TOKYO J. MATH.
VOL. 33, NO. 2, 2010

On the Extensions of Group Schemes Deforming $\mathbf{G}_a$ to $\mathbf{G}_m$

Takashi KONDO

Chuo University

(Communicated by S. Miyoshi)

Abstract. For given group schemes $\mathcal{G}^{(\lambda_i)}$ $(i = 1, 2, \dots)$ deforming the additive group scheme $\mathbf{G}_a$ to the multiplicative group scheme $\mathbf{G}_m$, T. Sekiguchi and N. Suwa constructed extensions:

$$0 \to \mathcal{G}^{(\lambda_2)} \to \mathcal{E}^{(\lambda_1,\lambda_2)} \to \mathcal{G}^{(\lambda_1)} \to 0,\ \dots,\ 0 \to \mathcal{G}^{(\lambda_n)} \to \mathcal{E}^{(\lambda_1,\dots,\lambda_n)} \to \mathcal{E}^{(\lambda_1,\dots,\lambda_{n-1})} \to 0, \dots$$

inductively, by calculating the group of extensions $\mathrm{Ext}^1(\mathcal{E}^{(\lambda_1,\dots,\lambda_{n-1})}, \mathcal{G}^{(\lambda_n)})$. Here we treat the group $\mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\dots,\lambda_n)})$ of extensions in the case of $n = 2,\ 3$. The case of $n = 2$ was studied by D. Horikawa.

Introduction

In [4], T. Sekiguchi and N. Suwa constructed the group schemes deforming the group schemes of Witt vectors to tori in order to unify the Kummer theory and the Artin-Schreier-Witt theory. Let A be a discrete valuation ring with the maximal ideal $\mathfrak{m}$. Then such group schemes of dimension 1 over A are known to be given only by $\mathcal{G}^{(\lambda)} = \mathrm{Spec}\, A[X, 1/(1 + \lambda X)]$ with $\lambda \in \mathfrak{m}\backslash\{0\}$ (compare [3], [8]). In higher dimensional case, Sekiguchi and Suwa determined the following successive groups of extensions:

$$\begin{aligned}
&\mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}) &&\ni\ 0 \to \mathcal{G}^{(\lambda_2)} \to \mathcal{E}^{(\lambda_1,\lambda_2)} \to \mathcal{G}^{(\lambda_1)} \to 0\,,\\
&\mathrm{Ext}^1(\mathcal{E}^{(\lambda_1,\lambda_2)}, \mathcal{G}^{(\lambda_3)}) &&\ni\ 0 \to \mathcal{G}^{(\lambda_3)} \to \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3)} \to \mathcal{E}^{(\lambda_1,\lambda_2)} \to 0\,,\\
&\mathrm{Ext}^1(\mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3)}, \mathcal{G}^{(\lambda_4)}) &&\ni\ 0 \to \mathcal{G}^{(\lambda_4)} \to \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} \to \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3)} \to 0\,,\\
&&&\dots
\end{aligned}$$

In this article, we will determine the group of extensions $\mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3)})$ (Theorem 6.2.1). The group $\mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2)})$ was determined by D. Horikawa [2]. We will

Received January 13, 2009; revised May 13, 2010
Mathematics Subject Classification: 22E27
Key words: group of extension, group scheme, Witt vector

use Horikawa's theorem essentially in our argument. For the readers' convenience, we will prove Horikawa's theorem (Theorem 5.1.1) in Section 5.

Notaions

In this article, ring means a commutative unitary ring.

- p : a fixed prime number
- $\mathbf{G}_{a,A} := \mathrm{Spec}\, A[T]$: the additive group scheme over a ring A
- $\mathbf{G}_{m,A} := \mathrm{Spec}\, A[T, 1/T]$: the multiplicative group scheme over a ring A
- $\mathbf{A}^n_A$: the affine space of dimension n over a ring A, endowed with the usual ring scheme structure
- $W(A)$: the ring of Witt vectors over a ring A
- $[a] := (a, 0, 0, \dots) \in W(A)$: the Teichmüller lifting of $a \in A$
- $\mathrm{Ext}^1(G, H)$: the group of extensions of abelian group schemes G and H
- $H^2_0(G, H)$: the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in H for group schemes G and H (cf. [1, Chap. II.3 and Chap. III.6]).

1. Witt Vectors

In this section, we recall the fundamental facts on Witt vectors.

1.1. For a non-negative integer n, we denote by $\Phi_n(\mathbf{X}) = \Phi_n(X_0, \dots, X_n)$ the Witt polynomial:

$$\Phi_n(\mathbf{X}) := X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^{n-1}X_{n-1}^p + p^n X_n$$

in $\mathbf{Z}[\mathbf{X}] = \mathbf{Z}[X_0, X_1, \dots]$. We put $\mathbf{W}_{n,\mathbf{Z}} := \mathrm{Spec}\, \mathbf{Z}[T_0, T_1, \dots, T_{n-1}]$ and define the map $\Phi^{(n)} : \mathbf{W}_{n,\mathbf{Z}} \to \mathbf{A}^n_{\mathbf{Z}}$ by

$$T_i \longmapsto \Phi_i(\mathbf{T}) = \Phi_i(T_0, \dots, T_i)\,.$$

PROPOSITION 1.1.1. *$\Phi^{(n)}$ induces the ring scheme structure on $\mathbf{W}_{n,\mathbf{Z}}$ uniquely so that it is a ring scheme homomorphism. In particular, $\mathbf{W}_{n,\mathbf{Q}} \simeq \mathbf{A}^n_{\mathbf{Q}}$.*

In fact, the addition σ and the multiplication π of $\mathbf{A}^n_{\mathbf{Z}}$ are given by

$$\sigma^* : T_i \mapsto X_i + Y_i\,,\ \pi^* : T_i \mapsto X_i \otimes Y_i$$

with $X_i := T_i \otimes 1$ and $Y_i := 1 \otimes T_i$. Suppose that Σ and Π are the addition and the multiplication which are induced by $\Phi^{(n)}$. Then $\Sigma^*(\Phi_i(\mathbf{T})) = \Phi_i(\mathbf{X}) + \Phi_i(\mathbf{Y})$, $\Pi^*(\Phi_i(\mathbf{T})) = \Phi_i(\mathbf{X})\Phi_i(\mathbf{Y})$ and the following diagrams are commutative:

$$\begin{array}{ccc} \mathbf{W}_{n,\mathbf{Z}} \times_{\operatorname{Spec}\mathbf{Z}} \mathbf{W}_{n,\mathbf{Z}} & \xrightarrow{\Phi^{(n)}\times\Phi^{(n)}} & \mathbf{A}^n_{\mathbf{Z}} \times_{\operatorname{Spec}\mathbf{Z}} \mathbf{A}^n_{\mathbf{Z}} \\ \Sigma\downarrow & & \downarrow\sigma \\ \mathbf{W}_{n,\mathbf{Z}} & \xrightarrow[\Phi^{(n)}]{} & \mathbf{A}^n_{\mathbf{Z}} \end{array}$$

$$\begin{array}{ccc} \mathbf{W}_{n,\mathbf{Z}} \times_{\operatorname{Spec}\mathbf{Z}} \mathbf{W}_{n,\mathbf{Z}} & \xrightarrow{\Phi^{(n)}\times\Phi^{(n)}} & \mathbf{A}^n_{\mathbf{Z}} \times_{\operatorname{Spec}\mathbf{Z}} \mathbf{A}^n_{\mathbf{Z}} \\ \Pi\downarrow & & \downarrow\pi \\ \mathbf{W}_{n,\mathbf{Z}} & \xrightarrow[\Phi^{(n)}]{} & \mathbf{A}^n_{\mathbf{Z}} \end{array} .$$

By induction on i, it can be seen that $\Sigma^*(T_i)$, $\Pi^*(T_i) \in \mathbf{Z}[\mathbf{X}, \mathbf{Y}]$, thus $\mathbf{W}_{n,\mathbf{Z}}$ is a ring scheme over $\mathbf{Z}$. We call $\mathbf{W}_{n,\mathbf{Z}}$ a ring scheme of Witt vectors over $\mathbf{Z}$ of length n. We also denote $\Sigma^*(T_i)$ and $\Pi^*(T_i)$ by $S_i(\mathbf{X}, \mathbf{Y})$ and $P_i(\mathbf{X}, \mathbf{Y})$, respectively.

We denote the ring of Witt vectors over a ring A by $W(A)$, and the formal completion of $W(A)$ along the zero section by $\widehat{W}(A)$. Then we have

$$\widehat{W}(A) = \left\{ (a_0, a_1, \dots) \in W(A) \;\middle|\; \begin{array}{l} a_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

1.2. In this subsection, we define some endomorphisms of the additive group $\mathbf{W}_{\mathbf{Z}} :=\operatorname{Spec}\mathbf{Z}[\mathbf{T}]$. We define the Verschiebung endomorphism $V : \mathbf{W}_{\mathbf{Z}} \to \mathbf{W}_{\mathbf{Z}}$ by

$$T_i \mapsto \begin{cases} T_{i-1} & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases} .$$

For $r \geq 0$, we define polynomials $F_r(\mathbf{T}) \in \mathbf{Q}[T_0, \dots, T_{r+1}]$ inductively by

$$\Phi_r(F_0(\mathbf{T}), \dots, F_r(\mathbf{T})) = \Phi_{r+1}(\mathbf{T}) .$$

Then $F_r(\mathbf{T}) \in \mathbf{Z}[T_0, \dots, T_{r+1}]$. We define the Frobenius endomorphism $F : \mathbf{W}_{\mathbf{Z}} \to \mathbf{W}_{\mathbf{Z}}$ by

$$T_i \mapsto F_i(\mathbf{T}) .$$

F is a ring scheme homomorphism and if A is a $\mathbf{F}_p$-algebra, then $F : W(A) \to W(A)$ is nothing but the usual Frobenius endomorphism. For a ring A and $\lambda \in A$, we define $F^{(\lambda)}$: $\widehat{W}(A) \to \widehat{W}(A)$ by

$$F^{(\lambda)}\boldsymbol{a} := (F - [\lambda^{p-1}])\boldsymbol{a} = F\boldsymbol{a} - [\lambda^{p-1}]\boldsymbol{a}$$

and denote the kernel and the cokernel of $F^{(\lambda)}$ by $\widehat{W}(A)^{F^{(\lambda)}}$ and $\widehat{W}(A)/F^{(\lambda)}$, respectively.

For a vector $\boldsymbol{a} := (a_0, a_1, \dots) \in W(A)$, we define a map $\langle \boldsymbol{a}, \cdot \rangle : W(A) \to W(A)$ by

$$\Phi_n(\langle \boldsymbol{a}, \boldsymbol{x} \rangle) = a_0^{p^n} \Phi_n(\boldsymbol{x}) + p a_1^{p^{n-1}} \Phi_{n-1}(\boldsymbol{x}) + \cdots + p^n a_n \Phi_0(\boldsymbol{x})$$

for $\boldsymbol{x} \in W(A)$. Then we have $\langle \boldsymbol{a}, \cdot \rangle = \sum_{k \geq 0} V^k[a_k]$ and it is an endomorphism (cf. [4, Remark 4.8]).

2. Artin-Hasse exponential series

In this section, we review some concepts on Artin-Hasse exponential series from [4].

2.1. We define a formal power series $E_p(T) \in \mathbf{Q}[[T]]$ by

$$E_p(T) := \exp\left(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots\right).$$

Then it can be seen $E_p(T) \in \mathbf{Z}_{(p)}[[T]]$ and we call it Artin-Hasse exponential series.

We define a formal power series $E_p(U, \Lambda; T) \in \mathbf{Q}[U, \Lambda][[T]]$ by

$$E_p(U, \Lambda; T) := (1 + \Lambda T)^{\frac{U}{\Lambda}} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} T^{p^k})^{\frac{1}{p^k}\{(\frac{U}{\Lambda})^{p^k} - (\frac{U}{\Lambda})^{p^{k-1}}\}}.$$

By Artin-Hasse exponential series, we have

$$E_p(U, \Lambda; T) = \begin{cases} \displaystyle\prod_{(k,p)=1} E_p(U\Lambda^{k-1}T^k)^{(-1)^{k-1}/k} & \text{if } p > 2 \\ \displaystyle\prod_{(k,2)=1} E_2(U\Lambda^{k-1}T^k)^{1/k} \left\{ \prod_{(k,2)=1} E_2(U\Lambda^{2k-1}T^{2k})^{1/k} \right\}^{-1} & \text{if } p = 2 \end{cases}$$

and $E_p(U, \Lambda; T) \in \mathbf{Z}_{(p)}[U, \Lambda][[T]]$.

Moreover, for an infinite sequence of indeterminates $\mathbf{U} := (U_0, U_1, \dots)$, we define formal power series $E_p(\mathbf{U}, \Lambda; T)$ by

$$E_p(\mathbf{U}, \Lambda; T) := \prod_{k=0}^{\infty} E_p(U_k, \Lambda^{p^k}, T^{p^k}) \in \mathbf{Z}_{(p)}[\mathbf{U}, \Lambda][[T]].$$

Then we have

$$E_p(\mathbf{U}, \Lambda; T) = (1+\Lambda T)^{\frac{U_0}{\Lambda}} \prod_{k=1}^{\infty} (1+\Lambda^{p^k} T^{p^k})^{\frac{1}{p^k \Lambda^{p^k}}(\Phi_k(\mathbf{U}) - \Lambda^{p^{k-1}(p-1)}\Phi_{k-1}(\mathbf{U}))}$$
$$= (1+\Lambda T)^{\frac{U_0}{\Lambda}} \prod_{k=1}^{\infty} (1+\Lambda^{p^k} T^{p^k})^{\frac{1}{p^k \Lambda^{p^k}}\Phi_{k-1}(F^{(\Lambda)}\mathbf{U})}.$$

Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda \in A$. Then for $\boldsymbol{a}, \boldsymbol{b} \in W(A)$, we have

$$E_p(\boldsymbol{a}, \lambda; T) E_p(\boldsymbol{b}, \lambda; T) = E_p(\boldsymbol{a} + \boldsymbol{b}, \lambda; T)$$

with $\boldsymbol{a} + \boldsymbol{b} = (S_0(\boldsymbol{a}, \boldsymbol{b}), S_1(\boldsymbol{a}, \boldsymbol{b}), \dots)$. Moreover, if $F^{(\lambda)}\boldsymbol{a} = \mathbf{0}$, then

$$E_p(\boldsymbol{a}, \lambda; T_0) E_p(\boldsymbol{a}, \lambda; T_1) = E_p(\boldsymbol{a}, \lambda; T_0 + T_1 + \lambda T_0 T_1).$$

We define a formal power series $F_p(\mathbf{U}, \Lambda; T_0, T_1) \in \mathbf{Q}[\mathbf{U}, \Lambda][[T_0, T_1]]$ by

$$F_p(\mathbf{U}, \Lambda; T_0, T_1) := \prod_{k=1}^{\infty} \left(\frac{(1+\Lambda^{p^k} T_0^{p^k})(1+\Lambda^{p^k} T_1^{p^k})}{1+\Lambda^{p^k}(T_0 + T_1 + \Lambda T_0 T_1)^{p^k}} \right)^{\frac{1}{p^k \Lambda^{p^k}}\Phi_{k-1}(\mathbf{U})}.$$

Then $F_p(\mathbf{U}, \Lambda; T_0, T_1) \in \mathbf{Z}_{(p)}[\mathbf{U}, \Lambda][[T_0, T_1]]$.

2.2. For $\boldsymbol{a} = (a_0, a_1, \dots) \in W(A)$, we define $\tilde{p} : W(A) \to W(A)$ by

$$\tilde{p}(\boldsymbol{a}) := (0, a_0^p, a_1^p, \dots).$$

Moreover, we define $\tilde{p}E_p(\mathbf{U}, \Lambda; X)$ and $\tilde{p}F_p(\mathbf{U}, \Lambda; X, Y)$ by

$$\tilde{p}E_p(\mathbf{U}, \Lambda; X) := E_p(\tilde{p}\mathbf{U}, \Lambda; X)$$

and

$$\tilde{p}F_p(\mathbf{U}, \Lambda; X, Y) := F_p(\tilde{p}\mathbf{U}, \Lambda; X, Y).$$

PROPOSITION 2.2.1 ([4, Lemma 4.10]). *For $k, \ell \in \mathbf{Z}$ with $k \geq 1$ and $\ell \geq 0$, we have*

$$(\tilde{p})^k E_p(\mathbf{U}, \Lambda; X) = E_p(\mathbf{U}^{(p^k)}, \Lambda^{p^k}; X^{p^k})$$
$$(\tilde{p})^{k+\ell} F_p(\mathbf{U}, \Lambda; X, Y) \equiv (\tilde{p})^{\ell} F_p(\mathbf{U}^{(p^k)}, \Lambda^{p^k}; X^{p^k}, Y^{p^k}) \bmod p^{\ell+1}$$

with $\mathbf{U}^{(p^k)} = (U_0^{p^k}, U_1^{p^k}, \dots)$.

We put

$$\mathbf{V} = (V_0, V_1, \dots) := \left(\frac{U_0}{\Lambda_2}, \frac{U_1}{\Lambda_2}, \dots \right),$$

and define formal power series $\tilde{E}_p(\mathbf{W}, \Lambda_2; E)$ and $\tilde{E}_p(\mathbf{W}, \Lambda_2; F)$ by

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; E) := E^{\frac{W_0}{\Lambda_2}} \prod_{k=1}^{\infty} ((\tilde{p})^k E)^{\frac{1}{p^k \Lambda_2^{p^k}} \Phi_{k-1}(F^{(\Lambda_2)}\mathbf{W})}$$

and

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; F) := F^{\frac{W_0}{\Lambda_2}} \prod_{k=1}^{\infty} ((\tilde{p})^k F)^{\frac{1}{p^k \Lambda_2^{p^k}} \Phi_{k-1}(F^{(\Lambda_2)}\mathbf{W})}$$

where $E := E_p(\mathbf{U}, \Lambda_1; X)$ and $F := F_p(\mathbf{U}, \Lambda_1; X, Y)$.

PROPOSITION 2.2.2 ([4, Proposition 4.11]). *Under the above notation, we have*

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; E) = E_p(\langle \mathbf{V}, \mathbf{W} \rangle, \Lambda_1, X)\,,$$
$$\tilde{E}_p(\mathbf{W}, \Lambda_2; F) = F_p(\langle \mathbf{V}, \mathbf{W} \rangle, \Lambda_1, X, Y)\,.$$

We define formal power series $G_p(\mathbf{W}, \Lambda_2; E)$ and $G_p(\mathbf{W}, \Lambda_2; F)$ by

$$G_p(\mathbf{W}, \Lambda_2; E) := \prod_{k=1}^{\infty} \left(\frac{1 + (E-1)^{p^k}}{(\tilde{p})^k E} \right)^{\frac{1}{p^k \Lambda_2^{p^k}} \Phi_{k-1}(\mathbf{W})}$$

and

$$G_p(\mathbf{W}, \Lambda_2; F) := \prod_{k=1}^{\infty} \left(\frac{1 + (F-1)^{p^k}}{(\tilde{p})^k F} \right)^{\frac{1}{p^k \Lambda_2^{p^k}} \Phi_{k-1}(\mathbf{W})}.$$

Then we have

$$G_p(F^{(\Lambda_2)}\mathbf{W}, \Lambda_2; E) = \frac{E_p(\mathbf{W}, \Lambda_2; \frac{1}{\Lambda_2}(E-1))}{\tilde{E}_p(\mathbf{W}, \Lambda_2; E)}$$

and

$$G_p(F^{(\Lambda_2)}\mathbf{W}, \Lambda_2; F) = \frac{E_p(\mathbf{W}, \Lambda_2; \frac{1}{\Lambda_2}(F-1))}{\tilde{E}_p(\mathbf{W}, \Lambda_2; F)}\,.$$

Moreover, we have $G_p(\mathbf{W}, \Lambda_2; E), G_p(\mathbf{W}, \Lambda_2; F) \in \mathbf{Z}_{(p)}[\mathbf{W}, \mathbf{U}/\Lambda_2, \Lambda_1, \Lambda_2][[X, Y]]$.

PROPOSITION 2.2.3 ([4, Proposition 4.13]). *Under the above notation, we have*

$$\tilde{E}_p(\mathbf{W}, \Lambda_3; G_p(\mathbf{A}, \Lambda_2; E)) = G_p\left(\left\langle \frac{\mathbf{A}}{\Lambda_3}, \mathbf{W} \right\rangle, \Lambda_2; E \right),$$

$$\tilde{E}_p(\mathbf{W}, \Lambda_3; G_p(\mathbf{A}, \Lambda_2; F)) = G_p\left(\left\langle \frac{\mathbf{A}}{\Lambda_3}, \mathbf{W} \right\rangle, \Lambda_2; F \right).$$

3. Extensions of commutative group schemes

In this section, we review some fundamental concepts on extensions of group schemes (cf. Serre [7]).

3.1.

DEFINITION 3.1.1. Let S be a scheme and G_i a commutative group scheme over S for $i = 1, 2, 3$. Then we say a sequence of the homomorphisms of commutative group schemes:

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

to be exact if the sequence is exact on flat site over S. We call G_2 an extension of G_3 by G_1.

DEFINITION 3.1.2. Let $0 \to G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \to 0$ and $0 \to G_1 \xrightarrow{i'} G_2' \xrightarrow{j'} G_3 \to 0$ be short exact sequences of commutative group schemes. If there exists a group homomorphism $\varphi : G_2 \to G_2'$ which makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \xrightarrow{i} & G_2 & \xrightarrow{j} & G_3 & \longrightarrow & 0 \\ & & \| & & \varphi\downarrow & & \| & & \\ 0 & \longrightarrow & G_1 & \underset{i'}{\longrightarrow} & G_2' & \underset{j'}{\longrightarrow} & G_3 & \longrightarrow & 0 \end{array}$$

commutative, then φ is an isomorphism. So in this case, we say the extensions G_2 and G_2' are isomorphic. Moreover, we denote by $\mathrm{Ext}^1(G_3, G_1)$ the set of isomorphy classes of extensions of G_3 by G_1.

PROPOSITION 3.1.3. *Let* $0 \to G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \to 0$ *be a short exact sequence of commutative group schemes. Then the following properties are equivalent*:

(1) $G_2 \simeq G_1 \times_S G_3$.
(2) *There exists a group scheme homomorphism* $s : G_2 \to G_1$ *such that* $s \circ i = id_{G_1}$.
(3) *There exists a group scheme homomorphism* $t : G_3 \to G_2$ *such that* $j \circ t = id_{G_3}$.

A short exact sequence is said to be split if one of the above conditions is satisfied.

3.2. pull-back and push-down

DEFINITION 3.2.1. For $0 \to G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \to 0 \in \mathrm{Ext}^1(G_3, G_1)$ and a group scheme homomorphism $f : G_3' \to G_3$, we define $f^*(G_2)$ by

$$f^*(G_2) := G_2 \times_{G_3} G_3' .$$

Then we have the exact sequence:

$$0 \to G_1 \to f^*(G_2) \to G_3' \to 0$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \longrightarrow & f^*(G_2) & \longrightarrow & G_3' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & G_1 & \underset{i}{\longrightarrow} & G_2 & \underset{j}{\longrightarrow} & G_3 & \longrightarrow & 0\,. \end{array}$$

Thus f induces the map $f^* : \mathrm{Ext}^1(G_3, G_1) \to \mathrm{Ext}^1(G_3', G_1)$. We call $f^*(G_2)$ the pull-back of G_2 by f.

DEFINITION 3.2.2. For $0 \to G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \to 0 \in \mathrm{Ext}^1(G_3, G_1)$ and a group scheme homomorphism $g : G_1 \to G_1'$, we define $g_*(G_2)$ by

$$g_*(G_2) := (G_1' \times_S G_2)/\{(g(x), -i(x))|x \in G_1\}\,.$$

Then we have the exact sequence:

$$0 \to G_1' \to g_*(G_2) \to G_3 \to 0$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \xrightarrow{i} & G_2 & \xrightarrow{j} & G_3 & \longrightarrow & 0 \\ & & {\scriptstyle g}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & G_1' & \longrightarrow & g_*(G_2) & \longrightarrow & G_3 & \longrightarrow & 0. \end{array}$$

Thus g induces the map $g_* : \mathrm{Ext}^1(G_3, G_1) \to \mathrm{Ext}^1(G_3, G_1')$. We call $g_*(G_2)$ the push-down of G_2 by g.

PROPOSITION 3.2.3. *For* $0 \to G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \to 0 \in \mathrm{Ext}^1(G_3, G_1)$ *and group scheme homomorphisms* $f : G_3' \to G_3$ *and* $g : G_1 \to G_1'$, *we have* $f^*(g_*(G_2)) \simeq g_*(f^*(G_2))$.

3.3. Group structure of the extensions

DEFINITION 3.3.1. Let $G_2, G_2' \in \mathrm{Ext}^1(G_3, G_1)$. Then we have the following diagram:

$$\begin{array}{ccccccccc} & & & & & & G_3 & & \\ & & & & & & \downarrow{\scriptstyle \Delta_{G_3}} & & \\ 0 & \longrightarrow & G_1 \times_S G_1 & \longrightarrow & G_2 \times_S G_2' & \longrightarrow & G_3 \times_S G_3 & \longrightarrow & 0 \\ & & {\scriptstyle m_{G_1}}\downarrow & & & & & & \\ & & G_1 & & & & & & \end{array}$$

with the diagonal morphism Δ_{G_3}, the addition m_{G_1} and the exact horizontal line. Then $\mathrm{Ext}^1(G_3, G_1)$ is an abelian group by

$$G_2 + G_2' := (\Delta_{G_3})^*(m_{G_1})_*(G_2 \times G_2') .$$

PROPOSITION 3.3.2. *Let* $0 \to G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \to 0 \in \mathrm{Ext}^1(G_3, G_1)$ *and* G_4 *a commutative group scheme. Then the following two sequences are exact*:

(1) $0 \to \mathrm{Hom}(G_4, G_1) \xrightarrow{i} \mathrm{Hom}(G_4, G_2) \xrightarrow{j} \mathrm{Hom}(G_4, G_3) \xrightarrow{\partial} \mathrm{Ext}^1(G_4, G_1)$
$\xrightarrow{i_*} Ext^1(G_4, G_2) \xrightarrow{j_*} \mathrm{Ext}^1(G_4, G_3)$ *with* $\partial(f) = f^*(G_2)$.

(2) $0 \to \mathrm{Hom}(G_3, G_4) \xrightarrow{j} \mathrm{Hom}(G_2, G_4) \xrightarrow{i} \mathrm{Hom}(G_1, G_4) \xrightarrow{\partial} \mathrm{Ext}^1(G_3, G_4)$
$\xrightarrow{j^*} \mathrm{Ext}^1(G_2, G_4) \xrightarrow{i^*} \mathrm{Ext}^1(G_1, G_4)$ *with* $\partial(g) = g_*(G_2)$.

4. Results on $\mathrm{Ext}^1(\mathcal{E}^{(\lambda_1,\cdots,\lambda_{n-1})}, \mathcal{G}^{(\lambda_n)})$

In [4], T. Sekiguchi and N. Suwa completely determined the extension groups $\mathrm{Ext}^1(\mathcal{E}^{(\lambda_1,\cdots,\lambda_{n-1})}, \mathcal{G}^{(\lambda_n)})$. In this section, we review some results of [4] which are needed in this paper.

4.1. Let $(A, \mathfrak{m})$ be a discrete valuation ring with the maximal ideal $\mathfrak{m}$ such that $\mathrm{ch}(\mathrm{Frac}(A)) = 0$ and $\mathrm{ch}(A/\mathfrak{m}) = p$. Then

$$\mathcal{G}^{(\lambda_1)} := \mathrm{Spec}\, A[X_1, 1/(1+\lambda_1 X_1)],\ \lambda_1 \in \mathfrak{m}\backslash\{0\}$$

is a group scheme over A with

$$\begin{array}{ll} \text{co-multiplication} & : X_1 \mapsto X_1 \otimes 1 + 1 \otimes X_1 + \lambda_1 X_1 \otimes X_1 , \\ \text{co-unit} & : X_1 \mapsto 0 , \\ \text{co-inverse} & : X_1 \mapsto -X_1/(1+\lambda_1 X_1) . \end{array}$$

Moreover, we have the following A-homomorphism $\alpha^{(\lambda_1)} : \mathcal{G}^{(\lambda_1)} \to \mathbf{G}_{m,A}$ by

$$\begin{array}{ccc} A[T, 1/T] & \to & A[X_1, 1/(1+\lambda_1 X_1)] \\ T & \mapsto & 1+\lambda_1 X_1 \end{array} .$$

In particular, for the generic point η and the special point s of $\mathrm{Spec}\, A$, $\alpha^{(\lambda_1)}$ induces $\alpha_\eta^{(\lambda_1)} : \mathcal{G}_\eta^{(\lambda_1)} \xrightarrow{\sim} \mathbf{G}_{m,K}$ and $\alpha_s^{(\lambda_1)} : \mathcal{G}_s^{(\lambda_1)} \xrightarrow{\sim} \mathbf{G}_{a,k}$ where $K := \mathrm{Frac}(A)$ and $k := A/\mathfrak{m}$.

4.2. Let $A_{\lambda_2} := A/\lambda_2 A$ for $\lambda_2 \in \mathfrak{m}\backslash\{0\}$ and $\iota : \mathrm{Spec}\, A_{\lambda_2} \hookrightarrow \mathrm{Spec}\, A$ the canonical closed immersion. Then the sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{G}^{(\lambda_2)} & \xrightarrow{\alpha^{(\lambda_2)}} & \mathbf{G}_{m,A} & \xrightarrow{r^{(\lambda_2)}} & \iota_*\mathbf{G}_{m,A_{\lambda_2}} & \longrightarrow 0 \\ & x & \longmapsto & 1+\lambda_2 x & & & \\ & & & t & \longmapsto & t \bmod \lambda_2 & \end{array} \qquad (*)$$

is exact on small flat site over Spec A and by Proposition 3.3.2 (1), we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\mathcal{G}^{(\lambda_1}, \mathcal{G}^{(\lambda_2)}) \xrightarrow{\alpha^{(\lambda_2)}} \mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \mathbf{G}_{m,A}) \xrightarrow{r^{(\lambda_2)}} \mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}})$$

$$\xrightarrow{\partial} \mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}) \xrightarrow{\alpha^{(\lambda_2)}} \mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathbf{G}_{m,A}) .$$

We have $\mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathbf{G}_{m,A}) = 0$ (cf. [4]), thus

$$\mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}) \simeq \mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}})/r^{(\lambda_2)}(\mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \mathbf{G}_{m,A})) .$$

Moreover, by $\mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \mathbf{G}_{m,A}) \simeq \{(1+\lambda_1 X_1)^n | n \in \mathbf{Z}\}$, we have

$$\mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}) \simeq \mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}})/\{(1+\lambda_1 X_1)^n \bmod \lambda_2 | n \in \mathbf{Z}\} .$$

The following theorem is crucial in the later argument.

THEOREM 4.2.1 ([5, Theorem 2.19.1]). *Let A be a $\mathbf{Z}_{(p)}$-algebra, and $\lambda \in A$ be a nilpotent. Then the group homomorphism*

$$\begin{array}{ccc} \widehat{W}(A)^{F^{(\lambda)}} & \longrightarrow & \mathrm{Hom}(\mathcal{G}^{(\lambda)}, \mathbf{G}_{m,A}) \\ \boldsymbol{a} & \longmapsto & E_p(\boldsymbol{a}, \lambda; X) \end{array}$$

and

$$\begin{array}{ccc} \widehat{W}(A)/F^{(\lambda)} & \longrightarrow & H_0^2(\mathcal{G}^{(\lambda)}, \mathbf{G}_{m,A}) \\ \boldsymbol{a} & \longmapsto & F_p(\boldsymbol{a}, \lambda; X, Y) \end{array}$$

are bijective.

By noting $E_p([\lambda_1], \lambda_1, X_1) = 1 + \lambda_1 X_1$, we have

$$\mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)}) \simeq \widehat{W}(A_{\lambda_2})^{F^{(\lambda_1)}} \big/ \langle [\lambda_1] \rangle .$$

This correspondence is given more explicitly as follows.

For $\boldsymbol{u}^1 \bmod \lambda_2 \in \widehat{W}(A_{\lambda_2})^{F^{(\lambda_1)}} \big/ \langle [\lambda_1] \rangle$, we put

$$D_1(X_1) := E_p(\boldsymbol{u}^1, \lambda_1; X_1) \bmod \lambda_2 .$$

Then D_1 is contained in $\mathrm{Hom}(\mathcal{G}^{(\lambda_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}})$, and $\partial D_1 \in \mathrm{Ext}^1(\mathcal{G}^{(\lambda_1)}, \mathcal{G}^{(\lambda_2)})$ is given by the pull-back of the exact sequence $(*)$ by D_1. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^{(\lambda_2)} & \longrightarrow & \partial D_1 & \longrightarrow & \mathcal{G}^{(\lambda_1)} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle D_1} & & \\ 0 & \longrightarrow & \mathcal{G}^{(\lambda_2)} & \xrightarrow[\alpha^{(\lambda_2)}]{} & \mathbf{G}_{m,A} & \xrightarrow[r^{(\lambda_2)}]{} & \iota_*\mathbf{G}_{m,A_{\lambda_2}} & \longrightarrow & 0. \end{array}$$

Let x_1 and t be local sections of $\mathcal{G}^{(\lambda_1)}$ and $\mathbf{G}_{m,A}$ respectively. Then by $D_1(x_1) = t \bmod \lambda_2$, there exists x_2 such that $t = D_1(x_1) + \lambda_2 x_2$ and we have

$$\partial D_1 = \text{the class of Spec}\, A\left[X_1, X_2, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}\right].$$

We put $\mathcal{E}^{(\lambda_1,\lambda_2;D_1)} := \operatorname{Spec} A\left[X_1, X_2, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}\right]$. Then it is an affine group scheme over A with

co-multiplication : $X_1 \mapsto X_1 \otimes 1 + 1 \otimes X_1 + \lambda_1 X_1 \otimes X_1,$

$$\begin{aligned} X_2 \mapsto\ & (X_2 \otimes 1)D_1(1 \otimes X_1) + D_1(X_1 \otimes 1)(1 \otimes X_2) + \lambda_2 X_2 \otimes X_2 \\ & + \frac{1}{\lambda_2}\{D_1(X_1 \otimes 1)D_1(1 \otimes X_1) - D_1(X_1 \otimes 1 + 1 \otimes X_1 \\ & \quad + \lambda_1 X \otimes X_1)\}, \end{aligned}$$

co-unit : $X_1 \mapsto 0,\ X_2 \mapsto 0$

co-inverse : $X_1 \mapsto -\dfrac{X_1}{1+\lambda_1 X_1}$,

$$X_2 \mapsto \frac{1}{\lambda_2}\left\{\frac{1}{D_1(X_1)+\lambda_2 X_2} - D_1\left(-\frac{X_1}{1+\lambda_1 X_1}\right)\right\}.$$

4.3. Replacing λ_2 by λ_3 in the exact sequence $(*)$ and apply the same argument to $\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}$, we have an exact sequence:

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \mathcal{G}^{(\lambda_3)}) &\xrightarrow{\alpha^{(\lambda_3)}} \operatorname{Hom}(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \mathbf{G}_{m,A}) \\ \xrightarrow{r^{(\lambda_3)}} \operatorname{Hom}(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}}) &\xrightarrow{\partial} \operatorname{Ext}^1(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \mathcal{G}^{(\lambda_3)}) \longrightarrow 0\,. \end{aligned}$$

By $\operatorname{Hom}(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \mathbf{G}_{m,A}) \simeq \{(1+\lambda_1 X_1)^n (D_1(X_1)+\lambda_2 X_2)^m | n, m \in \mathbf{Z}\}$, we have

$$\begin{aligned} &\operatorname{Ext}^1(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \mathbf{G}_{m,A}) \\ &\simeq \operatorname{Hom}(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}})/\{(1+\lambda_1 X_1)^n (D_1(X_1)+\lambda_2 X_2)^m \bmod \lambda_3 | n, m \in \mathbf{Z}\} \end{aligned}.$$

We put $\boldsymbol{b}_2^3 := \dfrac{1}{\lambda_2} F^{(\lambda_1)} \boldsymbol{u}^1$ and $U^2 := \begin{pmatrix} F^{(\lambda_1)} & -\langle \boldsymbol{b}_2^3, \cdot \rangle \\ 0 & F^{(\lambda_2)} \end{pmatrix}$. Then we have the following theorem.

THEOREM 4.3.1 (4, Theorem 5.1). *Let A be a $\mathbf{Z}_{(p)}$-algebra and $\lambda_1, \lambda_2, \lambda_3 \in A$. Suppose λ_1 and λ_2 are nilpotent in A_{λ_3}. Then the group homomorphism*

$$\begin{aligned} \operatorname{Ker}[U^2 : \widehat{W}(A_{\lambda_3})^2 \to \widehat{W}(A_{\lambda_3})^2] &\to \operatorname{Hom}(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}}) \\ (\boldsymbol{b}_1 \bmod \lambda_3, \boldsymbol{b}_2 \bmod \lambda_3) &\mapsto E_p(\boldsymbol{b}_1, \lambda_1; X_1) E_p\left(\boldsymbol{b}_2, \lambda_2; \frac{X_2}{D_1(X_1)}\right) \bmod \lambda_3 \end{aligned}.$$

is bijective.

In particular, under this correspondence, we have

$$([\lambda_1], \mathbf{0}) \mapsto 1 + \lambda_1 X_1$$

and

$$(\boldsymbol{u}^1, [\lambda_2]) \mapsto D_1(X_1) + \lambda_2 X_2 .$$

Therefore by Theorem 4.3.1, we have

$$\mathrm{Ext}^1(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \mathcal{G}^{(\lambda_3)}) \simeq \mathrm{Ker}[U^2 : \widehat{W}(A_{\lambda_3})^2 \to \widehat{W}(A_{\lambda_3})^2]/\langle([\lambda_1], \mathbf{0}), (\boldsymbol{u}^1, [\lambda_2])\rangle .$$

This correspondence is also given more explicitly as follows.

For $(\boldsymbol{u}_1^2, \boldsymbol{u}_2^2) \in \mathrm{Ker}[U^2 : \widehat{W}(A_{\lambda_3})^2 \to \widehat{W}(A_{\lambda_3})^2]$, we put

$$D_2(X_1, X_2) := E_p(\boldsymbol{u}_1^2, \lambda_1; X_1) E_p\left(\boldsymbol{u}_2^2, \lambda_2; \frac{X_2}{D_1(X_1)}\right) \bmod \lambda_3 .$$

Then D_2 is contained in $\mathrm{Hom}(\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}, \iota_* \mathbf{G}_{m,A_{\lambda_3}})$ and ∂D_2 is the pull-back of the exact sequence $(*)$ by D_2. Therefore, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G}^{(\lambda_3)} & \longrightarrow & \partial D_2 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle D_2} & & \\ 0 & \longrightarrow & \mathcal{G}^{(\lambda_3)} & \underset{\alpha^{(\lambda_3)}}{\longrightarrow} & \mathbf{G}_{m,A} & \underset{r^{(\lambda_3)}}{\longrightarrow} & \iota_* \mathbf{G}_{m,A_{\lambda_3}} & \longrightarrow & 0, \end{array}$$

where

$$\begin{aligned} \partial D_2 &= \text{the class of} \\ &\mathrm{Spec}\, A\left[X_1, X_2, X_3, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \frac{1}{D_2(X_1, X_2)+\lambda_3 X_3}\right]. \end{aligned}$$

We put

$$\begin{aligned} &\mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)} \\ &\quad := \mathrm{Spec}\, A\left[X_1, X_2, X_3, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \frac{1}{D_2(X_1, X_2)+\lambda_3 X_3}\right]. \end{aligned}$$

Then it is an affine group scheme over A with

$$\begin{aligned} \text{co-multiplication} : X_1 &\mapsto X_1 \otimes 1 + 1 \otimes X_1 + \lambda_1 X_1 \otimes X_1 , \\ X_2 &\mapsto (X_2 \otimes 1) D_1(1 \otimes X_1) + D_1(X_1 \otimes 1)(1 \otimes X_2) + \lambda_2 X_2 \otimes X_2 \\ &\quad + \frac{1}{\lambda_2}\{D_1(X_1 \otimes 1) D_1(1 \otimes X_1) - D_1(X_1 \otimes 1 + 1 \otimes X_1 \\ &\qquad + \lambda_1 X \otimes X_1)\} , \\ X_3 &\mapsto (X_3 \otimes 1) D_2(1 \otimes X_1, 1 \otimes X_2) + (1 \otimes X_3) D_2(X_1 \otimes 1, X_2 \otimes 1) \\ &\quad + \lambda_3 X_3 \otimes X_3 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\lambda_3}\Big[D_2(X_1\otimes 1, X_2\otimes 1)D_2(1\otimes X_1, 1\otimes X_2)\\
&\qquad - D_2\Big(X_1\otimes 1+1\otimes X_1+\lambda_1 X_1\otimes X_1,\ (X_1\otimes 1)D_1(1\otimes X_1)\\
&\qquad\qquad + D_1(X_1\otimes 1)(1\otimes X_2)+\lambda_2 X_2\otimes X_2\\
&\qquad\qquad + \frac{1}{\lambda_2}\{D_1(X_1\otimes 1)D_1(1\otimes X_1)\\
&\qquad\qquad - D_1(X_1\otimes 1+1\otimes X_1+\lambda_1 X_1\otimes X_1)\}\Big)\Big]
\end{aligned}
$$

co-unit $\quad : X_1 \mapsto 0,\ X_2 \mapsto 0,\ X_3 \mapsto 0$

co-inverse $\quad : X_1 \mapsto -\dfrac{X_1}{1+\lambda_1 X_1},$

$$
\begin{aligned}
X_2 &\mapsto \frac{1}{\lambda_2}\left\{\frac{1}{D_1(X_1)+\lambda_2 X_2} - D_1\Big(-\frac{X_1}{1+\lambda_1 X_1}\Big)\right\},\\
X_3 &\mapsto \frac{1}{\lambda_3}\left[\frac{1}{D_2(X_1, X_2)+\lambda_3 X_3}\right.\\
&\quad \left. - D_2\Big(-\frac{1}{1+\lambda_1 X_1}, \frac{1}{\lambda_2}\Big\{\frac{1}{D_1(X_1)+\lambda_2 X_2} - D_1\Big(\frac{1}{1+\lambda_1 X_1}\Big)\Big\}\Big)\right].
\end{aligned}
$$

5. Horikawa's theorem

Horikawa determined the group $\mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)})$ in his master thesis ([2]). Here we state Horikawa's theorem and give the proof.

5.1. Let $(A, \mathfrak{m})$ be a discrete valuation ring and $\lambda_0 \in \mathfrak{m}\backslash\{0\}$. Then by T. Sekiguchi, F. Oort and N. Suwa [3],

$$
\mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)}) \simeq \left\{\frac{1}{\lambda_1}\{(1+\lambda_0 X_0)^n - 1\}\ \middle|\ \lambda_1 \middle| \binom{n}{i}{\lambda_0}^i,\ \text{for } i = 1, 2, \dots, n\right\}.
$$

Moreover, we define the map $\Theta^{(\lambda_0,\lambda_1,\lambda_2)}$ by

$$
\begin{aligned}
\Theta^{(\lambda_0,\lambda_1,\lambda_2)} : \widehat{W}(A_{\lambda_1})^{F^{(\lambda_0)}}/\langle[\lambda_0]\rangle \times \widehat{W}(A_{\lambda_2})/\langle\boldsymbol{\ell}_2\rangle &\to \widehat{W}(A_{\lambda_2})\\
(\boldsymbol{a}_1, \boldsymbol{a}_2) &\mapsto F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle
\end{aligned}
$$

with $\ell_1, \ell_2 \in \mathbf{Z}$, $\boldsymbol{\ell}_1 := \ell_1[\lambda_0] \equiv \mathbf{0} \bmod \lambda_1$, $\boldsymbol{\ell}_2 := \ell_2[\lambda_0] - \big\langle \frac{\boldsymbol{\ell}_1}{\lambda_1}, \boldsymbol{u}^1\big\rangle$ and $\boldsymbol{b}_1^2 := \frac{1}{\lambda_1}F^{(\lambda_0)}\boldsymbol{a}_1$. Then $\Theta^{(\lambda_0,\lambda_1,\lambda_2)}$ is a homomorphism by Sekiguchi-Suwa [6] and we have the following theorem.

THEOREM 5.1.1 (Horikawa). *The group homomorphism*

$$
\Psi^{(\lambda_0,\lambda_1,\lambda_2)} : \mathrm{Ker}\,\Theta^{(\lambda_0,\lambda_1,\lambda_2)} \to \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)})
$$

defined by

$$(\boldsymbol{a}_1, \boldsymbol{a}_2) \mapsto \text{the class of}$$

$$\operatorname{Spec} A\left[X_0, X_1, X_2, \frac{1}{1+\lambda_0 X_0}, \frac{1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1}, \frac{1}{E_p(\boldsymbol{a}_2, \lambda_0; X_0) D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}\right) + \lambda_2 X_2}\right]$$

is bijective.

5.2. Here we give the proof of Theorem 5.1.1.

Let (x_1, x_2) be a local section of $\mathcal{E}^{(\lambda_1, \lambda_2; D_1)}$ and t a local section of $\mathbf{G}_{m,A}$. Let ρ and β be the homomorphisms defined by

$$\begin{aligned} \rho : \mathcal{E}^{(\lambda_1, \lambda_2; D_1)} &\to \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A} \\ (x_1, x_2) &\mapsto (x_1, D_1(x_1) + \lambda_2 x_2) \end{aligned}$$

and

$$\begin{aligned} \beta : \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A} &\to \iota_* \mathbf{G}_{m, A_{\lambda_2}} \\ (x_1, t) &\mapsto D_1(x_1)^{-1} t \bmod \lambda_2 \end{aligned} .$$

Then the sequence

$$0 \to \mathcal{E}^{(\lambda_1, \lambda_2; D_1)} \xrightarrow{\rho} \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A} \xrightarrow{\beta} \iota_* \mathbf{G}_{m, A_{\lambda_2}} \to 0 \qquad (**)$$

is exact on small flat site over Spec A and we have the following exact sequence:

$$\begin{aligned} 0 &\to \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1, \lambda_2; D_1)}) \xrightarrow{\rho} \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \xrightarrow{\beta} \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m, A_{\lambda_2}}) \\ &\xrightarrow{\partial} \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1, \lambda_2; D_1)}) \xrightarrow{\rho} \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \xrightarrow{\beta} \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m, A_{\lambda_2}}) \\ &\to \cdots . \end{aligned}$$

Combining this long exact sequence and the isomorphisms given in subsection 4.2, we have a diagram with the second horizontal exact line:

$$\begin{array}{ccc} \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) & \xrightarrow{\varphi_1} & \widehat{W}(A_{\lambda_2})^{F^{(\lambda_0)}} \\ \| & & \wr\downarrow \psi_1 \\ \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) & \xrightarrow[\beta]{} & \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \iota_* \mathbf{G}_{m, A_{\lambda_2}}) \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\varphi_2} & \operatorname{Ker} \Theta^{(\lambda_0, \lambda_1, \lambda_2)} & \xrightarrow{\varphi_3} & \widehat{W}(A_{\lambda_1})^{F^{(\lambda_0)}} / \langle [\lambda_0] \rangle \\ & \downarrow \Psi^{(\lambda_0, \lambda_1, \lambda_2)} & & \wr\downarrow \Psi^{(\lambda_0, \lambda_1)} \\ \xrightarrow[\partial]{} & \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1, \lambda_2; D_1)}) & \xrightarrow[\rho^*]{} & \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \end{array}$$

$$\xrightarrow{\varphi_4} \quad \widehat{W}(A_{\lambda_2})/F^{(\lambda_0)}$$
$$\wr\downarrow\psi_2$$
$$\xrightarrow[\beta^*]{} \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}})$$

where

$$\begin{aligned}
&\varphi_1 = \psi_1^{-1} \circ \beta \\
&\varphi_2 : \boldsymbol{a}_2 \mapsto (\mathbf{0}, \boldsymbol{a}_2) \\
&\varphi_3 : (\boldsymbol{a}_1, \boldsymbol{a}_2) \mapsto \boldsymbol{a}_1 \\
&\varphi_4 : \boldsymbol{a}_1 \mapsto -\langle \boldsymbol{b}_1^2, \boldsymbol{u}^1 \rangle
\end{aligned}$$

and $\psi_1$is an isomorphism. By the following lemma, ψ_2 is also an isomorphism.

LEMMA 5.2.1. $H_0^2(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}}) \simeq \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}})$.

PROOF. By

$$\operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_0}}) \simeq \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}_{A_{\lambda_2}}, \mathbf{G}_{m,A_{\lambda_2}})$$

with $\mathcal{G}^{(\lambda_0)}_{A_{\lambda_2}} = \operatorname{Spec} A_{\lambda_2}[X_0, 1/(1+\lambda_0 X_0)]$, it is sufficient to prove $\operatorname{Pic}(\mathcal{G}^{(\lambda_0)}_{A_{\lambda_2}}) = 0$.

We put $X := \mathcal{G}^{(\lambda_0)}_{A_{\lambda_2}}$ and let $\mathfrak{m}$ be the maximal ideal of A_{λ_2}. We have the exact sequence

$$0 \to \mathfrak{m}(\mathcal{O}_X/\mathfrak{m}^2\mathcal{O}_X) \to \mathcal{O}_X/\mathfrak{m}^2\mathcal{O}_X \to \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X \to 0$$

and it induces the exact sequences

$$1 \to 1 + \mathfrak{m}(\mathcal{O}_X/\mathfrak{m}^2\mathcal{O}_X) \to (\mathcal{O}_X/\mathfrak{m}^2\mathcal{O}_X)^\times \to (\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X)^\times \to 1$$

and

$$H^1(X, 1 + \mathfrak{m}(\mathcal{O}_X/\mathfrak{m}^2\mathcal{O}_X)) \to H^1(X, (\mathcal{O}_X/\mathfrak{m}^2\mathcal{O}_X)^\times) \to H^1(X, (\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X)^\times).$$

By $H^1(X, 1+\mathfrak{m}(\mathcal{O}_X/\mathfrak{m}^2\mathcal{O}_X)) \simeq H^1(X, \mathcal{O}_X/\mathfrak{m}\mathcal{O}_X) = 0$ and $H^1(X, (\mathcal{O}_X/\mathfrak{m}\mathcal{O}_X)^\times) = 0$, we have $H^1(X, (\mathcal{O}_X/\mathfrak{m}^2\mathcal{O}_X)^\times) = 0$.

For $n \in \mathbf{Z}$ with $n \geq 1$, we have $H^1(X, (\mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X)^\times) = 0$ by induction on n. Since A_{λ_2} is an Artinian local ring, there exists $r \in \mathbf{Z}$ with $r \geq 1$ such that $\mathfrak{m}^r\mathcal{O}_X = 0$. Thus we have $H^1(X, \mathcal{O}_X^\times) = \operatorname{Pic}(X) = 0$. □

Hence, in order to prove Theorem 5.1.1, it is sufficient to show that the first horizontal line of the above diagram is exact and the diagram is commutative. First, we show the exactness. By

$$\operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \simeq \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)}) \times \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathbf{G}_{m,A}),$$

we have

$$\begin{aligned}
&\beta\left(\frac{1}{\lambda_1}\{E_p(\boldsymbol{\ell}_1, \lambda_0; X_0) - 1\}, E_p(\ell_2[\lambda_0], \lambda_0; X_0)\right) \\
&= D_1\left(\frac{1}{\lambda_1}\{E_p(\boldsymbol{\ell}_1, \lambda_0; X_0) - 1\}\right)^{-1} E_p(\ell_2[\lambda_0], \lambda_0; X_0) \\
&= E_p\left(-\boldsymbol{u}^1, \lambda_1; \frac{1}{\lambda_1}\{E_p(\boldsymbol{\ell}_1, \lambda_0; X_0) - 1\}\right) E_p(\ell_2[\lambda_0], \lambda_0; X_0) \\
&= G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))\tilde{E}_p(-\boldsymbol{u}^1, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0)) E_p(\ell_2[\lambda_0], \lambda_0; X_0) \\
&= G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0)) E_p\left(-\left\langle \frac{\boldsymbol{\ell}_1}{\lambda_1}, \boldsymbol{u}^1\right\rangle, \lambda_0; X_0\right) E_p(\ell_2[\lambda_0], \lambda_0; X_0) \\
&\equiv E_p(\boldsymbol{\ell}_2, \lambda_0; X_0) \bmod \lambda_2 .
\end{aligned}$$

Therefore we have $\operatorname{Im}\varphi_1 = \operatorname{Ker}\varphi_2$. The equality $\operatorname{Im}\varphi_2 = \operatorname{Ker}\varphi_3$ is trivial.

For $(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \operatorname{Ker}\Theta^{(\lambda_0,\lambda_1,\lambda_2)}$, we have

$$\begin{aligned}
(\varphi_4 \circ \varphi_3)(\boldsymbol{a}_1, \boldsymbol{a}_2) &= -\langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle \\
&\equiv -F^{(\lambda_0)}\boldsymbol{a}_2 \bmod \lambda_2 .
\end{aligned}$$

Therefore, we have $\operatorname{Im}\varphi_3 \subset \operatorname{Ker}\varphi_4$. On the other hand, if there exists $\boldsymbol{a}_1 \in \widehat{W}(A_{\lambda_1})^{F^{(\lambda_0)}}/\langle[\lambda_0]\rangle$ such that $\varphi_4(\boldsymbol{a}_1) = \boldsymbol{0}$, then there exists $\boldsymbol{a}_2 \in \widehat{W}(A_{\lambda_2})$ such that $F^{(\lambda_0)}\boldsymbol{a}_2 \equiv \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle$ mod λ_2 and we have $\varphi_3(\boldsymbol{a}_1, \boldsymbol{a}_2) = \boldsymbol{a}_1$. Therefore, we have $\operatorname{Im}\varphi_3 = \operatorname{Ker}\varphi_4$.

Next we show the commutativity of the diagram. It is clear that $\psi_1 \circ \varphi_1 = \beta$. For $\boldsymbol{a}_2 \in \widehat{W}(A)^{F^{(\lambda_0)}}$, we have

$$(\Psi^{(\lambda_0,\lambda_1,\lambda_2)} \circ \varphi_2)(\boldsymbol{a}_2) = \text{ the class of } \operatorname{Spec} A\left[X_0, X_1, X_2, \frac{1}{1+\lambda_0 X_0}, \frac{1}{1+\lambda_1 X_1}, \frac{1}{E_p(\boldsymbol{a}_2, \lambda_0; X_0)D_1(X_1) + \lambda_2 X_2}\right]$$

and

$$\psi_1(\boldsymbol{a}_2) = E_p(\boldsymbol{a}_2, \lambda_0; X_0) \bmod \lambda_2 .$$

We put $E := E_p(\boldsymbol{a}_2, \lambda_0; X_0)$ mod λ_2. Then ∂E is the pull-back of $(**)$ by E and we have the commutative diagram with exact horizontal lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} & \longrightarrow & \partial E & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow E & & \\
0 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} & \underset{\rho}{\longrightarrow} & \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A} & \underset{\beta}{\longrightarrow} & \iota_*\mathbf{G}_{m,A_{\lambda_2}} & \longrightarrow & 0 .
\end{array}$$

Let x_1, t and x_0 be local sections of $\mathcal{G}^{(\lambda_1)}$, $\mathbf{G}_{m,A}$ and $\mathcal{G}^{(\lambda_0)}$, respectively. Then we have

$$E(x_0) \equiv D_1(x_1)^{-1} t \mod \lambda_2 .$$

Since it is equivalent to $E(x_0)D_1(x_1) \equiv t \mod \lambda_2$, there exists x_2 such that

$$t = E(x_0)D_1(x_1) + \lambda_2 x_2 .$$

Therefore we have

$$\partial E = \text{the class of Spec } A\left[\, X_0, X_1, X_2, \frac{1}{1+\lambda_0 X_0}, \frac{1}{1+\lambda_1 X_1}, \frac{1}{E_p(\boldsymbol{a}_2, \lambda_0; X_0)D_1(X_1) + \lambda_2 X_2}\right]$$

and $\partial \circ \psi_1 = \Psi^{(\lambda_0,\lambda_1,\lambda_2)} \circ \varphi_2$.

By

$$\mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \simeq \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)}) ,$$

ρ^* is the push-down map $\pi_* : \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)}) \to \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)})$ by the canonical projection $\pi : \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \to \mathcal{G}^{(\lambda_1)}$ and we have a commutative diagram with exact horizontal lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0 \\
 & & {\scriptstyle \pi}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{G}^{(\lambda_1)} & \longrightarrow & \pi_*\mathcal{E} & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0 .
\end{array}$$

Let $\overline{\mathcal{E}}$ be the class of $\mathcal{E}$. Then we have

$$\rho_*(\overline{\mathcal{E}}) = \overline{\pi_*\mathcal{E}} = \text{the class of Spec } A\left[X_0, X_1, \frac{1}{1+\lambda_0 X_0}, \frac{1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1}\right].$$

Therefore we have $\rho^* \circ \Psi^{(\lambda_0,\lambda_1,\lambda_2)} = \Psi^{(\lambda_0,\lambda_1)} \circ \varphi_3$.

Let $\mathcal{E} \in \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)})$. Then the exact sequence

$$0 \to \mathcal{G}^{(\lambda_1)} \to \mathcal{E} \to \mathcal{G}^{(\lambda_0)} \to 0$$

induces the exact sequence

$$0 \to \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A} \to \mathcal{E} \times \mathbf{G}_{m,A} \to \mathcal{G}^{(\lambda_0)} \to 0 ,$$

and we have a commutative diagram with exact horizontal lines:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A} & \longrightarrow & \mathcal{E} \times \mathbf{G}_{m,A} & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0 \\
 & & {\scriptstyle \beta}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \iota_*\mathbf{G}_{m,A_{\lambda_2}} & \longrightarrow & \beta_*(\mathcal{E} \times \mathbf{G}_{m,A}) & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0 .
\end{array}$$

Let $F : \mathcal{G}^{(\lambda_0)} \times \mathcal{G}^{(\lambda_0)} \to \iota_* \mathbf{G}_{m,A_{\lambda_2}}$ be the cocycle on $\mathcal{G}^{(\lambda_0)} \times \iota_* \mathbf{G}_{m,A_{\lambda_2}}$ giving $\beta_*(\mathcal{E} \times \mathbf{G}_{m,A})$. Then for local sections (x, t), $(x', t') \in \mathcal{G}^{(\lambda_0)} \times \iota_* \mathbf{G}_{m,A_{\lambda_2}}$, we have

$$(x, t)(x', t') = (x + x' + \lambda_0 x x', tt' F(x, x')) .$$

In particular, $(x, 1)(x', 1) = (x + x' + \lambda_0 x x', F(x, x'))$.

Since the group structure of $\mathcal{E}$ is induced by the group scheme homomorphism:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathbf{G}_{m,A} \times \mathbf{G}_{m,A} \\ (x_0, x_1) & \longmapsto & (1 + \lambda_0 x_0, E_p(\boldsymbol{a}_1, \lambda_0; x_0) + \lambda_1 x_1) \end{array} ,$$

we have

$$(x_0, x_1) = (x_0, 0) + \left(0, \frac{x_1}{E_p(\boldsymbol{a}_1, \lambda_0; x_0)}\right)$$

with a local section $(x_0, x_1) \in \mathcal{E}$ and

$$\begin{array}{ccc} \mathcal{E} \times \mathbf{G}_{m,A} & \longrightarrow & \beta_*(\mathcal{E} \times \mathbf{G}_{m,A}) \\ ((x_0, x_1), t) & \longmapsto & \left(x_0, D_1\left(\dfrac{x_1}{E_p(\boldsymbol{a}_1, \lambda_0; x_0)}\right)^{-1} t\right) \end{array} ,$$

with a local section $t \in \mathbf{G}_{m,A}$.

By the group scheme homomorphism $\mathcal{E} \to \mathbf{G}_{m,A} \times \mathbf{G}_{m,A}$, we have

$$((x_0, 0), 1) \mapsto ((1 + \lambda_0 x_0, E_p(\boldsymbol{a}_1, \lambda_0; x_0)), 1) .$$

We put $((X_0, X_1), 1) := ((x, 0), 1)((x', 0), 1)$. Then we have

$$\begin{aligned} &((1 + \lambda_0 X_0, E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1), 1) \\ &\quad = ((1 + \lambda_0(x + x' + \lambda_0 x x'), E_p(\boldsymbol{a}_1, \lambda_0; x) E_p(\boldsymbol{a}_1, \lambda_0; x')), 1) \end{aligned}$$

and

$$\begin{aligned} X_0 &= x + x' + \lambda_0 x x' \\ X_1 &= \frac{1}{\lambda_1}\{E_p(\boldsymbol{a}_1, \lambda_0; x) E_p(\boldsymbol{a}_1, \lambda_0; x') - E_p(\boldsymbol{a}_1, \lambda_0; X_0)\} . \end{aligned}$$

The cocycle F is given by

$$\begin{aligned} F(x, x') &= D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}\right)^{-1} \\ &= E_p\left(-\boldsymbol{u}^1, \lambda_1; \frac{1}{\lambda_1}\{F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x') - 1\}\right) \\ &= G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) \tilde{E}_p(-\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) \\ &\equiv F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}^1 \rangle, \lambda_0; x, x') \bmod \lambda_2 . \end{aligned}$$

Therefore we have $\beta_* \circ \Psi^{(\lambda_0,\lambda_1)} = \psi_2 \circ \varphi_4$. Hence, Theorem 5.1.1 is proved.

6. The main theorem

In this section, we state our main theorem (Theorem 6.2.1) and give the proof.

6.1. Let (x_1, x_2, x_3) be a local section of $\mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)}$ and t a local section of $\mathbf{G}_{m,A}$. We define group scheme homomorphisms ρ and β by

$$\begin{aligned}\rho : \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)} &\to \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A}\\ (x_1, x_2, x_3) &\mapsto ((x_1, x_2), D_2(x_1, x_2) + \lambda_3 x_3)\end{aligned}$$

and

$$\begin{aligned}\beta : \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A} &\to \iota_*\mathbf{G}_{m,A_{\lambda_3}}\\ ((x_1, x_2), t) &\mapsto D_2(x_1, x_2)^{-1} t \bmod \lambda_3\end{aligned}$$

Then the sequence

$$0 \to \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)} \xrightarrow{\rho} \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A} \xrightarrow{\beta} \iota_*\mathbf{G}_{m,A_{\lambda_3}} \to 0 \qquad (***)$$

is exact on small flat site over Spec A and we have an exact sequence

$$\begin{aligned}0 &\longrightarrow \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)}) \xrightarrow{\rho} \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A})\\ &\xrightarrow{\beta} \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}}) \xrightarrow{\partial} \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)})\\ &\xrightarrow{\rho} \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A}) \xrightarrow{\beta} \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}}) \to \cdots .\end{aligned}$$

6.2. For $\mathcal{E} \in \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)})$, by Theorem 5.1.1 we have

$$\begin{aligned}\mathcal{E} = \mathrm{Spec}\, A\Bigg[&X_0, X_1, X_2, \frac{1}{1+\lambda_0 X_0}, \frac{1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1},\\ &\frac{1}{E_p(\boldsymbol{a}_2, \lambda_0; X_0) D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}\right) + \lambda_2 X_2}\Bigg]\end{aligned}$$

with $(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \mathrm{Ker}\, \Theta^{(\lambda_0,\lambda_1,\lambda_2)}$.

We define the map $\Theta^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)}$ by

$$\begin{aligned}\mathrm{Ker}\, \Theta^{(\lambda_0,\lambda_1,\lambda_2)} \times \widehat{W}(A_{\lambda_3})/\langle \boldsymbol{\ell}_3 \rangle &\to \widehat{W}(A_{\lambda_3})\\ ((\boldsymbol{a}_1, \boldsymbol{a}_2), \boldsymbol{a}_3) &\mapsto F^{(\lambda_0)}\boldsymbol{a}_3 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2 \rangle - \langle \boldsymbol{b}_1^3, \boldsymbol{u}_2^2 \rangle\end{aligned}$$

with $\ell_1, \ell_2, \ell_3 \in \mathbf{Z}$, $\boldsymbol{\ell}_1 \equiv \mathbf{0} \bmod \lambda_1$, $\boldsymbol{\ell}_2 \equiv \mathbf{0} \bmod \lambda_2$, $\boldsymbol{\ell}_3 := \ell_3[\lambda_0] - \left\langle \frac{\boldsymbol{\ell}_1}{\lambda_1}, \boldsymbol{u}_1^2 \right\rangle - \left\langle \frac{\boldsymbol{\ell}_2}{\lambda_2}, \boldsymbol{u}_2^2 \right\rangle$ and $\boldsymbol{b}_1^3 = \frac{1}{\lambda_2}(F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1 \rangle)$. Then $\Theta^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)}$ is a homomorphism by Sekiguchi-Suwa [6]. Now we can state our main theorem.

THEOREM 6.2.1. *The group homomorphism*

$$\Psi^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} : \operatorname{Ker}\Theta^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} \to \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)})$$

defined by

$(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) \mapsto$ *the class of*

$$\operatorname{Spec} A\Bigg[X_0, X_1, X_2, X_3, \frac{1}{1+\lambda_0 X_0}, \frac{1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1}, \frac{1}{E_p(\boldsymbol{a}_2, \lambda_0; X_0) D_1(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}) + \lambda_2 X_2}, \frac{1}{E_p(\boldsymbol{a}_3, \lambda_0; X_0) D_2(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}, \frac{X_2}{E_p(\boldsymbol{a}_2,\lambda_0;X_0)}) + \lambda_3 X_3}\Bigg]$$

is bijective.

6.3. In order to prove Theorem 6.2.1, we use the following diagram of group homomorphisms

$$\begin{array}{ccc}
\operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A}) & \xrightarrow{\varphi_1} & \widehat{W}(A_{\lambda_3})^{F^{(\lambda_0)}} \\
\| & & \wr\downarrow \psi_1 \\
\operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A}) & \xrightarrow[\beta]{} & \operatorname{Hom}(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}})
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\varphi_2} & \operatorname{Ker}\Theta^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} & \xrightarrow{\varphi_3} & \operatorname{Ker}\Theta^{(\lambda_0,\lambda_1,\lambda_2)} \\
& \downarrow \Psi^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} & & \wr\downarrow \Psi^{(\lambda_0,\lambda_1,\lambda_2)} \\
\xrightarrow[\partial]{} & \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)}) & \xrightarrow[\rho^*]{} & \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A})
\end{array}$$

$$\begin{array}{cc}
\xrightarrow{\varphi_4} & \widehat{W}(A_{\lambda_3})/F^{(\lambda_0)} \\
& \wr\downarrow \psi_2 \\
\xrightarrow[\beta^*]{} & \operatorname{Ext}^1(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_3}})
\end{array}$$

where

$$\begin{aligned}
&\varphi_1 = \psi_1^{-1} \circ \beta \\
&\varphi_2 : \boldsymbol{a}_3 \mapsto (\mathbf{0}, \mathbf{0}, \boldsymbol{a}_3) \\
&\varphi_3 : (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) \mapsto (\boldsymbol{a}_1, \boldsymbol{a}_2) \\
&\varphi_4 : (\boldsymbol{a}_1, \boldsymbol{a}_2) \mapsto -\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2 \rangle - \langle \boldsymbol{b}_1^3, \boldsymbol{u}_2^2 \rangle\,.
\end{aligned}$$

As in the proof of Theorem 5.1.1, we will show that the diagram is exact and commutative.

DEFINITION 6.3.1. We put $E := E_p(\mathbf{W}_E, \Lambda_E; X_E)$ and $G := G_p(\mathbf{W}_G, \Lambda_G; X_G)$ and define $\tilde{p}(EG)$ by

$$\tilde{p}(EG) := E_p(\tilde{p}\mathbf{W}_E, \Lambda_E, X_E)G_p(\tilde{p}\mathbf{W}_G, \Lambda_G, X_G).$$

If $\mathbf{W}_G = \mathbf{0}$, then it coincides with the one in subsection 2.2.

Then we have

$$G_p(F^{(\Lambda_2)}\mathbf{W}, \Lambda_2; EG) = \frac{E_p(\mathbf{W}, \Lambda_2; \frac{1}{\Lambda_2}(EG-1))}{\tilde{E}_p(\mathbf{W}, \Lambda_2; EG)}$$

and

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; EG) = \tilde{E}_p(\mathbf{W}, \Lambda_2; E)\tilde{E}_p(\mathbf{W}, \Lambda_2; G).$$

Moreover, we put $F := F_p(\mathbf{W}_F, \Lambda_F; X_F, Y_F)$. Then by the definition

$$\tilde{p}(FG) := F_p(\tilde{p}\mathbf{W}_F, \Lambda_F, X_F, Y_F)G_p(\tilde{p}\mathbf{W}_G, \Lambda_G, X_G),$$

we have

$$G_p(F^{(\Lambda_2)}\mathbf{W}, \Lambda_2; FG) = \frac{E_p(\mathbf{W}, \Lambda_2; \frac{1}{\Lambda_2}(FG-1))}{\tilde{E}_p(\mathbf{W}, \Lambda_2; FG)}$$

and

$$\tilde{E}_p(\mathbf{W}, \Lambda_2; FG) = \tilde{E}_p(\mathbf{W}, \Lambda_2; F)\tilde{E}_p(\mathbf{W}, \Lambda_2; G).$$

Now we show the exactness of the first horizontal line of the diagram. By the exact sequence

$$0 \to \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)}) \to \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \to \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \iota_*\mathbf{G}_{m,A_{\lambda_2}})$$

and the canonical isomorphism

$$\mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)} \times \mathbf{G}_{m,A}) \simeq \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{G}^{(\lambda_1)}) \times \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathbf{G}_{m,A}),$$

we have

$$\mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)}) \simeq \left\{ (\Omega_1, \Omega_2) \;\middle|\; \begin{array}{l} \ell_1, \ell_2 \in \mathbf{Z} \\ \boldsymbol{\ell}_1 \equiv \mathbf{0} \bmod \lambda_1 \\ \boldsymbol{\ell}_2 \equiv \mathbf{0} \bmod \lambda_2 \end{array} \right\}$$

where $\varOmega_1 := \frac{1}{\lambda_1}\{E_p(\boldsymbol{\ell}_1, \lambda_0; X_0) - 1\}$ and $\varOmega_2 := \frac{1}{\lambda_2}\{E_p(\ell_2[\lambda_0], \lambda_0; X_0) - D_1(\varOmega_1)\}$. Therefore, we have

$$\begin{aligned}&\mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A})\\ &\quad\simeq \left\{ ((\varOmega_1, \varOmega_2), E_p(\ell_3[\lambda_0], \lambda_0; X_0)) \;\middle|\; \begin{array}{l} \ell_1, \ell_2, \ell_3 \in \mathbf{Z} \\ \boldsymbol{\ell}_1 \equiv \mathbf{0} \bmod \lambda_1 \\ \boldsymbol{\ell}_2 \equiv \mathbf{0} \bmod \lambda_2 \end{array} \right\}.\end{aligned}$$

For an element $((\varOmega_1, \varOmega_2), E_p(\ell_3[\lambda_0], \lambda_0; X_0)) \in \mathrm{Hom}(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A})$, we have the following equalities:

$$\begin{aligned}
&\beta((\varOmega_1, \varOmega_2))E_p(\ell_3[\lambda_0], \lambda_0; X_0)\\
&\equiv D_2(\varOmega_1, \varOmega_2)^{-1}E_p(\ell_3[\lambda_0], \lambda_0; X_0) \bmod \lambda_3\\
&= E_p\left(-\boldsymbol{u}_1^2, \lambda_1; \frac{1}{\lambda_1}\{E_p(\boldsymbol{\ell}_1, \lambda_0; X_0) - 1\}\right)\\
&\quad E_p\left(-\boldsymbol{u}_2^2, \lambda_2; \frac{1}{\lambda_2}\left[\frac{E_p(\ell_2[\lambda_0], \lambda_0; X_0)}{E_p(\boldsymbol{u}^1, \lambda_1; \frac{1}{\lambda_1}\{E_p(\boldsymbol{\ell}_1, \lambda_0; X_0) - 1\})} - 1\right]\right)E_p(\ell_3[\lambda_0], \lambda_0; X_0)\\
&= E_p\left(-\left\langle\frac{\boldsymbol{\ell}_1}{\lambda_1}, \boldsymbol{u}_1^2\right\rangle, \lambda_0; X_0\right)G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X))\\
&\quad E_p\left(-\boldsymbol{u}_2^2, \lambda_2; \frac{1}{\lambda_2}\{E_p(\boldsymbol{\ell}_2, \lambda_0; X_0)G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0)) - 1\}\right)\\
&\quad E_p(\ell_3[\lambda_0], \lambda_0; X_0)\\
&= E_p\left(\ell_3[\lambda_0] - \left\langle\frac{\boldsymbol{\ell}_1}{\lambda_1}, \boldsymbol{u}_1^2\right\rangle, \lambda_0; X_0\right)G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))\\
&\quad E_p\left(-\boldsymbol{u}_2^2, \lambda_2; \frac{1}{\lambda_2}\{E_p(\boldsymbol{\ell}_2, \lambda_0; X_0)G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0)) - 1\}\right)\\
&= E_p\left(\ell_3[\lambda_0] - \left\langle\frac{\boldsymbol{\ell}_1}{\lambda_1}, \boldsymbol{u}_1^2\right\rangle, \lambda_0; X_0\right)G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))\\
&\quad \tilde{E}_p\big(-\boldsymbol{u}_2^2, \lambda_2; E_p(\boldsymbol{\ell}_2, \lambda_0; X_0)G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))\big)\\
&\quad G_p\big(-F^{(\lambda_1)}\boldsymbol{u}_2^2, \lambda_2; E_p(\boldsymbol{\ell}_2, \lambda_0; X_0)G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))\big)\\
&\equiv E_p\left(\ell_3[\lambda_0] - \left\langle\frac{\boldsymbol{\ell}_1}{\lambda_1}, \boldsymbol{u}_1^2\right\rangle, \lambda_0; X_0\right)G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))\\
&\quad \tilde{E}_p(-\boldsymbol{u}_2^2, \lambda_2; E_p(\boldsymbol{\ell}_2, \lambda_0; X_0))\\
&\quad \tilde{E}_p(-\boldsymbol{u}_2^2, \lambda_2; G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))) \bmod \lambda_3\\
&= E_p\left(\ell_3[\lambda_0] - \left\langle\frac{\boldsymbol{\ell}_1}{\lambda_1}, \boldsymbol{u}_1^2\right\rangle, \lambda_0; X_0\right)G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))
\end{aligned}$$

$$E_p\left(-\left\langle\frac{\boldsymbol{\ell}_2}{\lambda_2}, \boldsymbol{u}_2^2\right\rangle, \lambda_0; X_0\right) G_p\left(\left\langle\frac{1}{\lambda_2} F^{(\lambda_1)}\boldsymbol{u}^1, \boldsymbol{u}_2^2\right\rangle, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0)\right)$$
$$= E_p(\boldsymbol{\ell}_3, \lambda_0; X_0) G_p(\langle \boldsymbol{b}_2^3, \boldsymbol{u}_2^2\rangle - F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; E_p(\boldsymbol{\ell}_1, \lambda_0; X_0))$$
$$\equiv E_p(\boldsymbol{\ell}_3, \lambda_0; X_0) \bmod \lambda_3 .$$

Therefore we have $\operatorname{Im}\varphi_1 = \operatorname{Ker}\varphi_2$. The equality $\operatorname{Im}\varphi_2 = \operatorname{Ker}\varphi_3$ is trivial. For $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) \in \operatorname{Ker}\Theta^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)}$, we have

$$(\varphi_4 \circ \varphi_3)(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = -\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle - \langle \boldsymbol{b}_1^3, \boldsymbol{u}_2^2\rangle$$
$$\equiv -F^{(\lambda_0)}\boldsymbol{a}_2 \bmod \lambda_3 .$$

Therefore, we have $\operatorname{Im}\varphi_3 \subset \operatorname{Ker}\varphi_4$. On the other hand, if there exists $(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \operatorname{Ker}\varphi_4$, then there exists $\boldsymbol{a}_3 \in \widehat{W}(A_{\lambda_3})$ such that $F^{(\lambda_0)}\boldsymbol{a}_3 \equiv \langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle + \langle \boldsymbol{b}_1^3, \boldsymbol{u}_2^2\rangle \bmod \lambda_3$ and we have $\varphi_3(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = (\boldsymbol{a}_1, \boldsymbol{a}_2)$. Therefore, we have $\operatorname{Im}\varphi_3 = \operatorname{Ker}\varphi_4$.

Next we show the commutativity of the diagram. It is clear that $\beta = \psi_1 \circ \varphi_1$. For $\boldsymbol{a}_3 \in \widehat{W}(A)^{F^{(\lambda_0)}}$, we have

$$(\Psi^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} \circ \varphi_2)(\boldsymbol{a}_3)$$
$$= \text{the class of } \operatorname{Spec} A\left[X_0, X_1, X_2, X_3, \frac{1}{1+\lambda_0 X_0}, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \frac{1}{E_p(\boldsymbol{a}_3, \lambda_0; X_0) D_2(X_1, X_2) + \lambda_3 X_3}\right]$$

and

$$\psi_1(\boldsymbol{a}_3) = E_p(\boldsymbol{a}_3, \lambda_0; X_0) \bmod \lambda_3 .$$

We put $E := E_p(\boldsymbol{a}_3, \lambda_0; X_0) \bmod \lambda_3$. Then ∂E is the pull-back of $(***)$ by E and we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)} & \longrightarrow & \partial E & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow E & & \\
0 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)} & \underset{\rho}{\longrightarrow} & \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A} & \underset{\beta}{\longrightarrow} & \iota_*\mathbf{G}_{m,A_{\lambda_3}} & \longrightarrow & 0 .
\end{array}$$

Let (x_1, x_2), t and x_0 be local sections of $\mathcal{E}^{(\lambda_1,\lambda_2;D_1)}$, $\mathbf{G}_{m,A}$ and $\mathcal{G}^{(\lambda_0)}$, respectively. Then we have

$$E(x_0) \equiv D_2(x_1, x_2)^{-1} t \bmod \lambda_3 .$$

Since it is equivalent to $E(x_0)D_2(x_1, x_2) \equiv t \bmod \lambda_3$, there exists x_3 such that

$$t = E(x_0)D_2(x_1, x_2) + \lambda_3 x_3 .$$

Therefore we have

$$\partial E = \text{the class of Spec } A\left[X_0, X_1, X_2, X_3, \frac{1}{1+\lambda_0 X_0}, \frac{1}{1+\lambda_1 X_1}, \frac{1}{D_1(X_1)+\lambda_2 X_2}, \frac{1}{E_p(\boldsymbol{a}_3, \lambda_0; X_0) D_2(X_1, X_2) + \lambda_3 X_3}\right]$$

and $\partial \circ \psi_1 = \Psi^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} \circ \varphi_2$.

By

$$\mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A}) \simeq \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)}),$$

ρ^* is the push-down map $\pi_* : \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)}) \to \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)})$ by the canonical projection $\pi : \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)} \to \mathcal{E}^{(\lambda_1,\lambda_2;D_1)}$ and we have a commutative diagram with the exact horizontal lines:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2,\lambda_3;D_1,D_2)} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0 \\ & & {\scriptstyle\pi}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} & \longrightarrow & \pi_*\mathcal{E} & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0. \end{array}$$

Let $\overline{\mathcal{E}}$ be the class of $\mathcal{E}$. Then we have

$$\begin{aligned} \rho^*(\overline{\mathcal{E}}) &= \overline{\pi_*\mathcal{E}} \\ &= \text{the class of Spec } A\left[X_0, X_1, X_2, \frac{1}{1+\lambda_0 X_0}, \frac{1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1}, \frac{1}{E_p(\boldsymbol{a}_2, \lambda_0; X_0) D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}\right) + \lambda_2 X_2}\right]. \end{aligned}$$

Therefore we have $\rho^* \circ \Psi^{(\lambda_0,\lambda_1,\lambda_2,\lambda_3)} = \Psi^{(\lambda_0,\lambda_1,\lambda_2)} \circ \varphi_3$ by Theorem 5.1.1.

Let $\mathcal{E} \in \mathrm{Ext}^1(\mathcal{G}^{(\lambda_0)}, \mathcal{E}^{(\lambda_1,\lambda_2;D_1)})$. Then the exact sequence

$$0 \to \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \to \mathcal{E} \to \mathcal{G}^{(\lambda_0)} \to 0$$

induces the exact sequence

$$0 \to \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A} \to \mathcal{E} \times \mathbf{G}_{m,A} \to \mathcal{G}^{(\lambda_0)} \to 0,$$

and we have a commutative diagram with exact horizontal lines:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}^{(\lambda_1,\lambda_2;D_1)} \times \mathbf{G}_{m,A} & \longrightarrow & \mathcal{E} \times \mathbf{G}_{m,A} & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0 \\ & & {\scriptstyle\beta}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \iota_*\mathbf{G}_{m,A_{\lambda_3}} & \longrightarrow & \beta_*(\mathcal{E} \times \mathbf{G}_{m,A}) & \longrightarrow & \mathcal{G}^{(\lambda_0)} & \longrightarrow & 0. \end{array}$$

Here, the vertical homomorphism is described as follows. Since the group structure of $\mathcal{E}$ is induced by the group scheme homomorphism

$$\begin{aligned}\mathcal{E} &\longrightarrow \mathbf{G}_{m,A} \times \mathbf{G}_{m,A} \times \mathbf{G}_{m,A} \\ (x_0, x_1, x_2) &\longmapsto \Bigg(1 + \lambda_0 x_0,\, E_p(\boldsymbol{a}_1, \lambda_0; x_0) + \lambda_1 x_1, \\ &\qquad E_p(\boldsymbol{a}_2, \lambda_0; x_0) D_1\left(\frac{x_1}{E_p(\boldsymbol{a}_1, \lambda_0; x_0)}\right) + \lambda_2 x_2\Bigg)\end{aligned}$$

we have

$$(x_0, x_1, x_2) = (x_0, 0, 0) + \left(0, \frac{x_1}{E_p(\boldsymbol{a}_1, \lambda_0; x_0)}, \frac{x_2}{E_p(\boldsymbol{a}_2, \lambda_0; x_0)}\right)$$

with a local section $(x_0, x_1, x_2) \in \mathcal{E}$ and the group scheme homomorphism is

$$\begin{aligned}\mathcal{E} \times \mathbf{G}_{m,A} &\longrightarrow \beta_*(\mathcal{E} \times \mathbf{G}_{m,A}) \\ ((x_0, x_1, x_2), t) &\longmapsto \left(x_0, D_2\left(\frac{x_1}{E_p(\boldsymbol{a}_1, \lambda_0; x_0)}, \frac{x_2}{E_p(\boldsymbol{a}_2, \lambda_0; x_0)}\right)^{-1} t\right)\end{aligned}$$

for a local section $t \in \mathbf{G}_{m,A}$.

By the group scheme homomorphism $\mathcal{E} \to \mathbf{G}_{m,A} \times \mathbf{G}_{m,A} \times \mathbf{G}_{m,A}$, we have

$$((x_0, 0, 0), 1) \mapsto ((1 + \lambda_0 x_0, E_p(\boldsymbol{a}_1, \lambda_0; x_0), E_p(\boldsymbol{a}_2, \lambda_0; x_0)), 1)\,.$$

We put $((X_0, X_1, X_2), 1) := ((x, 0, 0), 1)((x', 0, 0), 1)$. Then we have

$$\begin{aligned}&\Bigg(\Bigg(1 + \lambda_0 X_0,\, E_p(\boldsymbol{a}_1, \lambda_0; X_0) + \lambda_1 X_1, \\ &\qquad E_p(\boldsymbol{a}_2, \lambda_0; X_0) D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}\right) + \lambda_2 X_2\Bigg), 1\Bigg) \\ &= ((1 + \lambda_0(x + x' + \lambda_0 x x'), E_p(\boldsymbol{a}_1, \lambda_0; x_0) E_p(\boldsymbol{a}_1, \lambda_0; x')\,, \\ &\qquad E_p(\boldsymbol{a}_2, \lambda_0; x) E_p(\boldsymbol{a}_2, \lambda_0; x')), 1)\end{aligned}$$

and

$$\begin{aligned}X_0 &= x + x' + \lambda_0 x x' \\ X_1 &= \frac{1}{\lambda_1}\{E_p(\boldsymbol{a}_1, \lambda_0; x) E_p(\boldsymbol{a}, \xi; x') - E_p(\boldsymbol{a}_1, \lambda_0; X_0)\} \\ X_2 &= \frac{1}{\lambda_2}\left\{E_p(\boldsymbol{a}_2, \lambda_0; x) E_p(\boldsymbol{a}_2, \lambda_0; x') - E_p(\boldsymbol{a}_2, \lambda_0; X_0) D_1\left(\frac{X_1}{E_p(\boldsymbol{a}_1, \lambda_0; X_0)}\right)\right\}.\end{aligned}$$

The cocycle F on $\mathcal{G}^{(\lambda_0)} \times \iota_* \mathbf{G}_{m,A_{\lambda_3}}$ giving $\beta_*(\mathcal{E} \times \mathbf{G}_{m,A})$ is given by

$$\begin{aligned}
F(x,x') &= D_2\left(\frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}, \frac{X_2}{E_p(\boldsymbol{a}_2,\lambda_0;X_0)}\right)^{-1} \\
&= E_p\left(-\boldsymbol{u}_1^2, \lambda_1; \frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}\right) \\
&\quad E_p\left(-\boldsymbol{u}_2^2, \lambda_2; \frac{X_2}{E_p\big(\boldsymbol{u}^1, \lambda_1; \frac{X_1}{E_p(\boldsymbol{a}_1,\lambda_0;X_0)}\big)E_p(\boldsymbol{a}_2,\lambda_0;X_0)}\right) \\
&= E_p\left(-\boldsymbol{u}_1^2, \lambda_1; \frac{1}{\lambda_1}\{F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x') - 1\}\right) \\
&\quad E_p\left(-\boldsymbol{u}_2^2, \lambda_2; \frac{1}{\lambda_2}\left[\frac{F_p(F^{(\lambda_0)}\boldsymbol{a}_2, \lambda_0; x, x')}{E_p(\boldsymbol{u}^1, \lambda_1; \frac{1}{\lambda_1}\{F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x') - 1\})} - 1\right]\right) \\
&= \tilde{E}_p(-\boldsymbol{u}_1^2, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) \\
&\quad E_p\left(-\boldsymbol{u}_2^2, \lambda_2; \frac{1}{\lambda_2}\{F_p(F^{(\lambda_0)}\boldsymbol{a}_2, \lambda_0; x, x')\tilde{E}_p(-\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))\right. \\
&\quad \left. G_p(-F^{(\lambda_0)}\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) - 1\}\right) \\
&= F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle, \lambda_0; x, x')G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) \\
&\quad E_p\left(-\boldsymbol{u}_2^2, \lambda_2; \frac{1}{\lambda_2}\{F_p(F^{(\lambda_0)}\boldsymbol{a}_2, \lambda_0; x, x')F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle, \lambda_0; x, x')\right. \\
&\quad \left. G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) - 1\}\right) \\
&= F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle, \lambda_0; x, x')G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) \\
&\quad E_p\left(-\boldsymbol{u}_2^2, \lambda_2; \frac{1}{\lambda_2}\{F_p(F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle, \lambda_0; x, x')\right. \\
&\quad \left. G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) - 1\}\right) \\
&= F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle, \lambda_0; x, x')G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x')) \\
&\quad \tilde{E}_p(-\boldsymbol{u}_2^2, \lambda_2; F_p(F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle, \lambda_0; x, x') \\
&\quad G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))) \\
&\quad G_p(-F^{(\lambda_2)}\boldsymbol{u}_2^2, \lambda_2; F_p(F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle, \xi; x, x') \\
&\quad G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_2, \lambda_0; x, x')))
\end{aligned}$$

$$
\begin{aligned}
&\equiv F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle, \lambda_0; x, x')G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))\\
&\quad \tilde{E}_p(-\boldsymbol{u}_2^2, \lambda_2; F_p(F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle, \lambda_0; x, x'))\\
&\quad \tilde{E}_p(-\boldsymbol{u}_2^2, \lambda_2; G_p(-F^{(\lambda_1)}\boldsymbol{u}^1, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))) \bmod \lambda_3\\
&= F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle, \lambda_0; x, x')G_p(-F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))\\
&\quad F_p\left(-\left\langle \frac{1}{\lambda_2}(F^{(\lambda_0)}\boldsymbol{a}_2 - \langle \boldsymbol{b}_1^2, \boldsymbol{u}^1\rangle), \boldsymbol{u}_2^2\right\rangle, \lambda_0; x, x'\right)\\
&\quad G_p(\langle \boldsymbol{b}_2^3, \boldsymbol{u}_2^2\rangle, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))\\
&= F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle - \langle \boldsymbol{b}_1^3, \boldsymbol{u}_2^2\rangle, \lambda_0; x, x')\\
&\quad G_p(\langle \boldsymbol{b}_2^3, \boldsymbol{u}_2^2\rangle - F^{(\lambda_1)}\boldsymbol{u}_1^2, \lambda_1; F_p(F^{(\lambda_0)}\boldsymbol{a}_1, \lambda_0; x, x'))\\
&\equiv F_p(-\langle \boldsymbol{b}_1^2, \boldsymbol{u}_1^2\rangle - \langle \boldsymbol{b}_1^3, \boldsymbol{u}_2^2\rangle, \lambda_0; x, x') \bmod \lambda_3\,.
\end{aligned}
$$

Therefore we have $\beta^* \circ \Psi^{(\lambda_0,\lambda_1,\lambda_2)} = \psi_2 \circ \varphi_4$. Hence, Theorem 6.2.1 is proved.

ACKNOWLEDGMENT. The author is very grateful to his supervisor Professor Tsutomu Sekiguchi for his kind advice and encouragement. He could not complete this work without Professor Sekiguchi's support. He also thanks Professors Noriyuki Suwa and Fumiyuki Momose for their valuable suggestions. He also would like to express his gratitude to Mr. Daisuke Horikawa, Nobuhiro Aki, Michio Amano and Yuji Tsuno for discussions.

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Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING,
CHUO UNIVERSITY,
KASUGA, BUNKYO-KU, TOKYO, 112–8551 JAPAN.
e-mail: s88201@gug.math.chuo.ac.jp