TOKYO J. MATH.
VOL. 33, NO. 2, 2010

Properties of Minimal Charts and Their Applications III

Teruo NAGASE and Akiko SHIMA

Tokai University

Abstract. Charts are oriented labeled graphs in a disk which correspond to surface braids. C-moves are local modifications of charts in a disk, which induces an ambient isotopy between the closures of the corresponding two surface braids. A chart is minimal if its complexity is minimal among the charts which are modified from the chart by C-moves. We investigate a disk whose boundary consists of edges of the same label, called a k-angled disk, for a minimal chart. In this paper we investigate 2-angled disks and 3-angled disks containing at most one white vertex in their interiors for a minimal chart.

1. Introduction

Kamada introduced a method to describe surface braids as oriented labeled graphs in a disk, called charts ([2],[3],[4]) (see Section 2 for the definition of charts). In a chart there are three kinds of vertices; white vertices, crossings and black vertices. Kamada also introduced *C-moves* which are local modifications of charts in a disk. A C-move between two charts induces an ambient isotopy between the closures of the corresponding two surface braids. Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

In this paper, we investigate properties of minimal charts which we need to prove that there is no minimal chart with exactly seven white vertices. In particular we investigate a disk whose boundary consists of edges of the same label, called a k-angled disk.

Let Γ be a chart. For each label m, we denote by Γ_m the 'subgraph' of Γ consisting of edges of label m and their vertices. In this paper,

> *crossings are vertices of Γ but we do not consider crossings as vertices of Γ_m.* The vertices of Γ_m are white vertices and black vertices.

An *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all white vertices from Γ_m.

Let Γ be a chart. If an object consists of some edges of Γ, arcs in edges of Γ and arcs around white vertices, then the object is called *a pseudo chart*.

Received April 13, 2009; revised April 29, 2009; revised May 15, 2009
2000 *Mathematics Subject Classification*: 57Q45 (Primary), 57Q35 (Secondary)

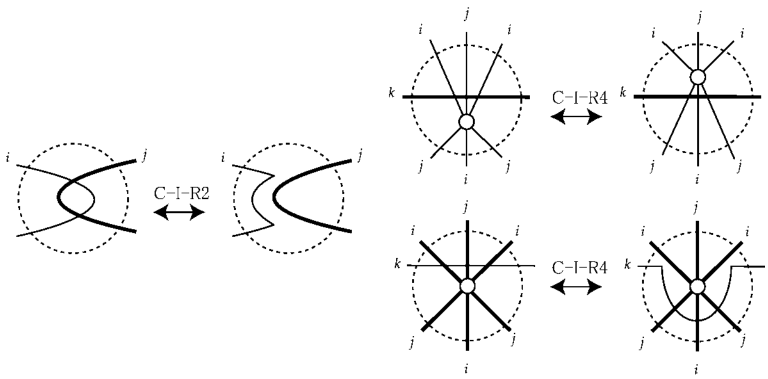

FIGURE 1. C-moves keeping thicken figures fixed.

Let Γ and Γ' be C-move equivalent charts. Suppose that a pseudo chart X of Γ is also a pseudo chart of Γ'. Then we say that Γ is modified to Γ' by *C-moves keeping X fixed*. In Figure 1, we give examples of C-moves keeping pseudo charts fixed.

In this paper for a set X we denote the interior of X, the boundary of X and the closure of X by $\mathrm{Int}\, X$, ∂X and $Cl(X)$ respectively.

Let Γ be a chart. Let D be a disk. If ∂D consists of k edges of the subgraph Γ_m, then D is called *a k-angled disk of Γ_m*. Let N be a boundary collar of D, i.e. a regular neighborhood of ∂D in D. If $(N - \partial D) \cap \Gamma_m$ consists of s arcs, then D is called a *k-angled disk with s feelers*. An edge of Γ_m is called a *feeler* of the k-angled disk D if the edge intersects $N - \partial D$.

Let D be a disk. Let

$$\begin{aligned} w(D) &= \text{the number of white vertices in Int}\, D\,, \\ c(D) &= \text{the number of crossings on } \partial D\,. \end{aligned}$$

Let D be a k-angled disk of Γ_m for a minimal chart Γ. The pair of integers $(w(D), c(D))$ is called the *local complexity with respect to D*, denoted by $\ell c(D; \Gamma)$. Let $\mathbb{S}$ be the set of all minimal charts each of which can be moved from Γ by C-moves in a regular neighborhood of D keeping ∂D fixed. The chart Γ is said to be *locally minimal with respect to D* if its local complexity with respect to D is minimal among the charts in $\mathbb{S}$ with respect to the lexicographic order.

Let Γ be a chart, D a k-angled disk of Γ_m, and G a pseudo chart with $\partial D \subset G$. Let $r : D \to D$ be a reflection of D, and G^* the pseudo chart obtained from G by changing the orientations of all of the edges. Then the set $\{G, G^*, r(G), r(G^*)\}$ is called the *RO-family of the pseudo chart G*.

The followings are main results in this paper:

THEOREM 1.1. *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m with at most one feeler such that Γ is locally minimal with respect to D. If $w(D) \leq 1$, then a regular*

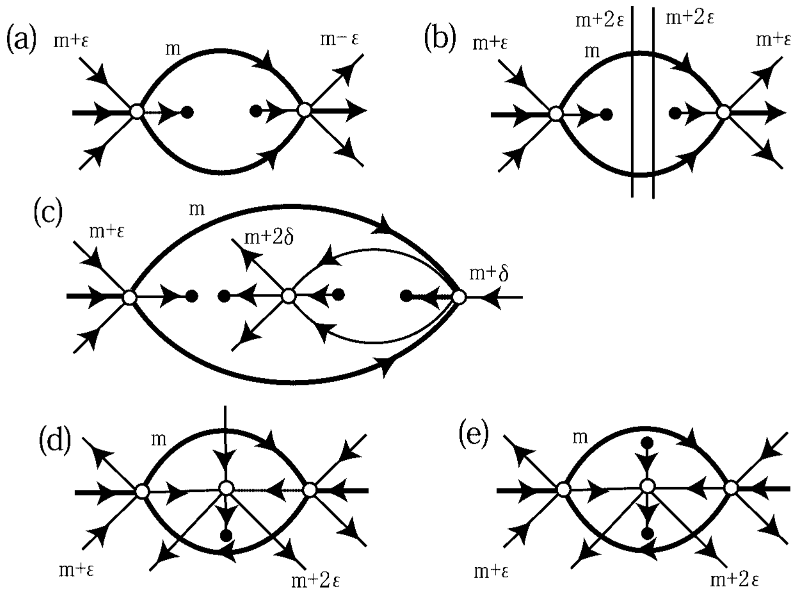

FIGURE 2. The 2-angled disk (c) has one feeler, the others do not have any feelers.

neighborhood of D contains an element in the RO-families of the five pseudo charts as shown in Figure 2*.*

Let Γ be a chart, and D a k-angled disk of Γ_m. Suppose that for any edge e of Γ_m if $e \cap \mathrm{Int}\, D \neq \emptyset$ and $e \cap \partial D \neq \emptyset$, then the edge e is a terminal edge. We say that D is a *special* k-angled disk. Note that for a special k-angled disk D, any feeler of D is a terminal edge.

THEOREM 1.2. *Let Γ be a minimal chart. Let D be a special* 3*-angled disk of Γ_m such that Γ is locally minimal with respect to D. If $w(D) \leq 1$, then a regular neighborhood of D contains an element in the RO-families of the eight pseudo charts as shown in Figure* 3*.*

This paper is organized as follows. In Section 2, we give notations and definitions. In Section 3, we prove Theorem 1.1. In Section 4, we investigate 3-angled disks without feelers. In Section 5, we prove Theorem 1.2. In Section 6, we investigate a closed edge of Γ_m containing a crossing but not containing any white vertices, called a ring.

2. Preliminaries

In this section, we define charts and notations.

Let n be a positive integer. An *n-chart* is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions:

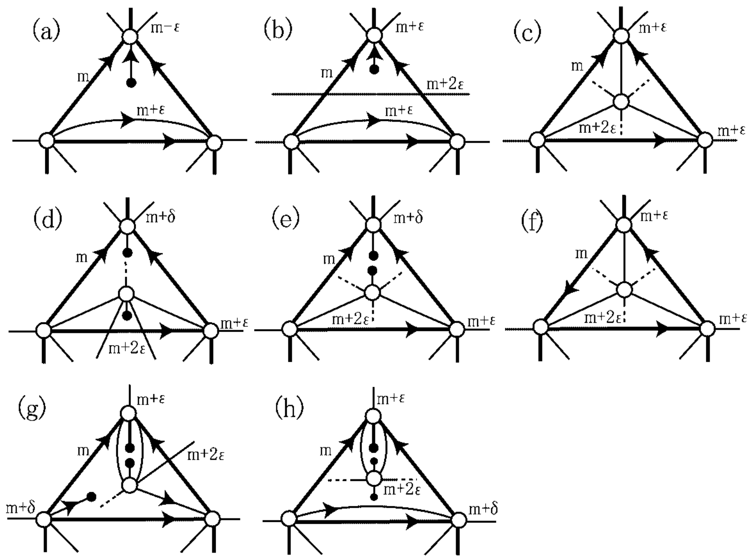

FIGURE 3. The 3-angled disks (g) and (h) have one feeler, the others do not have any feelers.

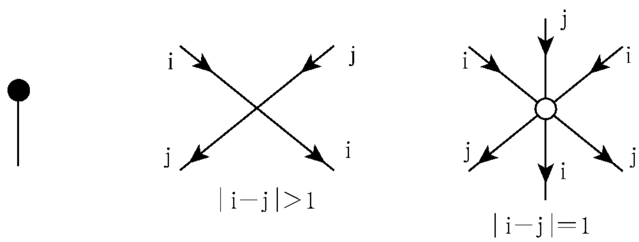

FIGURE 4

(1) Every vertex has degree 1, 4, or 6.
(2) The labels of edges are in $\{1, 2, \ldots, n-1\}$.
(3) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i, where the orientation and the label of each arc are inherited from the edge containing the arc.
(4) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

A vertex of degree 1, 4, and 6 is called a *black vertex,* a *crossing*, and a *white vertex* respectively (see Figure 4).

Among six short arcs in a small neighborhood of a white vertex, a center arc of each

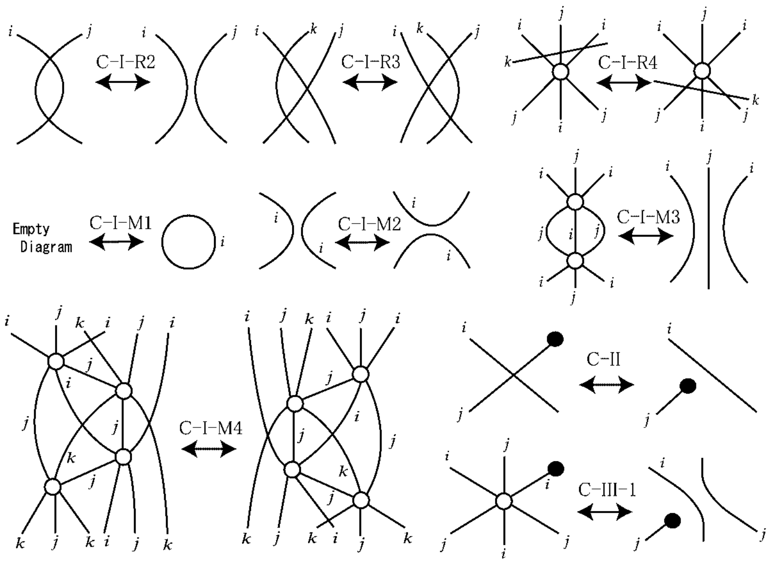

FIGURE 5. For the C-III-1 move, the edge containing the black vertex does not contain a middle arc in the left figure.

three consecutive arcs oriented inward or outward is called *a middle arc* at the white vertex (see Figure 4). There are two middle arcs in a small neighborhood of each white vertex.

C-moves are local modification of charts in a disk as shown in Figure 5 (see [1], [4] for the precise definition). Kamada originally defined CI-moves as follows (C-I-moves are special cases of CI-moves): A chart Γ is obtained from a chart Γ' by a *CI-move*, if there exists a disk D such that

(1) the two charts Γ and Γ' intersect the boundary of D transversely or do not intersect the boundary of D,

(2) $\Gamma \cap D^c = \Gamma' \cap D^c$, and

(3) neither $\Gamma \cap D$ nor $\Gamma' \cap D$ contains a black vertex,

where $(\cdots)^c$ is the complement of $(\cdots)$.

Let Γ be a chart. An edge of Γ or Γ_m is called a *free edge* if it has two black vertices. An edge of Γ or Γ_m is called a *terminal edge* if it has a white vertex and a black vertex. A closed edge of Γ or Γ_m is called a *loop* if it has only one white vertex. Note that free edges, terminal edges and loops may contain crossings of Γ.

For each chart Γ, let $w(\Gamma)$ and $f(\Gamma)$ be the number of white vertices, and the number of free edges respectively. The pair $(w(\Gamma), -f(\Gamma))$ is called the *complexity* of the chart. A chart is called a *minimal chart* if its complexity is minimal among the charts C-move equivalent to the chart with respect to the lexicographic order of pairs of integers.

The following lemma, we showed the difference of a chart in a disk and in a 2-sphere. This lemma follows from that there exists a natural one-to-one correspondence between {charts in S^2}/C-moves and {charts in D^2}/C-moves, conjugations ([4, Chapter 23 and Chapter 25]).

LEMMA 2.1 ([5, Lemma 2.1]). *Let Γ and Γ' be charts in a disk D. Suppose that Γ is ambient isotopic to Γ' in the one point compactification of the open disk* Int D, *i.e. the* 2*-sphere S^2. Then there exist hoops $C_1, C_2, \ldots, C_k$ in* Int D *such that*

(1) *the chart Γ is obtained from $\Gamma' \cup (\bigcup_{i=1}^{k} C_i)$ by C-moves in the disk D,*

(2) *the chart Γ' and hoops $C_1, C_2, \ldots, C_k$ are mutually disjoint, and*

(3) *each hoop C_i bounds a disk containing the chart Γ' in the disk D.*

Moreover the chart Γ is minimal if and only if Γ' is minimal. □

Lemma 2.1 says that we can move the point at infinity in S^2 to a complementary domain of the chart. To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk. In this paper,

all charts are contained in the 2*-sphere S^2.*

We have the special point in the 2-sphere S^2, called the point at infinity, denoted by ∞. In this paper, all charts are contained in a disk such that the disk does not contain the point at infinity ∞.

A *hoop* is a closed edge of a chart Γ without vertices (hence without crossings, neither). A *ring* is a closed edge of Γ_m containing a crossing but not containing any white vertices. A hoop is said to be *simple* if one of complementary domains of the hoop does not contain any white vertices.

As shown in [5], we assume that all minimal charts Γ satisfy the following six conditions:

ASSUMPTION 1. No terminal edge of Γ_m contains a crossing. Hence any terminal edge of Γ_m is a terminal edge of Γ and any terminal edge of Γ_m contains a middle arc.

ASSUMPTION 2. No free edge of Γ_m contains a crossing. Hence any free edge of Γ_m is a free edge of Γ.

ASSUMPTION 3. All free edges and simple hoops in Γ are moved into a small neighborhood U_∞ of the point at infinity ∞.

ASSUMPTION 4. Each complementary domain of any ring must contain at least one white vertex.

ASSUMPTION 5. Hence we assume that the subgraph obtained from Γ by omitting free edges and simple hoops does not meet the set U_∞. Also we assume that Γ does not contain free edges nor simple hoops, otherwise mentioned. Therefore we can assume that if an edge of Γ_m contains a black vertex, then it is not a free edge but a terminal edge and that each complementary domain of any hoops and rings of Γ contains a white vertex, otherwise mentioned.

ASSUMPTION 6. The point at infinity ∞ is moved in any complementary domain of Γ.

NOTATION. We use the following notation:

In our argument, we often need a name for an unnamed edge by using a given edge and a given white vertex. For the convenience, we use the following naming: Let e', e_i, e'' be three consecutive edges containing a white vertex w_j. Here, the two edges e' and e'' are unnamed edges. There are six arcs in a neighborhood U of the white vertex w_j. If the three arcs $e' \cap U$, $e_i \cap U$, $e'' \cap U$ lie anticlockwise around the white vertex w_j in this order, then e' and e'' are denoted by a_{ij} and b_{ij} respectively (see Figure 6). There is a possibility $a_{ij} = b_{ij}$ if they are contained in a loop.

Let α be a short arc of Γ in a small neighborhood of a vertex v with $v \in \partial\alpha$. If the arc α is oriented to v, then α is called *an inward arc*, and otherwise α is called *an outward arc*.

Let Γ be an n-chart. Let F be a closed domain with $\partial F \subset \Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$ for some integer m, where $\Gamma_0 = \emptyset$ and $\Gamma_n = \emptyset$. By the condition (3) for charts, in a small neighborhood of each white vertex, there are three inward arcs and three outward arcs. Also in a small neighborhood of each black vertex, there exists only one inward arc or one outward arc. We often use the following fact, when we fix (inward or outward) arcs near white vertices and black vertices:

> *The number of inward arcs contained in $F \cap \Gamma_m$ is equal to the number of outward arcs in $F \cap \Gamma_m$.*

When we use this fact, we say that we use *IO-Calculation with respect to Γ_m in F*. For

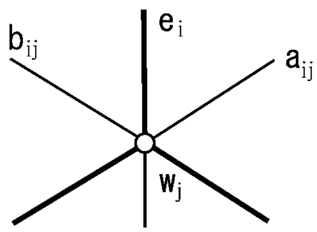

FIGURE 6

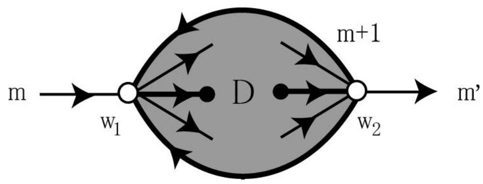

FIGURE 7

example, in a chart Γ, consider the pseudo chart as shown in Figure 7. Let D be the disk whose boundary is contained in Γ_{m+1} as shown in Figure 7. Suppose that Int D contains neither white vertices nor other black vertices. Then we have $m' = m$. For, if $m' \neq m$, then the number of inward arcs in $D \cap \Gamma_m$ is zero, but the number of outward arcs in $D \cap \Gamma_m$ is two. This is a contradiction. Instead of the above argument, we say that

we have $m' = m$ by IO-Calculation with respect to Γ_m in D.

3. 2-angled disks

In this section we give a proof of Theorem 1.1.

LEMMA 3.1 ([6, Corollary 5.8]). *Let Γ be a minimal chart. Let D be a* 2*-angled disk of Γ_m with at most one feeler. If $w(D) = 0$, then a regular neighborhood of D contains an element of the RO-families of the two pseudo charts as shown in Figure* 2a *and* b. □

LEMMA 3.2 ([6, Lemma 5.6 and 5.7]). *Let Γ be a minimal chart. Let D be a* 2*-angled disk of Γ_m with one feeler. Then $w(D) \geq 1$. If $w(D) = 1$, then a regular neighborhood of D contains an element of the RO-family of the pseudo chart as shown in Figure* 2c. □

LEMMA 3.3 ([6, Lemma 6.1]). *Let Γ be a minimal chart. Let G be a connected component of Γ_m containing a white vertex. Then G contains at least two white vertices.* □

Let α be an arc, and p, q points in α. We denote by $\alpha[p, q]$ the subarc of α whose end points are p and q.

Let Γ be a chart. Let α be an arc in an edge of Γ_m, and w a white vertex with $w \notin \alpha$. Suppose that there exists an arc β such that

(1) its end points are the white vertex w and an interior point p of the arc α, and

(2) the arc β is contained in Γ, or $\Gamma \cap \beta$ consists of at most finitely many points.

Then we say that *the white vertex w connects with the point p of α by the arc β.*

LEMMA 3.4 ([5, Lemma 4.2]) (Shifting Lemma). *Let Γ be a chart and α an arc in an edge of Γ_m. Let w be a white vertex of $\Gamma_k \cap \Gamma_h$ where $h = k + \varepsilon$, $\varepsilon \in \{+1, -1\}$. Suppose that the white vertex w connects with a point r of the arc α by an arc in an edge e of Γ_k. Suppose that one of the following two conditions is satisfied*:

(1) $h > k > m$.

(2) $h < k < m$.

*Then for any neighborhood V of the arc $e[w, r]$ we can shift the white vertex w to $e - e[w, r]$ along the edge e by C-I-R*2 *moves, C-I-R*3 *moves and C-I-R*4 *moves in V keeping $\bigcup_{i<0} \Gamma_{k+i\varepsilon}$ fixed* (*see Figure* 8). □

LEMMA 3.5. *Let Γ be a minimal chart. Let D be a disk with $w(D) = 1$ which is the closure of a complementary domain of Γ_m. Let w be the white vertex in* Int D. *If any edge containing w does not contain any white vertex in ∂D, then in any regular neighborhood of*

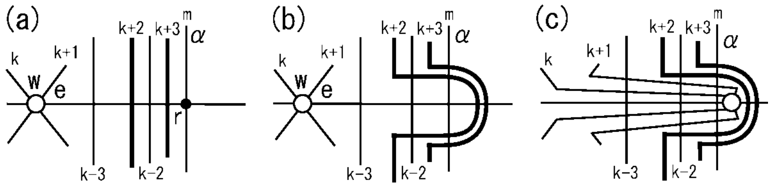

FIGURE 8. Lemma 3.4 Case (1): $k > m$ and $\varepsilon = +1$.

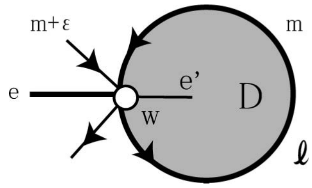

FIGURE 9

D we can shift the white vertex w to $S^2 - D$ by C-moves keeping ∂D fixed without increasing the complexity of Γ.

PROOF. Let k be the label with $w \in \Gamma_k \cap \Gamma_{k+1}$. If any edge of Γ_k containing w does not intersect ∂D, then the connected component of Γ_k containing w contains only one white vertex w. This contradicts Lemma 3.3. Hence there exists an edge of Γ_k containing w and intersecting ∂D. Thus $k \neq m$.

Similarly there exists an edge of Γ_{k+1} containing w and intersecting ∂D. Thus $k+1 \neq m$.

Hence $k + 1 > k > m$ or $k < k + 1 < m$. By Shifting Lemma (Lemma 3.4), we can shift the white vertex w to the exterior of the disk D. □

Let Γ be a chart. Let ℓ be a loop of label m, and w the white vertex in ℓ. Let e be the edge of Γ_m with $w \in e$ and $e \neq \ell$. Then the loop ℓ bounds two disks on the 2-sphere. One of the two disks does not contain the edge e. The disk is called *the associated disk of the loop ℓ* (see Figure 9).

LEMMA 3.6 ([6, Lemma 4.2]). *Let Γ be a minimal chart with a loop ℓ of label m. Then the associated disk D of the loop ℓ contains at least two white vertices in its interior. Hence $w(D) \geq 2$.* □

Let D be a 2-angled disk of Γ_m without feelers. Then a regular neighborhood of D contains an element of the RO-families of the three pseudo charts as shown in Figure 10.

LEMMA 3.7. *Let Γ be a minimal chart. Let D be a 2-angled disk of Γ_m of type* (0−a) *as shown in Figure* 10(0−a) *such that Γ is locally minimal with respect to D. Then $w(D) \neq 1$.*

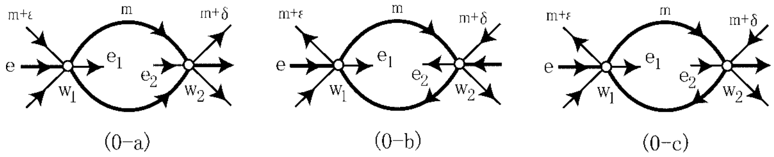

FIGURE 10. The white vertex w_1 is in $\Gamma_m \cap \Gamma_{m+\varepsilon}$ and the white vertex w_2 is in $\Gamma_m \cap \Gamma_{m+\delta}$ where $\varepsilon, \delta \in \{+1, -1\}$.

PROOF. Suppose $w(D) = 1$. Let w be the white vertex in $Int D$. We use the notations as shown in Figure 10(0−a).

We shall show that any edge containing w contains neither w_1 nor w_2. If $w \in e_1 \cap e_2$, then there exists a terminal edge of $\Gamma_{m+\varepsilon}$ containing w but not containing a middle arc at w. This contradicts Assumption 1. Hence $w \notin e_1 \cap e_2$.

If $w \in e_1$ or $w \in e_2$, then $w \notin e_1 \cap e_2$ implies that there exists a loop containing w whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.6. Hence $w \notin e_1$ and $w \notin e_2$. Therefore any edge containing w contains neither w_1 nor w_2.

Since any edge containing w does not contain any white vertex in ∂D, by Lemma 3.5 we can shift the white vertex w to the exterior of the disk D. This contradict the fact that Γ is locally minimal with respect to D. Hence $w(D) \neq 1$. □

LEMMA 3.8 ([6, Lemma 5.3]). *Let Γ be a minimal chart. Let D be a* 2*-angled disk of Γ_m of type* (0−b) *as shown in Figure* 10(0−b). *Then $w(D) \geq 1$. If $w(D) = 1$, then a regular neighborhood of D contains an element in the RO-families of the two pseudo charts as shown in Figure* 2d *and* e. □

LEMMA 3.9 ([6, Lemma 5.4]). *Let Γ be a minimal chart. Let D be a* 2*-angled disk of Γ_m of type* (0−c) *as shown in Figure* 10(0−c). *Then $w(D) \geq 2$.* □

PROOF OF THEOREM 1.1. Let D be a 2-angled disk of Γ_m with at most one feeler. If $w(D) = 0$, then we have the desired result from Lemma 3.1.

Suppose $w(D) = 1$. If D has one feeler, then we have the desired result from Lemma 3.2.

If D does not have any feelers, then a regular neighborhood of D contains an element of the RO-families of the three pseudo charts as shown in Figure 10. If D is of type (0−a), then $w(D) \neq 1$ from Lemma 3.7. This contradicts the fact $w(D) = 1$. Hence D is not of type (0−a). If D is of type (0−b), then we have the desired result from Lemma 3.8. If D is of type (0−c), then $w(D) \geq 2$ from Lemma 3.9. This contradicts the fact $w(D) = 1$. Hence D is not of type (0−c). Therefore we complete the proof of the first theorem (Theorem 1.1). □

4. 3-angled disks

In our argument we often construct a chart Γ. On the construction of a chart Γ, for a white vertex w, among the three edges of Γ_m containing w, if we have specified two edges and if the last edge of Γ_m containing w contains a black vertex (see Figure 11a and b), then

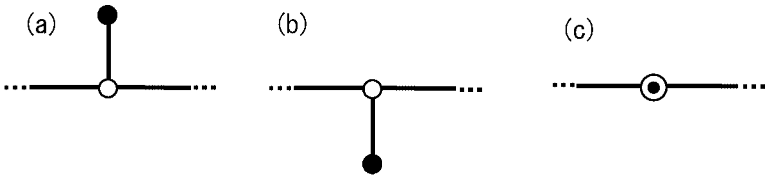

FIGURE 11

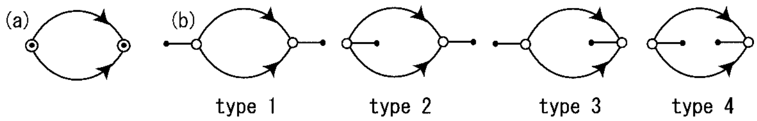

FIGURE 12

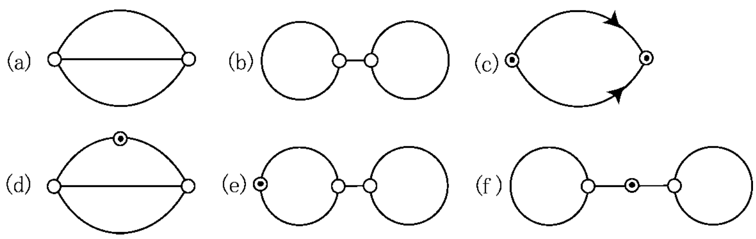

FIGURE 13

we remove the edge containing the black vertex and put a black dot at the center of the white vertex as shown in Figure 11c.

For example, the graph as shown in Figure 12a means one of the four graphs as shown in Figure 12b.

LEMMA 4.1 ([6, Lemma 6.2]). *Let Γ be a minimal chart. Let G be a connected component of Γ_m containing a white vertex. If G contains at most three white vertices, then it is one of six subgraphs as shown in Figure* 13. □

We call the subgraphs (a) and (c) in Figure 13 a *θ-curve* and an *oval* respectively.

Now a special k-angled disk is a k-angled disk of Γ_m such that any feeler is a terminal edge where a feeler is an edge of Γ_m intersecting ∂D and Int D.

LEMMA 4.2. *Let Γ be a minimal chart. Let D be a special* 3*-angled disk of Γ_m. Then a regular neighborhood of D contains an element in the RO-families of the four pseudo charts*

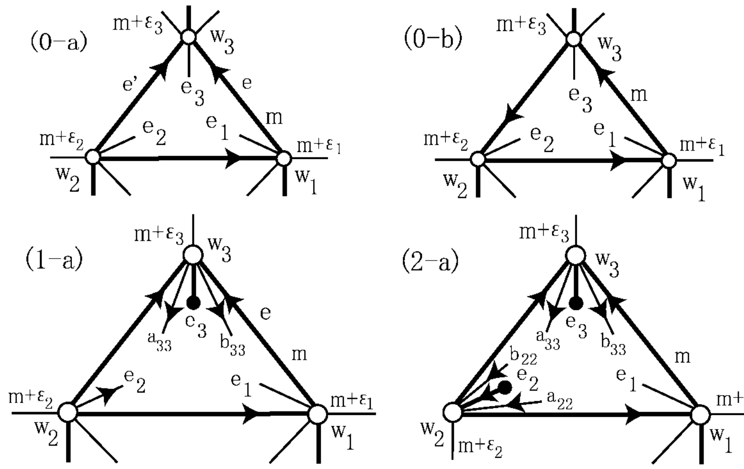

FIGURE 14. The white vertex w_i (i =1,2,3) is in $\Gamma_m \cap \Gamma_{m+\varepsilon_i}$ where $\varepsilon_i \in \{+1, -1\}$.

as shown in Figure 14.

PROOF. If D has three feelers, then the union of ∂D and the three feelers is a connected component of Γ_m. This component contains exactly three white vertices and three black vertices. This contradicts Lemma 4.1. Hence D has at most two feelers.

Suppose that D does not have any feelers. If necessary we take a reflection of D, we can assume that two of the three edges in ∂D are oriented anticlockwise. Hence we have two 3-angled disks as shown in Figure 14(0−a) and (0−b).

If D has a feeler, then we have the two 3-angled disks as shown in Figure 14(1−a) and (2−a). □

LEMMA 4.3. *Let Γ be a minimal chart. Let D be a* 3*-angled disk of type* (0−a) *of Γ_m as shown in Figure* 14(0−a). *If $w(D) = 0$, then a regular neighborhood of D contains an element of RO-families of the two pseudo charts as shown in Figure* 3a *and* b.

PROOF. We use the notations as shown in Figure 14(0−a). Since the edge e_1 of $\Gamma_{m+\varepsilon_1}$ does not contain a middle arc at w_1, it is not a terminal edge by Assumption 1. Hence $w(D) = 0$ implies that (1) $e_1 = e_2$ or (2) $e_1 = e_3$. When we change the orientations of all of the edges and we take a reflection of D, the case (1) changes the case (2). So we examine the case (1).

Since $w(D) = 0$, the edge e_3 is a terminal edge. If $\varepsilon_1 \neq \varepsilon_3$, then $\varepsilon_3 = -\varepsilon_1$ and we have the pseudo chart as shown in Figure 3a where we put $\varepsilon = \varepsilon_1 = \varepsilon_2$.

Now suppose $\varepsilon_1 = \varepsilon_3$. Put $\varepsilon = \varepsilon_1 = \varepsilon_2 = \varepsilon_3$. If there does not exist an edge of $\Gamma_{m+2\varepsilon}$ in the disk D intersecting each of the edges e and e', then there exists a simple arc α in D connecting the black vertex of e_3 and a point of e_1 with $\mathrm{Int}(\alpha) \cap (\Gamma_m \cup \Gamma_{m+\varepsilon} \cup \Gamma_{m+2\varepsilon}) = \emptyset$. Applying C-II moves for the edge e_3 along the arc α, we can elongate the edge e_3 so that the black vertex in e_3 situates near the edge e_1. Apply a C-I-M2 move between the terminal edge

e_3 and the edge e_1. Then we obtain a new terminal edge containing the white vertex w_1 but not containing a middle arc at the white vertex w_1. This contradicts Assumption 1. Therefore there exists an edge of $\Gamma_{m+2\varepsilon}$ in the disk D intersecting each of e and e'. Hence we have the pseudo chart as shown in Figure 3b. □

LEMMA 4.4. *Let Γ be a minimal chart. Let D be a* 3*-angled disk of type* (0−a) *of Γ_m as shown in Figure* 14(0−a) *such that Γ is locally minimal with respect to D. If $w(D) = 1$, then a regular neighborhood of D contains an element in the RO-families of the three pseudo charts as shown in Figure* 3c, d *and* e.

PROOF. We use the notations as shown in Figure 14(0−a). Let w be the white vertex in $\text{Int}\, D$. There are three cases: (1) $w \in \Gamma_m$, (2) $w \notin \Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$, (3) $w \in \Gamma_{m-1} \cup \Gamma_{m+1}$.

For the case (1), the connected component of Γ_m containing w contains only one white vertex. This contradicts Lemma 3.3. Hence the case (1) does not occur.

For the case (2), since any white vertex in ∂D contains $\Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$, any edge containing w does not contain any white vertex in ∂D. By Lemma 3.5, we can shift the white vertex w to $S^2 - D$ by C-moves keeping ∂D fixed. This contradict the fact that Γ is locally minimal with respect to D. Hence the case (2) does not occur.

For the case (3), if there exists a loop of containing w, by Lemma 3.6 we have $w(D) \geq 2$. This contradicts the fact $w(D) = 1$. Hence there exists no loop of containing w. Since the edge e_1 does not contain a middle arc at w_1, it is not a terminal edge by Assumption 1. Hence $w \in e_1$. Put $\varepsilon = \varepsilon_1$. Then $w \in \Gamma_{m+\varepsilon}$. Since there exist two edges of $\Gamma_{m+\varepsilon}$ containing w but not containing middle arcs at w, there exists an edge e'' of $\Gamma_{m+\varepsilon}$ with $\partial e'' = \{w, w_2\}$ or $\partial e'' = \{w, w_3\}$. Without loss of generality we can assume $\partial e'' = \{w, w_2\}$. Since $w \in \Gamma_{m+\varepsilon}$ and the case (1) does not occur, we have $w \in \Gamma_{m+2\varepsilon}$. Hence we have the three pseudo charts as shown in Figure 3c, d and e. □

LEMMA 4.5. *Let Γ be a minimal chart. Let D be a* 3*-angled disk of Γ_m of type* (0−*b*) *as shown in Figure* 14(0−b). *Then $w(D) \geq 1$. If $w(D) = 1$, then a regular neighborhood of D contains the pseudo chart as shown in Figure* 3f.

PROOF. We use the notations as shown in Figure 14(0−b).

Since the edge e_i $(i = 1, 2, 3)$ does not contain a middle arc at w_i, by Assumption 1 the edge e_i is not a terminal edge. By IO-Calculation with respect to $\Gamma_{m\pm1}$ in D, there exists a white vertex in $\text{Int}\, D$, say w. Hence $w(D) \geq 1$.

Suppose $w(D) = 1$. We shall show $w \in e_1$. If $w \notin e_1$, then $e_1 = e_2$ or $e_1 = e_3$. There exists a loop containing w in D whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.6. Hence $w \in e_1$. Similarly we have $w \in e_2 \cap e_3$.

The three edges e_1, e_2 and e_3 divide the disk D into three disks. Put $\varepsilon = \varepsilon_1$. We shall show $w \in \Gamma_{m+2\varepsilon}$. Since $w \in \Gamma_{m+\varepsilon}$, we have $w \in \Gamma_m$ or $w \in \Gamma_{m+2\varepsilon}$. If $w \in \Gamma_m$, then the connected component of Γ_m containing w contains only one white vertex w. This contradicts Lemma 3.3. Hence $w \in \Gamma_{m+2\varepsilon}$. Therefore we have the pseudo chart as shown in Figure 3f. □

5. Special 3-angled disks with feelers

In this section we give a proof of Theorem 1.2.

Let Γ be a chart. Let D be a disk such that ∂D consists of an edge e_1 of Γ_m and an edge e_2 of Γ_{m+1} and that any edge containing a white vertex in e_1 does not intersect the open disk Int D. Let w_1 and w_2 be the white vertices in e_1. If the disk D satisfies one of the following conditions, then D is called *a lens of type* $(m, m+1)$ (see Figure 15):

(1) Neither e_1 nor e_2 contains a middle arc.

(2) One of the two edges e_1 and e_2 contains middle arcs at both white vertices w_1 and w_2.

LEMMA 5.1 ([5, Theorem 1.1]). *Let Γ be a minimal chart. Then there exist at least three white vertices in the interior of any lens.* □

LEMMA 5.2. *Let Γ be a minimal chart. Let D be a special* 3*-angled disk of Γ_m with one feeler. Then $w(D) \geq 1$. If $w(D) = 1$, then a regular neighborhood of D contains an element in the RO-families one of the two pseudo charts as shown in Figure* 3g *and* h.

PROOF. By Lemma 4.2, the disk D contains the pseudo chart as shown in Figure 14(1−a). We use the notations as shown in Figure 14(1−a).

Since neither a_{33} nor b_{33} contains a middle arc at w_3, by Assumption 1 neither a_{33} nor b_{33} is a terminal edge. Put $\varepsilon = \varepsilon_3$. By IO-Calculation with respect to $\Gamma_{m+\varepsilon}$ in D, there exists a white vertex of $\Gamma_{m+\varepsilon}$ in Int D, say w. Hence $w(D) \geq 1$.

Suppose $w(D) = 1$. If $w \notin a_{33}$, then $a_{33} = e_1$. The edge e_1 divides D into two disks. Let D' be one of the two disks contains the edge b_{33}. Since the edge b_{33} is not a terminal edge, we have $w \in D'$ and D' contains a loop of label $m + \varepsilon$ containing w whose associated disk does not contain any white vertices in its interior. This contradicts Lemma 3.6. Hence $w \in a_{33}$.

If $w \notin b_{33}$, then $b_{33} = e_1$. The $e \cup b_{33}$ bounds a lens D' in D with $w(D') \leq 1$. This contradicts Lemma 5.1. Hence $w \in b_{33}$.

The set $a_{33} \cup b_{33}$ bounds a 2-angled disk D_1 of $\Gamma_{m+\varepsilon}$ with $w(D_1) = 0$. Since D_1 has at most one feeler, by Theorem 1.1 a regular neighborhood $N(D_1)$ of D_1 contains one of the two pseudo charts as shown in Figure 2a and b.

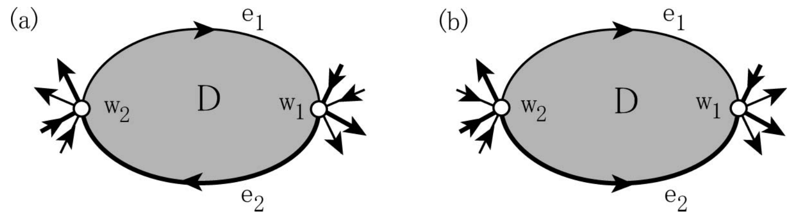

FIGURE 15

Since $w \in \Gamma_{m+\varepsilon}$, we have $w \in \Gamma_m$ or $w \in \Gamma_{m+2\varepsilon}$. By IO-Calculation with respect to Γ_m in $Cl(D - D_1)$, we have $w \in \Gamma_{m+2\varepsilon}$. Hence $N(D_1)$ contains the pseudo chart as shown in Figure 2a. Therefore we have the two pseudo charts as shown in Figure 3g and h. □

LEMMA 5.3. *Let Γ be a minimal chart. Let D be a special* 3*-angled disk of Γ_m with two feelers. Then $w(D) \geq 2$.*

PROOF. We use a contradiction. Suppose that $w(D) \leq 1$. By Lemma 4.2, the disk D contains the pseudo chart as shown in Figure 14(2−a). We use the notations as shown in Figure 14(2−a). If necessary we change the orientations of all of the edges and we take a reflection of D, we can assume that the edge e_1 contains an inward arc at w_1.

Since none of the five edges $e_1, a_{22}, b_{22}, a_{33}, b_{33}$ contain middle arcs at w_1, w_2, w_2, w_3, w_3 respectively, by Assumption 1 none of these edges are terminal edges. By IO-Calculation with respect to $\Gamma_{m\pm 1}$ in D, we have $w(D) \geq 1$. Thus $w(D) = 1$. Let w be the white vertex in Int D.

If $a_{33} = b_{22}$ or $b_{33} = e_1$, then there exists a lens D' in D. By Lemma 5.1, $w(D') \geq 3$. Hence $w(D) \geq 3$. This contradicts the fact $w(D) = 1$. Hence we have $a_{33} \neq b_{22}$ and $b_{33} \neq e_1$.

If $e_1 = a_{33}$, then e_1 splits D into two disks, say D_1 and D_2. By IO-Calculation with respect to $\Gamma_{m\pm 1}$ in D_1 and D_2, we have $w(D_1) \geq 1$ and $w(D_2) \geq 1$. Thus $w(D) \geq 2$. This contradicts the fact $w(D) = 1$. Hence we have $e_1 \neq a_{33}$. Now $e_1 \neq a_{33}$ and $e_1 \neq b_{33}$ imply $w \in e_1$.

If $a_{33} = a_{22}$, then $w(D) \geq 2$ by a similar way as above. Hence we have $a_{33} \neq a_{22}$. Now $a_{33} \neq a_{22}$ and $a_{33} \neq b_{22}$ imply that the edge a_{33} contains a white vertex w' different from w_1, w_2 and w_3. If $w \neq w'$ then $w(D) \geq 2$. This contradicts the fact $w(D) = 1$. Hence we have $w = w'$ and $w \in a_{33}$.

Since $w \in e_1 \cap a_{33}$, the arc $e_1 \cup a_{33}$ splits D into two disks. Let D_3 be one of the two disks containing the edge e_2. By IO-Calculation with respect to $\Gamma_{m\pm 1}$ in D_3, we have $w(D_3) \geq 1$. Therefore $w(D) \geq 2$. □

PROOF OF THEOREM 1.2. Let D be a special 3-angled disk of Γ_m with $w(D) \leq 1$. By Lemma 4.2, a regular neighborhood of D contains an element in the RO-families of the four pseudo charts as shown in Figure 14.

If D is of type (0−a), then we have the desired result from Lemma 4.3 and 4.4. If D is of type (0−b), then we have the desired result from Lemma 4.5. If D is of type (1−a), i.e. D has one feeler, then we have the desired result from Lemma 5.2. If D is of type (2−a), i.e. D has two feelers, then $w(D) \geq 2$ from Lemma 5.3. This contradicts the fact $w(D) \leq 1$. Hence D is not of type (2−a). Therefore we complete the proof of the second theorem (Theorem 1.2). □

6. Rings

In this section we investigate a ring such that one of the complementary domains of the ring contains two white vertices.

LEMMA 6.1. *Let Γ be a minimal chart. Let C be a ring or a non simple hoop, and D a disk with $\partial D = C$. If $w(D) = 1$, then Γ is C-move equivalent to the minimal chart $Cl(\Gamma - C)$.*

PROOF. Let w be the white vertex in the disk D. Since the curve C does not contain any white vertices, any edge containing w does not contain any white vertices in C. By Lemma 3.5 we can shift the white vertex w to the exterior of the disk D without increasing the number of rings and hoops.

Now D does not contain any white vertices. By Assumption 3 we can assume that D does not contain any free edges. Hence we can assume that D does not contain any black vertices. By a CI-move, Γ is C-move equivalent to $Cl(\Gamma - C)$. □

Let Γ be a chart, and D a disk. Let α be a simple arc in ∂D. We call a simple arc γ in an edge of Γ_k a (D, α)-*arc* of label k provided that $\partial\gamma \subset \operatorname{Int}\alpha$ and $\operatorname{Int}\gamma \subset \operatorname{Int} D$. If there is no (D, α)-arc in Γ, then the chart Γ is said to be (D, α)-*arc free*.

Let Γ be a chart and D a disk. Let α be a simple arc in ∂D. For each $k = 1, 2, \ldots$, let Σ_k be the pseudo chart which consists of all arcs in $D \cap \Gamma_k$ intersecting the set $Cl(\partial D - \alpha)$. Let $\Sigma_\alpha = \bigcup_k \Sigma_k$.

LEMMA 6.2 ([5, Lemma 3.2]) (Disk Lemma). *Let Γ be a minimal chart and D a disk. Let α be a simple arc in ∂D. Suppose that the interior of α contains neither white vertices, isolated points of $D \cap \Gamma$, nor arcs of $D \cap \Gamma$. If* $\operatorname{Int} D$ *does not contain white vertices of Γ, then for any neighborhood V of α, there exists a (D, α)-arc free minimal chart Γ' obtained from the chart Γ by C-moves in $V \cup D$ keeping Σ_α fixed (see Figure* 16). □

LEMMA 6.3 ([6, Lemma 6.7]). *Let Γ be a minimal chart. Let D be a* 2*-angled disk of Γ_m with two feelers such that ∂D is contained in an oval of Γ_m. Then $w(D) \geq 2$.* □

Let Γ be a chart. We consider the closure of a complementary domain of a ring or a hoop of Γ_m as a 0-angled disk.

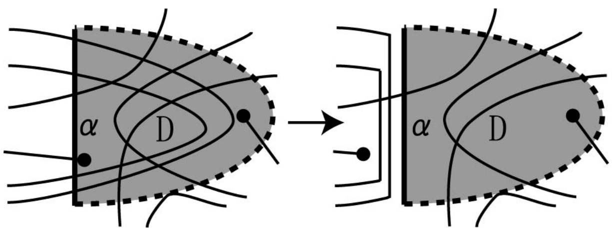

FIGURE 16. The disk D is a shaded area.

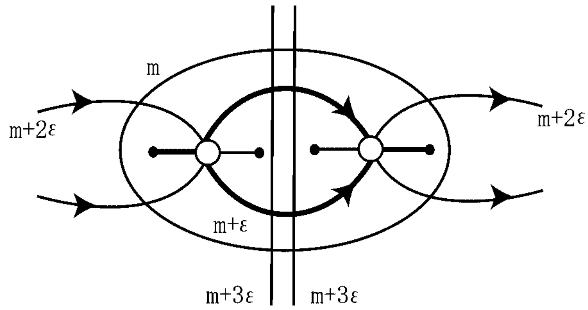

FIGURE 17

LEMMA 6.4. *Let Γ be a minimal chart. Let C be a ring or a non simple hoop of Γ_m. Let D be a disk with $\partial D = C$ such that Γ is locally minimal with respect to D where the disk D may contain the point at infinity ∞. Suppose $w(D) \leq 2$ and D contains a white vertex of $\Gamma_{m+\varepsilon}$ ($\varepsilon \in \{+1, -1\}$). If necessary we modify Γ by C-moves in a regular neighborhood $N(D)$ of D keeping ∂D fixed, then $N(D)$ contains the pseudo chart as shown in Figure* 17.

PROOF. We shall prove our lemma by four steps.

Step 1. We shall show that there does not exist any loop in D.

Suppose that D contains a loop. Then the associated disk of the loop contains at least two white vertices in its interior by Lemma 3.6, and the loop contains one white vertex. Thus we have $w(D) \geq 3$. This is a contradiction. Hence there does not exist any loop in D.

Step 2. We shall show that D contains an oval of type 1 of $\Gamma_{m+\varepsilon}$ (see Figure 12b).

Now $C \subset \Gamma_m$ implies that $D \cap \Gamma_{m+\varepsilon}$ consists of connected components of $\Gamma_{m+\varepsilon}$. Since D does not contain any loop and since $w(D) \leq 2$, the disk D contains a θ-curve or an oval of $\Gamma_{m+\varepsilon}$ by Lemma 4.1.

If D contains a θ-curve of $\Gamma_{m+\varepsilon}$, then there exists a 2-angled disk D' of $\Gamma_{m+\varepsilon}$ without feelers whose boundary is oriented clockwise or anticlockwise. Thus by Theorem 1.1, the disk D contains one of the two pseudo charts as shown in Figure 2d and e. Hence $w(D') \geq 1$. Hence $w(D) \geq 3$. This contradicts the fact that $w(D) \leq 2$. Hence D contains an oval of $\Gamma_{m+\varepsilon}$.

We shall show that the oval is of type 1. If the oval is of type 2 or 3, then D contains a 2-angled disk with one feeler. By Theorem 1.1, the 2-angled disk contains the pseudo chart as shown in Figure 2c. Hence $w(D) \geq 3$. This contradicts the fact $w(D) \leq 2$. If the oval is of type 4, then D contains a 2-angled disk with two feelers. By Lemma 6.3, the 2-angled disk contains at least two white vertices in its interior. Hence $w(D) \geq 4$. This contradicts the fact $w(D) \leq 2$. Therefore the oval is of type 1.

Step 3. We use the notations as shown in Figure 18a. We shall show that all of the four edges a_{11}, b_{11}, a_{22} and b_{22} are edges of $\Gamma_{m+2\varepsilon}$ and intersect ∂D.

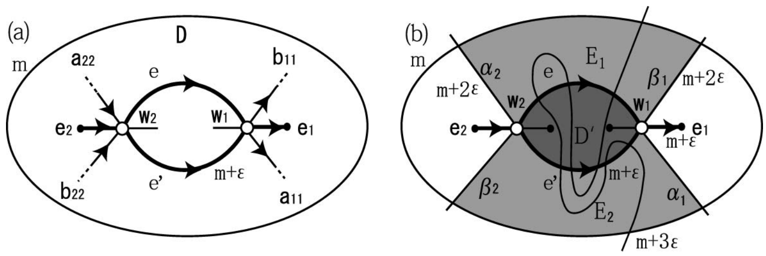

FIGURE 18

Now the two white vertices w_1 and w_2 are in $\Gamma_{m+\varepsilon} \cap D$, and e_1 and e_2 are the terminal edges of $\Gamma_{m+\varepsilon}$ with $w_1 \in e_1$ and $w_2 \in e_2$. Since none of the four edges a_{11}, b_{11}, a_{22} and b_{22} contain middle arcs at w_1, w_1, w_2 and w_2 respectively, none of them are terminal edges.

We shall show $a_{11} \cap \partial D \neq \emptyset$. Suppose $a_{11} \cap \partial D = \emptyset$. There are two cases: (1) $a_{11} = a_{22}$, (2) $a_{11} = b_{22}$. For the case (1), we have a contradiction by IO-Calculation with respect to Γ_m or $\Gamma_{m+2\varepsilon}$ in a disk bounded by $a_{11} \cup e$ or $a_{11} \cup e'$. Hence the case (1) does not occur. For the case (2), if $e' \cup a_{11}$ bounds a lens E with $w(E) = 0$, then this contradicts Lemma 5.1. If $e' \cup a_{11}$ bounds a disk containing the terminal edges e_1 and e_2, then $b_{11} = a_{22}$. Thus we have a lens bounded by $e \cup b_{11}$. We have the same contradiction as above. Hence the case (2) does not occur. Hence $a_{11} \cap \partial D \neq \emptyset$.

Similarly we can show that all of the three edges b_{11}, a_{22} and b_{22} intersect ∂D.

Since $w_i \in \Gamma_{m+\varepsilon}$ for $i = 1, 2$, we have $w_i \in \Gamma_m$ or $w_i \in \Gamma_{m+2\varepsilon}$. Hence a_{11}, b_{11}, a_{22} and b_{22} are edges of Γ_m or $\Gamma_{m+2\varepsilon}$. Since all of the four edges intersect ∂D, $\partial D \subset \Gamma_m$ implies that all of them are edges of $\Gamma_{m+2\varepsilon}$.

Step 4. Let α_i (resp. β_i) be the connected component of $a_{ii} \cap D$ (resp. $b_{ii} \cap D$) containing w_i (see Figure 18b). Let D' be the 2-angled disk of $\Gamma_{m+\varepsilon}$ without feelers in D. The set $\alpha_1 \cup \alpha_2 \cup \beta_1 \cup \beta_2$ separates the disk D into three disks. One contains the 2-angled disk D', say E. The boundary $\partial D'$ separates the disk E into three disks E_1, D' and E_2. Without loss of generality we can assume $e \subset E_1$ and $e' \subset E_2$.

Applying Disk Lemma (Lemma 6.2) for regular neighborhoods of E_1 and E_2, we can assume that Γ is (E_1, e)-arc free and (E_2, e')-arc free.

Since the oval of $\Gamma_{m+\varepsilon}$ is of type 1, by Theorem 1.1 we assume that there are two proper arcs ℓ'_1 and ℓ'_2 in D' of label $m + 3\varepsilon$ each of which intersects both of the two edges e and e' (see Figure 2b). Since Γ is (E_1, e)-arc free and (E_2, e')-arc free, for each $i = 1, 2$, ℓ'_i is contained in a proper arc ℓ_i in D such that ℓ_i is contained in an edge of $\Gamma_{m+3\varepsilon}$, and each of $\ell_i \cap E_1$, $\ell_i \cap D' = \ell'_i$ and $\ell_i \cap E_2$ is a proper arc. Therefore we have the pseudo chart as shown in Figure 17. □

LEMMA 6.5 ([2, Theorem 6]). *Any 3-chart is C-move equivalent to a chart without white vertices.* □

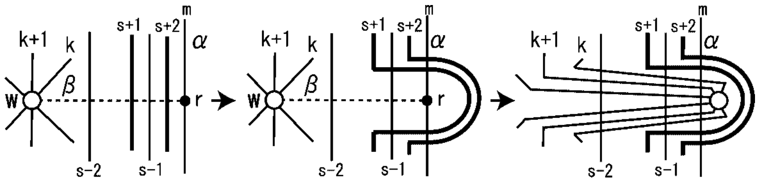

FIGURE 19. Lemma 6.6, Case (1) $k > s > m$.

LEMMA 6.6 ([5, Corollary 4.5]) (Shifting Lemma). *Let Γ be a chart and α an arc in an edge of Γ_m. Let w be a white vertex of $\Gamma_k \cap \Gamma_h$ where $h = k+\varepsilon$, $\varepsilon \in \{+1, -1\}$. Suppose that the white vertex w connects with a point r of the arc α by an arc β such that* Int β *intersects Γ transversely. Further suppose that one of the following two conditions is satisfied*:

(1) *$h > k > m$ and $\Gamma_s \cap \beta[w, r] = \emptyset$ for some integer s with $k > s > m$.*

(2) *$h < k < m$ and $\Gamma_s \cap \beta[w, r] = \emptyset$ for some integer s with $k < s < m$.*

*Then for any neighborhood V of the arc $\beta[w, r]$ we can shift the white vertex w to the other side of the arc α along the arc β by C-I-R*2 *moves*, *C-I-R*3 *moves and C-I-R*4 *moves in V keeping $\bigcup_{i \leq 0} \Gamma_{s+i\varepsilon}$ fixed* (*see Figure* 19). □

A chart Γ is of *type* $(m; n_1, n_2, \dots, n_k)$ or of *type* $(n_1, n_2, \dots, n_k)$ briefly if it satisfies the following three conditions:

(1) For each $i = 1, 2, \dots, k$, the chart Γ contains exactly n_i white vertices in $\Gamma_{m+i-1} \cap \Gamma_{m+i}$.

(2) If $i < 0$ or $i > k$, then Γ_{m+i} does not contain any white vertices.

(3) Both of the two subgraphs Γ_m and Γ_{m+k} contain at least one white vertex.

LEMMA 6.7. *Let Γ be a minimal chart of type $(m; n_1)$. Then there exists a ring or a non simple hoop of label $m-1$ or $m+2$.*

PROOF. By the condition (3) for the definition of type for charts, we have $n_1 > 0$.

Suppose that there do not exist any rings nor non simple hoops of Γ_{m-1} and Γ_{m+2}. By Assumption 5, Γ does not contain any free edges nor simple hoops. Hence we have $\Gamma_{m-1} = \emptyset$ and $\Gamma_{m+2} = \emptyset$.

Let $\mathbb{S}$ be the set of all minimal chart Γ' of type $(m; n_1)$ C-move equivalent to Γ such that $\Gamma'_{m-1} \cup \Gamma'_{m+2} = \emptyset$. For each $\Gamma' \in \mathbb{S}$, let $n(\Gamma')$ be the number of all rings and non simple hoops in $Cl(\Gamma' - (\Gamma'_m \cup \Gamma'_{m+1}))$. Let Γ^* be a minimal chart with $n(\Gamma^*) = \min\{n(\Gamma') \mid \Gamma' \in \mathbb{S}\}$. We shall show $n(\Gamma^*) = 0$.

Suppose $n(\Gamma^*) > 0$. Let C be a hoop or a ring of Γ^*_k with $k \neq m, m+1$, and D the disk with $\partial D = C$ and $D \not\ni \infty$. Since $k \neq m, m+1$ and since $\Gamma^*_{m-1} = \Gamma^*_{m+2} = \emptyset$, we have $m-1 > k$ or $m+2 < k$.

Suppose that D contains a white vertex w. Since Γ^* is of type $(m; n_1)$, we have $w \in \Gamma^*_m \cap \Gamma^*_{m+1}$. Let β be an arc in D connecting a point in C and the white vertex w such that Int β intersects Γ^* transversely. Since $\Gamma^*_{m-1} = \Gamma^*_{m+2} = \emptyset$, we have $(\Gamma^*_{m-1} \cup \Gamma^*_{m+2}) \cap \beta = \emptyset$.

If $m-1 > k$, then $m+1 > m > k$ and $\Gamma^*_{m-1} \cap \beta = \emptyset$. If $m+2 < k$, then $m < m+1 < k$ and $\Gamma^*_{m+2} \cap \beta = \emptyset$. By Shifting Lemma (Lemma 6.6) we can shift the white vertex w to $S^2 - D$ along the arc β by C-I-R2 moves, C-I-R3 moves and C-I-R4 moves.

Hence the number of white vertices in D can be reduced without increasing the number $n(\Gamma^*)$. By induction, we can assume that D does not contain any white vertices. By Assumption 3, the disk D does not contain any free edges. Since D does not contain any white vertices nor free edges, we can assume that D does not contain any black vertices. Hence Γ^* is C-move equivalent to $Cl(\Gamma^* - C)$ by a CI-move. Hence the number $n(\Gamma^*)$ is reduced. This contradicts the minimality of $n(\Gamma^*)$. Hence $n(\Gamma^*) = 0$.

Therefore we have $\Gamma^* = \Gamma^*_m \cup \Gamma^*_{m+1}$. Since Γ^* is like a 3-chart, the chart is C-move equivalent to a chart without white vertices by Lemma 6.5. This contradicts the fact that Γ^* is a minimal chart with $n_1 > 0$. Therefore there exists a ring or a non simple hoop of Γ_{m-1} or Γ_{m+2}. □

ACKNOWLEDGEMENT. The second author is partially supported by Grant-in-Aid for Scientific Research (No.20540093), Ministry of Education, Science and Culture, Japan.

References

[1] J. S. CARTER and M. SAITO, *Knotted surfaces and their diagrams*, Mathematical Surveys and Monographs, 55, American Mathematical Society, Providence, RI, 1998.

[2] S. KAMADA, Surfaces in R^4 of braid index three are ribbon, J. Knot Theory Ramifications 1 (1992), no. 2, 137–160.

[3] S. KAMADA, 2-dimensional braids and chart descriptions, *Topics in knot theory* (Erzurum, 1992), 277–287, NATO Adv. Sci. Inst. Ser. Math. Phys. Sci., 399, Kluwer Acad. Publ., Dordrecht, (1993).

[4] S. KAMADA, *Braid and Knot Theory in Dimension Four*, Mathematical Surveys and Monographs, Vol. 95, American Mathematical Society, 2002.

[5] T. NAGASE and A. SHIMA, Properties of minimal charts and their applications I, J. Math. Sci. Univ. Tokyo 14 (2007), 69–97.

[6] T. NAGASE and A. SHIMA, Properties of minimal charts and their applications II, Hiroshima Math. J., 39 (2009), 1–35.

Present Addresses:
TERUO NAGASE
DEPARTMENT OF MATHEMATICS,
TOKAI UNIVERSITY,
4–1–1 KITAKANAME, HIRATUKA KANAGAWA, 259–1292 JAPAN.
e-mail: nagase@keyaki.cc.u-tokai.ac.jp

AKIKO SHIMA
DEPARTMENT OF MATHEMATICS,
TOKAI UNIVERSITY,
4–1–1 KITAKANAME, HIRATUKA KANAGAWA, 259–1292 JAPAN.
e-mail: shima@keyaki.cc.u-tokai.ac.jp