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Invariant Functions on p-divisible Groups and the p-adic Corona Problem

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1. Introduction

In this note we are concerned with p-divisible groups $G = (G_\nu)$ over a complete discrete valuation ring R. We assume that the fraction field K of R has characteristic zero and that the residue field $k = R/\pi R$ is perfect of positive characteristic p.

Let C be the completion of an algebraic closure of K and denote by $\mathfrak{o} = \mathfrak{o}_C$ its ring of integers. The group $G_\nu(\mathfrak{o})$ acts on $G_\nu \otimes \mathfrak{o}$ by translation. Since $G_\nu \otimes K$ is étale the $G_\nu(C)$-invariant functions on $G_\nu \otimes C$ are just the constants. Using the counit it follows that the natural inclusion

$$\mathfrak{o} \xrightarrow{\sim} \Gamma(G_\nu \otimes \mathfrak{o}, \mathcal{O})^{G_\nu(\mathfrak{o})}$$

is an isomorphism. We are interested in an approximate $\bmod \pi^n$-version of this statement. Set $\mathfrak{o}_n = \mathfrak{o}/\pi^n \mathfrak{o}$ for $n \geq 1$. The group $G_\nu(\mathfrak{o})$ acts by translation on $G_\nu \otimes \mathfrak{o}_n$ for all n.

THEOREM 1. *Assume that the dual p-divisible group G' is at most one-dimensional and that the connected-étale exact sequence for G' splits over $\mathfrak{o}$. Then there is an integer $t \geq 1$ such that the cokernel of the natural inclusion*

$$\mathfrak{o}_n \hookrightarrow \Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})^{G_\nu(\mathfrak{o})}$$

is annihilated by p^t for all ν and n.

The example of $\boldsymbol{G}_m = (\mu_{p^\nu})$ in section 2 may be helpful to get a feeling for the assertion.

In the last section, at the suggestion of the referee we explain the reasoning which led to the statement of the theorem.

We expect theorem 1 to hold without any restriction on the dimension of G as will be explained later. Its assertion is somewhat technical but the proof may be of interest because it combines some of the main results of Tate on p-divisible groups with van der Put's solution of his one-dimensional p-adic Corona problem.

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The classic corona problem concerns the Banach algebra $H^\infty(D)$ of bounded analytic functions on the open unit disc D. The points of D give maximal ideals in $H^\infty(D)$ and hence points of the Gelfand spectrum $\hat{D} = \mathrm{sp}\, H^\infty(D)$. The question was whether D was dense in $\hat{D}$, (the set $\hat{D} \setminus \overline{D}$ being the "corona"). This was settled affirmatively by Carleson [C]. The analogous question for the polydisc D^d is still open for $d \geq 2$. An equivalent condition for D^d to be dense in $\mathrm{sp}\, H^\infty(D^d)$ is the following one, [H], Ch. 10:

CONDITION 2. If $f_1, \dots, f_n$ are bounded analytic functions in D^d such that for some $\delta > 0$ we have

$$\max_{1 \leq i \leq n} |f_i(z)| \geq \delta \quad \text{for all } z \in D^d ,$$

then $f_1, \dots, f_n$ generate the unit ideal of $H^\infty(D^d)$.

In [P] van der Put considered the analogue of condition 2 with $H^\infty(D^d)$ replaced by the algebra of bounded analytic C-valued functions on the p-adic open polydisc Δ^d in C^d, i.e. by the algebra

$$C\langle X_1, \dots, X_d \rangle = \mathfrak{o}[[X_1, \dots, X_d]] \otimes_{\mathfrak{o}} C .$$

He called this p-adic version of condition 2 the p-adic Corona problem and verified it for $d = 1$. The general case $d \geq 1$ was later treated by Bartenwerfer [B] using his earlier results on rigid cohomology with bounds.

In the proof of theorem 1 applying Tate's results from [T] we are led to a question about certain ideals in $C\langle X_1, \dots, X_d \rangle$, which for $d = 1$ can be reduced to van der Put's p-adic Corona problem. For $d \geq 2$, I did not succeed in such a reduction. However it seems possible that a generalization of Bartenwerfer's theory might settle that question.

It should be mentioned that van der Put's term "p-adic Corona problem" for the p-adic analogue of condition 2 is somewhat misleading. Namely as pointed out in [EM] a more natural analogue should be the question whether Δ^d was dense in the Berkovich space of $C\langle X_1, \dots, X_d \rangle$. This is not known, even for $d = 1$. The difference between the classic and the p-adic cases comes from the fact discovered by van der Put that contrary to $H^\infty(D^d)$ the algebra $C\langle X_1, \dots, X_d \rangle$ contains maximal ideals of infinite codimension.

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2. An example and other versions of the theorem

Consider an affine group scheme $\mathcal{G}$ over a ring S with Hopf-algebra $\mathcal{A} = \Gamma(\mathcal{G}, \mathcal{O})$, comultiplication $\mu : \mathcal{A} \to \mathcal{A} \otimes_S \mathcal{A}$ and counit $\varepsilon : \mathcal{A} \to S$. The operation of $\mathcal{G}(S) = \mathrm{Hom}_S(\mathcal{A}, S)$ on $\Gamma(\mathcal{G}, \mathcal{O})$ by translation is given by the map

$$\mathcal{G}(S) \times \mathcal{A} \to \mathcal{A} , \ (\chi, a) \mapsto (\chi \otimes \mathrm{id})\mu(a) \tag{1}$$

where $(\chi \otimes \mathrm{id})\mu$ is the composition

$$\mathcal{A} \xrightarrow{\mu} \mathcal{A} \otimes_S \mathcal{A} \xrightarrow{\chi \otimes \mathrm{id}} S \otimes_S \mathcal{A} = \mathcal{A}\,.$$

Given a homomorphism of groups $P \to \mathcal{G}(S)$ we may view $\mathcal{A}$ as a P-module. The composition $S \to \mathcal{A} \xrightarrow{\varepsilon} S$ being the identity we have an isomorphism

$$\mathrm{Ker}\,(\mathcal{A}^P \xrightarrow{\varepsilon} S) \xrightarrow{\sim} \mathcal{A}^P/S \quad \text{mapping } a \text{ to } a + S\,. \tag{2}$$

The inverse sends $a + S$ to $a - \varepsilon(a) \cdot 1$.

EXAMPLE. The theorem is true for $\boldsymbol{G}_m = (\mu_{p^\nu})$.

PROOF. Set $V = \mathfrak{o}_n[X, X^{-1}]/(X^{p^\nu} - 1)$. Applying formulas (1) and (2) with $\mathcal{G} = \mu_{p^\nu} \otimes \mathfrak{o}_n$ and $P = \mu_{p^\nu}(\mathfrak{o}) \to \mathcal{G}(\mathfrak{o}_n)$ we see that the cokernel of the map

$$\mathfrak{o}_n \to \Gamma(\mu_{p^\nu} \otimes \mathfrak{o}_n, \mathcal{O})^{\mu_{p^\nu}(\mathfrak{o})} \tag{3}$$

is isomorphic to the $\mathfrak{o}_n$-module:

$$\{\overline{Q} \in V \mid \overline{Q}(\zeta X) = \overline{Q}(X) \text{ for all } \zeta \in \mu_{p^\nu}(\mathfrak{o}) \text{ and } \overline{Q}(1) = 0\}\,.$$

Lift $\overline{Q}$ to a Laurent polynomial $Q = \sum_{\mu \in \mathcal{S}} a_\mu X^\mu$ in $\mathfrak{o}[X, X^{-1}]$ where $\mathcal{S} = \{0, \dots, p^\nu - 1\}$.

Then we have:

$$(\zeta^\mu - 1)a_\mu \equiv 0 \bmod \pi^n \quad \text{for } \mu \in \mathcal{S} \text{ and } \zeta \in \mu_{p^\nu}(\mathfrak{o}) \tag{4}$$

and

$$\sum_{\mu \in \mathcal{S}} a_\mu \equiv 0 \bmod \pi^n\,. \tag{5}$$

For any non-zero μ in $\mathcal{S}$ choose $\zeta \in \mu_{p^\nu}(\mathfrak{o})$ such that $\zeta^\mu \neq 1$. Then $\zeta^\mu - 1$ divides p in $\mathfrak{o}$ and hence (4) implies that $pa_\mu \equiv 0 \bmod \pi^n$ for all $\mu \neq 0$. Using (5) it follows that we have $pa_0 \equiv 0 \bmod \pi^n$ as well. Hence $pQ \bmod \pi^n$ is zero and therefore $p\overline{Q} = 0$ as well. Thus p annihilates the $\mathfrak{o}_n$-module (3) for all $\nu \geq 1$ and $n \geq 1$. □

Now assume that $S = R$ and that $\mathcal{G}/R$ is a finite, flat group scheme. Consider the Cartier dual $\mathcal{G}' = \mathrm{Spec}\,\mathcal{A}'$ where $\mathcal{A}' = \mathrm{Hom}_R(\mathcal{A}, R)$. The perfect pairing of finite free $\mathfrak{o}_n$-modules

$$(\mathcal{A} \otimes \mathfrak{o}_n) \times (\mathcal{A}' \otimes \mathfrak{o}_n) \to \mathfrak{o}_n \tag{6}$$

induces an isomorphism

$$\mathrm{Ker}\,((\mathcal{A} \otimes \mathfrak{o}_n)^{\mathcal{G}(\mathfrak{o})} \xrightarrow{\varepsilon} \mathfrak{o}_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{o}_n}((\mathcal{A}' \otimes \mathfrak{o}_n)_{\mathcal{G}(\mathfrak{o})}/\mathfrak{o}_n, \mathfrak{o}_n)\,. \tag{7}$$

Using (2) it follows that if p^t annihilates $(\mathcal{A}' \otimes \mathfrak{o}_n)_{\mathcal{G}(\mathfrak{o})}/\mathfrak{o}_n$ then p^t annihilates $(\mathcal{A} \otimes \mathfrak{o}_n)^{\mathcal{G}(\mathfrak{o})}/\mathfrak{o}_n$ as well. (The converse is not true in general.)

Hence theorem 1 follows from the next result (applied to the dual p-divisible group).

THEOREM 3. *Assume that the p-divisible group G is at most one-dimensional and that the connected-étale exact sequence for G splits over* $\mathfrak{o}$. *Then there is an integer* $t \geq 1$ *such that* p^t *annihilates the cokernel of the natural map*

$$\mathfrak{o}_n \to \Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})_{G'_\nu(\mathfrak{o})}$$

for all ν *and* n.

For a finite flat group scheme $\mathcal{G} = \operatorname{Spec} \mathcal{A}$ over a ring S, the group

$$\mathcal{G}'(S) = \operatorname{Hom}_{S-\mathrm{alg}}(\operatorname{Hom}_S(\mathcal{A}, S), S) \subset \mathcal{A}$$

consists of the group-like elements in $\mathcal{A}$ i.e. the units a in $\mathcal{A}$ with $\mu(a) = a \otimes a$. In this way $\mathcal{G}'(S)$ becomes a subgroup of the unit group $\mathcal{A}^*$ and hence $\mathcal{G}'(S)$ acts on $\mathcal{A}$ by multiplication. On the other hand $\mathcal{G}'(S)$ acts on $\mathcal{G}'$ by translation, hence on $\mathcal{A}' = \Gamma(\mathcal{G}', \mathcal{O})$ and hence on $\mathcal{A}'' = \mathcal{A}$. Using (1) one checks that the two actions of $\mathcal{G}'(S)$ on $\mathcal{A}$ are the same. This leads to the following description of the cofixed module in theorem 3. Set $A_\nu = \Gamma(G_\nu, \mathcal{O})$ and let J_ν be the ideal in $A_\nu \otimes_R \mathfrak{o}$ generated by the elements $h - 1$ with h group-like in this Hopf-algebra over $\mathfrak{o}$. Thus J_ν is also the $\mathfrak{o}$-submodule of $A_\nu \otimes_R \mathfrak{o}$ generated by the elements $ha - a$ for $h \in G'_\nu(\mathfrak{o})$ and $a \in A_\nu \otimes_R \mathfrak{o}$. Then we have the formula

$$\Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})_{G'_\nu(\mathfrak{o})} = (A_\nu \otimes_R \mathfrak{o}_n)/J_\nu(A_\nu \otimes_R \mathfrak{o}_n)\,. \tag{8}$$

This implies an isomorphism:

$$\operatorname{Coker}(\mathfrak{o}_n \to \Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})_{G'_\nu(\mathfrak{o})}) = \operatorname{Coker}(\mathfrak{o} \to (A_\nu \otimes_R \mathfrak{o})/J_\nu) \otimes_{\mathfrak{o}} \mathfrak{o}_n\,. \tag{9}$$

Hence theorem 3 and therefore also theorem 1 follow from the next claim:

CLAIM 4. *For a p-divisible group* $G = (G_\nu)$ *as in theorem 3 there exists an integer* $t \geq 1$ *such that* p^t *annihilates the cokernel of the natural map*
$\mathfrak{o} \to (A_\nu \otimes_R \mathfrak{o})/J_\nu$ *for all* $\nu \geq 1$.

As a first step in the proof of claim 4 we reduce to the case where G is either étale or connected. For simplicity set $\mathcal{G} = G_\nu \otimes_R \mathfrak{o} = \operatorname{Spec} \mathcal{A}$ and define $\mathcal{G}^0, \mathcal{G}^{\text{ét}}, \mathcal{A}^0, \mathcal{A}^{\text{ét}}$ similarly. By assumption we have isomorphisms $\mathcal{G} = \mathcal{G}^0 \times_{\mathfrak{o}} \mathcal{G}^{\text{ét}}$ and $\mathcal{A} = \mathcal{A}^0 \otimes_{\mathfrak{o}} \mathcal{A}^{\text{ét}}$ as group schemes, resp. Hopf-algebras over $\mathfrak{o}$. There is a compatible splitting of the group-like elements over $\mathfrak{o}$:

$$\mathcal{G}'(\mathfrak{o}) = \mathcal{G}^{0'}(\mathfrak{o}) \times \mathcal{G}^{\text{ét}'}(\mathfrak{o})\,.$$

For elements

$$h^0 \in \mathcal{G}^{0'}(\mathfrak{o}) \subset \mathcal{A}^0 \quad \text{and} \quad h^{\text{ét}} \in \mathcal{G}^{\text{ét}'}(\mathfrak{o}) \subset \mathcal{A}^{\text{ét}}$$

consider the identity:

$$h^0 \otimes h^{\text{ét}} - 1 = h^0 \otimes (h^{\text{ét}} - 1) + (h^0 - 1) \otimes 1 \quad \text{in } \mathcal{A}\,.$$

It implies that we have

$$J = \mathcal{A}^0 \otimes J^{\text{ét}} + J^0 \otimes \mathcal{A}^{\text{ét}} \quad \text{in } \mathcal{A}$$

where J is the ideal of $\mathcal{A}$ generated by the elements $h-1$ for $h \in \mathcal{G}'(\mathfrak{o})$ and J^0, $J^{\text{ét}}$ are defined similarly. Hence we have natural surjections

$$\mathcal{A}^0/J^0 \otimes \mathcal{A}^{\text{ét}}/J^{\text{ét}} \to \mathcal{A}/J$$

and

$$\operatorname{Coker}(\mathfrak{o} \to \mathcal{A}^0/J^0) \otimes \operatorname{Coker}(\mathfrak{o} \to \mathcal{A}^{\text{ét}}/J^{\text{ét}}) \to \operatorname{Coker}(\mathfrak{o} \to \mathcal{A}/J)\,.$$

Hence it suffices to prove claim 4 in the cases where G is either connected or étale. The étale case is straightforward: We have $G \otimes_R \mathfrak{o} = ((\mathbf{Z}/p^\nu)^h)_{\nu \geq 0}$. Hence $G'_\nu = \mu^h_{p^\nu}$ and $G'_\nu(\mathfrak{o}) = \mu_{p^\nu}(\mathfrak{o})^h$. The inclusion

$$\mu_{p^\nu}(\mathfrak{o})^h = \operatorname{Hom}((\mathbf{Z}/p^\nu)^h, \mathfrak{o}^*) \subset \operatorname{Maps}((\mathbf{Z}/p^\nu)^h, \mathfrak{o}) = A_\nu \otimes \mathfrak{o}$$

identifies $\mu_{p^\nu}(\mathfrak{o})^h$ with the group like elements in $A_\nu \otimes \mathfrak{o}$.

The ideal J_ν of $A_\nu \otimes \mathfrak{o}$ is given by:

$$J_\nu = (\chi_\zeta - 1 \mid \zeta \in \mu_{p^\nu}(\mathfrak{o})^h)$$

where χ_ζ is the character of $(\mathbf{Z}/p^\nu)^h$ defined by the equation

$$\chi_\zeta((a_1, \ldots, a_h)) = \zeta_1^{a_1} \cdots \zeta_h^{a_h} \quad \text{where } \zeta = (\zeta_1, \ldots, \zeta_h)\,.$$

The functions δ_a for $a \in (\mathbf{Z}/p^\nu)^h$ given by $\delta_a(a) = 1$ and $\delta_a(b) = 0$ if $b \neq a$ generate $A_\nu \otimes \mathfrak{o}$ as an $\mathfrak{o}$-module. For $a \neq 0$ choose $\zeta \in \mu_{p^\nu}(\mathfrak{o})^h$ with $\zeta^a \neq 1$. Then we have $p = (\zeta^a - 1)\beta$ for some $\beta \in \mathfrak{o}$. Define $f_a \in A_\nu \otimes \mathfrak{o}$ by setting

$$f_a(a) = \beta \quad \text{and} \quad f_a(b) = 0 \text{ for } b \neq a\,.$$

We then find:

$$f_a(\chi_\zeta - 1) = p\delta_a \quad \text{in } A_\nu \otimes \mathfrak{o}\,.$$

Hence we have $p\delta_a \in J_\nu$ for all $a \neq 0$ and therefore p annihilates $\operatorname{Coker}(\mathfrak{o} \to (A_\nu \otimes \mathfrak{o})/J_\nu)$.

The next two sections are devoted to the much more interesting case where G is connected.

3. The connected case I (p-adic Hodge theory)

In this section we reduce the assertion of claim 4 for connected p-divisible groups of arbitrary dimension to an assertion on ideals in $C\langle X_1, \ldots, X_d\rangle$. For this reduction we use theorems of Tate in [T].

Thus let $G = (G_\nu)$ be a connected p-divisible group of dimension d over R and set $A = \varprojlim_\nu A_\nu$ where $G_\nu = \operatorname{Spec} A_\nu$.

Consider the projective limit $A = \varprojlim A_n$ with the topology inherited from the product topology $\prod A_n$ where the A_n's are given the π-adic topology. This topology on A is the one defined by the R-submodules $K_n + \pi^k A$ for $n, k \geq 1$ where $K_n = \operatorname{Ker}(A \to A_n)$. Equivalently it is defined by the spaces $K_n + \pi^n A$ for $n \geq 1$. In [T] section (2.2) it is shown that A is isomorphic to $R[[X_1, \dots, X_d]]$ as a topological R-algebra. If M denotes the maximal ideal of A, then according to [T] Lemma 0 the topology of A coincides with the M-adic topology. Let $A \hat{\otimes}_R \mathfrak{o}$ be the completion of $A \otimes_R \mathfrak{o}$ with respect to the linear topology on $A \otimes_R \mathfrak{o}$ given by the subspaces $M^n \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o}$.

LEMMA 5. *We have*

$$\varprojlim (A_n \otimes_R \mathfrak{o}) = A \hat{\otimes}_R \mathfrak{o} = \mathfrak{o}[[X_1, \dots, X_d]]$$

as topological rings.

PROOF. Consider the isomorphisms

$$\begin{aligned}
\varprojlim_n (A_n \otimes_R \mathfrak{o}) &= \varprojlim_n (A_n \otimes_R (\varprojlim_k \mathfrak{o}/\pi^k \mathfrak{o})) \\
&\overset{(1)}{=} \varprojlim_n \varprojlim_k (A_n \otimes_R \mathfrak{o}/\pi^k \mathfrak{o}) \\
&= \varprojlim_n \varprojlim_k (A \otimes_R \mathfrak{o})/((K_n + \pi^k A) \otimes_R \mathfrak{o} + A \otimes_R \pi^k \mathfrak{o}) \\
&\overset{(2)}{=} \varprojlim_n (A \otimes_R \mathfrak{o})/((K_n + \pi^n A) \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o}) \\
&\overset{(3)}{=} \varprojlim_n (A \otimes_R \mathfrak{o})/(M^n \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o}) \\
&= A \hat{\otimes}_R \mathfrak{o} \\
&\overset{(4)}{=} \mathfrak{o}[[X_1, \dots, X_d]] .
\end{aligned}$$

Here (1) holds because $\varprojlim$ commutes with finite direct sums, (2) is true by cofinality, (3) holds because the topology on A can also be described as the M-adic topology. Finally (4) follows from the definition of $A \hat{\otimes}_R \mathfrak{o}$ and the fact that $A = R[[X_1, \dots, X_d]]$. □

The $\mathfrak{o}$-algebra $A \hat{\otimes}_R \mathfrak{o} = \varprojlim_\nu (A_\nu \otimes_R \mathfrak{o})$ contains the ideal $\tilde{J} = \varprojlim_\nu J_\nu$.

CLAIM 6. *We have*

$$A \hat{\otimes}_R \mathfrak{o}/(\mathfrak{o} + \tilde{J}) = \varprojlim_\nu (A_\nu \otimes_R \mathfrak{o}/(\mathfrak{o} + J_\nu)) .$$

PROOF. The inclusion $G_\nu \subset G_{\nu+1}$ corresponds to a surjection of Hopf-algebras $A_{\nu+1} \to A_\nu$. Hence $A_{\nu+1} \otimes_R \mathfrak{o} \to A_\nu \otimes_R \mathfrak{o}$ is surjective as well and group-like elements are mapped to group-like elements. The map on group-like elements is surjective because it corresponds to the surjective map $G'_{\nu+1}(\mathfrak{o}) \to G'_\nu(\mathfrak{o})$. Note here that $G'_\mu(\mathfrak{o}) = G'_\mu(C)$ for all μ. It follows that the map $J_{\nu+1} \to J_\nu$ is surjective as well. In the exact sequence of projective systems

$$0 \to (\mathfrak{o} + J_\nu) \to (A_\nu \otimes_R \mathfrak{o}) \to (A_\nu \otimes_R \mathfrak{o}/(\mathfrak{o} + J_\nu)) \to 0$$

the system $(\mathfrak{o} + J_\nu)$ is therefore Mittag–Leffler. Hence the sequence of projective limits is exact and the claim follows because the sum $\mathfrak{o} + J_\nu$ is direct: Group-like elements of $A_\nu \otimes_R \mathfrak{o}$ are mapped to 1 by the counit ε_ν. Therefore we have

$$J_\nu \subset I_\nu := \operatorname{Ker}(\varepsilon_\nu : A_\nu \otimes_R \mathfrak{o} \to \mathfrak{o}) . \tag{10}$$

The sum $\mathfrak{o} + I_\nu$ being direct we are done. □

Because of claim 6 and the surjectivity of the maps $A_{\nu+1} \otimes_R \mathfrak{o} \to A_\nu \otimes_R \mathfrak{o}$, claim 4 for connected groups is equivalent to the next assertion:

CLAIM 7. *Let G be a connected p-divisible group with* $\dim G \le 1$. *Then there is some* $t \ge 1$ *such that p^t annihilates*

$$\operatorname{Coker}(\mathfrak{o} \to A\hat{\otimes}_R \mathfrak{o}/\tilde{J}) .$$

For connected G of arbitrary dimension, consider the Tate module of G'

$$TG' = \varprojlim_\nu G'_\nu(C) = \varprojlim_\nu G'_\nu(\mathfrak{o}) \subset \varprojlim_\nu (A_\nu \otimes_R \mathfrak{o}) = A\hat{\otimes}_R \mathfrak{o} .$$

Let J be the ideal of $A\hat{\otimes}_R\mathfrak{o}$ generated by the elements $h - 1$ for $h \in TG'$. The image of J under the reduction map $A\hat{\otimes}_R\mathfrak{o} \to A_\nu \otimes_R \mathfrak{o}$ lies in J_ν. It follows that $J \subset \tilde{J}$. With I_ν as in (10) we set $I = \varprojlim_\nu I_\nu$, an ideal in $A\hat{\otimes}_R\mathfrak{o}$. We have $J \subset \tilde{J} \subset I$ because of (10). Since $A\hat{\otimes}_R\mathfrak{o} = \mathfrak{o} \oplus I$, we get a surjection

$$I/J \twoheadrightarrow \operatorname{Coker}(\mathfrak{o} \to A\hat{\otimes}_R\mathfrak{o}/\tilde{J}) . \tag{11}$$

Thus claim 7 will be proved if we can show that $p^t I \subset J$ at least for $\dim G = 1$. The construction in [T] section (2.2) shows that under the isomorphism of $\mathfrak{o}$-algebras

$$A\hat{\otimes}_R\mathfrak{o} = \mathfrak{o}[[X_1, \dots, X_d]] \quad \text{we have } I = (X_1, \dots, X_d) .$$

We will view the elements of $A\hat{\otimes}_R\mathfrak{o}$ and in particular those of J as analytic functions on the open d-dimensional polydisc

$$\Delta^d = \{x \in C^d \mid |x_i| < 1 \text{ for all } i\} .$$

Because of the inclusion $J \subset I$ all functions in J vanish at $0 \in \Delta^d$. There are no other common zeroes:

PROPOSITION 8 (Tate). *The zero set of J in Δ^d consists only of the origin $o \in \Delta^d$.*

PROOF. The $\mathfrak{o}$-valued points of the p-divisible group G,

$$G(\mathfrak{o}) = \varprojlim_i \varinjlim_\nu G_\nu(\mathfrak{o}/\pi^i \mathfrak{o})$$

can be identified with continuous $\mathfrak{o}$-algebra homomorphisms

$$G(\mathfrak{o}) = \mathrm{Hom}_{\mathrm{cont,alg}}(A, \mathfrak{o}) = \mathrm{Hom}_{\mathrm{cont,alg}}(A \hat{\otimes}_R \mathfrak{o}, \mathfrak{o}) .$$

Moreover we have a homeomorphism

$$\Delta^d \xrightarrow{\sim} G(\mathfrak{o}) \quad \text{via} \quad x \mapsto (f \mapsto f(x)) .$$

Here $f \in A \hat{\otimes}_R \mathfrak{o}$ is viewed as a formal power series over $\mathfrak{o}$. The group structure on $G(\mathfrak{o})$ induces a Lie group structure on Δ^d with $0 \in \Delta^d$ corresponding to $1 \in G(\mathfrak{o})$. Let U be the group of 1-units in $\mathfrak{o}$. Proposition 11 of [T] asserts that the homomorphism of Lie groups

$$\alpha : \Delta^d = G(\mathfrak{o}) \to \mathrm{Hom}_{\mathrm{cont}}(TG', U) \ , \ x \mapsto (h \mapsto h(x)) \tag{12}$$

is *injective*. Note here that $TG' \subset A \hat{\otimes}_R \mathfrak{o}$. Let $x \in \Delta^d$ be a point in the zero set of J. Then we have $(h-1)(x) = 0$ i.e. $h(x) = 1$ for all $h \in TG'$. Hence x maps to $1 \in \mathrm{Hom}_{\mathrm{cont}}(TG', U)$. Since α is injective, it follows that we have $x = 0$. □

If a Hilbert Nullstellensatz were true in $C\langle X_1, \dots, X_d\rangle$ we could conclude that we had $\sqrt{J \otimes C} = I \otimes C$. Then an inductive argument using the injectivity of α would imply $p^t I \subset J$. However the Nullstellensatz does not hold in the ring $C\langle X_1, \dots, X_d\rangle$.

In the next section we will provide a replacement which is proved for $d = 1$ and conjectured for $d \geq 2$. In order to apply it to the ideal $J \otimes C$ in $C\langle X_1, \dots, X_d\rangle$ we need to know the following assertion which is stronger than proposition 8. For $x \in C^m$ set $\|x\| = \max_i |x_i|$.

PROPOSITION 9. *Let $h_1, \dots, h_r$ be a $\mathbf{Z}_p$-basis of $TG' \subset \mathfrak{o}[[X_1, \dots, X_d]]$ and set $H(x) = (h_1(x), \dots, h_r(x))$ and $\mathbf{1} = (1, \dots, 1)$. Then there is a constant $\delta > 0$ such that we have:*

$$\|H(x) - \mathbf{1}\| \geq \delta \|x\| \quad \textit{for all } x \in \Delta^d .$$

PROOF. The $\mathbf{Z}_p$-rank r of TG' is the height of G' and hence we have $r \geq d = \dim G$. Consider the following diagram $(*)$ on p. 177 of [T]:

$$\begin{tikzcd}
1 \arrow[r] & G(\mathfrak{o})_{\mathrm{tors}} \arrow[r] \arrow[d, "\alpha_0", "\wr"'] & G(\mathfrak{o}) \arrow[r, "L"] \arrow[d, hook, "\alpha"] & t_G(C) \arrow[r] \arrow[d, hook, "d\alpha"] & 0 \\
1 \arrow[r] & \mathrm{Hom}(TG', U_{\mathrm{tors}}) \arrow[r] & \mathrm{Hom}(TG', U) \arrow[r, "\log_*"] & \mathrm{Hom}(TG', C) \arrow[r] & 0.
\end{tikzcd}$$

Here the Hom-groups refer to continuous homomorphisms and the map α was defined in equation (12) above. The map L is the logarithm map to the tangent space $t_G(C)$ of G and

$\log_*$ is induced by $\log : U \to C$. According to [T] proposition 11 the maps α and $d\alpha$ are injective and α_0 is bijective. It will suffice to prove the following two statements:

I) For any $\varepsilon > 0$ there is a constant $\delta(\varepsilon) > 0$ such that

$$\|H(x) - \mathbf{1}\| \geq \delta(\varepsilon) \text{ for all } x \in \Delta^d \text{ with } \|x\| \geq \varepsilon\,.$$

II) There are $\varepsilon > 0$ and $a > 0$ such that

$$\|H(x) - \mathbf{1}\| \geq a\|x\| \text{ for all } x \in \Delta^d \text{ with } \|x\| \leq \varepsilon\,.$$

Identifying $G(\mathfrak{o})$ with Δ^d where we write the induced group structure on Δ^d as $\oplus$, and identifying TG' with $\boldsymbol{Z}_p^r$ via the choice of the basis $h_1, \ldots, h_r$, the above diagram becomes the following one where $A = dH$ and H_0 is the restriction of H to $(\Delta^d)_{\mathrm{tors}}$

$$\begin{tikzcd}
0 \arrow[r] & (\Delta^d)_{\mathrm{tors}} \arrow[r] \arrow[d, "H_0"', "\wr"] & \Delta^d \arrow[r, "L"] \arrow[d, hook, "H"'] & C^d \arrow[r] \arrow[d, hook, "A=dH"] & 0 \\
1 \arrow[r] & U^r_{\mathrm{tors}} \arrow[r] & U^r \arrow[r, "\log"] & C^r \arrow[r] & 0.
\end{tikzcd}$$

Assume that assertion I is wrong for some $\varepsilon > 0$. Then there is a sequence $x^{(i)}$ of points in Δ^d with $\|x^{(i)}\| \geq \varepsilon$ such that $H(x^{(i)}) \to \mathbf{1}$ for $i \to \infty$. It follows that $A(L(x^{(i)}) = \log H(x^{(i)}) \to 0$ for $i \to \infty$. Since A is an injective linear map between finite dimensional C-vector spaces, there exists a constant $a > 0$ such that we have

$$\|A(v)\| \geq a\|v\| \quad \text{for all } v \in C^d\,. \tag{13}$$

Hence we see that $L(x^{(i)}) \to 0$ for $i \to \infty$. Since L is a local homeomorphism, there exists a sequence $y^{(i)} \to 0$ in Δ^d with $L(x^{(i)}) = L(y^{(i)})$ for all i. The sequence $z^{(i)} = x^{(i)} \ominus y^{(i)}$ in Δ^d satisfies $L(z^{(i)}) = 0$ and hence lies in $(\Delta^d)_{\mathrm{tors}}$. We have $H_0(z^{(i)}) = H(x^{(i)})H(y^{(i)})^{-1}$. Moreover $H(x^{(i)}) \to \mathbf{1}$ by assumption and $H(y^{(i)}) \to \mathbf{1}$ since $y^{(i)} \to 0$. Hence $H_0(z^{(i)}) \to 1$ and therefore $H_0(z^{(i)}) = 1$ for all $i \gg 0$ since the subspace topology on $U_{\mathrm{tors}} \subset U$ is the discrete topology. The map H_0 being bijective we find that $z^{(i)} = 0$ for $i \gg 0$ and therefore $x^{(i)} = y^{(i)}$ for $i \gg 0$. This implies that $x^{(i)} \to 0$ for $i \to \infty$ contradicting the assumption $\|x^{(i)}\| \geq \varepsilon$ for all i. Hence assertion I) is proved.

We now turn to assertion II). Set $X = (X_1, \ldots, X_d)$. Then we have

$$H(X) = \mathbf{1} + AX + (\deg \geq 2)\,.$$

Componentwise this gives for $1 \leq j \leq r$

$$h_j(x) - 1 = \sum_{i=1}^{d} a_{ij}x_i + (\deg \geq 2)_j\,.$$

Let a be the constant from equation (13) and choose $\varepsilon > 0$, such that for $\|x\| \le \varepsilon$ we have

$$\|(\deg \ge 2)_j\| \le \frac{a}{2}\|x\| \quad \text{for } 1 \le j \le r \, .$$

For any x with $\|x\| < \varepsilon$, according to (13) there is an index j with

$$\Big|\sum_{i=1}^{d} a_{ij} x_i\Big| \ge a\|x\| \, .$$

This implies that we have

$$|h_j(x) - 1| = \Big|\sum_{i=1}^{d} a_{ij} x_i + (\deg \ge 2)_j\Big| = \Big|\sum_{i=1}^{d} a_{ij} x_i\Big| \ge a\|x\|$$

and hence

$$\|H(x) - \mathbf{1}\| \ge a\|x\| \, .$$

□

4. The connected case II (the p-adic Corona problem)

As remarked in the previous section we need a version of the Hilbert Nullstellensatz in $C\langle X_1, \dots, X_d\rangle$ for the case where the zero set is $\{0\} \subset \Delta^d$. The only result for $C\langle X_1, \dots, X_d\rangle$ in the spirit of the Nullstellensatz that I am aware of concerns an empty zero set:

P-ADIC CORONA THEOREM 10 (van der Put, Bartenwerfer). *For $f_1, \dots, f_n$ in $C\langle X_1, \dots, X_d\rangle$ the following conditions are equivalent*:

(a) *The functions $f_1, \dots, f_n$ generate the C-algebra $C\langle X_1, \dots, X_d\rangle$.*

(b) *There is a constant $\delta > 0$ such that*

$$\max_{1 \le j \le n} |f_j(x)| \ge \delta \quad \textit{for all } x \in \Delta^d \, .$$

It is clear that the first condition implies the second. The non-trivial implication was proved by van der Put for $d = 1$ in [P] and by Bartenwerfer in general, c.f. [B]. Both authors give a more precise statement of the theorem where the norms of possible functions g_j with $\sum_j f_j g_j = 1$ are estimated.

Consider the following stronger conjecture which deals with the case where the zero set may contain $\{0\}$.

CONJECTURE 11. *For $g_1, \dots, g_n$ in $C\langle X_1, \dots, X_d\rangle$ the following conditions are equivalent*:

(a) $(g_1, \dots, g_n) \supset (X_1, \dots, X_d)$.

(b) *There is a constant $\delta > 0$ such that*

$$\max_{1 \le j \le n} |g_j(x)| \ge \delta \|x\| \quad \textit{for all } x \in \Delta^d . \tag{14}$$

As above, immediate estimates show that (a) implies (b). Note also that if (b) holds and in addition some g_j does not vanish at $x = 0$ we have

$$\max_{1 \le j \le n} |g_j(x)| \ge \delta' > 0 \text{ in a neighborhood of } x = 0 .$$

Together with (14) this implies that

$$\max_{1 \le j \le n} |g_j(x)| \ge \delta'' > 0 \quad \text{for all } x \in \Delta^d .$$

The p-adic Corona theorem 10 then gives $(g_1, \dots, g_n) = C\langle X_1, \dots, X_d\rangle$. Thus (b) implies (a) if $g_j(0) \neq 0$ for some j.

PROPOSITION 12. *The preceding conjecture* 11 *is true for* $d = 1$.

PROOF. As explained above, we may assume that all functions $g_1, \dots, g_n$ vanish at $x = 0$. Then $f_j(X) = X^{-1} g_j(X)$ is in $C\langle X\rangle$ for every $1 \le j \le n$ and estimate (14) implies the estimate

$$\max_{1 \le j \le n} |f_j(x)| \ge \delta \quad \text{for all } x \in \Delta .$$

The p-adic Corona theorem for $d = 1$ now shows that

$$(f_1, \dots, f_n) = (1) \quad \text{and hence} \quad (g_1, \dots, g_n) = (X) .$$

□

Let us now return to p-divisible groups and recall the surjection (11):

$$I/J \twoheadrightarrow \operatorname{Coker}(\mathfrak{o} \to A \hat{\otimes}_R \mathfrak{o}/\tilde{J}) . \tag{15}$$

Here $I = (X_1, \dots, X_d)$ in $\mathfrak{o}[[X_1, \dots, X_d]]$ and J is the ideal generated by the elements $h - 1$ for $h \in TG'$. Let $J_0 \subset J$ be the ideal generated by the elements $h_1 - 1, \dots, h_r - 1$ where $h_1, \dots, h_r$ form a $\mathbf{Z}_p$-basis of TG'. In proposition 9 we have seen that for some $\delta > 0$ we have

$$\max_{1 \le j \le r} |h_j(x) - 1| \ge \delta \|x\| \quad \text{for all } x \in \Delta^d .$$

Conjecture 11 (which is true for $d = 1$) would therefore imply

$$(h_1 - 1, \dots, h_r - 1) = (X_1, \dots, X_d) \quad \text{in } C\langle X_1, \dots, X_d\rangle .$$

Thus we would find some $t \ge 1$, such that we have

$$p^t X_i \in J_0 \subset \mathfrak{o}[[X_1, \dots, X_d]] \quad \text{for all } 1 \le j \le r$$

and hence also $p^t I \subset J_0 \subset J$. Using the surjection (15) this would prove claim 7 and hence theorem 3 without restriction on $\dim G$. Also theorem 1 would follow without restriction on $\dim G'$. However since we can prove conjecture 11 only for $d = 1$ we have to assume $\dim G \leq 1$ in claim 7 and theorem 3. Therefore we have to assume $\dim G' \leq 1$ in theorem 1.

5. Representations attached to homogenous vector bundles on abelian varieties

In this section we sketch the argument which led to the statement of theorem 1. With notations as in the introduction, assume that the residue field of R is finite. Let $\mathcal{A}$ be an abelian scheme over R and set $A = \mathcal{A} \otimes_R K$. A vector bundle E on $A_C = A \otimes_K C$ is called homogenous if for every point a in $A(C)$ we have $T_a^* E \cong E$ where T_a denotes translation by a. It is known that every homogenous bundle is a finite direct sum of vector bundles of the form $L \otimes U$ where L is a line bundle of degree zero on A_C and U a unipotent bundle i.e. a successive extension of the trivial line bundle by itself, cf. [Mi], [Mu]. By Atiyah's classification every vector bundle of degree zero on an elliptic curve A_C is homogenous but for abelian varieties of higher dimensions this is not true. Let E_0 be the fibre of E in the origin $0 \in A(C)$. By the results of [DW] section 1.2, a continuous representation $\rho_E : TA \to \mathrm{GL}\,(E_0)$ can be functorially attached to E. Here TA is the Tate module of A. The functor ρ mapping E to ρ_E is faithful as follows from the argument below.

We will now show that ρ is even fully faithful if the assertion of theorem 1 holds for the p-divisible group of $\mathcal{A}$. For this we briefly recall the construction of the representation ρ_E. Because of the structure of homogenous bundles recalled above, [DW] Theorem 1 implies that E can be extended to a vector bundle $\mathcal{E}$ on $\mathcal{A}_{\mathfrak{o}} = \mathcal{A} \otimes_R \mathfrak{o}$ which satisfies the following condition:
For every integer $n \geq 1$ there is an integer $N = N(n) \geq 1$ such that $N^*(\mathcal{E} \otimes \mathfrak{o}_n)$ is a trivial vector bundle on $\mathcal{A} \otimes \mathfrak{o}_n$. Here N also denotes the N-multiplication morphism on an abelian scheme.

Pulling back by the zero section 0 therefore induces an isomorphism

$$0^* : \Gamma(\mathcal{A} \otimes \mathfrak{o}_n, N^*(\mathcal{E} \otimes \mathfrak{o}_n)) \xrightarrow{\sim} \Gamma(\mathrm{Spec}\,\mathfrak{o}_n, 0^*(\mathcal{E} \otimes \mathfrak{o}_n)) = \mathcal{E}_0 \otimes \mathfrak{o}_n$$

where $\mathcal{E}_0 = \Gamma(\mathrm{Spec}\,\mathfrak{o}, 0^*\mathcal{E})$. The group $A_N(C) = \mathcal{A}_N(\mathfrak{o})$ operates naturally on the global sections of $N^*(\mathcal{E} \otimes \mathfrak{o}_n)$ and hence we get an induced action on $\mathcal{E}_0 \otimes \mathfrak{o}_n$. This gives a representation of TA on $\mathcal{E}_0 \otimes \mathfrak{o}_n$ which is independent of the integer N chosen above. For different n, the actions are compatible and one obtains a continuous representation of TA on $\varprojlim_n \mathcal{E}_0 \otimes \mathfrak{o}_n = \mathcal{E}_0$. The resulting representation ρ_E of TA on $E_0 = \mathcal{E}_0 \otimes C$ is independent of the choice of $\mathcal{E}$. The functor ρ is fully faithful if for any two homogenous vector bundles F and F' on A_C the map

$$\mathrm{Hom}(F, F') \longrightarrow \mathrm{Hom}_{TA}(F_0, F_0')$$

which sends a homomorphism of bundles φ to its fibre φ_0 is bijective. Since $E = \underline{\mathrm{Hom}}(F, F')$ is a homogenous bundle as well, this condition is equivalent to bijectivity of the following evaluation map sending a section s to $s(0)$:

$$0^* : \Gamma(A_C, E) \longrightarrow E_0^{TA} = \{e \in E_0 \mid \rho_E(\gamma)e = e \quad \text{for all } \gamma \in TA\}\,.$$

Thus it suffices to show that the corresponding map

$$0^* : \Gamma(\mathcal{A}_{\mathfrak{o}}, \mathcal{E}) \longrightarrow \mathcal{E}_0^{TA}$$

is injective and that its cokernel is annihilated by p^t for some integer $t \geq 1$. By the construction of the TA-action on $\mathcal{E}_0$ this will follow if for all $n \geq 1$ the map

$$N^* : \Gamma(\mathcal{A} \otimes \mathfrak{o}_n, \mathcal{E} \otimes \mathfrak{o}_n) \longrightarrow \Gamma(\mathcal{A} \otimes \mathfrak{o}_n, N^*(\mathcal{E} \otimes \mathfrak{o}_n))^{\mathcal{A}_N(\mathfrak{o})}$$

is injective and its cokernel is annihilated by p^t. Here it is understood that $N = N(n)$ varies with n as above and t should be independent of n. Therefore it suffices to show that the adjunction maps

$$\mathcal{E} \otimes \mathfrak{o}_n \longrightarrow (N_* N^*(\mathcal{E} \otimes \mathfrak{o}_n))^{\mathcal{A}_N(\mathfrak{o})}$$

are injective and that their cokernels are annihilated by some p^t for all n.

In fact, because of the projection formula it suffices to prove this for the trivial line bundle $\mathcal{E} = \mathcal{O}$. Let us write $N(n) = p^{\nu(n)} M(n)$ where $M = M(n)$ is prime to p. Since $\mathcal{A}_M$ is étale, we have $\mathcal{A}_M(\mathfrak{o}) = \mathcal{A}_M(\mathfrak{o}_n)$. Note also that M-multiplication on $\mathcal{A} \otimes \mathfrak{o}_n$ is étale and Galois with group $\mathcal{A}_M(\mathfrak{o})$. Hence it suffices to show that the following adjunction maps where $\nu = \nu(n)$:

$$\mathcal{O}_{\mathcal{A}\otimes\mathfrak{o}_n} \longrightarrow (p^\nu_* p^{\nu_*}(\mathcal{O}_{\mathcal{A}\otimes\mathfrak{o}_n}))^{\mathcal{A}_{p^\nu}(\mathfrak{o})}$$

are injective and their cokernels are annihilated by p^t for all n. Consider the base change of the faithfully flat morphism $p^\nu : T = \mathcal{A} \otimes \mathfrak{o}_n \to S = \mathcal{A} \otimes \mathfrak{o}_n$ by itself to a morphism $\pi : T \times_S T \to T$. It suffices to show that the adjunction maps

$$\mathcal{O}_T \longrightarrow (\pi_* \pi^* \mathcal{O}_T)^{\mathcal{A}_{p^\nu}(\mathfrak{o})} \tag{16}$$

are injective with cokernels killed by p^t for all n. Since

$$T \times_S T = (\mathcal{A} \otimes \mathfrak{o}_n) \times_{\mathfrak{o}_n} (\mathcal{A}_{p^\nu} \otimes \mathfrak{o}_n)$$

we have

$$\pi_* \pi^* \mathcal{O}_T = \pi_* \mathcal{O}_{T \times_S T} = \Gamma(\mathcal{A}_{p^\nu} \otimes \mathfrak{o}_n, \mathcal{O}) \otimes_{\mathfrak{o}_n} \mathcal{O}_T\,.$$

Hence the map (16) can be identified with the natural map

$$\mathcal{O}_T \longrightarrow \Gamma(\mathcal{A}_{p^\nu} \otimes \mathfrak{o}_n, \mathcal{O})^{\mathcal{A}_{p^\nu}(\mathfrak{o})} \otimes_{\mathfrak{o}_n} \mathcal{O}_T\,. \tag{17}$$

It is therefore injective and its cokernel is annihilated by p^t if the cokernel of the map

$$\mathfrak{o}_n \hookrightarrow \Gamma(\mathcal{A}_{p^\nu} \otimes \mathfrak{o}_n, \mathcal{O})^{\mathcal{A}_{p^\nu}(\mathfrak{o})}$$

is killed by p^t. The motivation for theorem 1 is now clear. It should be noted that the full faithfulness of ρ can be deduced for all A from Faltings' work [F]. However, I did not see how to prove theorem 1 using his methods.

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