$. Note that $\{N_x\}_{x \in \mathbf{Z}^2}$ are i.i.d. random variables subject to the Poisson distribution with parameter η. Also, let $\xi_x = \sqrt{2} + \sup\{r_i : m_i \in L_x\}$. If there is no Poisson point in L_x, we set $\sup\{r_i : m_i \in L_x\} = 0$. In the same way as the proof of Lemma 2.1, $\mathbf{E}[{r_n}^{4+\varepsilon}] < \infty$ implies $\mathbf{E}[{\xi_x}^{4+\varepsilon}] < \infty$.

Now, we define a new graph $\Gamma_1 = (G_1, E_1)$. Let the vertex set $G_1 = \mathbf{Z}^2$, and the edge set

$$E_1 = \{\langle x, y\rangle : x, y \in \mathbf{Z}^2, x \neq y, \xi_x + \xi_y \geq |x-y|_{\mathbf{R}^2}\}\,.$$

For each $\langle x, y\rangle \in E_1$, we give a weight $N_x N_y$. In the same way as the calculation in the proof of Lemma 2.2, we have

$$\mathbf{P}[\langle x, y\rangle \in E_1] \leq c_1 |x-y|_{\mathbf{Z}^2}^{-(4+\varepsilon)}\,.$$

Next, we construct another graph $\Gamma_2 = (G_2, E_2)$ from Γ_1 as follows. Let $G_2 = \mathbf{Z}^2$. We project the bonds of E_1 to nearest-neighbor bonds of $\mathbf{Z}^2$ in the following way.

(1) If a bond $\langle x, y\rangle \in E_1$ such that $x = (x_1, x_2), y = (y_1, y_2) \in \mathbf{Z}^2$, $|x-y|_{\mathbf{Z}^2} = |x_1 - y_1| + |x_2 - y_2| \geq 2$ exists, then, erase the bond, and to each nearest-neighbor bond in $[(x_1, x_2), (x_1, y_2)] \cup [(x_1, y_2), (x_2, y_2)]$ increase the weight by $N_x N_y |x-y|_{\mathbf{Z}^2}$.

(2) Repeat (1) for all bonds of E_1 except nearest-neighbor bonds.

By the shorting law, the effective resistance does not increase in the procedures. Hence, it is enough to show the recurrence of Γ_2. In this way, we have reduced the original model to the random conductance model in $\mathbf{Z}^2$.

Let $\mu_{0,1}$ be the weight of $\langle 0, 1\rangle$ in Γ_2. We have

$$\begin{aligned}
\mathbf{E}[\mu_{0,1}] &= \mathbf{E}\left[\sum_{x,y} 1_{\{\langle x,y\rangle \in E_1\}} N_x N_y |x-y|_{\mathbf{Z}^2}\right] \\
&= \sum_{x,y} |x-y|_{\mathbf{Z}^2} \mathbf{E}[1_{\{\langle x,y\rangle \in E_1\}} N_x N_y] \\
&\leq \sum_{x,y} |x-y|_{\mathbf{Z}^2} \mathbf{P}[\langle x, y\rangle \in E_1]^{1/p} \mathbf{E}[(N_x N_y)^q]^{1/q} \\
&\leq c_2 \sum_{x,y} |x-y|_{\mathbf{Z}^2}^{1-(4+\varepsilon)/p} \leq c_3 \sum_{n=1}^{\infty} n^{3-(4+\varepsilon)/p} < \infty.
\end{aligned}$$

In the above, the summation is taken over all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbf{Z}^2$ satisfying $\langle 0, 1\rangle \in [(x_1, y_2), (y_1, y_2)]$. We have used the Hölder inequality for p, q, such that $1 < p < 1 + \frac{\varepsilon}{4}$, $\frac{1}{p} + \frac{1}{q} = 1$. Thus, $\mu_{0,1}$ has the Cauchy tail. In other words, there exists $c_4 > 0$, such that $\mathbf{P}[\mu_{0,1} > c_4 n] \leq \frac{1}{n}$ holds for all $n \in \mathbf{N}$. Also, we note that the weights of the bonds in Γ_2 are identically distributed and stationary. So, by Theorem 3.9 in [4], we have that Γ_2 is recurrent, and the result follows. □

Appendix: Heat kernel estimates for strongly recurrent random walks on random graphs

In this appendix, we overview some of the results in [7] for strongly recurrent random walks on general random graphs. We consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ carrying a family of random graphs $\Gamma(\omega) = (G(\omega), E(\omega), \omega \in \Omega)$. Here, G is the vertex set and E is the edge set. Assume that the graph is infinite, locally finite, connected, and contains a marked vertex $0 \in G$. For $x \in G$, let μ_x be the number of bonds that contain x. We extend μ to a measure on G. Let $d(\cdot, \cdot)$ be a metric on G. We write $B(x, R)$ for the ball with center $x \in G$ and radius $R > 0$ under the metric d. Let $V(x, R) = \mu(B(x, R))$ be the volume of the ball. We denote $B_R = B(0, R)$, $V_R = V(0, R)$. Further, we write $X = (X_n, n \geq 0, P^x, x \in G)$ for the discrete-time simple random walk on Γ. For $n \geq 0$ and $x, y \in G$, we define the transition density of X with respect to μ by $p_n(x, y) = {\mu_y}^{-1} P^x(X_n = y)$. We also define $\tau_R = \min\{n \geq 0 : X_n \notin B_R\}$. To define X, we introduce a second measure space $(\overline{\Omega}, \overline{\mathcal{F}})$, and define X on the product $\Omega \times \overline{\Omega}$. We write $\overline{\omega}$ to denote elements of $\overline{\Omega}$. Let $R_{\mathrm{eff}}(A, B)$ be the effective resistance between A and B, which are disjoint subsets of G.

Let $v, r : \mathbf{N} \to [0, \infty)$ be strictly increasing functions with $v(1) = r(1) = 1$ which satisfy

$$C_1^{-1}\Big(\frac{R}{R'}\Big)^{d_1} \leq \frac{v(R)}{v(R')} \leq C_1\Big(\frac{R}{R'}\Big)^{d_2}, \quad C_2^{-1}\Big(\frac{R}{R'}\Big)^{\alpha_1} \leq \frac{r(R)}{r(R')} \leq C_2\Big(\frac{R}{R'}\Big)^{\alpha_2} \tag{3.1}$$

for all $0 < R' \leq R < \infty$, where $C_1, C_2 \geq 1$, $1 \leq d_1 \leq d_2$ and $0 < \alpha_1 \leq \alpha_2 \leq 1$. For convenience, set $v(0) = r(0) = 0$, $v(\infty) = r(\infty) = \infty$ and extend them to $v, r : [0, \infty] \to [0, \infty]$ such that v, r are continuous, strictly increasing, and satisfy (3.1).

DEFINITION 3.3. Let $\Gamma = (G, E)$ be as above. For $\lambda > 1$, define

$$\begin{aligned} J(\lambda) = \{R \in [1, \infty] : \lambda^{-1} v(R) \leq V_R \leq \lambda v(R)\,,\ R_{\mathrm{eff}}(0, B_R^c) \geq \lambda^{-1} r(R)\,,\\ R_{\mathrm{eff}}(0, y) \leq \lambda r(d(0, y)),\ \forall y \in B_R\}\,. \end{aligned}$$

As we see, $v(\cdot)$ gives the volume growth order, and $r(\cdot)$ gives the resistance growth order. We now make the following assumptions concerning the graphs $(\Gamma(\omega))$.

ASSUMPTION 3.4. (1) There exist $\lambda_0 > 1$ and $p(\lambda)$ which goes to 0 as $\lambda \to \infty$ such that

$$\mathbf{P}(R \in J(\lambda)) \geq 1 - p(\lambda) \qquad \text{for } R \geq 1\,, \lambda \geq \lambda_0\,.$$

(2) $\mathbf{E}[R_{\mathrm{eff}}(0, B_R^c) V_R] \leq c_1 v(R) r(R)$.
(3) There exist $q_0, c_2 > 0$ such that

$$p(\lambda) \leq \frac{c_2}{\lambda^{q_0}}\,.$$

We have the following consequences of Assumption 3.4 for random graphs. Let $\mathcal{I}(\cdot)$ be the inverse function of $(v \cdot r)(\cdot)$.

PROPOSITION 3.5. *Suppose that Assumption* 3.4(1) *and* (2) *hold. Then,*

$$c_1 v(R) r(R) \leq \mathbf{E}(E_\omega^0 \tau_R) \leq c_2 v(R) r(R) \textit{ for } R \geq 1\,,$$

$$\frac{c_3}{v(\mathcal{I}(n))} \leq \mathbf{E}(p_{2n}^\omega(0, 0)) \textit{ for } n \geq 1\,,$$

$$c_4\mathcal{I}(n) \le \mathbf{E}(E^0_\omega d(0, X_n)) \text{ for } n \ge 1\,.$$

THEOREM 3.6. *Suppose that Assumption* 3.4(1) *and* (3) *hold. Then, there exist* $\beta_1, \beta_2, \beta_3, \beta_4 < \infty$, *and a subset* Ω_0 *with* $\mathbf{P}(\Omega_0) = 1$ *such that the following statements hold.*

(a) *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$, *there exists* $N_x(\omega) < \infty$ *such that*

$$\frac{(\log n)^{-\beta_1}}{v(\mathcal{I}(n))} \le p^\omega_{2n}(x, x) \le \frac{(\log n)^{\beta_1}}{v(\mathcal{I}(n))}\,, \quad n \ge N_x(\omega)\,.$$

(b) *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$, *there exists* $R_x(\omega) < \infty$ *such that*

$$(\log R)^{-\beta_2} v(R)r(R) \le E^x_\omega \tau_R \le (\log R)^{\beta_2} v(R)r(R), \quad R \ge R_x(\omega)\,.$$

(c) *Let* $Y_n = \max_{0\le k\le n} d(0, X_k)$. *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$, *there exist* $N_x(\omega, \overline{\omega})$, $R_x(\omega, \overline{\omega})$ *such that* $P^x_\omega(N_x < \infty) = P^x_\omega(R_x < \infty) = 1$, *and*

$$(\log n)^{-\beta_3}\mathcal{I}(n) \le Y_n(\omega, \overline{\omega}) \le (\log n)^{\beta_3}\mathcal{I}(n)\,, \quad n \ge N_x(\omega, \overline{\omega})\,,$$
$$(\log R)^{-\beta_4} v(R)r(R) \le \tau_R(\omega, \overline{\omega}) \le (\log R)^{\beta_4} v(R)r(R), \quad R \ge R_x(\omega, \overline{\omega})\,.$$

Suppose further that v, r *satisfy the followings in addition to* (3.1);

$$C_3^{-1} R^D (\log R)^{-m_1} \le v(R) \le C_3 R^D (\log R)^{m_1}\,,$$
$$C_4^{-1} R^\alpha (\log R)^{-m_2} \le r(R) \le C_4 R^\alpha (\log R)^{m_2}\,,$$

where $C_3, C_4 \ge 1$, $D \ge 1$, $0 < \alpha \le 1$ *and* $m_1, m_2 > 0$. *Then, the following statements hold.*

(a) $d_s(G) := -2\lim_{n\to\infty} \frac{\log p^\omega_{2n}(x,x)}{\log n} = \frac{2D}{D+\alpha}$, $\mathbf{P}$*–a.s., and the random walk is recurrent.*

(b) $\lim_{R\to\infty} \frac{\log E^x_\omega \tau_R}{\log R} = D + \alpha$.

(c) *Let* $W_n = \{X_0, X_1, \dots, X_n\}$ *and let* $S_n = \mu(W_n) = \sum_{x\in W_n} \mu_x$. *For each* $\omega \in \Omega_0$ *and* $x \in G(\omega)$,

$$\lim_{n\to\infty} \frac{\log S_n}{\log n} = \frac{D}{D+\alpha}, \quad P^x_\omega\text{–a.s.}\,.$$

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