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A Characterization of the Tempered Distributions with Regular Closed Support by Bloch Equations

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Abstract. In this paper, we will establish the correspondence between the tempered distributions supported on a regular closed set and the space of the solutions of Bloch equations with some conditions on its support.

1. Introduction

The following equation is called Hermite heat equation, or in quantum statistical mechanics called Bloch equation,

$$\left(\frac{\partial}{\partial t} - \Delta_x + |x|^2\right) U(x,t) = 0\,, \quad x \in \mathbf{R}^d\,, \quad t > 0\,. \tag{1.1}$$

B. P. Dhungana et al. characterized the tempered distributions in [1] and the Fourier hyperfunctions in [2] by the solutions of (1.1).

In this paper we show the correspondence between the tempered distributions supported by a regular closed set and the space of the solutions of Bloch equations with some estimate on its support. Namely, we characterize the tempered distributions supported by a regular closed set. The definition and properties of a regular closed set will be given in section 3.

2. The Mehler kernel

First of all, we fix some notations. We use a multi-index $\alpha \in \mathbf{Z}_+^d$, namely, $\alpha = (\alpha_1, \ldots, \alpha_d)$ where $\alpha_i \in \mathbf{Z}$ and $\alpha_i \geq 0$. So, for $x \in \mathbf{R}^d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$, where $\partial_{x_j}^{\alpha_j} = (\frac{\partial}{\partial x_j})^{\alpha_j}$ and $\Delta_x = \sum_{i=1}^d \partial_{x_i}^2$. Moreover $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $\alpha! = \alpha_1! \cdots \alpha_d!$.

DEFINITION 1. The Fourier transform $\mathcal{F}$ for an integrable function f is defined by

$$\mathcal{F}f(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} e^{-ix\cdot\xi} f(x)dx$$

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and the inverse Fourier transform $\mathcal{F}^{-1}$ for an integrable function f is defined by

$$\mathcal{F}^{-1} f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} e^{ix\cdot\xi} f(\xi) d\xi ,$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_d\xi_d$.

We denote by $M(x, \xi, t)$ the Mehler kernel defined by

$$M(x, \xi, t) = \frac{e^{-dt}}{\pi^{d/2}(1 - e^{-4t})^{d/2}} e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(|x|^2+|\xi|^2)+\frac{2e^{-2t}}{1-e^{-4t}}x\cdot\xi} , \quad x , \ \xi \in \mathbf{R}^d , \quad t \in \mathbf{C} \text{ and } \operatorname{Re} t > 0 .$$

It is known (for instance, see: [3]) that

$$M(x, \xi, t) = \sum_{\gamma \in \mathbf{Z}_+^d} e^{-(2|\gamma|+d)t} h_\gamma(x) h_\gamma(\xi)$$

and

$$(-\Delta_x + |x|^2) h_\gamma(x) = (2|\gamma| + d) h_\gamma(x) ,$$

where the Hermite functions on $\mathbf{R}^1$ and $\mathbf{R}^d$ are defined by

$$h_n(x) = (2^n n!)^{-\frac{1}{2}} \pi^{-\frac{1}{4}} (-1)^n e^{\frac{x^2}{2}} \left(\frac{d}{dx}\right)^n e^{-x^2} , \quad x \in \mathbf{R}^1 , \ n = 0, 1, 2, \ldots$$

and

$$h_\gamma(x) = h_{\gamma_1}(x_1) \otimes \cdots \otimes h_{\gamma_d}(x_d) , \quad \gamma \in \mathbf{Z}_+^d , \quad x \in \mathbf{R}^d$$

respectively.

The Mehler kernel $M(x, \xi, t)$ satisfies Bloch equations (1.1) and

$$\lim_{t \to 0+} M(x, \xi, t) = \delta(x - \xi).$$

Moreover we obtain the following estimate on derivatives of the Mehler kernel:

PROPOSITION 1. *Let t_0 be the unique positive solution of* $\tanh(2t) = t$. *Then for any $\alpha \in \mathbf{Z}_+^d$, we obtain*

$$|\partial_\xi^\alpha M(x, \xi, t)| \le (\alpha!)^{1/2} t^{-|\alpha|} (1 + |x| + |\xi|)^{|\alpha|} M(x, \xi, t) , \quad x, \xi \in \mathbf{R}^d , \quad 0 < t < t_0 .$$

PROOF. Since the Fourier transform $\mathcal{F}$ of the Hermite function $h_\gamma(\xi)$ is

$$\mathcal{F}(h_\gamma)(y) = (-i)^{|\gamma|} h_\gamma(y) ,$$

we have

$$\mathcal{F}_\xi(M(x, \xi, t))(y) = \sum_{\gamma \in \mathbf{Z}_+^d} e^{-(2|\gamma|+d)t} h_\gamma(x) \mathcal{F}_\xi(h_\gamma(\xi))(y)$$

$$= \sum_{\gamma\in\mathbf{Z}_+^d} e^{-(2|\gamma|+d)t} h_\gamma(x)(-i)^{|\gamma|} h_\gamma(y)$$

$$= \sum_{\gamma\in\mathbf{Z}_+^d} e^{-\frac{\pi|\gamma|i}{2}} e^{-(2|\gamma|+d)t} h_\gamma(x) h_\gamma(y)$$

$$= e^{\frac{d\pi i}{4}} \sum_{\gamma\in\mathbf{Z}_+^d} e^{-(2|\gamma|+d)(t+\frac{\pi i}{4})} h_\gamma(x) h_\gamma(y)$$

$$= e^{\frac{d\pi i}{4}} M\left(x, y, t + \frac{\pi i}{4}\right), \tag{2.1}$$

where $\mathcal{F}_\xi$ is the partial Fourier transform on ξ variables. By (2.1),

$$\mathcal{F}_\xi(M(x,\xi,t))(y) = e^{\frac{d\pi i}{4}} \frac{e^{-d(t+\frac{\pi i}{4})}}{\pi^{d/2}(1 - e^{-4(t+\frac{\pi i}{4})})^{d/2}} e^{-\frac{1}{2}\frac{1+e^{-4(t+\frac{\pi i}{4})}}{1-e^{-4(t-\frac{\pi i}{4})}}(|x|^2+|y|^2) + \frac{2e^{-2(t+\frac{\pi i}{4})}}{1-e^{-4(t-\frac{\pi i}{4})}} x\cdot y}$$

$$= \frac{e^{-dt}}{\pi^{d/2}(1+e^{-4t})^{d/2}} e^{-\frac{1}{2}\frac{1-e^{-4t}}{1+e^{-4t}}(|x|^2+|y|^2) - \frac{2ie^{-2t}}{1+e^{-4t}} x\cdot y}$$

$$= \frac{e^{-dt}}{\pi^{d/2}(1+e^{-4t})^{d/2}} e^{-\frac{1}{2}\frac{1-e^{-4t}}{1+e^{-4t}}\left(y + \frac{2ie^{-2t}}{1-e^{-4t}}x\right)^2 - \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}|x|^2}.$$

Let

$$\mathcal{F}_\xi(M(x,\xi,t))(y) = \hat{M}(x,y,t),\ \ F(x,y,t) = e^{-\frac{1}{2}\frac{1-e^{-4t}}{1+e^{-4t}}\left(y + \frac{2ie^{-2t}}{1-e^{-4t}}x\right)^2}$$

and

$$G(x,t) = \frac{e^{-dt}}{\pi^{d/2}(1+e^{-4t})^{d/2}} e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}|x|^2}.$$

Then $\hat{M}(x,y,t) = F(x,y,t)G(x,t)$. By the inverse of the Fourier transform on y variables $\mathcal{F}_y^{-1}$, we have

$$\mathcal{F}_y^{-1}(\hat{M}(x,y,t))(\xi) = G(x,t)\mathcal{F}_y^{-1}(F(x,y,t))(\xi).$$

Now we have

$$\partial_\xi^\alpha \mathcal{F}_y^{-1}(F(x,y,t))(\xi) = \partial_\xi^\alpha (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} e^{i\xi\cdot y} e^{-\frac{1}{2}\frac{1-e^{-4t}}{1+e^{-4t}}\left(y + \frac{2ie^{-2t}}{1-e^{-4t}}x\right)^2} dy$$

$$= (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} (iy)^\alpha e^{i\xi\cdot y} e^{-\frac{1}{2}\frac{1-e^{-4t}}{1+e^{-4t}}\left(y + \frac{2ie^{-2t}}{1-e^{-4t}}x\right)^2} dy. \tag{2.2}$$

Let $A = \frac{1-e^{-4t}}{1+e^{-4t}}$ and $B = \frac{2e^{-2t}}{1-e^{-4t}}$. Then since $0 < t < t_0$, it is clear that

$$0 < A < 1,\ \frac{1}{A} \le \frac{1}{t} \text{ and } 0 < AB \le 1. \tag{2.3}$$

By (2.2),

$$(2.2) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} (iy)^\alpha e^{i\xi\cdot y} e^{\frac{1}{2}A(y+iBx)^2} dy$$
$$= (2\pi)^{-\frac{d}{2}} e^{-\frac{|\xi|^2}{2A}+B\xi\cdot x} \int_{\mathbf{R}^d} (iy)^\alpha e^{-\frac{A}{2}\{y-\left(\frac{\xi}{A}-Bx\right)i\}^2} dy$$

We set $I = \int_{\mathbf{R}^d} (iy)^\alpha e^{-\frac{A}{2}\{y-\left(\frac{\xi}{A}-Bx\right)i\}^2} dy$. If we put $\eta = \sqrt{\frac{A}{2}} \left\{ y - \left(\frac{\xi}{A} - Bx\right) i \right\}$, then we have

$$I = \int_{\mathbf{R}^d} i^{|\alpha|} \left(\sqrt{\frac{2}{A}}\eta + \left(\frac{\xi}{A} - Bx\right)\right)^\alpha e^{-\eta^2} \left(\frac{2}{A}\right)^{d/2} d\eta$$
$$= i^{|\alpha|} \left(\frac{2}{A}\right)^{d/2} \left(\frac{1}{A}\right)^{|\alpha|} \int_{\mathbf{R}^d} \left\{\sqrt{2A}\eta + (\xi - ABx)i\right\}^\alpha e^{-\eta^2} d\eta$$
$$= i^{|\alpha|} \left(\frac{2}{A}\right)^{d/2} \left(\frac{1}{A}\right)^{|\alpha|} \sum_{k\le\alpha} \binom{\alpha}{k} (\sqrt{2A})^{|k|} (\xi - ABx)^{\alpha-k} i^{|\alpha-k|} \int_{\mathbf{R}^d} \eta^k e^{-\eta^2} d\eta\,. \tag{2.4}$$

On the other hand, since

$$\int_{\mathbf{R}^d} \eta^k e^{-\eta^2} d\eta = \Pi_{j=1}^d \int_{\mathbf{R}} \eta_j^{k_j} e^{-\eta_j^2} d\eta_j = \begin{cases} \prod_{j=1}^d \Gamma\left(\frac{k_j+1}{2}\right), & k \in (2\mathbf{Z}_+)^d\,, \\ 0\,, & \text{otherwise} \end{cases} ,$$

we have

$$\left| \int_{\mathbf{R}^d} \eta^k e^{-\eta^2} d\eta \right| \le 2^{-|k|/2} (k!)^{1/2} \pi^{d/2}\,, \tag{2.5}$$

where Γ is the Euler Gamma function. Hence by (2.4) and (2.5), we obtain

$$|I| = \left| \left(\frac{2}{A}\right)^{d/2} \left(\frac{1}{A}\right)^{|\alpha|} \int_{\mathbf{R}^d} \left\{\sqrt{2A}\eta + (\xi - ABx)i\right\}^\alpha e^{-\eta^2} d\eta \right|$$
$$\le \left(\frac{2}{A}\right)^{d/2} \left(\frac{1}{A}\right)^{|\alpha|} \sum_{k\le\alpha} \binom{\alpha}{k} (\sqrt{2A})^{|k|} |(\xi - ABx)^{\alpha-k}| \int_{\mathbf{R}^d} |\eta|^k e^{-\eta^2} d\eta$$
$$\le \left(\frac{2}{A}\right)^{d/2} \left(\frac{1}{A}\right)^{|\alpha|} \sum_{k\le\alpha} \binom{\alpha}{k} (\sqrt{2A})^{|k|} |(\xi - ABx)^{\alpha-k}| \frac{(k!)^{1/2}}{2^{\frac{|k|}{2}}} \pi^{d/2}$$
$$\le \left(\frac{2}{A}\right)^{d/2} \left(\frac{1}{A}\right)^{|\alpha|} (\alpha!)^{\frac{1}{2}} (A^{\frac{1}{2}} + |\xi - ABx|)^{|\alpha|} \pi^{d/2}\,.$$

Since (2.3), we have

$$|I| \le \left(\frac{2\pi}{A}\right)^{d/2} t^{-|\alpha|}(\alpha!)^{1/2}(1+|x|+|\xi|)^{|\alpha|}\,. \tag{2.6}$$

Therefore by (2.6) we obtain

$$\begin{aligned}
|\partial_\xi^\alpha M(x,\xi,t)| &= |\partial_\xi^\alpha \mathcal{F}_y^{-1}\hat{M}(x,y,t)(\xi)| \\
&= |G(x,t)|\cdot|\partial_\xi^\alpha \mathcal{F}_y^{-1}(F(x,y,t))(\xi)| \\
&\le \frac{e^{-dt}}{\pi^{d/2}(1+e^{-4t})^{d/2}} e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}|x|^2}(2\pi)^{-\frac{d}{2}} e^{-\frac{|\xi|^2}{2A}+B\xi\cdot x}\cdot|I| \\
&\le (\alpha!)^{1/2} t^{-|\alpha|}(1+|x|+|\xi|)^{|\alpha|} M(x,y,t)\,. \qquad \square
\end{aligned}$$

COROLLARY 1. *Let $t > 0$. Then $M(x,\xi,t) \in \mathcal{S}(\mathbf{R}^d_\xi)$.*

B. P. Dhungana obtained the following characterization of the tempered distributions [1]:

THEOREM 1 ([1]). *Let $T > 0$ be fixed. For any v in $\mathcal{S}'(\mathbf{R}^d)$, put $U(x,t) = \langle v_\xi, M(x,\xi,t)\rangle$. Then $U(x,t)$ satisfies that*

$$U(x,t) \in C^\infty(\mathbf{R}^d \times (0,T))\,, \tag{2.7}$$

$$\left(\frac{\partial}{\partial t} - \Delta_x + |x|^2\right)U(x,t) = 0\,, \quad \text{on } \mathbf{R}^d \times (0,T) \tag{2.8}$$

and

$$|U(x,t)| \le C\left(1+t^{-\nu}\right) \tag{2.9}$$

for some $C > 0$, $\nu \in \mathbf{Z}_+$. Moreover for any $\varphi \in \mathcal{S}(\mathbf{R}^d)$,

$$\lim_{t\to 0+}\int_{\mathbf{R}^d} U(x,t)\varphi(x)dx = \langle v,\varphi\rangle\,.$$

Conversely for any $U(x,t) \in C^\infty(\mathbf{R}^d \times (0,T))$ satisfying (2.8) *and* (2.9), *there exists $v \in \mathcal{S}'(\mathbf{R}^d)$ such that $U(x,t) = \langle v_\xi, M(x,\xi,t)\rangle$.*

3. The structure of the tempered distributions supported by regular closed sets

DEFINITION 2 ([4]). Let A be a closed subset of $\mathbf{R}^d$. If there exist $d > 0$, $\omega > 0$ and $0 < q \le 1$ such that any x_1 and $x_2 \in A$ so that $|x_1 - x_2| \le d$ are linked by a curve in A whose length l satisfies $l \le \omega|x_1 - x_2|^q$, then we call A *a regular*.

For example, if A is a convex closed set, $\omega = q = 1$ and $d = d(A)$ and if A is a closure of the upper half-plane, $\omega = q = 1$ and $d = \infty$. Of course, a closure of the first quadrant (a proper convex cone) and the light cone are also a regular closed set.

Concerning on the tempered distributions supported on a regular closed set, the following result is known:

PROPOSITION 2 ([4]). *Let A be a regular closed set. If $f \in \mathcal{S}'(\mathbf{R}^d)$ and supp $f \subset A$, then there exist the tempered measures supported on A, μ_α $(|\alpha| \leq m)$, such that supp $\mu_\alpha \subset A$ and*

$$f = \sum_{|\alpha| \leq m} \partial^\alpha \mu_\alpha ,$$

where the tempered measure μ means that there exists $m \in \mathbf{Z}_+$ so that $\int \frac{|d\mu|(x)}{(1+|x|)^m} < \infty$.

Put $\mathcal{S}'_A = \{f \in \mathcal{S}' \mid \text{supp } f \subset A\}$. Now our main result is as follows:

THEOREM 2. *Let A be a regular closed set. For any v in $\mathcal{S}'_A(\mathbf{R}^d)$, set $U(x,t)$ be $U(x,t) = \langle v_\xi, M(x,\xi,t)\rangle$. Then $U(x,t)$ satisfies that*

$$U(x,t) \in C^\infty(\mathbf{R}^d \times (0,t_0)) , \tag{3.1}$$

$$\left(\frac{\partial}{\partial t} - \Delta_x + |x|^2\right) U(x,t) = 0 , \quad \text{on } \mathbf{R}^d \times (0,t_0) \tag{3.2}$$

and

$$|U(x,t)| \leq C(1+t^{-\nu}) e^{-\frac{1}{4}\frac{2e^{-2t}}{1-e^{-4t}} d(x,A)^2} \tag{3.3}$$

for some $C > 0$ and $\nu \in \mathbf{Z}_+$, where $d(x,A) = \inf_{\xi \in A} |x - \xi|$. Moreover for any $\varphi \in \mathcal{S}(\mathbf{R}^d)$,

$$\lim_{t \to 0+} \int_{\mathbf{R}^d} U(x,t)\varphi(x)dx = \langle v, \varphi \rangle .$$

Conversely for any $U(x,t) \in C^\infty(\mathbf{R}^d \times (0,t_0))$ satisfying (3.2) *and* (3.3) , *there exists $v \in \mathcal{S}'_A(\mathbf{R}^d)$ such that $U(x,t) = \langle v_\xi, M(x,\xi,t)\rangle$.*

PROOF. Let $u \in \mathcal{S}'_A$. If $U(x,t) = \langle u_\xi, M(x,\xi,t)\rangle$, then we have

$$\begin{aligned} |U(x,t)| &= |\langle u_\xi, M(x,\xi,t)\rangle| \\ &= \left| \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int_A \partial_\xi^\alpha M(x,\xi,t) d\mu_\alpha(\xi) \right| \\ &\leq \sum_{|\alpha| \leq m} \int_A |\partial_\xi^\alpha M(x,\xi,t)| |d\mu_\alpha|(\xi) . \end{aligned} \tag{3.4}$$

By Proposition 1, we have

$$(3.4) \leq \sum_{|\alpha| \leq m} (\alpha!)^{1/2} t^{-|\alpha|} \int_A (1+|x|+|\xi|)^{|\alpha|} M(x,\xi,t) |d\mu_\alpha|(\xi)$$

$$= \sum_{|\alpha|\le m} (\alpha!)^{1/2} t^{-|\alpha|} \sum_{l\le m} \binom{m}{l} (1+|x|)^l \int_A |\xi|^{m-l} M(x,\xi,t) |d\mu_\alpha|(\xi)$$

$$= \sum_{|\alpha|\le m} (\alpha!)^{1/2} t^{-|\alpha|} \sum_{l\le m} \binom{m}{l} (1+|x|)^l \int_A |\xi|^{m-l} \frac{e^{-dt}}{\pi^{-d/2}(1-e^{-4t})^{d/2}}$$

$$\times\, e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(|x|^2+|\xi|^2)+\frac{2e^{-2t}}{1-e^{-4t}}x\cdot\xi} |d\mu_\alpha|(\xi)\,. \tag{3.5}$$

Since $x \cdot \xi = \frac{-|x-\xi|^2+|x|^2+|\xi|^2}{2}$, we have

$$(3.5) \le \sum_{|\alpha|\le m} (\alpha!)^{1/2} t^{-|\alpha|} \sum_{l\le m} \binom{m}{l} (1+|x|)^l e^{-\frac{1}{2}\left(\frac{1+e^{-4t}}{1-e^{-4t}}-\frac{2e^{-2t}}{1-e^{-4t}}\right)|x|^2} e^{-\frac{1}{2}\frac{2e^{-2t}}{1-e^{-4t}}d(x,A)^2}$$

$$\times \int_A |\xi|^{m-l} e^{-\frac{1}{2}\left(\frac{1+e^{-4t}}{1-e^{-4t}}-\frac{2e^{-2t}}{1-e^{-4t}}\right)|\xi|^2} |d\mu_\alpha|(\xi)\,, \tag{3.6}$$

Since $0 < t < t_0$, for any $p \in \mathbf{Z}_+$,

$$|x|^p e^{-\frac{1}{2}\left(\frac{1+e^{-4t}}{1-e^{-4t}}-\frac{2e^{-2t}}{1-e^{-4t}}\right)|x|^2} \le (\tanh t)^{-p/2} p^{p/2} \le t^{-p/2} p^{p/2}$$

and μ_α is the tempered measure, by (3.6), there exist $r \in \mathbf{Z}_+$ and $C_r > 0$ such that

$$|U(x,t)| \le C_r (1+t^{-r}) e^{-\frac{1}{4}\frac{2e^{-2t}}{1-e^{-4t}}d(x,A)^2}\,.$$

Conversely for any $U(x,t) \in C^\infty(\mathbf{R}^d \times (0,t_0))$ satisfying (3.2) and (3.3), by Theorem 1, there exists $v \in \mathcal{S}'(\mathbf{R}^d)$ such that $U(x,t) = \langle v_\xi, M(x,\xi,t)\rangle$. Let $\varphi \in \mathcal{D}(\mathbf{R}^d)$ and $K = \operatorname{supp}\varphi \subset \mathbf{R}^d \setminus A$. Then we have

$$\begin{aligned}\left|\int_{\mathbf{R}^d} U(x,t)\varphi(x)dx\right| &\le \int_{\mathbf{R}^d} |U(x,t)||\varphi(x)|dx \\ &= \int_K |U(x,t)||\varphi(x)|dx \\ &\le C\left(1+t^{-\nu}\right) e^{-\frac{1}{4}\frac{2e^{-2t}}{1-e^{-4t}}d(K,A)^2} \\ &\to 0\,,\end{aligned}$$

as $t \to 0+$, where $d(K,A) = \inf_{x\in K} d(x,A)$. Hence we obtain

$$\lim_{t\to 0+} \int_{\mathbf{R}^d} U(x,t)\varphi(x)dx = 0\,.$$

On the other hand, by Theorem 1 we find that

$$\lim_{t\to 0+} U(x,t) = v(x) \text{ in } \mathcal{S}'(\mathbf{R}^d)\,.$$

Therefore we obtain that supp $v \subset A$. □

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