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H-Supermagic Strength of Some Graphs

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Abstract. A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs to a subgraph of G isomorphic to H. We say that G is H-magic if there is a total labeling $f : V \cup E \to \{1, 2, 3, \dots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of G isomorphic to H, $s(f) = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ is constant. When $f(V) = \{1, 2, \dots, |V|\}$, then G is said to be H-supermagic. In this case, the H-supermagic strength of G is defined as the minimum of all $s(f)$ where the minimum is taken over all H-supermagic labelings f of G, and is denoted by $SM_H(G)$. In this paper we find the C_k-supermagic strength of k-polygonal snakes of any length and H-supermagic strength of a chain of an arbitrary 2-connected simple graph H. Also we make a conjecture regarding the P_h-supermagic strength of P_n for $2 \le h \le n$.

1. Introduction

By a graph we mean a finite, simple, undirected,connected graph without loops or multiple edges. An edge-covering of a graph $G = (V, E)$ is a family of different subgraphs $H_1, H_2, H_3, \dots, H_k$ such that every edge of E belongs to at least one of the subgraphs H_i for $1 \le i \le k$. If every H_i is isomorphic to a given graph H, then we say that G admits an H-covering.

A total labeling is a one-to-one and onto function from the set of vertices union the set of edges of a given graph G to a set of labels. A graph $G = (V, E)$ that admits an H-covering is called H-magic if there exists a total labeling $f : V \cup E \to \{1, 2, 3, \dots, |V| + |E|\}$ such that for every subgraph $H' = (V', E')$ of G isomorphic to H, $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ is constant. If $f(V) = \{1, 2, 3, \dots, |V|\}$ then G is said to be H-supermagic.

The constant value that every copy of H takes under the labeling f is denoted by $m(f)$ in the magic case and by $s(f)$ in the supermagic case.

2. Main Results

DEFINITION 1 (H-magic strength). For any H-magic labeling f of G there is a constant $m(f)$ such that $m(f) = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e)$ for every subgraph $H' = (V', E')$

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of G isomorphic to H. The H-magic strength of G is defined as the minimum of all $m(f)$ where the minimum is taken over all H-magic labelings f of G and is denoted by $M_H(G)$. That is, $M_H(G)$=min$\{m(f) : f$ is a H-magic labeling of $G\}$.

DEFINITION 2 (H-supermagic strength). For any H-supermagic labeling f of G there is a constant $s(f)$ such that $s(f) = \sum_{v\in V'} f(v) + \sum_{e\in E'} f(e)$ for every subgraph $H' = (V', E')$ of G isomorphic to H. The H-supermagic strength of G is defined as the minimum of all $s(f)$ where the minimum is taken over all H-supermagic labelings f of G and is denoted by $SM_H(G)$. That is, $SM_H(G)$=min$\{s(f) : f$ is a H-supermagic labeling of $G\}$.

Let $G = (V, E)$ be a (ν, ϵ)-graph and let f be a H-supermagic labeling of G where H is a (p, q) subgraph of G. Let there be n subgraphs of G isomorphic to H in the H-supermagic covering of G. Then each edge and vertex occurs in at least one subgraph of the H-supermagic covering. We define the following

DEFINITION 3 (H-order of vertices and edges). The H-order of a vertex $v \in V(G)$ is defined as the number of subgraphs in the H-supermagic covering of G in which v occurs and is denoted by $O_H(v)$.
The H-order of an edge $e \in E(G)$ is defined as the number of subgraphs in the H-supermagic covering of G in which e occurs and is denoted by $O_H(e)$.

ILLUSTRATION 1. Given a 3-polygonal snake of length 6 and $H = C_3$. Here, $O_H(v_1) = 1, O_H(v_2) = 2, O_H(v_3) = 2, O_H(v_4) = 1, O_H(v_5) = 1, O_H(v_6) = 1$ and $O_H(v_7) = 1$. $O_H(e_1) = O_H(e_2) = O_H(e_3) = O_H(e_4) = O_H(e_5) = O_H(e_6) = O_H(e_7) = O_H(e_8) = O_H(e_9) = 1$.

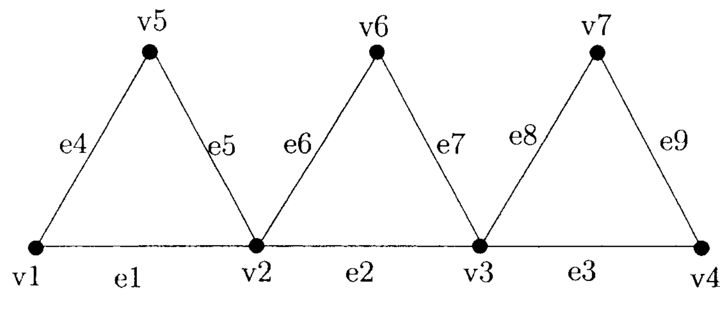

FIGURE 1. C_3-supermagic covering of the 3-polygonal snake of length 3

DEFINITION 4 (Descending Occurrence sequence of vertices and edges). The descending sequence of the H-orders of vertices in G is called the occurrence sequence of vertices of G. The descending sequence of the H-orders of edges in G is called the occurrence sequence of edges of G.

The concept of defining these sequences is to calculate the contribution of each vertex and edge in the maximum and minimum supermagic sum.

ILLUSTRATION 2. For the graph in illustration 1, the descending occurrence sequence of vertices is 2,2,1,1,1,1,1 and the descending occurrence sequence of edges is 1,1,1,1,1,1,1,1,1.

LEMMA 1. *Let $G = (V, E)$ be a (ν, ϵ)-graph and let f be a H-supermagic labeling of G where H is a (p, q) graph. Let there be n subsets of G isomorphic to H in the H-magic covering of G. Let $\{d_v^i\}_{i=1}^{\nu}$ be the descending occurrence sequence of vertices of G and $\{d_e^i\}_{i=1}^{\epsilon}$ be the descending occurrence sequence of edges of G. Then,* $\nu q + \frac{\sum_{i=1}^{\nu} i d_v^i + \sum_{i=1}^{\epsilon} i d_e^i}{n} \leq SM_H(G) \leq \nu q + \frac{\sum_{i=1}^{\nu}(v-i+1)d_v^i + \sum_{i=1}^{\epsilon}(\epsilon-i+1)d_e^i}{n}$.

PROOF. Note that, $n = \frac{\sum_{i=1}^{\nu} d_v^i}{p} = \frac{\sum_{i=1}^{\epsilon} d_e^i}{q}$. Let f be a H-supermagic labeling of G. Then the lower limit of $s(f)$ can be obtained by assigning least labels to vertices and edges with largest H-orders. Thus,

$$\begin{aligned} ns(f) &\geq \sum_{i=1}^{\nu} i d_v^i + \sum_{i=1}^{\epsilon} (\nu + i) d_e^i \\ &= \nu n q + \sum_{i=1}^{\nu} i d_v^i + \sum_{i=1}^{\epsilon} i d_e^i . \end{aligned}$$

Therefore, $s(f) \geq \nu q + \frac{\sum_{i=1}^{\nu} i d_v^i + \sum_{i=1}^{\epsilon} i d_e^i}{n}$.

The upper limit can be obtained by assigning largest labels to vertices and edges with largest H-orders. Thus,

$$\begin{aligned} ns(f) &\leq \sum_{i=1}^{\nu} (v - i + 1) d_v^i + \sum_{i=1}^{\epsilon} (\nu + \epsilon - i + 1) d_e^i \\ &= \nu n q + \sum_{i=1}^{\nu} (v - i + 1) d_v^i + \sum_{i=1}^{\epsilon} (\epsilon - i + 1) d_e^i . \end{aligned}$$

Therefore, $s(f) \leq \nu q + \frac{\sum_{i=1}^{\nu}(v-i+1)d_v^i + \sum_{i=1}^{\epsilon}(\epsilon-i+1)d_e^i}{n}$.

Hence, $\nu q + \frac{\sum_{i=1}^{\nu} i d_v^i + \sum_{i=1}^{\epsilon} i d_e^i}{n} \leq SM_H(G) \leq \nu q + \frac{\sum_{i=1}^{\nu}(v-i+1)d_v^i + \sum_{i=1}^{\epsilon}(\epsilon-i+1)d_e^i}{n}$. $\square$

In [2] a chain graph Hn of H of length n is defined as follows: Let $H_1, H_2, \ldots, H_n$ be copies of a graph H. Let u_i and v_i be two distinct vertices of H_i for $i = 1, 2, \ldots, n$. We construct a chain graph Hn of H of length n by identifying two vertices u_i and v_{i+1} for $i = 1, 2, \ldots, n-1$. And it is proved that if H is a 2-connected (p, q) simple graph then Hn admits a H-supermagic labeling f with supermagic sum $s(f) = \frac{n(p+q)^2+3(p+q)-2n(p+q)+2n-2}{2}$ if either $p+q$ is even or $p+q+n$ is even. With the help of Lemma 1, it can be easily verified that it is the least supermagic sum. Hence we have the following corollary.

COROLLARY 2. *Let H be a 2-connected (p, q) simple graph. Then $SM_H(Hn) = \frac{n(p+q)^2+3(p+q)-2n(p+q)+2n-2}{2}$ if either $p+q$ is even or $p+q+n$ is even.*

ILLUSTRATION 3

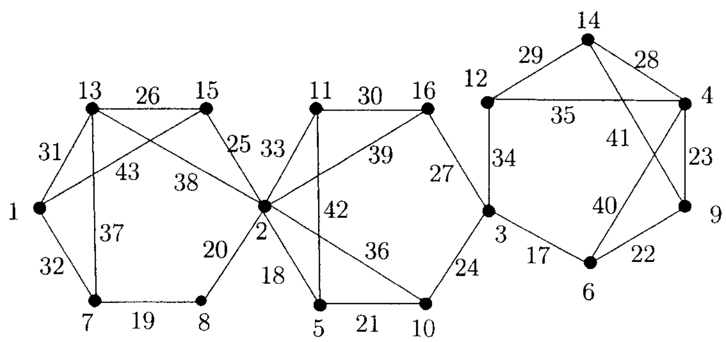

FIGURE 2. H- supermagic covering of a chain of H of length 3. $SM_H(G)$=317

A k-polygonal snake of length n is nothing but a chain of n cycles C_k. Hence we have the following corollary.

COROLLARY 3. *The supermagic strength of k-polygonal snake of length n is $2kn(k-1)+3k+n-1$.*

ILLUSTRATION 4

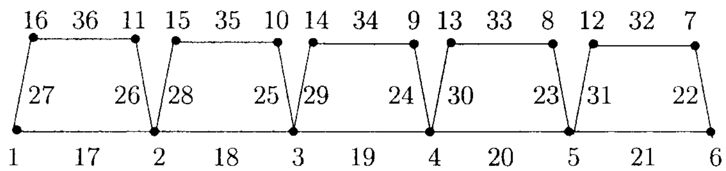

FIGURE 3. C_4-supermagic covering of 4-polygonal snake of length 5. $SM_{C_4}(G)$=136

DEFINITION 5 (H-dual supermagic labeling). Let $G = (V, E)$ be a (ν, ϵ)-graph and let f be a H-supermagic labeling of G where H is a $(p.q)$ graph. Define $f' : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$ as follows,
$f'(v) = \nu + 1 - f(v)$, $v \in V$ and $f'(e) = 2\nu + \epsilon + 1 - f(e)$, $e \in E$. Then f' is called the H-dual of f.

ILLUSTRATION 5. A P_4-supermagic labeling of P_7 is shown in Figure 4(a) and its P_4-dual supermagic labeling is shown in figure 4(b).

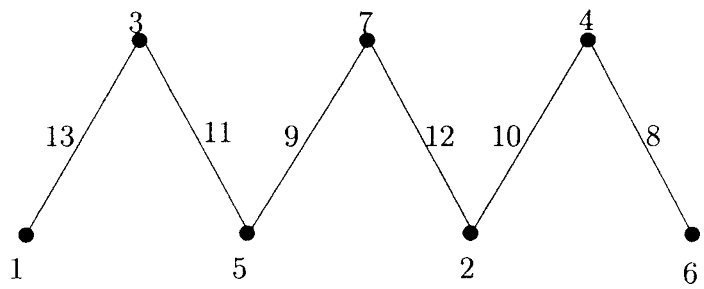

FIGURE 4(a). A P_4-supermagic covering of P_7

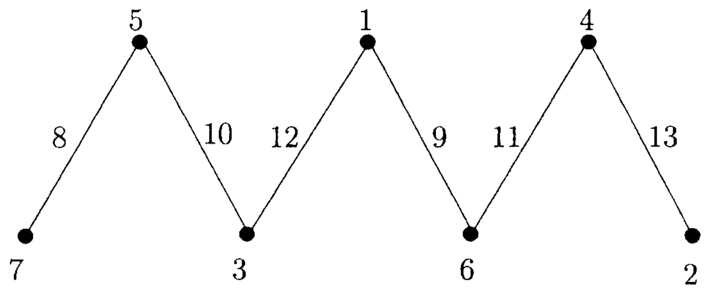

FIGURE 4(b). The P_4-dual of the P_4-supermagic covering of P_7 in Figure 4(a)

LEMMA 4. *The H-dual of an H-supermagic labeling f of G is also an H-supermagic labeling with $s(f') = \nu(p + 2q) + p + q + q\epsilon - s(f)$.*

3. Bounds of supermagic strength of Path-supermagic graphs

Let P_n be the path graph with vertex set $V = \{v_i : 1 \le i \le n\}$ and edge set $E = \{e_i = v_i v_{i+1} : 1 \le i \le n-1\}$. For any integer $2 \le h \le n$ and $1 \le i \le n-h+1$, $P_h^{(i)}$ is the subpath of P_n with vertex set $V_i = \{v_i, v_{i+1}, \ldots, v_{i+h-1}\}$ and edge set $E_i = \{e_i, e_{i+1}, \ldots, e_{i+h-2}\}$. Then P_n admits an $(P_h^{(1)}, P_h^{(2)}, \ldots, P_h^{(n-h+1)})$-covering. Since each $P_h^{(i)}$ is isomorphic to P_h, P_n admits a P_h-covering.

In [1], A. Gutierrez, A. Llado proved that the path P_n is P_h-supermagic for any integer $2 \le h \le n$.(Theorem 3) In the proof they defined a H-supermagic labeling f and under this labeling the supermagic sum is $s(f) = \frac{4nh-3n+h-(th-lh+l^2-t^2-t)}{2}$ where $1 \le l \le h$, $1 \le t \le h-1$ such that $n \equiv l(\text{mod } h)$ and $n-1 \equiv t(\text{mod}(h-1))$. In this section we find yet another P_h-supermagic labeling for P_n and prove that the supermagic sum of its dual labeling is a better upper bound for the P_h-supermagic strength of P_n. We use the proof of Theorem 3 in [1] with a little modification.

THEOREM 5. *$SM_{P_h}(P_n) \le \frac{4nh-3n+h-(lh+th-l^2-t^2-t)}{2}$ for any integer $2 \le h \le n$ and $1 \le l \le h$, $1 \le t \le h-1$ such that $n \equiv l(\text{mod } h)$ and $n-1 \equiv t(\text{mod}(h-1))$.*

PROOF. Let P_n be the path graph with vertex set $V = \{v_i : 1 \le i \le n\}$ and edge set $E = \{e_i = v_i v_{i+1} : 1 \le i \le n-1\}$. For any integer $2 \le h \le n$ and $1 \le i \le n-h+1$, $P_h^{(i)}$ is the subpath of P_n with vertex set $V_i = \{v_i, v_{i+1}, \ldots, v_{i+h-1}\}$ and edge set $E_i = \{e_i, e_{i+1}, \ldots, e_{i+h-2}\}$.

For each $1 \le i \le n$ consider the decomposition $i = i_1 + i_2 h$, with $1 \le i_1 \le h$ and $0 \le i_2 \le \frac{n}{h}$ and write $\alpha(i) = (i_1, i_2)$. Let f_1 be the lexicographic ordering of the pairs $\alpha(i)$.

Similarly, for each $1 \le i \le n-1$ consider the decomposition $i = i_1 + i_2(h-1)$, with $1 \le i_1 \le h-1$ and $0 \le i_2 \le \frac{n-1}{h-1}$ and write $\beta(i) = (i_1, i_2)$. Let f_2 be the lexicographic ordering of the pairs $\beta(i)$.

Consider the total labeling $f : V \cup E \to [1, 2n-1]$ defined as follows: $f(v_i) = f_1(\alpha(i))$ on the set of vertices and $f(e_{n-i}) = n + f_2(\beta(i))$ on the set of edges. It is clear that $f(V) = [1, n]$ and $f(E) = [n+1, 2n-1]$.

For $1 \le i \le n-h$ we have, $f(P_h^{(i+1)}) - f(P_h^{(i)}) = f(v_{i+h}) - f(v_i) + f(e_{i+h-1}) - f(e_i)$. Note that, if $\alpha(i) = (i_1, i_2)$ then $\alpha(i+h) = (i_1, i_2+1)$ and hence $f(v_{i+h}) - f(v_i) = 1$. Also if $\beta(i) = (i_1, i_2)$ then $\beta(i+h-1) = (i_1, i_2+1)$. Now,

$$\begin{aligned} f(e_{i+h-1}) - f(e_i) &= n + f_2(\beta(n-(i+h-1)) - n - f_2(\beta(n-i) \\ &= f_2(\beta(n-i) - h - 1)) - f_2(\beta(n-i)) \\ &= -1\,. \end{aligned}$$

Hence, $f(P_h^{(i+1)}) - f(P_h^{(i)}) = 0$ for $1 \le i \le n-h$. Hence f is P_h-supermagic.

Let us find the supermagic sum $s(f)$ of the supermagic labeling f. Let $1 \le l \le h$, $1 \le t \le h-1$ such that $n \equiv l(\text{mod } h)$ and $n-1 \equiv t(\text{mod}(h-1))$. Let $n = kh + l$ and

$n-1 = sh+t$.

We find the labelings of the vertices and edges of P_n under f.

For $1 \le i \le n$, if $i = mh + j$ the vertex labelings are given by
$f(v_i) = (j-1)(k+1) + m + 1$ for $0 \le m \le k$ and $1 \le j \le l$ and
$f(v_i) = l(k+1) + (j-l-1)k + m + 1$ for $0 \le m \le k-1$ and $l+1 \le j \le h$.

For $1 \le i \le n-1$, if $i = m(h-1) + j$ the edge labelings are given by
$f(e_{n-i}) = n + (j-1)(s+1) + m + 1$ for $0 \le m \le s$, $1 \le j \le t$ and
$f(e_{n-i}) = n + t(s+1) + (j-t-1)s + m + 1$ for $0 \le m \le s-1$, $t+1 \le j \le h-1$.

$$\begin{aligned}
f(P_h^{(1)}) &= \sum_{j=1}^{l} f(v_j) + \sum_{j=1}^{h-1} f(e_j) \\
&= \sum_{j=1}^{l} f(v_j) + \sum_{j=l+1}^{h} f(v_j) + \sum_{j=1}^{t} f(e_j) + \sum_{j=t+1}^{h-1} f(e_j) \\
&= \sum_{j=1}^{l} [(j-1)(k+1)+1] + \sum_{j=l+1}^{h} [l(k+1) + (j-l-1)k + 1] \\
&\quad + \sum_{j=1}^{t} [n + (t-j+1)(s+1)] + \sum_{j=t+1}^{h-1} [n + t(s+1) + (h-j)s] \\
&= \frac{nh + 2h - n + lh - l^2}{2} + (h-1)n + \frac{h(n+t-1) - t^2 - t}{2} \\
&= \frac{4nh - 3n + h + (lh + th - l^2 - t^2 - t)}{2}.
\end{aligned}$$

Hence $s(f) = \frac{4nh-3n+h+(lh+th-l^2-t^2-t)}{2}$.
Let f' be the P_h-dual of f. By Theorem 4,

$$\begin{aligned}
s(f') &= (n+1)(2h-1) + (h-1)(2n-1) - s(f) \\
&= \frac{4nh - 3n + h - (lh + th - l^2 - t^2 - t)}{2}.
\end{aligned}$$

The supermagic sum of the the supermagic labeling g defined in Theorem 3 of [1] is, $s(g) = \frac{4nh-3n+h-(th-lh+l^2-t^2-t)}{2}$. It can be easily verified that $lh+th-l^2-t^2-t \ge th-lh+l^2-t^2-t$ and consequently, $s(f') \le s(g)$.

Hence, $SM_{P_h}(P_n) \le \frac{4nh-3n+h-(lh+th-l^2-t^2-t)}{2}$. □

CONJECTURE 1. *$SM_{P_h}(P_n) = \frac{4nh-3n+h-(lh+th-l^2-t^2-t)}{2}$ for any integer $2 \le h \le n$ and $1 \le l \le h$, $1 \le t \le h-1$ such that $n \equiv l (\mathrm{mod}\ h)$ and $n-1 \equiv t (\mathrm{mod} (h-1))$.*

ILLUSTRATION 6. Various P_4-supermagic labelings of P_9 and their P_4-duals are given below.

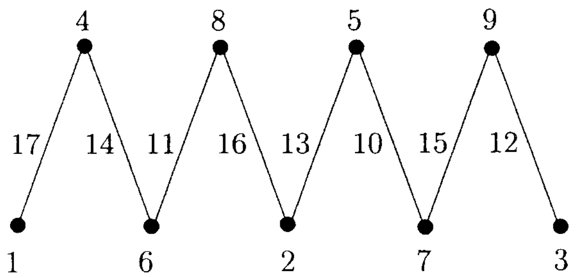

FIGURE 5(a). P_4-supermagic labeling g of P_9 with $s(g) = 61$

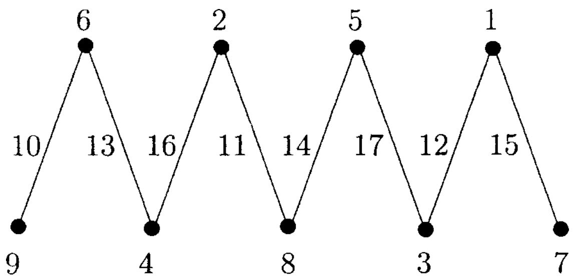

FIGURE 5(b). P_4- dual supermagic labeling $g'$of g with $s(g') = 60$

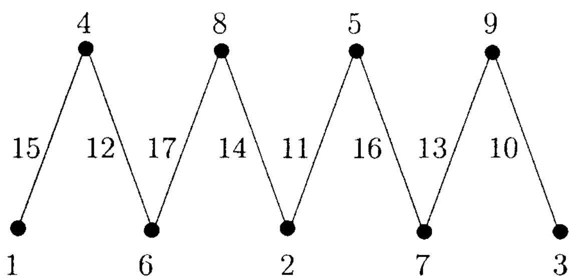

FIGURE 5(c). P_4-supermagic labeling f of P_9 with $s(f) = 63$

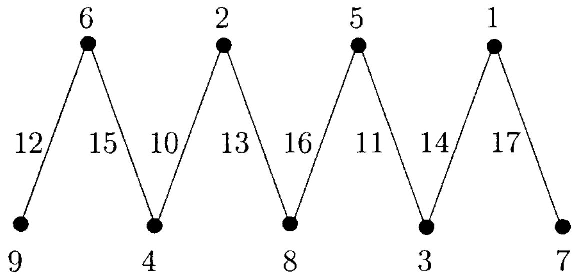

FIGURE 5(d). P_4- dual supermagic labeling f' of f with $s(f') = 58$

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