TOKYO J. MATH.
VOL. 33, NO. 1, 2010

Asymptotic Behavior of Positive Solutions of $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ in the Sublinear Case

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(Communicated by H. Nakada)

Abstract. In this paper, we shall consider an initial value problem of the differential equation written in the title where

$$' = d/dt\,, \quad -(2\lambda+1)^2/4\lambda(\lambda+1) < \alpha < 0\,, \quad \lambda < -1\,.$$

For every initial condition, we shall obtain analytical expressions of the solution of the initial value problem. These analytical expressions are valid in neighborhoods of both ends of the domain of the solution, and hence show asymptotic behavior of the solution.

1. Introduction

Let us consider a second order nonlinear differential equation

$$x'' = t^{\alpha\lambda-2}x^{1+\alpha} \quad ('= d/dt) \tag{E}$$

in a region

$$0 < t < \infty\,, \quad 0 < x < \infty\,.$$

Here α, λ are real parameters with

$$-\frac{(2\lambda+1)^2}{4\lambda(\lambda+1)} < \alpha < 0\,, \quad \lambda < -1 \text{ or } \lambda > 0\,.$$

(E) relates to various fields – mathematical physics, variational problems, theory of partial differential equations and so on (cf. [1], [19]). Moreover many authors treated (E) (cf. [3], [7], [8], [9], [22] etc.). For example, in the recent papers [3], [7], and [9], transformations of (E) were discussed. However an initial value problem of (E) was not considered in these papers. So, given an initial condition

$$x(t_0) = A\,, \quad x'(t_0) = B \tag{I}$$

$$(0 \le t_0 < \infty\,, \quad 0 < A < \infty\,, \quad -\infty < B < \infty)\,,$$

we shall treat the initial value problem (E) and (I), and show asymptotic behavior of the solution.

Received November 5, 2008

Since the case $\alpha > 0$ has been already treated in our previous papers [10], [11], [17], and [19], we supposed $\alpha < 0$ as above. For showing the asymptotic behavior, we shall transform (E) into a 2 dimensional autonomous system for which $(1, 0)$ is a critical point. If α, λ satisfy the above condition, then the point $(1, 0)$ is a node, but if α, λ do not, then this is another kind of a critical point, or the transformation becomes meaningless.

In the present paper we shall suppose $\lambda < -1$ and treat the case $0 < t_0 < \infty$. The cases $t_0 = 0$ and $\lambda > 0$ will be discussed in another paper. Our discussion will be carried out as follows. In Section 2 of this paper, we shall transform (E) into a first order rational differential equation and rewrite this as the 2 dimensional autonomous system stated above. Moreover we shall state Theorems 1, 2. For the proof of these theorems, we shall consider the 2 dimensional autonomous system in Sections 3, 4, 5 and complete the proofs in Section 6. In Section 7 we shall consider the first order rational differential equation in the neighborhood of infinity. We shall state Theorem 3 and its proof, Theorems 4, 5 and their proofs, and Theorems 6, 7 and their proofs in Sections 8, 9, and 10 respectively. These proofs will be done with the aid of lemmas obtained in Sections 3, 4, 5, and 7.

In the previous papers, we described theorems in the way that we fixed t_0, A arbitrarily and varied B. However in this paper we shall state these in another way for the sake of preventing us from repeating the same conclusion several times.

2. Reduction of (E) and statements of Theorems 1, 2

First of all, use a transformation

$$y = \psi(t)^{-\alpha} x^{\alpha} \text{ (namely } x = \psi(t) y^{1/\alpha}), \quad z = t y' \tag{T}$$

where

$$\psi(t) = \{\lambda(\lambda + 1)\}^{1/\alpha} t^{-\lambda}$$

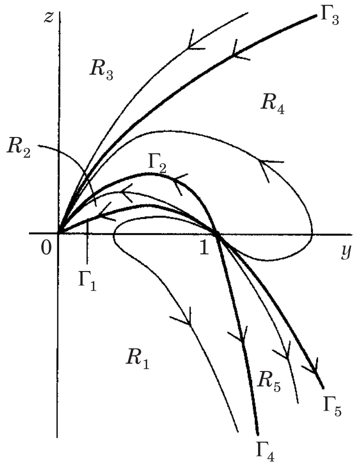
is a particular solution of (E). Then as in the papers [10], [11], this transforms (E) into a first order rational differential equation

$$\frac{dz}{dy} = \frac{(\alpha - 1)z^2 + \alpha(2\lambda + 1)yz + \alpha^2\lambda(\lambda + 1)y^2(y - 1)}{\alpha yz} . \tag{R}$$

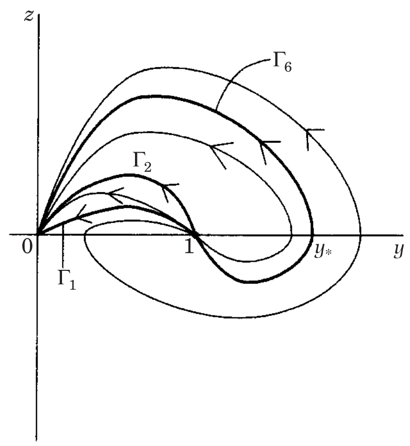
Using a parameter s, we write this as a 2 dimensional autonomous system

$$\begin{aligned} \frac{dy}{ds} &= \alpha yz , \\ \frac{dz}{ds} &= (\alpha - 1)z^2 + \alpha(2\lambda + 1)yz + \alpha^2\lambda(\lambda + 1)y^2(y - 1) . \end{aligned} \tag{S}$$

This has two critical points $(0, 0)$, $(1, 0)$.



The case (2.1) — $-2<\alpha<0$



The case (2.2) — $\alpha\leq-2$

FIGURE 1. The phase portrait of (S)

Hereafter, suppose

$$\lambda < -1 .$$

The phase portrait of (S) depends on whether α is greater than -2 or not. Compare $-(2\lambda+1)^2/4\lambda(\lambda+1)$ and -2. Then we find that the case $-2<\alpha<0$ occurs if

$$\begin{aligned} &-2<\alpha<0\,, \quad -\frac{1+\sqrt{2}}{2}<\lambda<-1 \\ \text{or } &-\frac{(2\lambda+1)^2}{4\lambda(\lambda+1)}<\alpha<0\,, \quad \lambda\leq-\frac{1+\sqrt{2}}{2} \end{aligned} \tag{2.1}$$

and the other case $\alpha\leq-2$ does if

$$-\frac{(2\lambda+1)^2}{4\lambda(\lambda+1)}<\alpha\leq-2\,, \quad -\frac{1+\sqrt{2}}{2}<\lambda\leq-1\,. \tag{2.2}$$

Indeed, an inequality

$$-\frac{(2\lambda+1)^2}{4\lambda(\lambda+1)}=-1-\frac{1}{4\lambda(\lambda+1)}<-2$$

has a solution

$$-\frac{1+\sqrt{2}}{2}<\lambda<-1\,.$$

As proved below, the phase portrait of (S) is as in Figure 1. Here we judge the direction of the orbit as s increases from the sign of $dy/ds = \alpha yz$ of (S). Moreover $\Gamma_1, \Gamma_2, \ldots, \Gamma_6$ denote the orbits and cut the phase plane into the regions $R_1, R_2, \ldots, R_5$. We shall define these orbits and these regions in detail during our discussion.

Now from (T) we have

$$z = \alpha y\left(\lambda + \frac{tx'}{x}\right). \tag{2.3}$$

Here, put $t = t_0$, $(y, z) = (y_0, z_0)$. Then from (I) we get

$$y_0 = \psi(t_0)^{-\alpha} A^{\alpha}, \quad z_0 = \alpha y_0\left(\lambda + \frac{t_0 B}{A}\right). \tag{2.4}$$

If equations $z = z_j(y)$ represent orbits Γ_j $(j = 1, 2, \ldots, 6)$, then $\Gamma_1 : z = z_1(y)$ is the unique orbit defined for $0 < y < 1$ such that

$$\lim_{y \to 0} z = 0, \quad \lim_{y \to 0} \frac{z}{y} = \alpha(\lambda + 1). \tag{2.5}$$

Moreover Γ_1 satisfies

$$\lim_{y \to 1} z = 0, \quad \lim_{y \to 1} \frac{z}{y - 1} = \frac{\mu}{\alpha} \tag{2.6}$$

where

$$\mu = \alpha\left\{2\lambda + 1 + \sqrt{(2\lambda + 1)^2 + 4\alpha\lambda(\lambda + 1)}\right\}/2.$$

In addition, we define

$$\nu = \alpha\left\{2\lambda + 1 - \sqrt{(2\lambda + 1)^2 + 4\alpha\lambda(\lambda + 1)}\right\}/2.$$

Notice that $0 < \mu < \nu$.

Here, let $x(t)$ be a solution of the initial value problem (E), (I). Then in both cases (2.1) and (2.2) we state Theorems 1, 2 as follows:

THEOREM 1. *If we take t_0, A, B in* (I) *such that $(y_0, z_0) \in \Gamma_1$, then $x(t)$ is defined for $0 < t < \infty$ and represented as follows*:

In the neighborhood of $t = 0$, we get

$$x(t) = Kt\left\{1 + \sum_{n=1}^{\infty} x_n t^{\alpha(\lambda+1)n}\right\} \tag{2.7}$$

where K is a positive constant and x_n are constants. Moreover in the neighborhood of $t = \infty$, we obtain

$$x(t) = \{\lambda(\lambda + 1)\}^{1/\alpha} t^{-\lambda}\left\{1 + \sum_{m+n>0} x_{mn} t^{(\mu/\alpha)m + (\nu/\alpha)n}\right\} \tag{2.8}$$

if $\nu/\mu \notin \boldsymbol{N}$, *and*

$$x(t) = \{\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda} \left\{ 1 + \sum_{k=1}^{\infty} t^{(\mu/\alpha)k} p_k(\log t) \right\} \tag{2.9}$$

if $\nu/\mu = N \in \boldsymbol{N}$, *where* x_{mn} *are constants and* p_k *are polynomials with*

$$\deg p_k \leq \left[\frac{k}{N} \right],$$

[ ] *denoting Gaussian symbol.*

$\Gamma_2 : z = z_2(y)$ is the unique orbit defined for $0 < y < 1$ such that

$$\lim_{y \to 1} z = 0, \quad \lim_{y \to 1} \frac{z}{y-1} = \frac{\nu}{\alpha}. \tag{2.10}$$

Furthermore Γ_2 satisfies

$$\lim_{y \to 0} z = 0, \quad \lim_{y \to 0} \frac{z}{y} = \alpha\lambda. \tag{2.11}$$

THEOREM 2. *If we take* t_0, A, B *such that* $(y_0, z_0) \in \Gamma_2$, *then* $x(t)$ *is defined for* $0 < t < \infty$ *and represented as follows:*

In the neighborhood of $t = 0$, *we get*

$$x(t) = K\left(1 + \sum_{m+n>0} x_{mn} t^{\alpha\lambda m + n}\right) \tag{2.12}$$

if $1/\alpha\lambda \notin \boldsymbol{N}$, *and*

$$x(t) = K\left\{1 + \sum_{k=1}^{\infty} t^{\alpha\lambda k} q_k(\log t)\right\} \tag{2.13}$$

if $1/\alpha\lambda \in \boldsymbol{N}$, *where* K *is a positive constant*, x_{mn} *are constants*, *and* q_k *are polynomials with*

$$\deg q_k \leq [\alpha\lambda k].$$

Moreover in the neighborhood of $t = \infty$, *we obtain*

$$x(t) = \{\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda} \left\{ 1 + \sum_{n=1}^{\infty} x_n t^{(\nu/\alpha)n} \right\} \tag{2.14}$$

where x_n *are constants.*

3. The research of (S) around the critical point (0, 0)

We show the following three lemmas on the orbits tending to (0, 0).

LEMMA 3.1. *Let $z = z(y)$ be an orbit of* (S) *such that*

$$\lim_{y \to 0} z(y) = 0 .$$

Then we have

$$\lim_{y \to 0} \frac{z(y)}{y} = \alpha\lambda,\ \alpha(\lambda + 1) .$$

PROOF. From Lemma 1 of [17], we have

$$\lim_{y \to 0} \frac{z(y)}{y} = \alpha\lambda,\ \alpha(\lambda + 1),\ \ \pm\infty .$$

However if the orbit $z = z(y)$ satisfies

$$\lim_{y \to 0} \frac{z(y)}{y} = \pm\infty , \tag{3.1}$$

then we put

$$w = yz(y)^{-1}$$

and obtain

$$w \to 0 \quad \text{as } y \to 0$$

$$y\frac{dw}{dy} = \frac{1}{\alpha}w - (2\lambda + 1)w^2 - \alpha\lambda(\lambda + 1)(y - 1)w^3 .$$

Hence from $1/\alpha < 0$ and Lemma 2.5 of [16] we get a contradiction

$$w \equiv 0 .$$

Namely (3.1) does not occur and the proof is complete. □

LEMMA 3.2. *There exists an orbit $z = z(y)$ such that*

$$\lim_{y \to 0} \frac{z(y)}{y} = \alpha\lambda . \tag{3.2}$$

If we apply (T) *to this, then we obtain a solution of* (E) *represented as* (2.12), (2.13) *in the neighborhood of $t = 0$.*

PROOF. As done in [15], put

$$v = y^{-1}z - \alpha\lambda$$

in (R). Then we get

$$y\frac{dv}{dy} = (\lambda + 1)y + \frac{1}{\alpha\lambda}v + \cdots .$$

Here let us determine a solution v of this such that

$$v \to 0 \quad \text{as } y \to 0\,.$$

Then since $1/\alpha\lambda > 0$, v is given as

$$v = \sum_{m+n>0} v_{mn} y^m \{y^{1/\alpha\lambda}(h \log y + C)\}^n$$

where v_{mn}, C, h are constants and if $1/\alpha\lambda \notin \boldsymbol{N}$, then we obtain $h = 0$. Therefore we get

$$z = \alpha\lambda y \left[1 + \sum_{m+n>0} \tilde{v}_{mn} y^m \{y^{1/\alpha\lambda}(h \log y + C)\}^n\right] \tag{3.3}$$

in the neighborhood of $y = 0$, where

$$\tilde{v}_{mn} = \frac{v_{mn}}{\alpha\lambda}\,.$$

This implies the existence of the orbit with (3.2).

Applying (T) to (3.3), we get a differential equation

$$ty' = \alpha\lambda y \left[1 + \sum_{m+n>0} \tilde{v}_{mn} y^m \{y^{1/\alpha\lambda}(h \log y + C)\}^n\right].$$

Solving this, we have

$$y \left[1 + \sum_{m+n>0} a_{mn} y^m \{y^{1/\alpha\lambda}(h \log y + C)\}^n\right] = \Gamma t^{\alpha\lambda}$$

where a_{mn}, Γ are constants. Using Smith's lemma (cf. Lemma 1 of [11]) to this and rewriting $\alpha\lambda h$, $h \log \Gamma + C$ as h, C, we obtain

$$y = \Gamma t^{\alpha\lambda} \left[1 + \sum_{m+n>0} b_{mn} t^{\alpha\lambda m} \{t(h \log t + C)\}^n\right]$$

b_{mn} being constants. Hence from (T) we get a solution of (E) represented as

$$x(t) = K \left[1 + \sum_{m+n>0} x_{mn} t^{\alpha\lambda m} \{t(h \log t + C)\}^n\right] \tag{3.4}$$

$$(K = \{\lambda(\lambda + 1)\}^{1/\alpha} \Gamma^{1/\alpha})$$

in the neighborhood of $t = 0$. Moreover if $1/\alpha\lambda \notin \boldsymbol{N}$, then since $h = 0$ we have (2.12) from replacing $x_{mn} C^n$ with x_{mn}. If $1/\alpha\lambda \in \boldsymbol{N}$, then from (3.4) we obtain

$$x(t) = K \left\{1 + \sum_{m+n>0} t^{\alpha\lambda m + n} Q_{mn}(\log t)\right\}$$

where Q_{mn} are polynomials with *deg* $Q_{mn} \leq n$. So if we put

$$m + \frac{n}{\alpha\lambda} = k\,, \quad q_k = Q_{mn}\,,$$

then q_k are polynomials with *deg* $q_k \leq [\alpha\lambda k]$. Furthermore we get (2.13), which completes the proof. □

LEMMA 3.3. *There exists the unique orbit* $z = z(y)$ *of* (S) *such that*

$$\lim_{y\to 0} \frac{z(y)}{y} = \alpha(\lambda + 1)\,. \tag{3.5}$$

If we apply (T) *to this*, *then we have a solution of* (E) *represented as* (2.7) *in the neighborhood of* $t = 0$.

PROOF. Put

$$v = y^{-1}z - \alpha(\lambda + 1)$$

in (R). Then we obtain

$$y\frac{dv}{dy} = \lambda y - \frac{1}{\alpha(\lambda + 1)}v + \cdots\,.$$

Now let us consider a solution of this such that

$$v \to 0 \quad \text{as } y \to 0\,. \tag{3.6}$$

Then from $-1/\alpha(\lambda + 1) < 0$ and Lemma 2.5 of [16] there exists uniquely the solution such that (3.6) holds. Moreover this is holomorphic in the neighborhood of $y = 0$ and represented as

$$v = \sum_{n=1}^{\infty} v_n y^n \quad (v_n : \text{constants})\,.$$

Therefore we get the unique orbit of (S) such that (3.5) holds. Furthermore this is represented as

$$z = \alpha(\lambda + 1)y\left(1 + \sum_{n=1}^{\infty} \tilde{v} y^n\right) \quad \left(\tilde{v}_n = \frac{v_n}{\alpha(\lambda + 1)}\right) \tag{3.7}$$

in the neighborhood of $y = 0$. If we apply (T) to this and follow the line of the proof of Lemma 3.1, then we have (2.7). This completes the proof. □

Γ_1 is the orbit satisfying (3.5) and represented as (3.7).

4. The research of (S) around the critical point (1, 0)

Let us consider orbits of (S) tending to (1, 0).

LEMMA 4.1. *There exist only two orbits* (y, z) *of* (S) *tending to* $(1, 0)$ *as* $s \to -\infty$ *such that*

$$\lim_{s\to-\infty} \frac{z}{y-1} = \frac{\nu}{\alpha}\,. \tag{4.1}$$

If (y, z) *is another orbit of* (S) *tending to* $(1, 0)$, *then this tends to* $(1, 0)$ *as* $s \to -\infty$ *and satisfies*

$$\lim_{s\to-\infty} \frac{z}{y-1} = \frac{\mu}{\alpha}\,. \tag{4.2}$$

PROOF. For the sake of obtaining a solution of (S) tending to $(1, 0)$, we use the discussion done in Section 2 of [16].

Put

$$y = 1 + \eta\,, \quad z = \zeta\,. \tag{4.3}$$

Then from (S) we get

$$\frac{d\eta}{ds} = \alpha\zeta + \cdots\,, \quad \frac{d\zeta}{ds} = \alpha^2\lambda(\lambda+1)\eta + \alpha(2\lambda+1)\zeta + \cdots \tag{4.4}$$

where ... denotes terms whose degrees are greater than the degree of the previous term. In the righthand sides of this, the coefficient matrix of the linear terms is given as

$$\begin{pmatrix} 0 & \alpha \\ \alpha^2\lambda(\lambda+1) & \alpha(2\lambda+1) \end{pmatrix}$$

whose eigenvalues are μ, ν. Adopting a transformation

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \alpha & \alpha \\ \mu & \nu \end{pmatrix} \begin{pmatrix} \tilde\eta \\ \tilde\zeta \end{pmatrix} \tag{4.5}$$

and diagonalizing this matrix, we get

$$\frac{d\tilde\eta}{ds} = \mu\tilde\eta + \cdots\,, \quad \frac{d\tilde\zeta}{ds} = \nu\tilde\zeta + \cdots\,. \tag{4.6}$$

Hence it follows from Theorem A of [2] and its proof that if we use a transformation

$$\tilde\eta = w_1 + \cdots\,, \quad \tilde\zeta = w_2 + \cdots \tag{4.7}$$

valid in the neighborhood of $(w_1, w_2) = (0, 0)$, we transform (4.6) into

$$\frac{dw_1}{ds} = \mu w_1\,, \quad \frac{dw_2}{ds} = \nu w_2 + g w_1^N$$

where g is a constant such that if $g \neq 0$, then we have

$$\frac{\nu}{\mu} = N \in \boldsymbol{N}\,.$$

Therefore we obtain

$$w_1 = Le^{\mu s}\,, \quad w_2 = (M + gL^N s)e^{\nu s} \tag{4.8}$$

where L, M are constants. Notice that $(w_1, w_2) \to (0, 0)$ as $s \to -\infty$, since $0 < \mu < \nu$.

Here if we substitute (4.7) into (4.5) and (4.5) into (4.3), then we get

$$\begin{aligned} y &= 1 + \alpha w_1 + \alpha w_2 + \sum_{m+n\geq 2} a_{mn} w_1^m w_2^n \\ z &= \mu w_1 + \nu w_2 + \sum_{m+n\geq 2} b_{mn} w_1^m w_2^n \end{aligned} \tag{4.9}$$

where a_{mn}, b_{mn} are constants. Therefore the orbit (y, z) tends to $(1, 0)$ as $s \to -\infty$.

If $L = 0$ in (4.8), then we have

$$w_1 = 0\,, \quad w_2 = Me^{\nu s}$$

and from (4.9)

$$y = 1 + \alpha w_2 + \sum_{n=2}^{\infty} a_{0n} w_2^n\,, \quad z = \nu w_2 + \sum_{n=2}^{\infty} b_{0n} w_2^n\,.$$

Eliminating w_2, we obtain

$$z = \frac{\nu}{\alpha}(y - 1) + \sum_{n=2}^{\infty} c_n (y - 1)^n \tag{4.10}$$

in the neighborhood of $y = 1$ where c_n are constants. As easily checked, this satisfies (4.1).

On the other hand, if $L \neq 0$ then we get

$$\frac{w_2}{w_1} = \frac{M + gL^N s}{L} e^{(\nu-\mu)s} \to 0$$

as $s \to -\infty$, and

$$\begin{aligned} &\lim_{s\to-\infty} \frac{z}{y-1} \\ &\quad = \lim_{s\to-\infty} \frac{\mu + \nu(w_2/w_1) + (1/w_1)\sum_{m+n\geq 2} b_{mn} w_1^m w_2^n}{\alpha + \alpha(w_2/w_1) + (1/w_1)\sum_{m+n\geq 2} a_{mn} w_1^m w_2^n} = \frac{\mu}{\alpha}\,. \end{aligned}$$

Namely we have (4.2).

Since (4.10) represents two orbits lying in the regions $y > 1$ and $y < 1$, we conclude that there exist only two orbits satisfying (4.1) and the other orbits tending to $(1, 0)$ satisfy (4.2). This completes the proof. □

Γ_2, Γ_4, Γ_6 in Figure 1 are orbits represented as (4.10) and satisfy (4.1). Γ_2 lies in $y < 1$ and Γ_4, Γ_6 lie in $y > 1$. Γ_4, Γ_6 appear in the cases (2.1), (2.2) respectively.

Let (y, z) be an orbit of (S) tending to $(1, 0)$ again. Then applying (T) to (4.9), we obtain the analytical expressions (2.8), (2.9), (2.14) appearing in Theorems 1, 2 as follows:

LEMMA 4.2. *If* (y, z) *satisfies* (4.2), *then we get* (2.8), (2.9) *in the neighborhood of* $t = \infty$, *and if* (y, z) *satisfies* (4.1), then we have (2.14) in the neighborhood of $t = \infty$.

PROOF. It follows from the proof of Lemma 4.1 that (y, z) is represented as (4.9). From (4.9) and $z = ty'$ in (T), we obtain

$$\mu w_1 + \nu w_2 + \cdots = t\{\alpha\mu w_1 + \alpha(\nu w_2 + g w_1^N) + \cdots\}s' .$$

Comparing coefficients of both sides, we get

$$t\alpha s' = 1$$

namely

$$s = \frac{1}{\alpha}\log t + C \tag{4.11}$$

C being a constant.

Now, substituting (4.11) into (4.8) and rewriting $Le^{\mu C}$, g/α, $(M + gL^N C)e^{\nu s}$ as L, g, M respectively, we have

$$w_1 = Lt^{\mu/\alpha} , \quad w_2 = (M + gL^N \log t)t^{\nu/\alpha} . \tag{4.12}$$

Therefore from (4.9) we obtain

$$\begin{aligned} y = {} & 1 + \alpha L t^{\mu/\alpha} + \alpha(M + gL^N \log t)t^{\nu/\alpha} \\ & + \sum_{m+n\geq 2} a_{mn}(Lt^{\mu/\alpha})^m((M + gL^N \log t)t^{\nu/\alpha})^n \end{aligned}$$

where a_{mn} are constants and from (T) we obtain a solution of (E) represented as

$$\begin{aligned} x(t) = {} & \{\lambda(\lambda+1)\}^{1/\alpha}t^{-\lambda}\{1 + Lt^{\mu/\alpha} + (M + gL^N \log t)t^{\nu/\alpha} \\ & + \sum_{m+n\geq 2} x_{mn}(Lt^{\mu/\alpha})^m((M + gL^N \log t)t^{\nu/\alpha})^n\} . \end{aligned} \tag{4.13}$$

Recall that (4.9) is valid in the neighborhood of $(w_1, w_2) = (0, 0)$. As (w_1, w_2) tends to $(0, 0)$, we get $s \to -\infty$ from (4.8) and $0 < \mu < \nu$. Hence from (4.11) we have $t \to \infty$. Therefore (4.13) is valid in the neighborhood of $t = \infty$.

Let us simplify the form of (4.13). Since $g = 0$ if $\nu/\mu \notin \boldsymbol{N}$, we obtain (2.8) from (4.13). Here we need to rewrite x_{mn} suitably. If $\nu/\mu = N \in \boldsymbol{N}$, then from (4.13) we get

$$x(t) = \{\lambda(\lambda+1)\}^{1/\alpha}t^{-\lambda}\left\{1 + \sum_{m+n>0} t^{(\mu/\alpha)(m+Nn)}P_{mn}(\log t)\right\}$$

where P_{mn} are polynomials whose degree is not greater than n. Put

$$m + Nn = k\,.$$

Then we have

$$n = \frac{k-m}{N} \leq \left[\frac{k}{N}\right].$$

Moreover, put

$$p_k = P_{mn}\,.$$

Then we obtain (2.9).

Especially if (4.9) satisfies (4.2), then we get (2.8), (2.9).

Finally, suppose that (4.9) satisfies (4.1). Then we have $L = 0$. Hence substituting this into (4.13), we obtain (2.14). This completes the proof. □

5. On orbits Γ_1, Γ_2

In this section, we show that Γ_1 satisfies (2.6), namely

$$\lim_{y\to 1} z = 0\,, \quad \lim_{y\to 1} \frac{z}{y-1} = \frac{\mu}{\alpha}$$

and Γ_2 satisfies (2.11), namely

$$\lim_{y\to 0} z = 0\,, \quad \lim_{y\to 0} \frac{z}{y} = \alpha\lambda\,.$$

First, we conclude the following:

LEMMA 5.1. Γ_1 *satisfies* (2.6).

PROOF. Put

$$f(y) = \sigma(y - y^2)$$

where σ is a constant. Then on the curve $z = f(y)$, we get from (S)

$$\begin{aligned}\frac{d}{ds}(z - f(y)) = y^2(1-y)\{&(\alpha+1)\sigma^2 y \\ &- (\sigma - \alpha\lambda)(\sigma - \alpha(\lambda+1))\}\,.\end{aligned} \tag{5.1}$$

Here, suppose

$$-1 < \alpha < 0\,.$$

Then if $\sigma = \alpha\lambda$ and $0 < y < 1$, from (5.1) we have

$$\frac{d}{ds}(z - f(y)) = (\alpha+1)\alpha^2\lambda^2 y^3(1-y) > 0\,. \tag{5.2}$$

Moreover on the open segment $0 < y < 1$, $z = 0$, we obtain from (S)

$$\frac{dy}{ds} = 0\,, \quad \frac{dz}{ds} = \alpha^2\lambda(\lambda+1)y^2(y-1) < 0\,. \tag{5.3}$$

Since

$$f'(0) = \alpha\lambda > \alpha(\lambda+1)\,,$$

Γ_1 is contained in the region which the curve $z = f(y)$ and the segment $0 < y < 1$, $z = 0$ surround, and from Poincaré-Bendixon's theorem Γ_1 tends to $(1, 0)$ as $s \to -\infty$.

On the other hand, we get

$$\frac{\nu}{\alpha} < -\alpha\lambda < -\alpha(\lambda+1) < \frac{\mu}{\alpha}\,.$$

Hence from $f'(1) = -\alpha\lambda$ we have

$$\frac{\nu}{\alpha} < f'(1) < \frac{\mu}{\alpha}$$

and conclude from Lemma 4.1 that

$$\frac{z}{y-1} \to \frac{\mu}{\alpha} \quad \text{as } s \to -\infty\,.$$

Namely Γ_1 satisfies (2.6).

Next, suppose

$$-\frac{(2\lambda+1)^2}{4\lambda(\lambda+1)} < \alpha < -1\,.$$

Then we have

$$4(\alpha+1)\lambda(\lambda+1) + 1 > 0\,.$$

Furthermore if $0 < y < 1$, then in (5.1) we obtain

$$\begin{aligned}&(\alpha+1)\sigma^2 y - (\sigma - \alpha\lambda)(\sigma - \alpha(\lambda+1))\\ &\quad \geq \alpha\left(\sigma + \lambda + \frac{1}{2}\right)^2 - \frac{\alpha\{4(\alpha+1)\lambda(\lambda+1)+1\}}{4}\,.\end{aligned}$$

Therefore if we take $\sigma = -\lambda - 1/2$, then we get

$$(\alpha+1)\sigma^2 y - (\sigma - \alpha\lambda)(\sigma - \alpha(\lambda+1)) > 0$$

and

$$\frac{d}{ds}(z - f(y)) > 0$$

on $z = f(y)$ and $0 < y < 1$. Moreover we have

$$\frac{\nu}{\alpha} < f'(1) = \lambda + \frac{1}{2} < \frac{\mu}{\alpha} \tag{5.4}$$

$$f'(0) = -\lambda - \frac{1}{2} > \alpha(\lambda + 1)\,. \tag{5.5}$$

Hence Γ_1 is contained in the region which the curve $z = f(y)$ and the segment $0 < y < 1$, $z = 0$ surround, and from the same reasoning as in the case $-1 < \alpha < 0$, Γ_1 satisfies (2.6). This completes the proof. □

Since $\mu/\alpha > \nu/\alpha$, it follows from Lemma 5.1 just shown that Γ_2 lies above Γ_1.

Next, let us consider the sign of dz/dy of (S). If we put $dz/ds = 0$, then we get

$$(\alpha - 1)z^2 + \alpha(2\lambda + 1)yz + \alpha^2\lambda(\lambda + 1)y^2(y - 1) = 0$$

and

$$z = Z_{\pm}(y)$$

where

$$Z_{\pm}(y) = \frac{\alpha}{2(\alpha - 1)} y\{-(2\lambda + 1) \pm \sqrt{(2\lambda + 1)^2 - 4\lambda(\lambda + 1)(\alpha - 1)(y - 1)}\}\,.$$

This exists for

$$y \geq \frac{4\alpha\lambda(\lambda + 1) + 1}{4(\alpha - 1)\lambda(\lambda + 1)}$$

and

$$Z'_{\pm}(y) = \frac{\alpha}{2(\alpha - 1)} \times \frac{\pm 3R - 2(2\lambda + 1)\sqrt{R} \mp \{4\alpha\lambda(\lambda + 1) + 1\}}{2\sqrt{R}}$$

where

$$R = (2\lambda + 1)^2 - 4\lambda(\lambda + 1)(\alpha - 1)(y - 1)\,.$$

Hence the sign of dz/ds is as in Figure 2.

If an orbit (y, z) of (S) exists in $0 < y < 1$ and tends to $(1, 0)$, then it follows from Lemma 4.1 that (y, z) satisfies

$$\lim_{y \to 1} \frac{z}{y - 1} = \frac{\mu}{\alpha} \text{ or } \frac{\nu}{\alpha} < 0\,.$$

Therefore in the neighborhood of $y = 1$, (y, z) exists in the region $z > 0$ and z can decrease as y increases. However since $dy/ds = \alpha yz < 0$ from (S), y decreases as s increases. Hence in the neighborhood of $y = 1$, z can increase as s increases and (y, z) is in the region where $dz/ds > 0$.

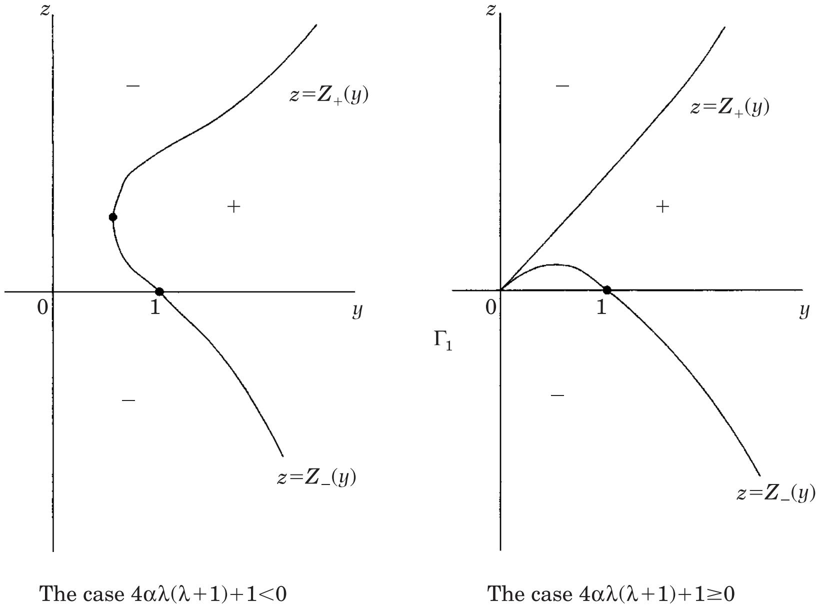


FIGURE 2. The sing of dz/ds

Moreover, suppose that (y, z) lies above Γ_1. Then as s increases to ∞, (y, z) gets out of the region where $dz/ds > 0$ and tends to $(0, 0)$ from Poincaré-Bendixon's theorem. In fact, the positive z axis is an orbit. Especially Γ_2 tends to $(0, 0)$ as $s \to \infty$. Furthermore we conclude the following:

LEMMA 5.2. *Γ_2 satisfies* (2.11).

PROOF. From the lemmas of Section 3, if (y, z) is the orbit tending to $(0, 0)$ then we get

$$\lim_{y \to 0} \frac{z}{y} = \alpha\lambda$$

except Γ_1. This completes the proof. □

6. Proofs of Theorems 1, 2

Here we prove Theorems 1, 2. For this we use the following:

LEMMA 6.1. *If an interval (ω_-, ω_+) is a domain of a solution $x = x(t)$ of* (E), *then as $t \to \omega_+$ or ω_-, (y, z) defined as* (T) *does not tend to a regular point of* (S) *situated in* $y > 0$.

This is Lemma 2 of [17] and the proof is omitted.

Now, define (y_0, z_0) as (2.4) for (t_0, A, B) in (I) and let Γ be an orbit of (S) passing (y_0, z_0). Moreover, let $x(t)$ be a solution of (E), (I) and define an orbit (y, z) of (S) from applying (T) to $x(t)$. Then from Lemma 6.1, (y, z) moves all over Γ as t increases all over the domain of $x(t)$.

PROOF OF THEOREM 1. Take (t_0, A, B) such that $(y_0, z_0) \in \Gamma_1$. Then from the discussion just above, (y, z) moves all over Γ_1. Since Γ_1 satisfies (3.5), from Lemma 3.3 we get a representation (2.7) of $x(t)$ in the neighborhood of $t = 0$. Furthermore due to Lemma 5.1, Γ_1 satisfies (2.6) and (4.2). Therefore from Lemma 4.2, $x(t)$ is represented as (2.8), (2.9). Since (2.8), (2.9) are valid in the neighborhood of $t = \infty$, the domain of $x(t)$ is $(0, \infty)$. Now the proof is complete. □

PROOF OF THEOREM 2. If we take (t_0, A, B) such that $(y_0, z_0) \in \Gamma_2$, then (y, z) moves all over Γ_2 as t takes all values of the domain of $x(t)$. Since Γ_2 tends to $(1, 0)$ and satisfies (4.1), we get (2.14) in the neighborhood of $t = \infty$ from Lemma 4.2. On the other hand, Γ_2 has (2.11) (namely (3.2)) from Lemma 5.2. Therefore due to Lemma 3.2 we obtain (2.12), (2.13) in the neighborhood of $t = 0$. Thus the domain of $x(t)$ is $(0, \infty)$ and the proof is complete. □

7. The research of (R) around $y = \infty$

Let us consider (R) in the neighborhood of $y = \infty$. For this, put $y = 1/\eta$. Then we get

$$\frac{dz}{d\eta} = -\frac{(\alpha-1)\eta^3 z^2 + \alpha(2\lambda+1)\eta^2 z + \alpha^2\lambda(\lambda+1)(1-\eta)}{\alpha\eta^4 z}. \tag{7.1}$$

Moreover, put $z = 1/\zeta$. Then we have

$$\frac{d\zeta}{d\eta} = \frac{(\alpha-1)\eta^3\zeta + \alpha(2\lambda+1)\eta^2\zeta^2 + \alpha^2\lambda(\lambda+1)(1-\eta)\zeta^3}{\alpha\eta^4}. \tag{7.2}$$

Put $w = \eta^{-3/2}\zeta$, $\xi = \eta^{1/2}$ here. Then we obtain

$$\xi\frac{dw}{d\xi} = -\frac{\alpha+2}{\alpha}w + 2(2\lambda+1)\xi w^2 + 2\alpha\lambda(\lambda+1)(1-\xi^2)w^3. \tag{7.3}$$

If we get a solution of (7.3) from applying these changes of variables to a solution of (R) continuable to ∞, then this solution is continuable to $\xi = 0$.

Now, let γ be a cluster point of a solution w of (7.3) as $\xi \to 0$. Notice that if the righthand side of (7.3) vanishes, then we get

$$w = 0, \pm\rho$$

where

$$\rho = \frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2\lambda(\lambda+1)}}$$

which is a positive number only if $\alpha > -2$. Furthermore, recall that (2.1) implies $-2 < \alpha < 0$, and (2.2), $\alpha \leq -2$. Then we conclude the following:

LEMMA 7.1. *In the case* (2.1), *we have*

$$\gamma = 0, \pm\rho\,.$$

Moreover in the neighborhood of $\xi = 0$, *the solution* w *of* (7.3) *is represented as*

$$w = C\xi^{-(\alpha+2)/\alpha}\left\{1 + \sum_{m+n>0} w_{mn}\xi^m (C\xi^{-(\alpha+2)/\alpha})^n\right\} \tag{7.4}$$

$$(C,\ w_{mn} : constants)$$

if $\gamma = 0$, *and as*

$$w = \gamma + \sum_{n=1}^{\infty} a_n \xi^n \quad (a_n : constants) \tag{7.5}$$

if $\gamma = \pm\rho$.

In the case (2.2), *there does not exist* γ.

PROOF. If $\gamma \neq 0, \pm\rho, \pm\infty$, then from (7.3) we obtain

$$\frac{d\xi}{dw} = \frac{\xi}{-((\alpha+2)/\alpha)w + 2(2\lambda+1)\xi w^2 + 2\alpha\lambda(\lambda+1)(1-\xi^2)w^3}$$

which implies a contradiction $\xi \equiv 0$. Hence γ is the limit and

$$\gamma = 0, \pm\rho, \pm\infty\,.$$

Suppose $\gamma = 0$. Then if $-2 < \alpha < 0$, we get $-(\alpha+2)/\alpha > 0$ and hence (7.4) from (7.3). This implies that the case $\gamma = 0$ really arises. If $\alpha = -2$, then (7.3) becomes

$$\xi\frac{dw}{d\xi} = 2(2\lambda+1)\xi w^2 - 4\lambda(\lambda+1)(1-\xi^2)w^3$$

and from the theories of [5] we have

$$\begin{aligned} w = &\{8\lambda(\lambda+1)(\log\xi + C)\}^{-1/2}\Bigg[1 + \sum_{2j+k<2(N+1)} p_{jk}\xi^j \\ &\{8\lambda(\lambda+1)(\log\xi + C)\}^{-k/2} + \Omega\Bigg] \quad (p_{jk} : \text{constants}) \end{aligned}$$

where Ω is a function satisfying

$$|\Omega| \leq K_N|\log\xi|^{-N} \quad (K_N : \text{a constant})\,.$$

Therefore w is not a real variable and the case $\gamma = 0$ does not occur. If $\alpha < -2$, then since $-(\alpha+2)/\alpha < 0$ it follows from Lemma 2.5 of [16] that (7.3) has only a unique holomorphic solution which is $w \equiv 0$. This implies a contradiction $z \equiv \infty$. Hence the case $\gamma = 0$ does not also occur.

If $\gamma = \pm\rho$, then we must have $-2 < \alpha < 0$. Here, put

$$u = w - \gamma\,.$$

Then we obtain

$$\xi\frac{du}{d\xi} = \frac{(\alpha+2)(2\lambda+1)}{\alpha^2\lambda(\lambda+1)}\xi + \frac{2(\alpha+2)}{\alpha}u + \cdots\,. \tag{7.6}$$

Since $2(\alpha+2)/\alpha < 0$, we get a unique holomorphic solution

$$u = \sum_{n=1}^{\infty} a_n\xi^n \quad (a_n : \text{constants})$$

of (7.6) from Lemma 2.5 of [16]. From this we have (7.5).

Finally if $\gamma = \pm\infty$, then we put $w = 1/u$ and obtain

$$\frac{d\xi}{du} = \frac{\xi u}{((\alpha+2)/\alpha)u^2 - 2(2\lambda+1)\xi u - 2\alpha\lambda(\lambda+1)(1-\xi^2)}\,.$$

This implies a contradiction

$$\xi \equiv 0$$

and the cases $\gamma = \pm\infty$ never occur. Now the proof is complete. □

Now, change ξ, w for η, ζ, and η, ζ for y, z. Then in the neighborhood of $y = \infty$ we get

$$z^{-1} = Cy^{(1-\alpha)/\alpha}\left\{1 + \sum_{m+n>0} w_{mn}y^{-m/2}(Cy^{(\alpha+2)/2\alpha})^n\right\} \tag{7.7}$$

from (7.4) and

$$z^{-1} = y^{-3/2}\left(\gamma + \sum_{n=1}^{\infty} a_n y^{-n/2}\right) \tag{7.8}$$

from (7.5). These are only orbits continuable to ∞. Since (7.8) was obtained from the unique solution, (7.8) exists uniquely and we name (7.8) as Γ_3 if $\gamma = \rho$ and as Γ_5 if $\gamma = -\rho$. (7.8) satisfies

$$\lim_{y\to\infty} z = \infty\,, \quad \lim_{y\to\infty} y^{-3/2}z = \frac{1}{\rho} \tag{7.9}$$

if $\gamma = \rho$, and

$$\lim_{y\to\infty} z = -\infty\,, \quad \lim_{y\to\infty} y^{-3/2} z = -\frac{1}{\rho} \tag{7.10}$$

if $\gamma = -\rho$. Since (7.7), (7.8) satisfy

$$z = O(y^{(\alpha-1)/\alpha})\,, \quad z = O(y^{3/2}) \quad \text{as } y \to \infty$$

respectively and $(\alpha - 1)/\alpha > 3/2$ in the case (2.1), (7.7) represents an orbit lying above Γ_3 if $\gamma = \rho$, $z > 0$, and below Γ_5 if $\gamma = -\rho$, $z < 0$. Applying (T) to (7.7), (7.8) or (7.4), (7.5), we conclude the following:

LEMMA 7.2. *From* (7.7), *we get a solution of* (E) *represented as*

$$x(t) = L(\omega_+ - t)\left\{1 + \sum_{j+k+l>0} d_{jkl}(\omega_+ - t)^j(\omega_+ - t)^{-(\alpha/2)k}(\omega_+ - t)^{((\alpha+2)/2)l}\right\} \tag{7.11}$$

if $z > 0$, *and as*

$$x(t) = L(t - \omega_-)\left\{1 + \sum_{j+k+l>0} d_{jkl}(t - \omega_-)^j(t - \omega_-)^{-(\alpha/2)k}(t - \omega_-)^{((\alpha+2)/2)l}\right\} \tag{7.12}$$

if $z < 0$. *Here* L *is a positive constant and* d_{jkl} *are constants. Moreover from* (7.8), *we have a solution of* (E) *expressed as*

$$x(t) = \left\{\frac{2(\alpha+2)}{\alpha^2\omega_+^{\alpha\lambda-2}}\right\}^{1/\alpha}(\omega_+ - t)^{-2/\alpha}\left\{1 + \sum_{n=1}^{\infty} x_n(\omega_+ - t)^n\right\} \tag{7.13}$$

if $\gamma = \rho$, *and as*

$$x(t) = \left\{\frac{2(\alpha+2)}{\alpha^2\omega_-^{\alpha\lambda-2}}\right\}^{1/\alpha}(t - \omega_-)^{-2/\alpha}\left\{1 + \sum_{n=1}^{\infty} x_n(t - \omega_-)^n\right\} \tag{7.14}$$

if $\gamma = -\rho$. *Here* x_n *are constants.*

PROOF. Reduce (7.7) to (7.4). Then from

$$w = \eta^{-3/2}\zeta\,, \quad \zeta = 1/z\,, \quad z = ty'$$

we obtain

$$C\xi^{-(\alpha+2)/\alpha+3}\left\{1 + \sum_{m+n>0} w_{mn}\xi^m(C\xi^{-(\alpha+2)/\alpha})^n\right\}ty' = 1\,.$$

On the other hand, we get

$$y = \eta^{-1} = \xi^{-2}\,, \quad y' = -2\xi^{-3}\xi'\,.$$

Hence we have

$$\xi^{-(\alpha+2)/\alpha}\left\{1+\sum_{m+n>0} w_{mn}\xi^m (C\xi^{-(\alpha+2)/\alpha})^n\right\}\xi' = -\frac{1}{2Ct}$$

and integrating both sides

$$-\frac{\alpha}{2}\xi^{-2/\alpha}\left(1+\sum_{m+n>0}\tilde{w}_{mn}\xi^{m-((\alpha+2)/\alpha)n}\right) = -\frac{1}{2C}\log t + D$$

where $\tilde{w}_{mn}$, D are constants. Here, suppose that $t \to \tau$ as $y \to \infty$. Then if we let ξ tend to 0, we obtain $y \to \infty$ and

$$D = \frac{1}{2C}\log\tau$$

from this. Therefore we get $|\tau| < \infty$ and

$$\xi^{-2/\alpha}\left(1+\sum_{m+n>0}\tilde{w}_{mn}\xi^{m-((\alpha+2)/\alpha)n}\right) = \frac{1}{\alpha C}\log\frac{t}{\tau}\,.$$

Raise both sides to the power $-\alpha/2$. Then we have

$$\xi\left\{1+\sum_{m+n>0}\hat{w}_{mn}\xi^m(\xi^{-(\alpha+2)/\alpha})^n\right\} = \left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{-\alpha/2}$$

where $\hat{w}_{mn}$ are constants. Apply Smith's lemma (cf. Lemma 1 of [11]) to this. Then we obtain

$$\xi = \left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{-\alpha/2}\left\{1+\sum_{m+n>0}a_{mn}\left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{-(\alpha/2)m}\left(\frac{1}{\alpha C}\log\frac{t}{\tau}\right)^{((\alpha+2)/2)n}\right\}.$$

Now, let an interval (ω_-, ω_+) denote the domain of ξ. Then (ω_-, ω_+) is the domain of y and of a solution $x(t)$ obtained from y and (T). If $z = ty' > 0$, then y is an increasing function of t and $\tau = \omega_+$. Hence from (T), namely

$$x(t) = \{\lambda(\lambda+1)\}^{1/\alpha}t^{-\lambda}\xi^{-2/\alpha}$$

and expanding $t^{-\lambda}$, $\log t/\omega_+$ around $t = \omega_+$, we get (7.11).

Similarly in the case $z < 0$ we have (7.12).

Moreover from (7.8) we obtain the second conclusion in the same way. Now the proof is complete. □

LEMMA 7.3. *In the case* (2.2), *every orbit* $z = z(y)$ *of* (S) *is not continuable to* $y = \infty$.

PROOF. It is obvious, since γ does not exist from Lemma 7.1. □

Finally we show the following:

LEMMA 7.4. *Every solution* $z = z(y)$ *of* (R) *does not diverge as* $y \to c$, *if* c *denotes a nonnegative finite number in the closure of the domain of* $z(y)$.

PROOF. Suppose the contrary. Then there exist a sequence $\{y_n\}$ and some constant c $(0 \le c < \infty)$ such that

$$y_n \to c\,, \quad z(y_n) \to \pm\infty \quad \text{as}\; n \to \infty\,.$$

On the other hand, putting $z = 1/\zeta$ in (R) we get

$$\frac{d\zeta}{dy} = -\frac{(\alpha-1)\zeta + \alpha(2\lambda+1)y\zeta^2 + \alpha^2\lambda(\lambda+1)y^2(y-1)\zeta^3}{\alpha y} \tag{7.15}$$

or

$$y\frac{d\zeta}{dy} = -\frac{\alpha-1}{\alpha}\zeta - (2\lambda+1)y\zeta^2 - \alpha\lambda(\lambda+1)y^2(y-1)\zeta^3\,. \tag{7.16}$$

Hence we get a contradiction $\zeta \equiv 0$ from (7.15) if $0 < c < \infty$, and from (7.16) if $c = 0$, since $-(\alpha-1)/\alpha < 0$. This completes the proof. □

8. Obtaining Theorem 3 from Γ_3

Now, suppose $-2 < \alpha < 0$, namely (2.1). In the last section, we introduced $\Gamma_3 : z = z_3(y)$. This is represented as (7.8) $(\gamma = \rho)$ in the neighborhood of $y = \infty$, and satisfies (7.9). Moreover in Figure 2 this cannot enter the region "+" as s increases. Indeed if (y, z) represents Γ_3, then as s increases y decreases and z increases in this region. Therefore (y, z) is in the region "−" and z decreases. Furthermore Γ_3 lies above Γ_2. Hence from Poincaré-Bendixon's theorem, (y, z) tends to $(0, 0)$ as $s \to \infty$. Since only Γ_1 gets (3.5), namely

$$\lim_{y\to 0}\frac{z(y)}{y} = \alpha(\lambda+1)\,,$$

it follows from Lemma 3.1 that Γ_3 satisfies (3.2), namely

$$\lim_{y\to 0}\frac{z(y)}{y} = \alpha\lambda\,.$$

In the end, Γ_3 is the unique orbit defined for $0 < y < \infty$ such that (7.9) holds, and Γ_3 satisfies (3.2), namely (2.11).

THEOREM 3. *If we take* t_0, A, B *such that* $(y_0, z_0) \in \Gamma_3$, *then* $x(t)$ *is defined for* $0 < t < \omega_+$ *where* $0 < \omega_+ < \infty$. *Moreover* $x(t)$ *is represented as* (2.12), (2.13) *in the neighborhood of* $t = 0$, *and as* (7.13) *in the neighborhood of* $t = \omega_+$.

PROOF. Take (t_0, A, B) such that $(y_0, z_0) \in \Gamma_3$. Then as stated just before the proof of Theorem 1, (y, z) moves all over Γ_3. As stated above, Γ_3 is represented as (7.8) $(\gamma = \rho)$ in the neighborhood of $y = \infty$ and from Lemma 7.2 we get a solution $x(t)$ of (E) represented as (7.13) in the neighborhood of $t = \omega_+$ $(0 < \omega_+ < \infty)$. Furthermore from the above, Γ_3 satisfies (3.2) and from Lemma 3.2, $x(t)$ is represented as (2.12), (2.13) in the neighborhood of $t = 0$. Therefore the domain of $x(t)$ is $(0, \omega_+)$. Now the proof is complete. □

9. Obtaining Theorems 4, 5 from Γ_4, Γ_5

In this section, let us treat the case (2.1). In Section 4, we showed that Γ_4 was represented as (4.10):

$$z = \frac{\nu}{\alpha}(y-1) + \sum_{n=2}^{\infty} c_n (y-1)^n .$$

This implies that Γ_4 satisfies (2.10):

$$\lim_{y\to 1} z = 0\,, \quad \lim_{y\to 1} \frac{z}{y-1} = \frac{\nu}{\alpha}\,.$$

Moreover in Section 7, we proved that Γ_5 satisfies (7.8). Now we show the following:

LEMMA 9.1. *If* $-1 < \alpha < 0$, *then* Γ_4 *satisfies*

$$\lim_{y\to\infty} z = -\infty\,, \quad \lim_{y\to\infty} y^{-3/2} z = -\infty \tag{9.1}$$

and Γ_5 *satisfies* (2.6) :

$$\lim_{y\to 1} z = 0\,, \quad \lim_{y\to 1} \frac{z}{y-1} = \frac{\mu}{\alpha}\,.$$

PROOF. If we put $\sigma = \alpha\lambda,\ \alpha(\lambda+1)$ in (5.1), then from (S) we get

$$\frac{d}{ds}(z - f(y)) = (\alpha+1)\sigma^2 y^3 (1-y) < 0 \tag{9.2}$$

on $z = f(y) = \sigma(y - y^2)$, $y > 1$. On the other hand, we have

$$f'(1) = -\sigma = -\alpha\lambda,\ \ -\alpha(\lambda+1)\,,$$

$$\frac{\nu}{\alpha} < -\alpha\lambda < -\alpha(\lambda+1) < \frac{\mu}{\alpha}\,.$$

Therefore Γ_4 lies below $z = f(y)$ and $(y, z) \in \Gamma_4$ satisfies

$$(y, z) \to (\infty, -\infty)$$

as $s \to \infty$ from Lemma 7.4. Indeed we have $dy/ds = \alpha yz > 0$ in $y > 0$, $z < 0$ from (T). However as stated in Section 7, (7.7), (7.8) are only orbits continuable to $y = \infty$. In this section, we take $\gamma = -\rho$ in (7.8). Since $(\alpha - 1)/\alpha > 2$, (7.7) lies below $z = f(y)$ in the neighborhood of $y = \infty$. Moreover (7.8) lies above $z = f(y)$ from $3/2 < 2$ and (7.9). Hence Γ_4 satisfies (7.7) and (9.1).

Here from (S) we have

$$\frac{dy}{ds} = 0\,, \quad \frac{dz}{ds} = \alpha^2\lambda(\lambda+1)y^2(y-1) > 0$$

on $y > 1$, $z = 0$. Therefore the orbit (7.8) never passes the y axis and tends to $(1, 0)$ as $s \to -\infty$. Since orbits tending to $(1, 0)$ have (2.6) or (2.10) and only Γ_4 satisfies (2.10) in $y > 1$, (7.8) admits (2.6). Namely Γ_5 gets (2.6), which completes the proof. □

Next we show the same conclusion in the other case as follows:

LEMMA 9.2. *If* $-2 < \alpha \leq -1$, *then* Γ_4 *satisfies* (9.1) *and* Γ_5, (2.6).

PROOF. First we consider (7.5) from which we obtain (7.8). So, suppose that a solution w of (7.3) is unbounded as $\xi \to \xi_*$ $(0 < |\xi_*| < 1)$. Then putting $w = 1/u$, u accumulates to 0 and from (7.3) we get

$$\frac{d\xi}{du} = \frac{\xi u}{((\alpha+2)/\alpha)u^2 - 2(2\lambda+1)\xi u - 2\alpha\lambda(\lambda+1)(1-\xi^2)}$$

which implies a contradiction $\xi \equiv 0$, since the righthand side of this is holomorphic at $(u, \xi) = (0, \xi_*)$. Hence w is bounded as $\xi \to \xi_*$. Therefore (7.5) converges in $|\xi| < 1$, since the righthand side of (7.3) is holomorphic.

Next, suppose that a nontrivial solution w of (7.3) accumulates to 0 as $\xi \to \xi_*$. Then from (7.3) we have a contradiction $w \equiv 0$. Hence w does not accumulate to 0 and if w denotes (7.5) in particular, then $1/w$ is holomorphic at $\xi = \xi_*$. Therefore z got from (7.8) is represented as

$$z = \frac{\xi^{-3}}{w} = -\rho^{-1}\xi^{-3}\left(1 + \sum_{n=1}^{\infty} b_n\xi^n\right) \quad (b_n : \text{constants}) \tag{9.3}$$

since $w = \eta^{-3/2}\zeta$, $\xi = \eta^{1/2}$, $\zeta = 1/z$. The power series of the righthand side converges in $|\xi| < 1$.

Now, regard equations of orbits of (S) as functions of ξ. Then it follows from Lemma 9.1 that if $-1 < \alpha < 0$, there exists an orbit $z = z(\xi, \alpha, \lambda)$ of (S) lying in $y > 1$, $z < 0$ and satisfying (2.6) such that

$$-\rho^{-1}\xi^{-3}\left(1 + \sum_{n=1}^{\infty} b_n\xi^n\right) = z(\xi, \alpha, \lambda) \tag{9.4}$$

in $1 - \epsilon < \xi < 1$ where ϵ is some positive constant. On the other hand, if we put $\xi = y^{-1/2}$

in (R) then we get

$$\frac{dz}{d\xi} = -\frac{2\{(\alpha-1)\xi^6 z^2 + \alpha(2\lambda+1)\xi^4 z + \alpha^2\lambda(\lambda+1)(1-\xi^2)\}}{\alpha\xi^7 z}\,.$$

The righthand side of this is holomorphic in the neighborhood of $(\xi, z) = (1, z_*)$ where z_* is a nonzero constant. Hence (9.3), $z(\xi, \alpha, \lambda)$ are holomorphic in a region $1-\epsilon < \xi < 1$, $\alpha < 0$. Therefore if we regard (9.3) and $z(\xi, \alpha, \lambda)$ as functions of α and use the monodromy theorem, then we have (9.4) even in the case $-2 < \alpha \leq -1$. That is, Γ_5 satisfies (2.6).

Consequently Γ_4 lies below Γ_5. Moreover, recall that the orbits continuable to $y = \infty$ are only (7.7) and (7.8), and only Γ_5 satisfies (7.8). Then we conclude that Γ_4 satisfies (7.7) and hence (9.1). Now the proof is complete. □

Eventually $\Gamma_4 : z = z_4(y)$ is the unique orbit defined for $1 < y < \infty$ such that (2.10) holds. In addition, this satisfies

$$\lim_{y\to\infty} z = -\infty\,, \quad \lim_{y\to\infty} y^{-3/2} z = -\infty\,. \tag{9.5}$$

THEOREM 4. *If we take* t_0, A, B *such that* $(y_0, z_0) \in \Gamma_4$, *then* $x(t)$ *is defined for* $\omega_- < t < \infty$ *where* $0 < \omega_- < \infty$ *and represented as follows:*

In the neighborhood of $t = \omega_-$, *we get* (7.12) *and in the neighborhood of* $t = \infty$, (2.14).

PROOF. Taking (t_0, A, B) such that $(y_0, z_0) \in \Gamma_4$, (y, z) moves all over Γ_4. Since Γ_4 satisfies (2.10), namely (4.1), from Lemma 4.2 we have a solution $x(t)$ of (E) represented as (2.14) in the neighborhood of $t = \infty$. Moreover from the proofs of Lemmas 9.1, 9.2, Γ_4 is represented as (7.7) and from Lemma 7.2, $x(t)$ is represented as (7.12) in the neighborhood of $t = \omega_-$ $(0 < \omega_- < \infty)$. Hence the domain of $x(t)$ is (ω_-, ∞). □

From Lemmas 9.1, 9.2, $\Gamma_5 : z = z_5(y)$ is the unique orbit defined for $1 < y < \infty$ such that

$$\lim_{y\to\infty} z = -\infty\,, \quad \lim_{y\to\infty} y^{-3/2} z = -\frac{1}{\rho}$$

and satisfies (2.6).

THEOREM 5. *If we take* t_0, A, B *such that* $(y_0, z_0) \in \Gamma_5$, *then* $x(t)$ *is defined for* $\omega_- < t < \infty$ *where* $0 < \omega_- < \infty$ *and represented as follows:*

In the neighborhood of $t = \omega_-$, *we obtain* (7.14) *and in the neighborhood of* $t = \infty$, (2.8), (2.9).

PROOF. If we take (t_0, A, B) such that $(y_0, z_0) \in \Gamma_5$, (y, z) moves all over Γ_5. From the proof of Lemmas 9.1, 9.2, Γ_5 satisfies (7.8) $(\gamma = -\rho)$ and from Lemma 7.2 we get a solution $x(t)$ of (E) represented as (7.14) in the neighborhood of $t = \omega_-$ $(0 < \omega_- < \infty)$. Furthermore from Lemmas 9.1, 9.2, Γ_5 has (2.6) namely (4.2), and from Lemma 4.2, $x(t)$ is represented as (2.8), (2.9) in the neighborhood of $t = \infty$. The domain of $x(t)$ is thus (ω_-, ∞). □

10. The other theorems and their proofs

First, suppose (2.1). Then the regions $R_1, R_2, \dots, R_5$ of Figure 1 is as follows:

R_1: the region lying below Γ_1, Γ_4,
R_2: the region which Γ_1, Γ_2 surround,
R_3: the region lying above Γ_3,
R_4: the region lying between the orbits Γ_2, Γ_5 and the orbit Γ_3,
R_5: the region lying between Γ_4 and Γ_5.

THEOREM 6. (i) *If we take t_0, A, B such that $(y_0, z_0) \in R_1 \cup R_5$, then $x(t)$ is defined for $\omega_- < t < \infty$ where $0 < \omega_- < \infty$, and represented as follows:*

In the neighborhood of $t = \omega_-$, we get (7.12), *and in the neighborhood of $t = \infty$,* (2.8), (2.9).

(ii) *If we take t_0, A, B such that $(y_0, z_0) \in R_2 \cup R_4$, then $x(t)$ is defined for $0 < t < \infty$ and represented as follows:*

In the neighborhood of $t = 0$, we get (2.12), (2.13), *and in the neighborhood of $t = \infty$,* (2.8), (2.9).

(iii) *If we take t_0, A, B such that $(y_0, z_0) \in R_3$, then $x(t)$ is defined for $0 < t < \omega_+$ where $0 < \omega_+ < \infty$ and represented as follows:*

In the neighborhood of $t = 0$, we get (2.12), (2.13), *and in the neighborhood of $t = \omega_+$,* (7.11).

PROOF. (i). Take (t_0, A, B) such that $(y_0, z_0) \in R_1 \cup R_5$. Then (y, z) draws an orbit starting from $(1, 0)$ and tending to $(\infty, -\infty)$, and satisfies (2.6) (namely (4.2)), (7.7). Hence from Lemmas 4.2, 7.2 we obtain a solution of (E) represented as (2.8), (2.9) in the neighborhood of $t = \infty$ and as (7.12) in the neighborhood of $t = \omega_-$ ($0 < \omega_- < \infty$). Hence the domain of $x(t)$ is (ω_-, ∞).

(ii). Now, take (t_0, A, B) such that $(y_0, z_0) \in R_2 \cup R_4$. Then (y, z) draws an orbit connecting $(0, 0)$ and $(1, 0)$ such that we get (2.6), (2.11) (namely (4.2), (3.2)) from the uniqueness of Γ_1, Γ_2. Therefore from Lemmas 4.2, 3.2 we have a solution of (E) represented as (2.8), (2.9) in the neighborhood of $t = \infty$ and as (2.12), (2.13) in the neighborhood of $t = 0$. Thus the domain of $x(t)$ is $(0, \infty)$.

(iii). If we take (t_0, A, B) such that $(y_0, z_0) \in R_3$, then (y, z) draws an orbit connecting (∞, ∞) and $(0, 0)$ as in Figure 1. Due to the uniqueness of Γ_1, Γ_3, (y, z) satisfies (2.11), (9.1) (namely (3.2), (7.7)) and from Lemmas 3.2, 7.2 we obtain a solution $x(t)$ of (E) represented as (2.12), (2.13) in the neighborhood of $t = 0$ and as (7.11) in the neighborhood of $t = \omega_+$ ($0 < \omega_+ < \infty$). Hence the domain of $x(t)$ is $(0, \omega_+)$. □

Finally, suppose $\alpha \leq -2$, namely (2.2). Then as introduced in Section 4, there appears the orbit $\Gamma_6 : z = z_6(y)$ which satisfies (2.10) for $y > 1$ uniquely. Owing to Lemmas 7.3, 7.4 and Figure 2, $z_6(y)$ is bounded and Γ_6 behaves as in Figure 1. Here, notice that $z_6(y)$ is a multivalued function for $1 \leq y < y_*$ where y_* is a constant. Moreover Γ_6 satisfies (2.11) from the uniqueness of Γ_1.

THEOREM 7. (i) *If we take* t_0, A, B *such that* $(y_0, z_0) \in \Gamma_6$, *then the conclusion of Theorem* 2 *follows.*

(ii) *If we take* t_0, A, B *such that* $(y_0, z_0) \notin \Gamma_1 \cup \Gamma_2 \cup \Gamma_6$, *then the conclusion of Theorem* 6 (ii) *follows.*

PROOF. (i). Take (t_0, A, B) such that $(y_0, z_0) \in \Gamma_6$. Then (y, z) draws Γ_6 itself. Therefore in the neighborhoods of $y = 0$, $y = 1$, (y, z) has the same properties as in the proof of Theorem 2 and we get the same conclusions.

(ii). If we take (t_0, A, B) such that $(y_0, z_0) \notin \Gamma_1 \cup \Gamma_2 \cup \Gamma_6$, then from the above reasoning used for Γ_6, (y, z) draws an orbit leaving $(1, 0)$ and reaching $(0, 0)$. Furthermore in the neighborhoods of $y = 0$, $y = 1$, (y, z) admits the same properties as in the proof of Theorem 6 (ii). Therefore we have the same conclusions. □

The case $(y_0, z_0) = (1, 0)$ is lacking in the above theorems. However if we take t_0, A, B such that this case arises, then we get the trivial conclusion

$$x(t) = \psi(t)$$

from (T). Therefore we have just shown the asymptotic behavior for all t_0, A, B.

If we fix t_0, A and vary B, then (y_0, z_0) draws a line parallel to the z axis from (2.4). Hence as in the previous papers we can restate the asymptotic behavior for every B.

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