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On Transformations that Preserve Fixed Anharmonic Ratio

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Abstract. O. Kobayashi [6] in 2007 proved that C^1-mappings preserving anharmonic ratio are Moebius transformations. We strengthen his theorem and prove that the requirement of differentiability and even of injectivity can be omitted.

1. Introduction

A concept of Apollonian tetrad in the complex plane **C** was introduced in the paper ([3], Def. 1, p.15): any ordered quadruple of distinct points $\{z_1, z_2, z_3, z_4\} \subset \mathbf{C}$ is called an *Apollonian* tetrad, if

$$|z_2 - z_3| \cdot |z_1 - z_4| = |z_3 - z_1| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| . \qquad (1)$$

It was proved in ([3], Main Theorem, p. 19) that any univalent analytic function in the domain $D \subset \mathbf{C}$ is linear-fractional iff the image of any Apollonian tetrad in D is also Apollonian tetrad.

Since that time some articles appeared, in which different generalizations of the mentioned property of tetrad to finite sets of points were introduced: Haruki H. and Rassias T.M. in [4] considered Apollonian triangles and hexagons, Bulut S. and Yilmaz Özgür N. in [5] considered Apollonian set consisting of $2n$ pairwise different points and proved that the analytic univalent function is linear-fractional iff the image of any Apollonian set in D is also an Apollonian set.

O. Kobayashi [6] noticed that relation (1) is equivalent to the equality $[z_1 : z_2 : z_3 : z_4] = (1 \pm i\sqrt{3})/2$, where $[z_1 : z_2 : z_3 : z_4]$ is the anharmonic ratio of the quadruple $\{z_1, z_2, z_3, z_4\}$, and he obtained the following result:

THEOREM 1 ([6], Theorem 2.1, p. 118). *Let $\lambda \in \boldsymbol{C} \setminus \boldsymbol{R}$ and $U \subset \boldsymbol{C}$ be a domain. Suppose $f : U \to \boldsymbol{C}$ is an injective C^1- mapping. If for any quadruple of pairwise distinct points $\{z_1, z_2, z_3, z_4\} \in U$ with anharmonic ratio $[z_1 : z_2 : z_3 : z_4] = \lambda$ the equality $[f(z_1) : f(z_2) : f(z_3) : f(z_4)] = \lambda$ holds, then f is a Moebius transformation.*

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In this paper we show that the theorem is valid for any $\lambda \in \mathbf{C} \setminus \{0, 1\}$ and for any continuous non-constant mapping $f : U \to \overline{\mathbf{C}}$ of the domain $U \subset \overline{\mathbf{C}}$ without the requirement of injectivity and differentiability of f.

2. Some definitions and the main result

Let $\mathbf{T}$ be set of all ordered quadruples (tetrads) $T = \{z_1, z_2, z_3, z_4\}$ in the extended complex plane $\overline{\mathbf{C}}$ such that do not contain any three coincident elements.

A tetrad with four pairwise different elements is called *nonsingular*.

For any tetrad we define its anharmonic ratio (see [1], §44) $A(T) = [z_1 : z_2 : z_3 : z_4]$.

If all points in a tetrad T are finite and pairwise different, then

$$A(T) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_3 - z_2)(z_4 - z_1)} .$$

Under these conditions $A(T)$ is different from 0, 1 and ∞. For tetrads $s_{12}(T)$, $s_{13}(T)$, $s_{14}(T)$, obtained from T by permutation of first and second, of first and third, of first and fourth elements respectively the following equalities hold:

$$A(s_{12}(T)) = \frac{1}{A(T)}; \; A(s_{13}(T)) = \frac{A(T)}{A(T) - 1}; \; A(s_{14}(T)) = 1 - A(T) , \qquad (2)$$

If in the tetrad $T = \{z_1, z_2, z_3, z_4\}$ neither three elements coincide, then there exists a finite or infinite limit

$$A(T) = \lim_{w_1 \to z_1,\, w_2 \to z_2,\, w_3 \to z_3,\, w_4 \to z_4} \frac{(w_1 - w_3)(w_2 - w_4)}{(w_3 - w_2)(w_4 - w_1)} .$$

where the limit is taken on the set of all nonsingular tetrads $\{w_1, w_2, w_3, w_4\}$. This limit defines the anharmonic ratio of the tetrad T in general case. Particularly, for nonsingular tetrad $T = \{z_1, z_2, z_3, \infty\}$ we have $A(T) = -(z_1 - z_3)/(z_3 - z_2)$. We notice that the tetrad $T \in \mathbf{T}$ is nonsingular iff $A(T)$ is different from 0, 1 and ∞.

A Moebius transformation $\mu : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is defined as a superposition of finite number of reflections (or inversions) with respect to generalized circles in $\overline{\mathbf{C}}$ (see [1], Def 3.1.1, p. 25) and its realized either by the linear-fractional function or by its conjugate.

If μ is a linear-fractional mapping, then for any tetrad $T \in \mathbf{T}$ the equality $A(\mu(T)) = A(T)$ (the invariance of anharmonic ratio by the linear-fractional mappings) holds. The opposite is also true: a bijective mapping $\mu : \mathbf{C} \to \mathbf{C}$, which preserves an anharmonic ratio of all nonsingular tetrads, is realized by linear-fractional function (see [1], §4.4).

For given complex number $\alpha \notin \{0, 1, \infty\}$, we denote by $\mathbf{T}(\alpha)$ the set of all tetrads $T \in \mathbf{T}$ with $A(T) = \alpha$ (all that tetrads are nonsingular) and by $\mathbf{T}(\alpha, f)$ the set of all tetrads of $\mathbf{T}(\alpha)$, for which $f(T) \in \mathbf{T}$.

Denote by $\mathrm{Bd}(U)$ a boundary of a domain U.

DEFINITION 1. We say that a mapping $f : U \to \overline{\mathbf{C}}$ of a domain $U \subset \overline{\mathbf{C}}$ satisfies the condition (α), if for any tetrad $T \in \mathbf{T}(\alpha,\ f)$ the equality $A(f(T)) = \alpha = A(T)$ holds.

The main result of the paper is the following

THEOREM 2. *If $\alpha \notin \{0,\ 1,\ \infty\}$, then any continuous mapping $f : U \to \overline{\mathbf{C}}$ of a domain $U \subset \overline{\mathbf{C}}$, which satisfies the condition (α), is either a constant or the Moebius transformation; and if α is not a real number, then a function f is either a constant or a linear-fractional function.*

3. The injectivity lemma

PROPOSITION 1. *If a continuous mapping $f : U \to \overline{\mathbf{C}}$ satisfies the condition (α), $\alpha \notin \{0,\ 1,\ \infty\}$, then it satisfies the condition (β), where β is taken from the set*

$$P_\alpha = \left\{\alpha,\ \frac{1}{\alpha},\ 1-\alpha,\ \frac{1}{1-\alpha},\ \frac{\alpha}{\alpha-1},\ \frac{\alpha-1}{\alpha}\right\}.$$

PROOF. The proof follows immediately from relations (2) and from the fact that the equality $f(s(T)) = s(f(T))$ is preserved by the permutations s of the elements of the tetrad, and therefore the relations $f(T) \in \mathbf{T}$ and $f(s(T)) \in \mathbf{T}$ are equivalent. ■

LEMMA 1. *Suppose $\alpha \notin \{0,\ 1,\ \infty\}$ and let $f : U \to \overline{\mathbf{C}}$ be a continuous mapping of a domain $U \subset \overline{\mathbf{C}}$ satisfying the condition (α). Then f is either injective in U or f is a constant map.*

PROOF. We assume that f is not injective. Then we can find different points $\zeta_0,\ \zeta_\infty \in U$, for which $f(\zeta_0) = f(\zeta_\infty) = a \in \overline{\mathbf{C}}$. Take such linear-fractional mappings μ and ν that $\mu(\zeta_\infty) = \nu(a) = \infty$. Evidently $\infty \in \mu(U) = U'$. A continuous mapping $g = \nu \circ f \circ \mu^{-1} : U' \to \overline{\mathbf{C}}$ satisfies the condition (γ) for any constant $\gamma \in P_\alpha$; whereas $g(\infty) = g(\mu(\zeta_\infty)) = \infty$.

Consider the set $M = g^{-1}(\{\infty\}) \cap \mathbf{C} = \{z \in \mathbf{C} : g(z) = \infty\}$.

Since g is continuous, the set M is closed in $U' \backslash \{\infty\}$. We show that M is an open set.

Let z_0 be a point in M.

Build an open disc $B = B(z_0,\ r) \subset U'$ with radius

$$r = \min\{1,\ \operatorname{dist}(z_0,\ \operatorname{Bd}(U'))\}/(2+|\alpha|)\,. \tag{3}$$

If $g(z) \equiv \infty$ in B, then $B \subset M$, and z_0 is an interior point of the set M. Consider the point $z_1 \in B$, in which $g(z_1) \neq \infty$. The point $z_2 = z_0 + \alpha(z_1 - z_0)$ lies in U', whereas

$$|z_2 - z_0| = |\alpha| \cdot |z_1 - z_0| \leq \operatorname{dist}(z_0,\ \operatorname{Bd}(U)')\frac{|\alpha|}{2+|\alpha|} < \operatorname{dist}(z_0,\ \operatorname{Bd}(U'))\,.$$

For $T = \{z_2,\ z_1,\ z_0,\ \infty\} \subset U'$ we have $A(T) = -(z_2 - z_0)/(z_0 - z_1) = \alpha$. The assumption that $f(T) = \{g(z_2),\ g(z_1),\ \infty,\ \infty\} \in \mathbf{T}$, contradicts the condition (α), because in this case $A(f(T)) \in \{0,\ 1,\ \infty\}$, that is $f(A) \neq \alpha$.

Therefore the tetrad $f(T)$ has three equal elements. But $g(z_1) \neq \infty$ implies that $g(z_2) = \infty$, that is $z_2 \in M$.

Since $g(z_1) \neq \infty$ and f is continuous in the point z_1, take such $\varepsilon > 0$ that the disc $B(z_1,\ \varepsilon) \subset B$ and a function $g(z) \neq \infty$ in $B(z_1,\ \varepsilon)$. From the inequality $\varepsilon < r$ and from (3) it follows that

$$B\left(z_0, \varepsilon\frac{|\alpha|}{1+|\alpha|}\right) \subset B\,. \tag{4}$$

Putting $\delta = \varepsilon\frac{|\alpha|}{1+|\alpha|}$, consider any point $z \in\ B\ (z_0,\ \delta)$. For $w = z+\alpha^{-1}(z_2-z)$ the inequality holds:

$$|w - z_1| = |z - z_1 + z_1 - z_0 + \alpha^{-1}(z_0 - z)| < |z - z_0|\left|1 - \frac{1}{\alpha}\right| < \varepsilon\,,$$

which means that $w \in B(z_1,\ \varepsilon)$ and therefore $g(w) \neq \infty$. For $T' = \{z_2,\ w,\ z,\ \infty\} \subset U'$ we have $A(T') = -(z_2 - z)/(z - w) = \alpha$. The assumption that $g(T') = \{g(z_2) = \infty,\ g(w),\ g(z),\ \infty\} \in \mathbf{T}$, contradicts the condition (α), because $A(g(T')) \in \{0,\ 1,\ \infty\}$ and therefore $A(g(T')) \neq \alpha$. Then in the tetrad $g(T')$ there are three equal elements. But $g(w) \neq \infty$ and therefore $g(z) = \infty$. Thus we see that $g(z) \equiv \infty$ in the disc $B(z_0,\ \delta)$. So $B(z_0,\ \delta) \subset M$ and z_0 is an interior point of the set M. Since any point of the set M it is an interior point, the set M is open.

Since the set $U'\backslash\{\infty\}$ is connected, the open-and-closed set M is an empty set or $U'\backslash\{\infty\}$. Since $\mu(\zeta_0) \in M$, we conclude that $M \neq \varnothing$. Therefore $M = U'\backslash\{\infty\}$, and it means that $g(z) \equiv \infty$ in U'. Thus $f(\zeta) \equiv a$ in U. ■

4. Proof of Theorem 2

We prove Theorem 2 in several steps arranging them as independent propositions.

PROPOSITION 2. *Suppose the domain* $U \subset \overline{\mathbf{C}}$ *contains* ∞, *and a continuous injective mapping* $f : U \to \overline{\mathbf{C}}$ *with* $f(\infty) = \infty$, *satisfies the condition* (α) *for* $\alpha \notin \{0,\ 1,\ \infty\}$. *Then for any* $z_0 \in U$ *and* $a_0 = z_0 + w_0,\ b_0 = z_0 - w_0$, *where* $|w_0| < \operatorname{dist}(z_0,\ \operatorname{Bd}(U))$, *the equality holds:*

$$f(z_0) = \frac{f(a_0) + f(b_0)}{2}\,. \tag{5}$$

PROOF. Let $\alpha \in \{2,\ 1/2,\ -1\}$. Then by Proposition 1 the mapping f satisfies the condition (α) with $\alpha = 2$. For $T = \{a_0,\ b_0,\ z_0,\ \infty\}$ we obtain

$$2 = A(T) = [f(a_0) : f(z_0) : f(b_0) : \infty] = -\frac{f(a_0) - f(b_0)}{f(b_0) - f(z_0)}\,,$$

from which the desired equation (5) immediately follows.

Suppose now $\alpha \notin \{0,\ 1,\ \infty,\ 2,\ 1/2,\ -1\}$.

If $|\alpha| > 1$ we take $\beta = \alpha$; in otherwise we take $\beta = 1/\alpha$. Then f satisfies the condition (β), $\beta \notin \{0,\ 1,\ \infty,\ 1/2\}$ (see Proposition 1). For $\beta' = (1-\beta)/(1-2\beta)$ we have the equation $(1-2\beta)/(1-2\beta') = -1$. As $\beta \notin \overline{B}(1/2,\ 1/2)$, then $|\beta - 1/2| > 1/2$, that is $|1-2\beta| > 1$. Therefore $|1-2\beta'| = q < 1$.

Put $R = \operatorname{dist}(z_0,\ \operatorname{Bd}(U))$ and $w_k = (2\beta'-1)^k w_0,\ k = 1,\ 2, \ldots$

As $|w_k| = q^k |w_0| < R$, then all points $a_k = z_0 + w_k$ and $b_k = z_0 - w_k$ lie in the disc $B(z_0,\ R) \subset U$. We show that for any $k = 0,\ 1,\ 2, \ldots$ the follow equality holds:

$$f(a_k) + f(b_k) = f(a_0) + f(b_0)\,. \tag{6}$$

For $k = 0$ condition (6) is trivial. We suggest that it holds for some k and show that $f(a_{k+1}) + f(b_{k+1}) = f(a_0) + f(b_0)$.

For the tetrad $T_1 = \{b_k,\ a_{k+1},\ b_{k+1},\ \infty\}$ we have

$$A(T_1) = -\frac{b_k - b_{k+1}}{b_{k+1} - a_{k+1}} = -\frac{w_{k+1} - w_k}{-2w_{k+1}} = \frac{\beta' - 1}{2\beta' - 1} = \beta\,.$$

The equality

$$A(T_1) = -\frac{f(b_k) - f(b_{k+1})}{f(b_{k+1}) - f(a_{k+1})} = \beta$$

follows from the condition (β). For $T_2 = \{a_k,\ b_{k+1},\ a_{k+1},\ \infty\}$ we have

$$A(T_2) = -\frac{a_k - a_{k+1}}{a_{k+1} - b_{k+1}} = -\frac{w_k - w_{k+1}}{2w_{k+1}} = \beta\,.$$

Therefore

$$A(T_2) = -\frac{f(a_k) - f(a_{k+1})}{f(a_{k+1}) - f(b_{k+1})} = \beta\,.$$

Thus we come to the equality

$$\frac{f(b_{k+1}) - f(b_k)}{f(b_{k+1}) - f(a_{k+1})} = \frac{f(a_k) - f(a_{k+1})}{f(b_{k+1}) - f(a_{k+1})}\,,$$

from which it follows that $f(b_{k+1}) - f(b_k) = f(a_k) - f(a_{k+1})$.

Then by induction we obtain

$$f(a_{k+1}) + f(b_{k+1}) = f(a_k) + f(b_k) = f(a_0) + f(b_0)\,.$$

Thus we have proved (6) for all $k = 0,\ 1,\ 2, \ldots$.

Since $w_k \to 0$ as $k \to \infty$, we have $a_k \to z_0$ and $b_k \to z_0$.

From (6) we obtain the desired relation (5). ∎

PROPOSITION 3. *Suppose a domain* $U \subset \overline{\mathbf{C}}$ *contains* ∞, *and a continuous mapping* $f : U \to \overline{\mathbf{C}}$ *with* $f(\infty) = \infty$ *satisfies the condition* (α), $\alpha \notin \{0, 1, \infty\}$. *Then the mapping* f *moves any linear segment* $L \subset U\backslash\{\infty\}$ *to a linear segment*; *any ray* $P \subset U\backslash\{\infty\}$ *to some ray*; *any line* $Q \subset U\backslash\{\infty\}$ *to some line.*

PROOF. We show that for any point $c \in U\backslash\{\infty\}$ the mapping f is linear on any linear segment $L \subset B(c, r)$, where $r = \mathrm{dist}(c, \mathrm{Bd}(U))/3$. Let a, b be the endpoints of the segment L. For any point $z \in L$, $\max\{|z-a|, |z-b|\} \le 2r < \mathrm{dist}(z, \mathrm{Bd}(U))$. That is, we may apply Proposition 2 to any point $z \in L$ and any w such that $|w| < \max\{|z-a|, |z-b|\}$. Therefore for any pair of points $t_1, t_2 \in L$ we have $f((t_1+t_2)/2) = (f(t_1)+f(t_2))/2$. We conclude that the function f is linear on a dense subset of the segment L and so, by continuity of f, it is linear on L.

Thus the mapping f is locally linear on any connected subset $S \subset U\backslash\{\infty\}$, which lies on a straight line. Therefore by connectedness of S, f is linear on all of S. Particularly, an image of any segment is some segment, the image of any ray is some ray, the image of any line is some line. ■

Next we use a criterion of Moebiusness for mappings of n-dimensional domains, proposed by Y. B. Zelinsky (we take this theorem for case $n = 2$).

THEOREM 3 ([2], Th. 8, p. 35). *Suppose a continuous mapping* $f : D \to \overline{\mathbf{C}}$ *of a domain* $D \subset \overline{\mathbf{C}}$ *moves any set* $P \subset D$, *which lies on a generalized circle to a set on a generalized circle. If* $f(D)$ *does not lie on a generalized circle*, *then* f *is a Moebius transformation.*

LEMMA 2. *Any continuous injective mapping* $f : U \to \overline{\mathbf{C}}$ *of a domain* $U \subset \overline{\mathbf{C}}$, *which satisfies the condition* (α), $\alpha \notin \{0, 1, \infty\}$, *is a Moebius transformation.*

PROOF. Take any open disc $D \subset U$ such that $\overline{D} \subset U$ and a set $P \subset D$ lies on a generalized circle $S \subset \overline{\mathbf{C}}$. Then $S \cap D$ is a connected subset of a generalized circle S.

The situation 1.

Suppose $S \cap \mathrm{Bd}(D) \neq \varnothing$ and $a \in S \cap \mathrm{Bd}(D)$. We build linear-fractional mappings $\mu : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ and $\eta : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that $\mu(a) = \infty$, $\eta(f(a)) = \infty$ and consider the mapping $g = \eta \circ f \circ \mu^{-1} : \mu(U) \to \overline{\mathbf{C}}$. This mapping satisfies the condition (α) and $\infty \in \mu(U)$, $g(\infty) = \infty$. The set $L' = \mu((S \cap \overline{D})\backslash\{a\}) \subset \mu(U)\backslash\{\infty\}$ is either a ray or a line. By Proposition 3 the set $g(L')$ is also either some ray or a line. Therefore $f = \eta^{-1} \circ g \circ \mu$ maps a set $S \cap D$ (and any subset P also) to a subset of a generalized circle.

The situation 2.

Take $S \subset D$ and $a \in S$. We build linear-fractional mappings μ and η by analogy with the situation 1 and consider the mapping $g = \eta \circ f \circ \mu^{-1}$. By Proposition 3 the mapping g moves the line $L' = \mu((S\backslash\{a\} \subset \mu(U)\backslash\{\infty\}$ to some line. Therefore $g(L')$ is a generalized circle and f moves the generalized circle $S = \mu^{-1}(L')$ to a generalized circle, and any subset $P \subset S$ to a subset of a generalized circle. Thus f satisfies the conditions of Theorem 3 in the disc D and is injective. Therefore f is a Moebius transformation on the disc D. Since our choice of the disc $D \subset U$ is arbitrary, the mapping f is locally Moebius on a domain U. Moebius transformations are explicitly defined by their values on some quadruple, that does not lie on a generalized circle. From local Moebiusness of f it follows that f is Moebius on a domain U. ∎

LEMMA 3. *Any continuous injective mapping $f : U \to \overline{\boldsymbol{C}}$ of a domain $U \subset \overline{\boldsymbol{C}}$, which satisfies the condition (α), $\alpha \in \boldsymbol{C} \setminus \boldsymbol{R}$, is a linear-fractional function.*

PROOF. By Lemma 2 the mapping f is a Moebius transformation, so f is either a linear-fractional function or its conjugate. We show that if $\alpha = a + ib$ and $b \neq 0$, then the second is impossible. Let $f(z) = \overline{\mu(z)}$, where $\mu(z)$ is a linear-fractional function. Take any tetrad $T = \{z_1, z_2, z_3, z_4\} \subset U$ with anharmonic ratio $A(T) = \alpha = a + ib$. Then $A(\mu(T)) = A(T) = a + ib$ and $A(f(T)) = A(\overline{\mu}(T)) = \overline{A(\mu(T))} = a - ib \neq \alpha$. This contradicts the condition (α). Therefore f is a linear-fractional function. ∎

The proof of Theorem 2 follows immediately from Lemma 1, Lemma 2 and Lemma 3.

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